Received: 16 July 2020 | Revised: 8 November 2020 | Accepted: 8 November 2020

DOI: 10.1002/rnc.5343

RESEARCH ARTICLE

WILEY

Dynamic $\mathcal{L}_2$ output feedback control of delayed linear parameter varying systems with piecewise constant parameters: A clock-dependent L–K approach

Muhammad Zakwan[1] | **Saeed Ahmed**[2] | **Naim Bajcinca**[2]

[1]Institute of Mechanical Engineering, École Polytechnique Fédérale de Lausanne, Lausanne, Switzerland

[2]Department of Mechanical and Process Engineering, University of Kaiserslautern, Kaiserslautern, Germany

Correspondence
Saeed Ahmed, Department of Mechanical and Process Engineering, University of Kaiserslautern, Gottlieb-Daimler-Straße, Kaiserslautern 67663, Germany.
Email: saeed.ahmed@mv.uni-kl.de

Abstract

This article proposes a new clock-dependent gain-scheduled dynamic output feedback controller for delayed linear parameter varying systems with piecewise constant parameters. The proposed controller guarantees $\mathcal{L}_2$-performance. By employing a clock-dependent Lyapunov–Krasovskii functional, a sufficient condition for the existence of the controller is provided in terms of clock- and parameter-dependent linear matrix inequalities. A case study on output feedback control of delayed switched systems is also provided. To illustrate the efficacy of the result, it is applied to a practical VTOL helicopter model.

KEYWORDS

clock-dependent L–K functional, delay, dwell-time, LPV systems, output feedback

1 | INTRODUCTION

Since the advent of gain-scheduled controllers in the late eighties,[1,2] researchers have been showing keen interest in developing and applying gain-scheduled control. On the other hand, linear parameter varying (LPV) systems offer the plausibility to synthesize these gain-scheduled controllers in a very systematic way.[1,2] LPV systems find their applications in many real-world processes such as automotive systems,[3,4] turbofan engines,[5] aperiodic sampled-data systems,[6] robotics,[7] and so forth. Motivated by these real-world applications, various aspects related to LPV systems have been extensively studied in the literature; see, for example, References 8-15, and the references therein. In this article, we consider a class of LPV systems with piecewise constant parameters[16] that can be considered a generalization of switched linear systems where the subsystems take values within some interval instead of some finite set. This class of LPV systems can model large-scale interconnected systems, networked control systems, and multiagent systems where switching topology is an inevitable phenomenon because the interconnection links may change over time due to various reasons.[17] For example, communicating mobile agents may lose an existing connection due to the presence of an obstacle. On the other hand, a new connection may be established between the agents when they come close to each other in an effective range of detection.[17] LPV systems with piecewise constant parameters also arise naturally in synchronous buck converters with piecewise constant loads[18] and as a simplifying assumption in the control of LPV sampled-data systems.[19] In the stochastic framework, LPV systems with piecewise constant parameters have been discussed in References 16 and 20. As the class of LPV systems with piecewise constant parameters is closely related to the class of switched systems, it seems natural to use tools developed for switched systems[21,22] to obtain sufficient stability conditions for such LPV systems by introducing dwell-time constraints.

Time delays often affect real-world applications. They can degrade the dynamical systems' performance, or in the worst case, they can cause instability.[23] Time delays frequently appear in communication networks, milling machines, aircraft, robotized teleoperation, and many other domains; see, for example, References 22,24. Time delays can also adversely affect the stability of the LPV systems.[8,25-32] This fact motivated us to consider delayed LPV systems with piecewise constant parameters. Note that LPV systems can be categorized into two main groups, that is, LPV systems with arbitrarily fast varying parameters (possibly including discontinuities), and LPV systems with continuously differentiable parameters. When the parameters vary arbitrarily fast, the quadratic stability notion based on parameter-independent Lyapunov functions is usually employed. On the other hand, when the parameters have bounded variation rates, the notion of parameter-dependent Lyapunov functions is preferred because quadratic stability, in this case, yields conservative results. Since our considered systems' parameters belong to the class of arbitrarily fast varying parameters, considering quadratic stability based on (quadratic) parameter-independent Lyapunov functionals seems a natural choice. However, by doing so, we would fail to take advantage of the fact that the parameters are constant between the jumps. Therefore, quadratic stability would lead to restrictive results if applied to LPV systems with piecewise constant parameters. This remark motivated us to consider the concept of *minimum dwell time*,[11] that is, we ensure that the jumps in parameter trajectory should not occur too often. To this aim, we employ the clock-dependent Lyapunov–Krasovskii functional. These functionals inherit a clock that measures the time elapsed since the last jump in the parameters' trajectory yielding *clock-dependent stability conditions*, which are less restrictive compared with quadratic stability conditions. However, these conditions result in infinite-dimensional semidefinite programs that are intractable. Fortunately, several techniques, such as the gridding method [8, appendix C], robust optimization module of YALMIP,[33] and sum-of-squares (SOS) polynomials,[34,35] are available to approximate semiinfinite constraint linear matrix inequality (LMI) by a finite number of LMIs. Note also that induced $\mathcal{L}_2$-gain performance is a vital performance index in control systems because it measures the disturbance attenuation ability needed in many engineering applications.[36] Motivated by this fact, we aim to design a controller that stabilizes the system and provides disturbance attenuation measured in terms of induced $\mathcal{L}_2$-norm of the system.

This article's main objective is to provide a sufficient condition for the existence of a full-order clock-dependent gain-scheduled dynamic output feedback controller for delayed LPV systems with piecewise constant parameters. Considering a time-varying delay in the dynamics of LPV systems with piecewise constant parameters renders the output feedback control problem challenging. To the best of authors' knowledge, dynamic clock-dependent output feedback $\mathcal{L}_2$ control of delayed LPV systems with piecewise constant parameters has not been studied before.

Relevant articles on the control of LPV systems with piecewise constant parameters are References 11 and 37. There are three main differences between our work and Reference 11. First, no delay is present in Reference 11. Here, we extend the results of Reference 11 to the challenging case where a time-varying delay is present in the dynamics of LPV systems with piecewise constant parameters. Second, our controller provides disturbance attenuation measured in terms of induced $\mathcal{L}_2$-norm of the system, which was not considered in Reference 11. Third, we design a dynamic output feedback controller where the states' information is not required as opposed to Reference 11, where states' information is mandatory to stabilize the system. Note that our earlier work[37] provides a state feedback controller for delayed LPV systems with piecewise constant parameters, whereas, in the present article, we study output feedback control of these systems. We wish to emphasize that the main focus of this article is the output feedback control problem, and the output feedback controller structure proposed in this article is different from the state feedback controller proposed in Reference 37. The output feedback controller design poses additional challenges to construct tractable convex design conditions than the state feedback controller design. Therefore, the output feedback control result is entirely new and is not available yet. Moreover, as opposed to Reference 37, the case study presented in this article to demonstrate how our result applies to delayed switched systems is an important result for its own sake and can be regarded as one of the contributions of this article. One additional feature of our work is that the system and the controller matrices can have a polynomial parameter-dependent representation of an arbitrary degree rather than the particular polytopic one.

This article includes several contributions. First, we solve the output feedback control problem considering a time-varying delay in the dynamics of the LPV systems with piecewise constant parameters. The study of output feedback $\mathcal{L}_2$ problem for LPV systems with piecewise constant parameters in the presence of a time-varying delay in the dynamics is new. Second, we provide a new clock-dependent synthesis condition for the existence of an output feedback controller for the considered class of systems. This condition is less conservative than the classical synthesis techniques relying on (quadratic) parameter-independent Lyapunov functionals developed for LPV systems with arbitrarily fast varying parameters. Third, we employ a recent integral based Wirtinger's inequality[38] to bound an integral term in the derivative of Lyapunov–Krasovskii functional. This inequality is less conservative as compared with Jensen's inequality.[39] To the best of our knowledge, Wirtinger's inequality has not been used before for obtaining synthesis conditions for the existence of an output feedback controller for delayed LPV systems. Finally, our proposed controller stabilizes the delayed LPV system with piecewise constant parameters and provides disturbance attenuation measured in terms of induced $\mathcal{L}_2$-norm of the system.

The article unfolds as follows. The general framework of delayed LPV systems with piecewise constant parameters and some related preliminary stability results are presented in Section 2. The main results on clock-dependent gain-scheduled dynamic $\mathcal{L}_2$ output feedback controller synthesis for our considered class of LPV systems are provided in Section 3. This section also provides a case study on output feedback stabilization of delayed switched systems. Section 4 illustrates our approach by applying it to a practical VTOL helicopter model. Finally, Section 5 concludes the article.

We employ standard notation throughout the article. The notation will be simplified whenever no confusion can arise from the context. The sets of positive and nonnegative integers are denoted by $\mathbb{N}$ and $\mathbb{Z}_{\geq 0} := \mathbb{N} \cup \{0\}$, respectively. The sets of real and nonnegative real numbers are denoted by $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$, respectively. The identity and null matrices of dimension n are denoted by I_n and $\mathcal{O}_n$, respectively. We write $M \succ 0$ (resp. $M \preccurlyeq 0$) to indicate that M is a symmetric positive definite (resp. negative semidefinite) matrix. For some square matrix A, $A + A^T$ will be denoted by Sym$[A]$. The asterisk symbol (*) denotes the complex conjugate transpose of a matrix. The set of real symmetric positive definite matrices of dimension $n \times n$ is denoted by $\mathbb{S}^n_{\succ 0}$ and the and the set of real symmetric positive semidefinite matrices of dimension $n \times n$ is denoted by $\mathbb{S}^n_{\succcurlyeq 0}$. Given any constant $h > 0$, we let $\mathscr{C}([-h, 0], \mathbb{R}^n)$ denote the set of all continuous $\mathbb{R}^n$-valued functions that are defined on $[-h, 0]$. We call the set $\mathscr{C}([-h, 0], \mathbb{R}^n)$ as the set of all *initial functions*. In addition, for any continuous function $x : [-h, \infty) \to \mathbb{R}^n$ and all $t \geq 0$, we define x_t by $x_t(\theta) := x(t + \theta)$ for all $\theta \in [-h, 0]$, that is, $x_t \in \mathscr{C}([-h, 0], \mathbb{R}^n)$ is the translation operator. The operator ∂_x denotes differentiation with respect to the variable x.

2 | SYSTEM DESCRIPTION AND PRELIMINARIES

We consider in this article delayed LPV systems with piecewise constant parameters that can be described as

$$\Sigma_p : \begin{cases} \dot{x}(t) = A(\rho)x(t) + A_d(\rho)x(t - d(t)) + B(\rho)u(t) + E(\rho)w(t) \\ z(t) = C(\rho)x(t) + C_d(\rho)x(t - d(t)) + D(\rho)u(t) + F(\rho)w(t) \\ y(t) = C_y(\rho)x(t) + C_{dy}(\rho)x(t - d(t)) + F_y(\rho)w(t) \\ x(\theta) = \phi(\theta),\ \forall\theta \in [-h, 0], \end{cases}$$

where $x \in \mathbb{R}^n$ is the system state, $w \in \mathbb{R}^{n_w}$ is the exogenous input, $u \in \mathbb{R}^{n_u}$ is the control input, $z \in \mathbb{R}^{n_z}$ is the controlled output, $y \in \mathbb{R}^{n_y}$ is the measured output, and $\phi \in \mathscr{C}([-h, 0], \mathbb{R}^n)$ is the initial condition. The time-varying delay $d(t)$ is assumed to belong to the set $\mathscr{D} := \{d : \mathbb{R}_{\geq 0} \to [0, h],\ 0 \leq \dot{d} \leq \mu < 1\}$ with $h < +\infty$. The parameter vector trajectory $\rho : \mathbb{R}_{\geq 0} \to \mathscr{P} \subset \mathbb{R}^s$, $\mathscr{P}$ compact and connected, is assumed to be piecewise constant and measurable, and that the matrix-valued functions $A(\cdot)$, $A_d(\cdot)$, $B(\cdot)$, $C(\cdot)$, $C_d(\cdot)$, $D(\cdot)$, $E(\cdot)$, $F(\cdot)$, $C_y(\cdot)$, $C_{dy}(\cdot)$, and $F_y(\cdot)$ are bounded and continuous on $\mathscr{P}$. We assume that the matrices in Σ_p are polynomial functions of ρ. We define the sequence $\{t_k\}_{k \in \mathbb{Z}_{\geq 0}}$, $t_0 = 0$, of time instants where the parameters change values. We assume that there exists a constant $T_D > 0$ such that

$$T_k := t_{k+1} - t_k \geq T_D,\ \forall k \in \mathbb{Z}_{\geq 0}, \tag{1}$$

where T_D is referred to as *minimum dwell-time*. Since we are interested in induced $\mathcal{L}_2$-gain performance for the closed-loop system. Therefore, we define the worst-case $\mathcal{L}_2$-gain[25] for system Σ_p from the exogenous input w to the desired controlled output z (with $u \equiv 0$) as

$$J = \sup_{\rho \in \mathscr{P}} \sup_{|w|_2 \neq 0} \frac{|z|_2}{|w|_2}, \tag{2}$$

where $|\cdot|_2$ denotes the Euclidean norm.

2.1 Wirtinger's inequality

We now provide the following integral based Wirtinger's inequality to be used in the proof of our preliminary stability results to bound the derivative of the Lyapunov–Krasovskii functional. The Wirtinger's inequality is proposed in Reference 38 to analyze the stability of linear time-delay systems.

Lemma 1 (38). *For a given matrix $R > 0$, the following inequality holds for all continuously differentiable function $x \in [a, b] \mapsto \mathbb{R}^n$:*

$$-(b-a)\int_a^b \dot{x}(s)^T R\dot{x}(s)ds \leq -(x(b)-x(a))^T R(x(b)-x(a)) - 3\Theta^T R\Theta,$$

where $\Theta = x(b) + x(a) - \frac{2}{b-a}\int_a^b x(s)ds$.

2.2 Preliminary stability results

This section provides dwell-time based stability results that are modified versions of the lemmas in our recent work.[37] We will use them to prove our main result. To this aim, we define the set

$$\mathscr{P}_{\geq T_D} = \left\{ \begin{matrix} \rho : \mathbb{R}_{\geq 0} \to \mathscr{P} \ : \rho(t) = \alpha_k \in \mathscr{P}, \\ t \in [t_k, t_{k+1}), t_{k+1} \geq t_k + T_D, k \in \mathbb{N}_0 \end{matrix} \right\},$$

which contains all the possible parameter trajectories.

Lemma 2. *For given constants $h \geq 0$, $\mu \in [0, 1]$, $\kappa > 0$, and $T_D > 0$, if there exist matrix-valued functions $P : [0,\ T_D] \times \mathscr{P} \mapsto \mathbb{S}^{2n}_{>0}$, $Q_1 : [0,\ T_D] \times \mathscr{P} \mapsto \mathbb{S}^n_{>0}$, $Q_2 : [0,\ T_D] \times \mathscr{P} \mapsto \mathbb{S}^n_{>0}$, $R : [0,\ T_D] \times \mathscr{P} \mapsto \mathbb{S}^n_{>0}$, $M : [0,\ T_D] \times \mathscr{P} \mapsto \mathbb{R}^n$, and $Z : [0,\ T_D] \times \mathscr{P} \mapsto \mathbb{R}^n$ such that the LMIs*

$$\Gamma(\tau,\rho) = \begin{bmatrix} \Gamma_{11}(\tau,\rho) & M(\tau,\rho)A_d(\rho) & M(\tau,\rho)E(\rho) & C^T(\rho) & \Gamma_{15}(\tau,\rho) & \Gamma_{16}(\tau,\rho) & \Gamma_{17}(\tau,\rho) \\ * & \Gamma_{22}(\tau,\rho) & 0 & C_d^T(\rho) & 0 & 0 & A_d^T(\rho)Z^T(\tau,\rho) \\ * & * & -\gamma^2 I & F^T(\rho) & 0 & 0 & E^T(\rho)Z^T(\tau,\rho) \\ * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & \Gamma_{55}(\tau,\rho) & \Gamma_{56}(\tau,\rho) & 0 \\ * & * & * & * & * & \Gamma_{66}(\tau,\rho) & P_{12}^T(\tau,\rho) \\ * & * & * & * & * & * & \Gamma_{77}(\tau,\rho) \end{bmatrix} \prec 0 \quad (3)$$

$$\Gamma(T_D^+,\rho) \prec 0 \quad (4)$$

$$P(T_D,\rho) - P(0,\eta) \succcurlyeq 0 \quad (5)$$

$$Q_1(T_D,\rho) - Q_1(0,\eta) \succcurlyeq 0 \quad (6)$$

$$Q_2(T_D,\rho) - Q_2(0,\eta) \succcurlyeq 0 \quad (7)$$

$$R(T_D,\rho) - R(0,\eta) \succcurlyeq 0 \quad (8)$$

$$\kappa Q_1(T_D,\rho) - \partial_\tau Q_1(0,\rho) \succcurlyeq 0 \quad (9)$$

$$\kappa Q_2(T_D,\rho) - \partial_\tau Q_2(0,\rho) \succcurlyeq 0 \quad (10)$$

$$\kappa R(T_D,\rho) - \partial_\tau R(0,\rho) \succcurlyeq 0 \quad (11)$$

hold for all $\tau \in [0,\ T_D]$ *and all* $\rho, \eta \in \mathscr{P}$ *, where*

$$P(\tau,\rho) = \begin{bmatrix} P_{11}(\tau,\rho) & P_{12}(\tau,\rho) \\ * & P_{22}(\tau,\rho) \end{bmatrix}$$
$$\Gamma_{11}(\tau,\rho) = \mathrm{Sym}[M(\tau,\rho)A(\rho) + P_{12}(\tau,\rho)] + \partial_\tau P_{11}(\tau,\rho) + Q_1(\tau,\rho) + Q_2(\tau,\rho) - 4e^{-\kappa h}R(\tau,\rho)$$
$$\Gamma_{15}(\tau,\rho) = -P_{12}(\tau,\rho) - 2e^{-\kappa h}R(\tau,\rho)$$
$$\Gamma_{16}(\tau,\rho) = P_{22}^T(\tau,\rho) + \partial_\tau P_{12}(\tau,\rho) + 6h^{-1}e^{-\kappa h}R(\tau,\rho)$$
$$\Gamma_{17}(\tau,\rho) = P_{11}(\tau,\rho) - M(\tau,\rho) + A^T(\rho)M^T(\tau,\rho)$$
$$\Gamma_{22}(\tau,\rho) = -(1-\mu)e^{-\kappa h}Q_1(\tau,\rho)$$
$$\Gamma_{55}(\tau,\rho) = -4e^{-\kappa h}R(\tau,\rho) - e^{-\kappa h}Q_2(\tau,\rho)$$
$$\Gamma_{56}(\tau,\rho) = -P_{22}^T(\tau,\rho) + 6h^{-1}e^{-\kappa h}R(\tau,\rho)$$
$$\Gamma_{66}(\tau,\rho) = -12h^{-2}e^{-\kappa h}R(\tau,\rho) + \partial_\tau P_{22}(\tau,\rho)$$
$$\Gamma_{77}(\tau,\rho) = -\mathrm{Sym}[Z(\tau,\rho)] + h^2R(\tau,\rho)$$

then the system Σ_p *is uniformly asymptotically stable in the absence of disturbance* w *with* $u \equiv 0$*, and* $\rho \in \mathscr{P}_{\geq T_D}$*. Moreover, the* $\mathcal{L}_2$*-gain of the map* $w \mapsto z$ *is at most* γ*, that is,* $J \leq \gamma$*.*

Remark 1. Note that the matrix-valued functions $P(\tau,\rho)$, $Q_1(\tau,\rho)$, $Q_2(\tau,\rho)$, and $R(\tau,\rho)$ have polynomial dependence on the clock τ and the parameter ρ during the holding time $t \in [t_k, t_k + T_D]$, and have only polynomial dependence on the parameter ρ for $t > t_k + T_D$.

Proof. Let us define the clock-dependent Lyapunov–Krasovskii functional

$$V(t,x_t,\rho) = V_1(t,x,\rho) + V_2(t,x_t,\rho) + V_3(t,x_t,\rho) + V_4(t,x_t,\rho) \tag{12}$$

where

$$V_1(t,x,\rho) = \tilde{x}^T P(\tau,\rho)\tilde{x}$$
$$V_2(t,x,\rho) = \int_{t-d(t)}^{t} e^{\kappa(s-t)}x^T(s)Q_1(\tau,\rho)x(s)ds$$
$$V_3(t,x,\rho) = \int_{t-h}^{t} e^{\kappa(s-t)}x^T(s)Q_2(\tau,\rho)x(s)ds$$
$$V_4(t,x,\rho) = h\int_{-h}^{0}\int_{t+\theta}^{t} e^{\kappa(s-t)}\dot{x}^T(s)R(\tau,\rho)\dot{x}(s)dsd\theta, \tag{13}$$

where $\tilde{x} = [x^T(t)\ \ \int_{t-h}^{t} x^T(s)ds]^T$, $\rho \in \mathscr{P}_{\geq T_D}$ is the parameter vector, $\tau = \min\{t - t_k, T_D\}$ is the clock , t_k is the instant where ρ changes its value with a finite jump, and T_D is the minimum dwell-time defined in (1).

Taking the time derivative of V along the trajectories of the system Σ_p and using the Leibniz integral rule give

$$\dot{V}_1(t,x,\rho) = 2\tilde{x}^T(t)\begin{bmatrix} P_{11}(\tau,\rho) & P_{12}(\tau,\rho) \\ * & P_{22}(\tau,\rho) \end{bmatrix}\begin{bmatrix} \dot{x}(t) \\ x(t) - x(t-h) \end{bmatrix} + \tilde{x}^T(t)\partial_\tau P(\tau,\rho)\tilde{x}(t)$$
$$\dot{V}_2(t,x_t,\rho) = x^T(t)Q_1(\tau,\rho)x(t) - (1-\mu)e^{-\kappa h}x^T(t-d(t))Q_1(\tau,\rho)x(t-d(t)) + \int_{t-d(t)}^{t} e^{\kappa(s-t)}x^T(s)\left(\partial_\tau Q_1(\tau,\rho) - \kappa Q_1(\tau,\rho)\right)x(s)ds$$
$$\dot{V}_3(t,x_t,\rho) = x^T(t)Q_2(\tau,\rho)x(t) - e^{-\kappa h}x^T(t-h)Q_2(\tau,\rho)x(t-h) + \int_{t-h}^{t} e^{\kappa(s-t)}x^T(s)\left(\partial_\tau Q_2(\tau,\rho) - \kappa Q_2(\tau,\rho)\right)x(s)ds$$
$$\dot{V}_4(t,x_t,\rho) = h^2\dot{x}^T(t)R(\tau,\rho)\dot{x}(t) - h\int_{t-h}^{t}\dot{x}^T(s)e^{-\kappa h}R(\tau,\rho)\dot{x}(s)ds + h\int_{-h}^{0}\int_{t+\theta}^{t} e^{\kappa(s-t)}\dot{x}^T(s)\left(\partial_\tau R(\tau,\rho) - \kappa R(\tau,\rho)\right)\dot{x}(s)dsd\theta. \tag{14}$$

Collecting all the derivative terms yields

$$
\begin{aligned}
\dot{V}(t,x_t,\rho) &= \dot{V}_1(t,x,\rho) + \dot{V}_2(t,x_t,\rho) + \dot{V}_3(t,x_t,\rho) + \dot{V}_4(t,x_t,\rho) \\
&= 2\tilde{x}^T(t)\begin{bmatrix} P_{11}(\tau,\rho) & P_{12}(\tau,\rho) \\ * & P_{22}(\tau,\rho) \end{bmatrix}\begin{bmatrix} \dot{x}(t) \\ x(t)-x(t-h) \end{bmatrix} + \tilde{x}^T(t)\partial_\tau P(\tau,\rho)\tilde{x}(t) + x^T(t)Q_1(\tau,\rho)x(t) \\
&\quad - (1-\mu)e^{-\kappa h}x^T(t-d(t))Q_1(\tau,\rho)x(t-d(t)) + \int_{t-d(t)}^{t} e^{\kappa(s-t)}x^T(s)\left(\partial_\tau Q_1(\tau,\rho) - \kappa Q_1(\tau,\rho)\right)x(s)ds \\
&\quad + x^T(t)Q_2(\tau,\rho)x(t) - e^{-\kappa h}x^T(t-h)Q_2(\tau,\rho)x(t-h) + \int_{t-h}^{t} e^{\kappa(s-t)}x^T(s)\left(\partial_\tau Q_2(\tau,\rho) - \kappa Q_2(\tau,\rho)\right)x(s)ds \\
&\quad + h^2\dot{x}^T(t)R(\tau,\rho)\dot{x}(t) - h\int_{t-h}^{t}\dot{x}^T(s)e^{-\kappa h}R(\tau,\rho)\dot{x}(s)ds + h\int_{-h}^{0}\int_{t+\theta}^{t} e^{\kappa(s-t)}\dot{x}^T(s)\left(\partial_\tau R(\tau,\rho) - \kappa R(\tau,\rho)\right)\dot{x}(s)dsd\theta.
\end{aligned}
\tag{15}
$$

Since (9)–(11) hold, the terms $h\int_{-h}^{0}\int_{t+\theta}^{t} e^{\kappa(s-t)}\dot{x}^T(s)\left(\partial_\tau R(\tau,\rho) - \kappa R(\tau,\rho)\right)\dot{x}(s)dsd\theta < 0$, $\int_{t-d(t)}^{t} e^{\kappa(s-t)}x^T(s)$ $(\partial_\tau Q_1(\tau,\rho) - \kappa Q_1(\tau,\rho))\,x(s)ds < 0$, and $\int_{t-h}^{t} e^{\kappa(s-t)}x^T(s)\left(\partial_\tau Q_2(\tau,\rho) - \kappa Q_2(\tau,\rho)\right)x(s)ds < 0$ can be dropped from inequality (15) yielding

$$
\begin{aligned}
\dot{V}(t,x_t,\rho) &\preccurlyeq 2\tilde{x}^T(t)\begin{bmatrix} P_{11}(\tau,\rho) & P_{12}(\tau,\rho) \\ * & P_{22}(\tau,\rho) \end{bmatrix}\begin{bmatrix} \dot{x}(t) \\ x(t)-x(t-h) \end{bmatrix} + \tilde{x}^T(t)\partial_\tau P(\tau,\rho)\tilde{x}(t) + x^T(t)Q_1(\tau,\rho)x(t) + x^T(t)Q_2(\tau,\rho)x(t) \\
&\quad - (1-\mu)e^{-\kappa h}x^T(t-d(t))Q_1(\tau,\rho)x(t-d(t)) - e^{-\kappa h}x^T(t-h)Q_2(\tau,\rho)x(t-h) + h^2\dot{x}^T(t)R(\tau,\rho)\dot{x}(t) \\
&\quad - h\int_{t-h}^{t}\dot{x}^T(s)e^{-\kappa h}R(\tau,\rho)\dot{x}(s)ds.
\end{aligned}
\tag{16}
$$

Employing Wirtinger's inequality from Lemma 1 with $\Theta = x(t) + x(t-h) - \frac{2}{h}\int_{t-h}^{t} x(s)ds$ to upper bound the last term in (16) yields

$$
\begin{aligned}
\dot{V}(t,x_t,\rho) &\preccurlyeq 2\tilde{x}^T(t)\begin{bmatrix} P_{11}(\tau,\rho) & P_{12}(\tau,\rho) \\ * & P_{22}(\tau,\rho) \end{bmatrix}\begin{bmatrix} \dot{x}(t) \\ x(t)-x(t-h) \end{bmatrix} + \tilde{x}^T(t)\partial_\tau P(\tau,\rho)\tilde{x}(t) + x^T(t)Q_1(\tau,\rho)x(t) + x^T(t)Q_2(\tau,\rho)x(t) \\
&\quad + h^2\dot{x}^T(t)R(\tau,\rho)\dot{x}(t) - (1-\mu)e^{-\kappa h}x^T(t-d(t))Q_1(\tau,\rho)x(t-d(t)) - e^{-\kappa h}x^T(t-h)Q_2(\tau,\rho)x(t-h) \\
&\quad - [x(t)-x(t-h)]^T e^{-\kappa h}R(\tau,\rho)[x(t)-x(t-h)] \\
&\quad - 3e^{-\kappa h}\left[x(t)+x(t-h)-\frac{2}{h}\int_{t-h}^{t}x(s)ds\right]^T R(\tau,\rho)\times\left[x(t)+x(t-h)-\frac{2}{h}\int_{t-h}^{t}x(s)ds\right].
\end{aligned}
\tag{17}
$$

To ensure the prescribed induced $\mathcal{L}_2$-norm performance level of γ, we further require

$$
\dot{V}(t,x_t,\rho) - \gamma^2 w^T(t)w(t) + z^T(t)z(t) \prec 0. \tag{18}
$$

Note that

$$
2[x^T(t)M(\tau,\rho) + \dot{x}^T(t)Z(\tau,\rho)][-\dot{x}(t) + A(\rho)x + A_d(\rho)x(t-d(t)) + E(\rho)w(t)] = 0, \tag{19}
$$

where $M(\tau,\rho)$ and $Z(\tau,\rho)$ are arbitrary matrix-valued functions with appropriate dimensions.

Combining (17)–(19), we arrive at

$$
\xi^T(t)\begin{bmatrix}
\Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} & \Phi_{15} & \Phi_{16} \\
* & \Phi_{22} & \Phi_{23} & 0 & 0 & \Phi_{26} \\
* & * & \Phi_{33} & 0 & 0 & \Phi_{36} \\
* & * & * & \Phi_{44} & \Phi_{45} & \Phi_{46} \\
* & * & * & * & \Phi_{55} & \Phi_{56} \\
* & * & * & * & * & \Phi_{66}
\end{bmatrix}\xi(t) \prec 0, \tag{20}
$$

where

$$\xi(t) = \left[x^T(t)\ \ x^T(t-d(t))\ \ w^T(t)\ \ x^T(t-h)\ \ \int_{t-h}^{t} x^T(s)ds\ \ \dot{x}^T(t)\right]^T$$
$$\Phi_{11} = \mathrm{Sym}[M(\tau,\rho)A(\rho) + P_{12}(\tau,\rho)] + \partial_\tau P_{11}(\tau,\rho) + Q_1(\tau,\rho) + Q_2(\tau,\rho) - 4e^{-\kappa h}R(\tau,\rho) + C^T(\rho)C(\rho)$$
$$\Phi_{12} = M(\tau,\rho)A_d(\rho) + C^T(\rho)C_d(\rho)$$
$$\Phi_{13} = M(\tau,\rho)E(\rho) + C^T(\rho)D(\rho)$$
$$\Phi_{14} = -P_{12}(\tau,\rho) - 2e^{-\kappa h}R(\tau,\rho)$$
$$\Phi_{15} = P_{22}^T(\tau,\rho) + \partial_\tau P_{12}(\tau,\rho) + 6h^{-1}e^{-\kappa h}R(\tau,\rho)$$
$$\Phi_{16} = P_{11}(\tau,\rho) - M(\tau,\rho) + A^T(\rho)M^T(\tau,\rho)$$
$$\Phi_{22} = -(1-\mu)e^{-\kappa h}Q_1(\tau,\rho) + C_d^T(\rho)C_d(\rho)$$
$$\Phi_{23} = C_d^T(\rho)D(\rho), \Phi_{26} = A_d^T(\rho)Z^T(\tau,\rho)$$
$$\Phi_{33} = -\gamma^2 I + F^T(\rho)F(\rho), \Phi_{36} = E^T(\rho)Z^T(\tau,\rho)$$
$$\Phi_{44} = -4e^{-\kappa h}R(\tau,\rho) - e^{-\kappa h}Q_2(\tau,\rho)$$
$$\Phi_{45} = -P_{22}^T(\tau,\rho) + 6h^{-1}e^{-\kappa h}R(\tau,\rho)$$
$$\Phi_{55} = -12h^{-2}e^{-\kappa h}R(\tau,\rho) + \partial_\tau P_{22}(\tau,\rho), \Phi_{56} = P_{12}^T(\tau,\rho)$$
$$\Phi_{66} = -\mathrm{Sym}[Z(\tau,\rho)] + h^2R(\tau,\rho).$$

Taking Schur complement of (20) yields the LMI (3). This condition ensures that Lyapunov–Krasovskii functional is decreasing between two consecutive jumps of the parameter vector ρ.

We next analyze the change in Lypaunov–Krasovskii functional at the jumping instant t_k of the parameters' trajectory which is given as

$$\begin{aligned} V(t_k^-, x_t, \rho) - V(t_k^+, x_t, \eta) &= \tilde{x}^T(t)[P(T_D,\rho) - P(0,\eta)]\tilde{x}(t) + \int_{t_k-d(t_k)}^{t_k} e^{-\kappa(s-t_k)}x^T(s)[Q_1(T_D,\rho) - Q_1(0,\eta)]x(s)ds \\ &+ \int_{t_k-h}^{t_k} e^{-\kappa(s-t_k)}x^T(s)[Q_2(T_D,\rho) - Q_2(0,\eta)]x(s)ds \\ &+ h\int_{-h}^{0}\int_{t_k+\theta}^{t_k} e^{-\kappa(s-t_k)}\dot{x}^T(s)[R(T_D,\rho) - R(0,\eta)]\dot{x}(s)dsd\theta, \end{aligned}$$

where t_k^+ and t_k^- are the right-hand limit and left-hand limit of t_k, respectively. Since (5)–(8) hold, the Lyapunov–Krasovskii functional cannot increase at the time instant t_k as $V(t_k^-, x_t, \rho) - V(t_k^+, x_t, \eta) \geqslant 0$. Therefore, Σ_p is uniformly asymptotically stable. This concludes the proof. ■

Remark 2. The parameter- and clock-dependent conditions derived in Lemma 2 ensure that $V(t, x_t, \rho)$ is decreasing between the consecutive parameter jumps and the jumps are nonincreasing at the switching instants when they occur after the dwell time. Hence, uniform asymptotic stability holds.

The LMI characterization in Lemma 2 involves multiple product terms such as $M(\tau,\rho)A(\rho)$ and $Z(\tau,\rho)A(\rho)$, which make it difficult to employ linearizing change of variables to derive the synthesis conditions. Therefore, we use a slack variable approach[31] to derive a relaxed stability condition. The following lemma presents this condition, and it will be used to derive the synthesis conditions in the following section.

Lemma 3. *For given constants $h \geq 0$, $\mu \in [0,1]$, $\kappa > 0$, and $T_D > 0$, if there exist matrix-valued functions $P : [0,\ T_D] \times \mathscr{P} \mapsto \mathbb{S}^{2n}_{>0}$, $Q_1 : [0,\ T_D] \times \mathscr{P} \mapsto \mathbb{S}^n_{>0}$, $Q_2 : [0,\ T_D] \times \mathscr{P} \mapsto \mathbb{S}^n_{>0}$, $R : [0,\ T_D] \times \mathscr{P} \mapsto \mathbb{S}^n_{>0}$, $M : [0,\ T_D] \times \mathscr{P} \mapsto \mathbb{R}^n$, $Z : [0,\ T_D] \times \mathscr{P} \mapsto \mathbb{R}^n$, and $L : \mathscr{P} \mapsto \mathbb{R}^n$ such that the LMIs*

$$\Lambda(\tau,\rho) := \begin{bmatrix} -\mathrm{Sym}[L(\rho)] & \Lambda_{12}(\tau,\rho) & L^T(\rho)A_d(\rho) & L^T(\rho)E(\rho) & 0 & 0 & 0 & L^T(\rho) & Z(\tau,\rho) \\ * & \Lambda_{22}(\tau,\rho) & 0 & 0 & C^T(\rho) & \Lambda_{26}(\tau,\rho) & \Lambda_{27}(\tau,\rho) & 0 & \Lambda_{29}(\tau,\rho) \\ * & * & \Lambda_{33}(\tau,\rho) & 0 & C_d^T(\rho) & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma^2 I & F^T(\rho) & 0 & 0 & 0 & 0 \\ * & * & * & * & -I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Lambda_{66}(\tau,\rho) & \Lambda_{67}(\tau,\rho) & 0 & 0 \\ * & * & * & * & * & * & \Lambda_{77}(\tau,\rho) & 0 & P_{12}^T(\tau,\rho) \\ * & * & * & * & * & * & * & -P_{11}(\tau,\rho) & Z(\tau,\rho) \\ * & * & * & * & * & * & * & * & \Lambda_{99}(\tau,\rho) \end{bmatrix} \prec 0 \tag{21}$$

$$\Lambda(T_D^+,\rho) \prec 0 \tag{22}$$

$$P(T_D,\rho) - P(0,\eta) \succcurlyeq 0 \tag{23}$$

$$Q_1(T_D,\rho) - Q_1(0,\eta) \succcurlyeq 0 \tag{24}$$

$$Q_2(T_D,\rho) - Q_2(0,\eta) \succcurlyeq 0 \tag{25}$$

$$R(T_D,\rho) - R(0,\eta) \succcurlyeq 0 \tag{26}$$

$$\kappa Q_1(T_D,\rho) - \partial_\tau Q_1(0,\rho) \succcurlyeq 0 \tag{27}$$

$$\kappa Q_2(T_D,\rho) - \partial_\tau Q_2(0,\rho) \succcurlyeq 0 \tag{28}$$

$$\kappa R(T_D,\rho) - \partial_\tau R(0,\rho) \succcurlyeq 0 \tag{29}$$

for all $\tau \in [0, T_D]$ and all $\rho, \eta \in \mathscr{P}$, where

$$P(\tau,\rho) = \begin{bmatrix} P_{11}(\tau,\rho) & P_{12}(\tau,\rho) \\ * & P_{22}(\tau,\rho) \end{bmatrix}$$
$$\Lambda_{12}(\tau,\rho) = M(\tau,\rho) + L^T(\rho)A(\rho)$$
$$\Lambda_{22}(\tau,\rho) = \mathrm{Sym}[P_{12}(\tau,\rho)] - P_{11}(\tau,\rho) + \partial_\tau P_{11}(\tau,\rho) + Q_1(\tau,\rho) + Q_2(\tau,\rho) - 4e^{-\kappa h}R(\tau,\rho)$$
$$\Lambda_{26}(\tau,\rho) = -P_{12}(\tau,\rho) - 2e^{-\kappa h}R(\tau,\rho)$$
$$\Lambda_{27}(\tau,\rho) = P_{22}^T(\tau,\rho) + \partial_\tau P_{12}(\tau,\rho) + 6h^{-1}e^{-\kappa h}R(\tau,\rho)$$
$$\Lambda_{29}(\tau,\rho) = P_{11}(\tau,\rho) - M(\tau,\rho)$$
$$\Lambda_{33}(\tau,\rho) = -(1-\mu)Q_1(\tau,\rho)e^{-\kappa h}$$
$$\Lambda_{66}(\tau,\rho) = -4e^{-\kappa h}R(\tau,\rho) - Q_2(\tau,\rho)e^{-\kappa h}$$
$$\Lambda_{67}(\tau,\rho) = -P_{22}^T(\tau,\rho) + 6h^{-1}e^{-\kappa h}R(\tau,\rho)$$
$$\Lambda_{77}(\tau,\rho) = -12h^{-2}e^{-\kappa h}R(\tau,\rho) + \partial_\tau P_{22}(\tau,\rho)$$
$$\Lambda_{99}(\tau,\rho) = -\mathrm{Sym}[Z(\tau,\rho)] + h^2 R(\tau,\rho)$$

then the system Σ_p is uniformly asymptotically stable in the absence of disturbance w with $u \equiv 0$, and $\rho \in \mathscr{P}_{\geq T_D}$. Moreover, the $\mathcal{L}_2$-gain of the map $w \mapsto z$ is at most γ, that is, $J \leq \gamma$.

Proof. We first prove that the feasibility of (21) guarantees the feasibility of (3). To this end, we decompose $\Lambda(\tau,\rho)$ in (21) as follows:

$$\Lambda(\tau,\rho)=\Lambda(\tau,\rho)|_{L(\rho)=0}+\mathscr{U}^T(\rho)L(\rho)\mathscr{V}(\rho)+\mathscr{V}^T(\rho)L^T(\rho)\mathscr{U}(\rho),$$

where $\mathscr{U}(\rho)=[-I_n\ A(\rho)\ A_d(\rho)\ E(\rho)\ 0\ 0\ 0\ I_n\ 0]$ and $\mathscr{V}(\rho)=[I_n\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ ]$. Then, invoking the projection lemma,[40] the feasibility of $\Lambda(\tau,\rho)\prec 0$ implies the feasibility of the LMIs

$$\mathscr{N}_{\mathscr{U}}^T(\rho)\Lambda(\tau,\rho)|_{L(\rho)=0}\mathscr{N}_{\mathscr{U}}(\rho)\prec 0 \tag{30a}$$

$$\mathscr{N}_{\mathscr{V}}^T(\rho)\Lambda(\tau,\rho)|_{L(\rho)=0}\mathscr{N}_{\mathscr{V}}(\rho)\prec 0, \tag{30b}$$

where $\mathscr{N}_{\mathscr{U}}(\rho)$ and $\mathscr{N}_{\mathscr{V}}(\rho)$ are basis of the null space of $\mathscr{U}(\rho)$ and $\mathscr{V}(\rho)$. The inequality (30a) yields (3) and the inequality (30b) yields $-\text{Sym}[Z(\tau,\rho)]+h^2R(\tau,\rho)\prec 0$ for all $\rho\in\mathscr{P}_{\geq T_D}$ and all $\tau\in[0,T_D]$. Note that this inequality is a relaxed form of the right bottom 1×1 block of the inequality (3) and is always satisfied . Hence, the feasibility of (21) implies the feasibility of (3). This concludes the proof. ■

3 | MAIN RESULT

In this section, we aim at obtaining synthesis conditions for the clock-dependent gain-scheduled dynamic output feedback controller Σ_c of the form:

$$\Sigma_c:\begin{cases}\dot{x}_c(t)=\mathcal{A}(t-t_k,\rho(t_k))x_c(t)+\mathcal{A}_d(t-t_k,\rho(t_k))x_c(t-d(t))+\mathcal{B}(t-t_k,\rho(t_k))y(t) & \text{for } t\in[t_k,t_k+T_D)\\ u(t)=\mathcal{C}(t-t_k,\rho(t_k))x_c(t)+\mathcal{C}_d(t-t_k,\rho(t_k))x_c(t-d(t))+\mathcal{D}(t-t_k,\rho(t_k))y(t) & \\ \\ \dot{x}_c(t)=\mathcal{A}(T_D,\rho(t_k))x_c(t)+\mathcal{A}_d(T_D,\rho(t_k))x_c(t-d(t))+\mathcal{B}(T_D,\rho(t_k))y(t) & \text{for } t\in[t_k+T_D,t_{k+1})\\ u(t)=\mathcal{C}(T_D,\rho(t_k))x_c(t)+\mathcal{C}_d(T_D,\rho(t_k))x_c(t-d(t))+\mathcal{D}(T_D,\rho(t_k))y(t) & \end{cases}$$

where $x_k(\theta_k)=\phi(\theta_k)$ for all $\theta_k\in[-h,0]$, $\rho\in\mathscr{P}_{\geq T_D}$, $\tau\in[0,T_D]$, $\tau=\min\{t-t_k,T_D\}$, $\mathcal{A}:[0,T_D]\times\mathscr{P}\mapsto\mathbb{R}^{n\times n}$, $\mathcal{A}_d:[0,T_D]\times\mathscr{P}\mapsto\mathbb{R}^{n\times n}$, $\mathcal{B}:[0,T_D]\times\mathscr{P}\mapsto\mathbb{R}^{n\times n_y}$, $\mathcal{C}:[0,T_D]\times\mathscr{P}\mapsto\mathbb{R}^{n_u\times n}$, $\mathcal{C}_d:[0,T_D]\times\mathscr{P}\mapsto\mathbb{R}^{n_u\times n}$, and $\mathcal{D}:[0,T_D]\times\mathscr{P}\mapsto\mathbb{R}^{n_u\times n_y}$ are clock- and parameter-dependent matrix-valued functions. The controller matrices $\mathcal{A}$, $\mathcal{A}_d$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{C}_d$, and $\mathcal{D}$ are chosen to be clock- and parameter-dependent during the holding time $t\in[t_k,t_k+T_D)$. and are chosen to be only parameter dependent for $t\in[t_k+T_D,t_{k+1})$. The motivation to employ this specific structure of the controller Σ_c is twofold. First, this parameterization of the controller ensures bounded matrix valued functions appearing in Σ_c. Second, this type of controller structure is compatible with the clock-dependent stability conditions resulting in nonrestrictive synthesis conditions.

We augment the controller Σ_c with the plant Σ_p to form the closed-loop system

$$\Sigma_{cl}:\begin{cases}\dot{x}_{cl}(t)=\mathcal{A}_{cl}(\tau,\rho)x_{cl}(t)+\mathcal{A}_{dcl}(\tau,\rho)x_{cl}(t-d(t))+\mathcal{E}_{cl}(\tau,\rho)w(t)\\ z(t)=\mathcal{C}_{cl}(\tau,\rho)x_{cl}(t)+\mathcal{C}_{dcl}(\tau,\rho)x_{cl}(t-d(t))+\mathcal{F}_{cl}(\tau,\rho)w(t),\end{cases}$$

where $x_{cl}(t)^T=[x(t)^Tx_c(t)^T]^T$ is the augmented closed-loop state with $x_{cl}\in\mathbb{R}^{2n}$. The matrices of the closed-loop system Σ_{cl} for all $\rho\in\mathscr{P}_{\geq T_D}$ and all $\tau\in[0,T_D]$, are parameterized as

$$\mathcal{A}_{cl}(\tau,\rho)=\begin{bmatrix}A(\rho)+B(\rho)\mathcal{D}(\tau,\rho)C_y(\rho) & B(\rho)\mathcal{C}(\tau,\rho)\\ \mathcal{B}(\tau,\rho)C_y(\rho) & \mathcal{A}(\tau,\rho)\end{bmatrix}$$

$$\mathcal{A}_{dcl}(\tau,\rho)=\begin{bmatrix}A_d(\rho)+B(\rho)\mathcal{D}(\tau,\rho)C_{dy}(\rho) & B(\rho)\mathcal{C}_d(\tau,\rho)\\ \mathcal{B}(\tau,\rho)C_{dy}(\rho) & \mathcal{A}_d(\tau,\rho)\end{bmatrix}$$

$$\mathcal{E}_{cl}(\tau,\rho) = \begin{bmatrix} E(\rho) + B(\rho)\mathcal{D}(\tau,\rho)F_y(\rho) \\ \mathcal{B}(\tau,\rho)F_y(\rho) \end{bmatrix}$$
$$\mathcal{C}_{cl}(\tau,\rho) = \begin{bmatrix} C(\rho) + D(\rho)\mathcal{D}(\tau,\rho)C_y(\rho) & D(\rho)\mathcal{C}(\tau,\rho) \end{bmatrix}$$
$$\mathcal{C}_{dcl}(\tau,\rho) = \begin{bmatrix} C_d(\rho) + D(\rho)\mathcal{D}(\tau,\rho)C_{dy}(\rho) & D(\rho)\mathcal{C}_d(\tau,\rho) \end{bmatrix}$$
$$\mathcal{F}_{cl}(\tau,\rho) = \begin{bmatrix} F(\rho) + D(\rho)\mathcal{D}(\tau,\rho)F_y(\rho) \end{bmatrix}.$$

Our objective is to find sufficient synthesis conditions for clock-dependent gain-scheduled dynamic output feedback controller such that the closed-loop system Σ_{cl} is asymptotically stable in the absence of disturbance w and that the map $w \mapsto z$ has a guaranteed $\mathcal{L}_2$-gain of at most γ, that is, $J \le \gamma$.

Now we are ready to state and prove the main result of this article.

Theorem 1. *For given constants $h \geq 0$, $0 \le \mu < 1$, $\kappa > 0$, and $T_D > 0$, if there exist matrix-valued functions $\tilde{\mathcal{P}} : [0, T_D] \times \mathscr{P} \mapsto \mathbb{S}^{4n}_{>0}$, $\tilde{\mathcal{Q}}_1 : [0, T_D] \times \mathscr{P} \mapsto \mathbb{S}^{2n}_{>0}$, $\tilde{\mathcal{Q}}_2 : [0, T_D] \times \mathscr{P} \mapsto \mathbb{S}^{2n}_{>0}$, $\tilde{\mathcal{R}} : [0, T_D] \times \mathscr{P} \mapsto \mathbb{S}^{2n}_{>0}$, $\tilde{\mathcal{M}} : [0, T_D] \times \mathscr{P} \mapsto \mathbb{R}^{2n}$, $\tilde{\mathcal{Z}} : [0, T_D] \times \mathscr{P} \mapsto \mathbb{R}^{2n}$, $X : \mathscr{P} \mapsto \mathbb{R}^n$, and $W : \mathscr{P} \mapsto \mathbb{R}^n$ such that the LMIs*

$$\tilde{\Lambda}(\tau,\rho) := \begin{bmatrix} -\mathrm{Sym}[\tilde{X}(\rho)] & \tilde{\Lambda}_{12}(\tau,\rho) & \tilde{\Lambda}_{13}(\tau,\rho) & \tilde{\Lambda}_{14}(\tau,\rho) & 0 & 0 & 0 & \tilde{X}^T(\rho) & \tilde{\mathcal{Z}}(\tau,\rho) \\ * & \tilde{\Lambda}_{22}(\tau,\rho) & 0 & 0 & \tilde{\Lambda}_{25}(\tau,\rho) & \tilde{\Lambda}_{26}(\tau,\rho) & \tilde{\Lambda}_{27}(\tau,\rho) & 0 & \tilde{\Lambda}_{29}(\tau,\rho) \\ * & * & \tilde{\Lambda}_{33}(\tau,\rho) & 0 & \tilde{\Lambda}_{35}(\tau,\rho) & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma^2 I & \tilde{\Lambda}_{45}(\tau,\rho) & 0 & 0 & 0 & 0 \\ * & * & * & * & -I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \tilde{\Lambda}_{66}(\tau,\rho) & \tilde{\Lambda}_{67}(\tau,\rho) & 0 & 0 \\ * & * & * & * & * & * & \tilde{\Lambda}_{77}(\tau,\rho) & 0 & \tilde{\mathcal{P}}_{12}^T(\tau,\rho) \\ * & * & * & * & * & * & * & -\tilde{\mathcal{P}}_{11}(\tau,\rho) & \tilde{\mathcal{Z}}(\tau,\rho) \\ * & * & * & * & * & * & * & * & \tilde{\Lambda}_{99}(\tau,\rho) \end{bmatrix} \prec 0 \tag{31}$$

$$\tilde{\Lambda}(T_D^+,\rho) \prec 0 \tag{32}$$

$$\tilde{\mathcal{P}}(T_D,\rho) - \tilde{\mathcal{P}}(0,\eta) \succcurlyeq 0 \tag{33}$$

$$\tilde{\mathcal{Q}}_1(T_D,\rho) - \tilde{\mathcal{Q}}_1(0,\eta) \succcurlyeq 0 \tag{34}$$

$$\tilde{\mathcal{Q}}_2(T_D,\rho) - \tilde{\mathcal{Q}}_2(0,\eta) \succcurlyeq 0 \tag{35}$$

$$\tilde{\mathcal{R}}(T_D,\rho) - \tilde{\mathcal{R}}(0,\eta) \succcurlyeq 0 \tag{36}$$

$$\kappa\tilde{\mathcal{Q}}_1(T_D,\rho) - \partial_\tau\tilde{\mathcal{Q}}_1(0,\rho) \succcurlyeq 0 \tag{37}$$

$$\kappa\tilde{\mathcal{Q}}_2(T_D,\rho) - \partial_\tau\tilde{\mathcal{Q}}_2(0,\rho) \succcurlyeq 0 \tag{38}$$

$$\kappa\tilde{\mathcal{R}}(T_D,\rho) - \partial_\tau\tilde{\mathcal{R}}(0,\rho) \succcurlyeq 0 \tag{39}$$

$$\tilde{X}(\rho) = \begin{bmatrix} W(\rho) & I_n \\ I_n & X(\rho) \end{bmatrix} \succ 0 \tag{40}$$

hold for all $\tau \in [0, T_D]$ *and all* $\rho, \eta \in \mathscr{P}$ *, where*

$$\tilde{\mathcal{P}}(\tau,\rho) = \begin{bmatrix} \tilde{\mathcal{P}}_{11}(\tau,\rho) & \tilde{\mathcal{P}}_{12}(\tau,\rho) \\ * & \tilde{\mathcal{P}}_{22}(\tau,\rho) \end{bmatrix}$$

$$\tilde{\Lambda}_{12}(\tau,\rho) = \begin{bmatrix} A(\rho)W(\rho) + B(\rho)\hat{C}(\tau,\rho) & A(\rho) + B(\rho)\hat{\mathcal{D}}(\tau,\rho)C_y(\rho) \\ \hat{\mathcal{A}}(\tau,\rho) & X(\rho)A(\rho) + \hat{\mathcal{B}}(\tau,\rho)C_y(\rho) \end{bmatrix} + \tilde{\mathcal{M}}(\tau,\rho)$$

$$\tilde{\Lambda}_{13}(\tau,\rho) = \begin{bmatrix} A_d(\rho)W(\rho) + B(\rho)\hat{C}(\tau,\rho) & A(\rho) + B(\rho)\hat{\mathcal{D}}(\tau,\rho)C_{dy}(\rho) \\ \hat{\mathcal{A}}_d(\tau,\rho) & X(\rho)A_d(\rho) + \hat{\mathcal{B}}(\tau,\rho)C_{dy}(\rho) \end{bmatrix}$$

$$\tilde{\Lambda}_{14}(\tau,\rho) = \begin{bmatrix} E(\rho) + B(\rho)\hat{\mathcal{D}}(\tau,\rho)F_y(\rho) \\ X(\rho)E(\rho) + \hat{\mathcal{B}}(\tau,\rho)F_y(\rho) \end{bmatrix}$$

$$\tilde{\Lambda}_{22}(\tau,\rho) = \mathrm{Sym}[\tilde{\mathcal{P}}_{12}(\tau,\rho)] - \tilde{\mathcal{P}}_{11}(\tau,\rho) + \partial_\tau \tilde{\mathcal{P}}_{11}(\tau,\rho) + \tilde{\mathcal{Q}}_1(\tau,\rho) + \tilde{\mathcal{Q}}_2(\tau,\rho) - 4e^{-\kappa h}\tilde{\mathcal{R}}(\tau,\rho)$$

$$\tilde{\Lambda}_{25}(\tau,\rho) = \begin{bmatrix} C(\rho)W(\rho) + D(\rho)\hat{C}(\tau,\rho) & C(\rho) + D(\rho)\hat{\mathcal{D}}(\tau,\rho)C_y(\rho) \end{bmatrix}^T$$

$$\tilde{\Lambda}_{26}(\tau,\rho) = -\tilde{\mathcal{P}}_{12}(\tau,\rho) - 2e^{-\kappa h}\tilde{\mathcal{R}}(\tau,\rho)$$

$$\tilde{\Lambda}_{27}(\tau,\rho) = \tilde{\mathcal{P}}_{22}^T(\tau,\rho) + \partial_\tau \tilde{\mathcal{P}}_{12}(\tau,\rho) + 6h^{-1}e^{-\kappa h}\tilde{\mathcal{R}}(\tau,\rho)$$

$$\tilde{\Lambda}_{29}(\tau,\rho) = \tilde{\mathcal{P}}_{11}(\tau,\rho) - \tilde{\mathcal{M}}(\tau,\rho)$$

$$\tilde{\Lambda}_{33}(\tau,\rho) = -(1-\mu)\tilde{\mathcal{Q}}_1(\tau,\rho)e^{-\kappa h}$$

$$\tilde{\Lambda}_{35}(\tau,\rho) = \begin{bmatrix} C_d(\rho)W(\rho) + D(\rho)\hat{C}_d(\tau,\rho) & C_d(\rho) + D(\rho)\hat{\mathcal{D}}(\tau,\rho)C_{dy}(\rho) \end{bmatrix}^T$$

$$\tilde{\Lambda}_{45}(\tau,\rho) = \begin{bmatrix} F(\rho) + D(\rho)\hat{\mathcal{D}}(\tau,\rho)F_y(\rho) \end{bmatrix}^T$$

$$\tilde{\Lambda}_{66}(\tau,\rho) = -4e^{-\kappa h}\tilde{\mathcal{R}}(\tau,\rho) - \tilde{\mathcal{Q}}_2(\tau,\rho)e^{-\kappa h}$$

$$\tilde{\Lambda}_{67}(\tau,\rho) = -\tilde{\mathcal{P}}_{22}^T(\tau,\rho) + 6h^{-1}e^{-\kappa h}\tilde{\mathcal{R}}(\tau,\rho)$$

$$\tilde{\Lambda}_{77}(\tau,\rho) = -12h^{-2}e^{-\kappa h}\tilde{\mathcal{R}}(\tau,\rho) + \partial_\tau \tilde{\mathcal{P}}_{22}(\tau,\rho)$$

$$\tilde{\Lambda}_{99}(\tau,\rho) = -\mathrm{Sym}[\tilde{\mathcal{Z}}(\tau,\rho)] + h^2\tilde{\mathcal{R}}(\tau,\rho),$$

then the closed-loop system Σ_{cl} *with* $\rho \in \mathscr{P}_{\geq T_D}$ *is uniformly asymptotically stable in the absence of disturbance* w *and the* $\mathcal{L}_2$*-gain of the map* $w \mapsto z$ *is at most* γ*, that is,* $J \leq \gamma$.

Moreover, once the unknown matrix-valued functions satisfying the LMIs conditions are obtained, the clock-dependent output feedback control matrices can be computed by first obtaining two matrices $T : \mathscr{P} \mapsto \mathbb{R}^{n\times n}$ *and* $S : \mathscr{P} \mapsto \mathbb{R}^{n\times n}$ *from the factorization problem*

$$I_n - X(\rho)W(\rho) = T(\rho)S^T(\rho), \tag{41}$$

and then by reversing the following transformation

$$\begin{aligned}
\hat{\mathcal{A}}(\tau,\rho) &= X(\rho)A(\rho)W(\rho) + X(\rho)B(\rho)\mathcal{D}(\tau,\rho)C_y(\rho)W(\rho) + T(\rho)\mathcal{B}(\tau,\rho)C_y(\rho)W(\rho) \\
&\quad + X(\rho)B(\rho)\mathcal{C}(\tau,\rho)S^T(\rho) + T(\rho)\mathcal{A}(\tau,\rho)S^T(\rho) \\
\hat{\mathcal{A}}_d(\tau,\rho) &= X(\rho)A_d(\rho)W(\rho) + X(\rho)B(\rho)\mathcal{D}(\tau,\rho)C_{dy}(\rho)W(\rho) + T(\rho)\mathcal{B}(\tau,\rho)C_{dy}(\rho)W(\rho) \\
&\quad + X(\rho)B(\rho)\mathcal{C}_d(\tau,\rho)S^T(\rho) + T(\rho)\mathcal{A}_d(\tau,\rho)S^T(\rho) \\
\hat{\mathcal{B}}(\tau,\rho) &= X(\rho)B(\rho)\mathcal{D}(\tau,\rho) + T(\rho)\mathcal{B}(\tau,\rho) \\
\hat{C}(\tau,\rho) &= \mathcal{D}(\tau,\rho)C_y(\rho)W(\rho) + \mathcal{C}(\tau,\rho)S^T(\rho) \\
\hat{C}_d(\tau,\rho) &= \mathcal{D}(\tau,\rho)C_{dy}(\rho)W(\rho) + \mathcal{C}_d(\tau,\rho)S^T(\rho) \\
\hat{\mathcal{D}}(\tau,\rho) &= \mathcal{D}(\tau,\rho).
\end{aligned}$$

Proof. First, we rewrite the closed-loop system Σ_{cl} as

$$\begin{bmatrix} \mathcal{A}_{cl}(\tau,\rho) & \mathcal{A}_{dcl}(\tau,\rho) & \mathcal{E}_{cl}(\tau,\rho) \\ \mathcal{C}_{cl}(\tau,\rho) & \mathcal{C}_{dcl}(\tau,\rho) & \mathcal{F}_{cl}(\tau,\rho) \end{bmatrix} = \Pi_1(\rho) + \begin{bmatrix} 0 & B(\rho) \\ I_n & 0 \\ 0 & D(\rho) \end{bmatrix} \mathcal{K}(\tau,\rho)\Pi_2(\rho), \tag{42}$$

where

$$\Pi_1(\rho) := \left[\begin{array}{cc|cc|c} A(\rho) & 0 & A_d(\rho) & 0 & E(\rho) \\ 0 & 0 & 0 & 0 & 0 \\ \hline C(\rho) & 0 & C_d(\rho) & 0 & F(\rho) \end{array}\right], \ \mathcal{K}(\tau,\rho) := \begin{bmatrix} \mathcal{A}(\tau,\rho) & \mathcal{A}_d(\tau,\rho) & \mathcal{B}(\tau,\rho) \\ \mathcal{C}(\tau,\rho) & \mathcal{C}_d(\tau,\rho) & \mathcal{D}(\tau,\rho) \end{bmatrix},$$

$$\Pi_2(\rho) := \left[\begin{array}{cc|cc|c} 0 & I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n & 0 \\ C_y(\rho) & 0 & C_{dy}(\rho) & 0 & F_y(\rho) \end{array}\right].$$

Restricting the matrix $L : \mathscr{P} \mapsto \mathbb{S}^{2n}_{>0}$ in Lemma 3 its inverse as

$$L(\rho) = \begin{bmatrix} X(\rho) & T(\rho) \\ T^T(\rho) & \star \end{bmatrix}, \ L^{-1}(\rho) = \begin{bmatrix} W(\rho) & S(\rho) \\ S^T(\rho) & \star \end{bmatrix},$$

and substituting the closed-loop matrices (42) in the LMI (21) yields

$$\begin{bmatrix}
-\mathrm{Sym}[L(\rho)] & \Upsilon_{12}(\tau,\rho) & L^T(\rho)\mathcal{A}_{dcl}(\tau,\rho) & L^T(\rho)\mathcal{E}_{cl}(\tau,\rho) & 0 & 0 & 0 & L^T(\rho) & \mathcal{Z}(\tau,\rho) \\
* & \Upsilon_{22}(\tau,\rho) & 0 & 0 & \mathcal{C}_{cl}^T(\tau,\rho) & \Upsilon_{26}(\tau,\rho) & \Upsilon_{27}(\tau,\rho) & 0 & \Upsilon_{29}(\tau,\rho) \\
* & * & \Upsilon_{33}(\tau,\rho) & 0 & \mathcal{C}_{dcl}^T(\tau,\rho) & 0 & 0 & 0 & 0 \\
* & * & * & -\gamma^2 I & \mathcal{F}_{cl}^T(\tau,\rho) & 0 & 0 & 0 & 0 \\
* & * & * & * & -I & 0 & 0 & 0 & 0 \\
* & * & * & * & * & \Upsilon_{66}(\tau,\rho) & \Upsilon_{67}(\tau,\rho) & 0 & 0 \\
* & * & * & * & * & * & \Upsilon_{77}(\tau,\rho) & 0 & \mathcal{P}_{12}^T(\tau,\rho) \\
* & * & * & * & * & * & * & -\mathcal{P}_{11}(\tau,\rho) & \mathcal{Z}(\tau,\rho) \\
* & * & * & * & * & * & * & * & \Upsilon_{99}(\tau,\rho)
\end{bmatrix} \prec 0 \tag{43}$$

where

$$\mathcal{P}(\tau,\rho) = \begin{bmatrix} \mathcal{P}_{11}(\tau,\rho) & \mathcal{P}_{12}(\tau,\rho) \\ * & \mathcal{P}_{22}(\tau,\rho) \end{bmatrix}$$

$$\Upsilon_{12}(\tau,\rho) = \mathcal{M}(\tau,\rho) + L^T(\rho)\mathcal{A}_{cl}(\tau,\rho)$$

$$\Upsilon_{22}(\tau,\rho) = \mathrm{Sym}[\mathcal{P}_{12}(\tau,\rho)] - \mathcal{P}_{11}(\tau,\rho) + \partial_\tau\mathcal{P}_{11}(\tau,\rho) + \mathcal{Q}_1(\tau,\rho) + \mathcal{Q}_2(\tau,\rho) - 4e^{-\kappa h}\mathcal{R}(\tau,\rho)$$

$$\Upsilon_{26}(\tau,\rho) = -\mathcal{P}_{12}(\tau,\rho) - 2e^{-\kappa h}\mathcal{R}(\tau,\rho)$$

$$\Upsilon_{27}(\tau,\rho) = \mathcal{P}_{22}^T(\tau,\rho) + \partial_\tau\mathcal{P}_{12}(\tau,\rho) + 6h^{-1}e^{-\kappa h}\mathcal{R}(\tau,\rho)$$

$$\Upsilon_{29}(\tau,\rho) = \mathcal{P}_{11}(\tau,\rho) - \mathcal{M}(\tau,\rho)$$

$$\Upsilon_{33}(\tau,\rho) = -(1-\mu)\mathcal{Q}_1(\tau,\rho)e^{-\kappa h}$$

$$\Upsilon_{66}(\tau,\rho) = -4e^{-\kappa h}\mathcal{R}(\tau,\rho) - \mathcal{Q}_2(\tau,\rho)e^{-\kappa h}$$

$$\Upsilon_{67}(\tau,\rho) = -\mathcal{P}_{22}^T(\tau,\rho) + 6h^{-1}e^{-\kappa h}\mathcal{R}(\tau,\rho)$$

$$\Upsilon_{77}(\tau,\rho) = -12h^{-2}e^{-\kappa h}\mathcal{R}(\tau,\rho) + \partial_\tau\mathcal{P}_{22}(\tau,\rho)$$

$$\Upsilon_{99}(\tau,\rho) = -\mathrm{Sym}[\mathcal{Z}(\tau,\rho)] + h^2\mathcal{R}(\tau,\rho).$$

To linearize the inequality (43), a congruence transformation is performed with respect to the matrix $diag(Y(\rho), Y(\rho), Y(\rho), I_{n_w}, I_{n_z}, Y(\rho), Y(\rho), Y(\rho), Y(\rho))$, where

$$Y(\rho) := \begin{bmatrix} W(\rho) & I_n \\ S^T(\rho) & 0 \end{bmatrix}.$$

This leads to

$$\begin{bmatrix} \tilde{\Upsilon}_{11}(\rho) & \tilde{\Upsilon}_{12}(\tau,\rho) & \tilde{\Upsilon}_{13}(\tau,\rho) & \tilde{\Upsilon}_{14}(\tau,\rho) & 0 & 0 & 0 & Y^T(\rho)L(\rho)Y(\rho) & \tilde{\mathcal{Z}}(\tau,\rho) \\ * & \tilde{\Upsilon}_{22}(\tau,\rho) & 0 & 0 & Y^T(\rho)C_{cl}^T(\tau,\rho) & \tilde{\Upsilon}_{26}(\tau,\rho) & \tilde{\Upsilon}_{27}(\tau,\rho) & 0 & \tilde{\Upsilon}_{29}(\tau,\rho) \\ * & * & \tilde{\Upsilon}_{33}(\tau,\rho) & 0 & Y^T(\rho)C_{dcl}^T(\tau,\rho) & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma^2 I & \mathcal{F}_{cl}^T(\tau,\rho) & 0 & 0 & 0 & 0 \\ * & * & * & * & -I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \tilde{\Upsilon}_{66}(\tau,\rho) & \tilde{\Upsilon}_{67}(\tau,\rho) & 0 & 0 \\ * & * & * & * & * & * & \tilde{\Upsilon}_{77}(\tau,\rho) & 0 & \tilde{\mathcal{P}}_{12}^T(\tau,\rho) \\ * & * & * & * & * & * & * & -\tilde{\mathcal{P}}_{11}(\tau,\rho) & \tilde{\mathcal{Z}}(\tau,\rho) \\ * & * & * & * & * & * & * & * & \tilde{\Upsilon}_{99}(\tau,\rho) \end{bmatrix} \prec 0 \quad (44)$$

with

$$\begin{aligned} \tilde{\Upsilon}_{11}(\rho) &= -2Y^T(\rho)L(\rho)Y(\rho) \\ \tilde{\Upsilon}_{12}(\tau,\rho) &= \tilde{\mathcal{M}}(\tau,\rho) + Y(\rho)L^T(\rho)\mathcal{A}_{cl}(\tau,\rho)Y(\rho) \\ \tilde{\Upsilon}_{13}(\tau,\rho) &= Y^T(\rho)L^T(\rho)\mathcal{A}_{dcl}(\tau,\rho)Y(\rho) \\ \tilde{\Upsilon}_{14}(\tau,\rho) &= Y^T(\rho)L^T(\rho)\mathcal{E}_{cl}(\tau,\rho) \\ \tilde{\Upsilon}_{22}(\tau,\rho) &= \mathrm{Sym}[\tilde{\mathcal{P}}_{12}(\tau,\rho)] - \tilde{\mathcal{P}}_{11}(\tau,\rho) + \partial_\tau\tilde{\mathcal{P}}_{11}(\tau,\rho) + \tilde{\mathcal{Q}}_1(\tau,\rho) + \tilde{\mathcal{Q}}_2(\tau,\rho) - 4e^{-\kappa h}\tilde{\mathcal{R}}(\tau,\rho) \\ \tilde{\Upsilon}_{26}(\tau,\rho) &= -\tilde{\mathcal{P}}_{12}(\tau,\rho) - 2e^{-\kappa h}\tilde{\mathcal{R}}(\tau,\rho) \\ \tilde{\Upsilon}_{27}(\tau,\rho) &= \tilde{\mathcal{P}}_{22}^T(\tau,\rho) + \partial_\tau\tilde{\mathcal{P}}_{12}(\tau,\rho) + 6h^{-1}e^{-\kappa h}\tilde{\mathcal{R}}(\tau,\rho) \\ \tilde{\Upsilon}_{29}(\tau,\rho) &= \tilde{\mathcal{P}}_{11}(\tau,\rho) - \tilde{\mathcal{M}}(\tau,\rho) \\ \tilde{\Upsilon}_{33}(\tau,\rho) &= -(1-\mu)\tilde{\mathcal{Q}}_1 e^{-\kappa h} \\ \tilde{\Upsilon}_{66}(\tau,\rho) &= -4e^{-\kappa h}\tilde{\mathcal{R}}(\tau,\rho) - \tilde{\mathcal{Q}}_2 e^{-\kappa h} \\ \tilde{\Upsilon}_{67}(\tau,\rho) &= -\tilde{\mathcal{P}}_{22}^T(\tau,\rho) + 6h^{-1}e^{-\kappa h}\tilde{\mathcal{R}}(\tau,\rho) \\ \tilde{\Upsilon}_{77}(\tau,\rho) &= -12h^{-2}e^{-\kappa h}\tilde{\mathcal{R}}(\tau,\rho) + \partial_\tau\tilde{\mathcal{P}}_{22}(\tau,\rho) \\ \tilde{\Upsilon}_{99}(\tau,\rho) &= -\mathrm{Sym}[\tilde{\mathcal{Z}}(\tau,\rho)] + h^2\tilde{\mathcal{R}}(\tau,\rho) \\ \tilde{\mathcal{P}}(\tau,\rho) &= \mathrm{diag}(Y^T(\rho), Y^T(\rho))\mathcal{P}(\tau,\rho)diag(Y(\rho), Y(\rho)) \\ \tilde{\mathcal{Q}}_1(\tau,\rho) &= Y^T(\rho)\mathcal{Q}_1(\tau,\rho)Y(\rho) \\ \tilde{\mathcal{Q}}_2(\tau,\rho) &= Y^T(\rho)\mathcal{Q}_2(\tau,\rho)Y(\rho) \\ \tilde{\mathcal{M}}(\tau,\rho) &= Y^T(\rho)\mathcal{M}(\tau,\rho)Y(\rho) \\ \tilde{\mathcal{Z}}(\tau,\rho) &= Y^T(\rho)\mathcal{Z}(\tau,\rho)Y(\rho) \\ \tilde{\mathcal{R}}(\tau,\rho) &= Y^T(\rho)\mathcal{R}(\tau,\rho)Y(\rho)\,. \end{aligned} \quad (45)$$

Noting that

$$Y^T(\rho)L(\rho) = \begin{bmatrix} I_n & 0 \\ X(\rho) & T(\rho) \end{bmatrix} \text{ and } Y^T(\rho)L(\rho)Y(\rho) = \begin{bmatrix} W(\rho) & I_n \\ I_n & X(\rho) \end{bmatrix},$$

we obtain

$$\mathscr{Z}(\tau,\rho) := \left[\begin{array}{c|c|c} Y^T(\rho)L(\rho)\mathcal{A}_{cl}(\tau,\rho)Y(\rho) & Y^T(\rho)L(\rho)\mathcal{A}_{dcl}(\tau,\rho)Y(\rho) & Y^T(\rho)L(\rho)\mathcal{E}_{cl}(\tau,\rho) \\ \hline C_{cl}(\tau,\rho)Y(\rho) & C_{dcl}(\tau,\rho)Y(\rho) & \mathcal{F}_{cl}(\tau,\rho) \end{array}\right]$$

$$= \left[\begin{array}{cc|cc|c} A(\rho)W(\rho) & A(\rho) & A_d(\rho)W(\rho) & A(\rho) & E(\rho) \\ 0 & X(\rho)A(\rho) & 0 & X(\rho)A_d(\rho) & X(\rho)E(\rho) \\ \hline C(\rho)W(\rho) & C(\rho) & C_d(\rho)W(\rho) & C_d(\rho) & F(\rho) \end{array}\right] + \Theta_1(\rho)\begin{bmatrix} \hat{\mathcal{A}}(\tau,\rho) & \hat{\mathcal{A}}_d(\tau,\rho) & \hat{\mathcal{B}}(\tau,\rho) \\ \hat{\mathcal{C}}(\tau,\rho) & \hat{\mathcal{C}}_d(\tau,\rho) & \hat{\mathcal{D}}(\tau,\rho) \end{bmatrix}\Theta_2(\rho),$$

where

$$\Theta_1(\rho) = \left[\begin{array}{cc} 0 & B(\rho) \\ I_n & 0 \\ \hline 0 & D(\rho) \end{array}\right],\ \Theta_2(\rho) = \left[\begin{array}{cc|cc|c} I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 \\ 0 & C_y(\rho) & 0 & C_{dy}(\rho) & F_y(\rho) \end{array}\right],$$

$$\begin{bmatrix} \hat{\mathcal{A}}(\tau,\rho) & \hat{\mathcal{A}}_d(\tau,\rho) & \hat{\mathcal{B}}(\tau,\rho) \\ \hat{\mathcal{C}}(\tau,\rho) & \hat{\mathcal{C}}_d(\tau,\rho) & \hat{\mathcal{D}}(\tau,\rho) \end{bmatrix} = \begin{bmatrix} X(\rho)A(\rho)W(\rho) & X(\rho)A_d(\rho)W(\rho) & 0 \\ 0 & 0 & 0 \end{bmatrix} + \Theta_3(\rho)\mathcal{K}(\tau,\rho)\Theta_4(\rho),$$

$$\Theta_3(\rho) = \begin{bmatrix} T(\rho) & X(\rho)B(\rho) \\ 0 & I_{n_u} \end{bmatrix},\ \text{and}\ \Theta_4(\rho) = \begin{bmatrix} S^T(\rho) & 0 & 0 \\ 0 & S^T(\rho) & 0 \\ C_y(\rho)W(\rho) & C_{dy}(\rho)W(\rho) & I_{n_y} \end{bmatrix}.$$

Finally, we get

$$\begin{aligned} \mathcal{F}_{cl}(\tau,\rho) &= F(\rho) + D(\rho)\hat{\mathcal{D}}(\tau,\rho)F_y(\rho) \\ Y^T(\rho)L(\rho)\mathcal{A}_{cl}(\tau,\rho)Y(\rho) &= \begin{bmatrix} A(\rho)W(\rho) + B(\rho)\hat{\mathcal{C}}(\tau,\rho) & A(\rho) + B(\rho)\hat{\mathcal{D}}(\tau,\rho)C_y(\rho) \\ \hat{\mathcal{A}}(\tau,\rho) & X(\rho)A(\rho) + \hat{\mathcal{B}}(\tau,\rho)C_y(\rho) \end{bmatrix} \\ Y^T(\rho)L(\rho)\mathcal{A}_{dcl}(\tau,\rho)Y(\rho) &= \begin{bmatrix} A_d(\rho)W(\rho) + B(\rho)\hat{\mathcal{C}}(\tau,\rho) & A(\rho) + B(\rho)\hat{\mathcal{D}}(\tau,\rho)C_{dy}(\rho) \\ \hat{\mathcal{A}}_d(\tau,\rho) & X(\rho)A_d(\rho) + \hat{\mathcal{B}}(\tau,\rho)C_{dy}(\rho) \end{bmatrix} \\ Y^T(\rho)L(\rho)\mathcal{E}_{cl}(\tau,\rho) &= \begin{bmatrix} E(\rho) + B(\rho)\hat{\mathcal{D}}(\tau,\rho)F_y(\rho) \\ X(\rho)E(\rho) + \hat{\mathcal{B}}(\tau,\rho)F_y(\rho) \end{bmatrix} \\ C_{cl}(\tau,\rho)Y(\rho) &= \begin{bmatrix} C(\rho)W(\rho) + D(\rho)\hat{\mathcal{C}}(\tau,\rho) & C(\rho) + D(\rho)\hat{\mathcal{D}}(\tau,\rho)C_y(\rho) \end{bmatrix}^T \\ C_{dcl}(\tau,\rho)Y(\rho) &= \begin{bmatrix} C_d(\rho)W(\rho) + D(\rho)\hat{\mathcal{C}}_d(\tau,\rho) & C_d(\rho) + D(\rho)\hat{\mathcal{D}}(\tau,\rho)C_{dy}(\rho) \end{bmatrix}^T \end{aligned} \tag{46}$$

from which it is obvious that the expressions are affine in the variables $X(\rho)$, $W(\rho)$, $\hat{\mathcal{A}}(\tau,\rho)$, $\hat{\mathcal{A}}_d(\tau,\rho)$, $\hat{\mathcal{B}}(\tau,\rho)$, $\hat{\mathcal{C}}(\tau,\rho)$, $\hat{\mathcal{C}}_d(\tau,\rho)$, and $\hat{\mathcal{D}}(\tau,\rho)$. Substitution (46) in the inequality (44) yields the LMI (31). This completes the proof. ■

Remark 3. Theorem 1 presents a new result for output feedback $\mathcal{L}_2$ controller design for delayed LPV systems with piecewise constant parameters. It is essential to mention that the proposed design conditions (31)–(40) result in infinite-dimensional semidefinite programs that cannot be solved directly via standard LMI solvers. One way to resolve this issue is first to employ relaxation techniques like the gridding method [8, appendix C], robust optimization module of YALMIP,[33] or SOS polynomials[34,35] to approximate semiinfinite constraint LMI by a finite number of LMIs. This finite-dimensional approximation of the infinite-dimensional semidefinite program can then be solved via standard solvers such as SeDuMi.[41]

3.1 | Case study: Delayed switched systems

LPV systems with piecewise constant parameters represent a class of switched linear systems with infinite modes in a bounded compact set. Here, we provide a case study to demonstrate the application of our main result to a class of delayed switched systems. This result is important for its own sake and can be regarded as a contribution of this article.

Let $\mathcal{P} = \{1, \dots, \mathcal{S}\}$ for some $\mathcal{S} \in \mathbb{N}$, and

$$\begin{aligned} A(\rho) &= \sum_{i=1}^{\mathcal{S}} \delta_{i,\rho} A_i,\ A_d(\rho) = \sum_{i=1}^{\mathcal{S}} \delta_{i,\rho} A_{di},\ B(\rho) = \sum_{i=1}^{\mathcal{S}} \delta_{i,\rho} B_i, E(\rho) = \sum_{i=1}^{\mathcal{S}} \delta_{i,\rho} E_i, \\ C(\rho) &= \sum_{i=1}^{\mathcal{S}} \delta_{i,\rho} C_i,\ C_d(\rho) = \sum_{i=1}^{\mathcal{S}} \delta_{i,\rho} C_{di},\ D(\rho) = \sum_{i=1}^{\mathcal{S}} \delta_{i,\rho} D_i,\ F(\rho) = \sum_{i=1}^{\mathcal{S}} \delta_{i,\rho} F_i, \\ C_y(\rho) &= \sum_{i=1}^{\mathcal{S}} \delta_{i,\rho} C_{yi},\ C_{dy}(\rho) = \sum_{i=1}^{\mathcal{S}} \delta_{i,\rho} C_{dyi},\ F_y(\rho) = \sum_{i=1}^{\mathcal{S}} \delta_{i,\rho} F_{yi}, \end{aligned} \tag{47}$$

where $\delta_{i,j}$ is the Kronecker delta, that is, $\delta_{i,j} = 1$ if $i = j$, and 0 otherwise.

We then have the following corollary.

Corollary 1. *For given constants $h \geq 0$, $0 \leq \mu < 1$, $\kappa > 0$, and $T_D > 0$, if there exist matrix-valued functions $\tilde{\mathcal{P}}_i : [0, T_D] \mapsto \mathbb{S}^{4n}_{>0}$, $\tilde{\mathcal{Q}}_{1,i} : [0, T_D] \mapsto \mathbb{S}^{2n}_{>0}$, $\tilde{\mathcal{Q}}_{2,i} : [0, T_D] \mapsto \mathbb{S}^{2n}_{>0}$, $\tilde{\mathcal{R}}_i : [0, T_D] \mapsto \mathbb{S}^{2n}_{>0}$, $\tilde{\mathcal{M}}_i : [0, T_D] \mapsto \mathbb{R}^{2n}$, $\tilde{\mathcal{Z}}_i : [0, T_D] \mapsto \mathbb{R}^{2n}$, $X_i \in \mathbb{R}^n$, and $W_i \in \mathbb{R}^n$ such that the LMIs*

$$\tilde{\Lambda}_i(\tau) := \begin{bmatrix} -\mathrm{Sym}[\tilde{X}_i] & \tilde{\Lambda}_{12,i}(\tau) & \tilde{\Lambda}_{13,i}(\tau) & \tilde{\Lambda}_{14,i}(\tau) & 0 & 0 & 0 & \tilde{X}_i^T & \tilde{\mathcal{Z}}_i(\tau) \\ * & \tilde{\Lambda}_{22,i}(\tau) & 0 & 0 & \tilde{\Lambda}_{25,i}(\tau) & \tilde{\Lambda}_{26,i}(\tau) & \tilde{\Lambda}_{27,i}(\tau) & 0 & \tilde{\Lambda}_{29,i}(\tau) \\ * & * & \tilde{\Lambda}_{33,i}(\tau) & 0 & \tilde{\Lambda}_{35,i}(\tau) & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma^2 I & \tilde{\Lambda}_{45,i}(\tau) & 0 & 0 & 0 & 0 \\ * & * & * & * & -I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \tilde{\Lambda}_{66,i}(\tau) & \tilde{\Lambda}_{67,i}(\tau) & 0 & 0 \\ * & * & * & * & * & * & \tilde{\Lambda}_{77,i}(\tau) & 0 & \tilde{\mathcal{P}}^T_{12,i}(\tau) \\ * & * & * & * & * & * & * & -\tilde{\mathcal{P}}_{11,i}(\tau) & \tilde{\mathcal{Z}}_i(\tau) \\ * & * & * & * & * & * & * & * & \tilde{\Lambda}_{99,i}(\tau) \end{bmatrix} \prec 0 \tag{48}$$

$$\tilde{\Lambda}_i(T_D^+) \prec 0 \tag{49}$$

$$\tilde{\mathcal{P}}_i(T_D) - \tilde{\mathcal{P}}_j(0) \succcurlyeq 0 \tag{50}$$

$$\tilde{\mathcal{Q}}_{1,i}(T_D) - \tilde{\mathcal{Q}}_{1,j}(0) \succcurlyeq 0 \tag{51}$$

$$\tilde{\mathcal{Q}}_{2,i}(T_D) - \tilde{\mathcal{Q}}_{2,j}(0) \succcurlyeq 0 \tag{52}$$

$$\tilde{\mathcal{R}}_i(T_D) - \tilde{\mathcal{R}}_j(0) \succcurlyeq 0 \tag{53}$$

$$\kappa\tilde{\mathcal{Q}}_{1,i}(T_D) - \dot{\tilde{\mathcal{Q}}}_{1,i}(0) \succcurlyeq 0 \tag{54}$$

$$\kappa\tilde{\mathcal{Q}}_{2,i}(T_D) - \dot{\tilde{\mathcal{Q}}}_{2,i}(0) \succcurlyeq 0 \tag{55}$$

$$\kappa\tilde{\mathcal{R}}_i(T_D) - \dot{\tilde{\mathcal{R}}}_i(0) \succcurlyeq 0 \tag{56}$$

$$\tilde{X}_i = \begin{bmatrix} W_i & I_n \\ I_n & X_i \end{bmatrix} \succ 0 \tag{57}$$

hold for all $\tau \in [0, T_D]$ and all $i,j \in \{1, \dots, \mathcal{S}\}, i \neq j$, where

$$\tilde{\mathcal{P}}_i(\tau) = \begin{bmatrix} \tilde{\mathcal{P}}_{11,i}(\tau) & \tilde{\mathcal{P}}_{12,i}(\tau) \\ * & \tilde{\mathcal{P}}_{22,i}(\tau) \end{bmatrix}$$
$$\tilde{\Lambda}_{12,i}(\tau) = \begin{bmatrix} A_iW_i + B_i\hat{\mathcal{C}}_i(\tau) & A_i + B_i\hat{\mathcal{D}}_i(\tau)C_{yi} \\ \hat{\mathcal{A}}_i(\tau) & X_iA_i + \hat{\mathcal{B}}_i(\tau)C_{yi} \end{bmatrix} + \tilde{\mathcal{M}}_i(\tau)$$
$$\tilde{\Lambda}_{13,i}(\tau) = \begin{bmatrix} A_{di}W_i + B_i\hat{\mathcal{C}}_i(\tau) & A_i + B_i\hat{\mathcal{D}}_i(\tau)C_{dyi} \\ \hat{\mathcal{A}}_{di}(\tau) & X_iA_{di} + \hat{\mathcal{B}}_i(\tau)C_{dyi} \end{bmatrix}$$
$$\tilde{\Lambda}_{14,i}(\tau) = \begin{bmatrix} E_i + B_i\hat{\mathcal{D}}_i(\tau)F_{yi} \\ X_iE_i + \hat{\mathcal{B}}_i(\tau)F_{yi} \end{bmatrix}$$
$$\tilde{\Lambda}_{22,i}(\tau) = \mathrm{Sym}[\tilde{\mathcal{P}}_{12,i}(\tau)] - \tilde{\mathcal{P}}_{11,i}(\tau) + \partial_\tau\tilde{\mathcal{P}}_{11,i}(\tau) + \tilde{\mathcal{Q}}_{1,i}(\tau) + \tilde{\mathcal{Q}}_{2,i}(\tau) - 4e^{-\kappa h}\tilde{\mathcal{R}}_i(\tau)$$
$$\tilde{\Lambda}_{25,i}(\tau) = \begin{bmatrix} C_iW_i + D_i\hat{\mathcal{C}}_i(\tau) & C_i + D_i\hat{\mathcal{D}}_i(\tau)C_{yi} \end{bmatrix}^T$$
$$\tilde{\Lambda}_{26,i}(\tau) = -\tilde{\mathcal{P}}_{12,i}(\tau) - 2e^{-\kappa h}\tilde{\mathcal{R}}_i(\tau)$$
$$\tilde{\Lambda}_{27,i}(\tau) = \tilde{\mathcal{P}}_{22,i}^T(\tau) + \partial_\tau\tilde{\mathcal{P}}_{12,i}(\tau) + 6h^{-1}e^{-\kappa h}\tilde{\mathcal{R}}_i(\tau)$$
$$\tilde{\Lambda}_{29,i}(\tau) = \tilde{\mathcal{P}}_{11,i}(\tau) - \tilde{\mathcal{M}}_i(\tau)$$
$$\tilde{\Lambda}_{33,i}(\tau) = -(1-\mu)\tilde{\mathcal{Q}}_{1,i}(\tau)e^{-\kappa h}$$
$$\tilde{\Lambda}_{35,i}(\tau) = \begin{bmatrix} C_{di}W_i + D_i\hat{\mathcal{C}}_{di}(\tau) & C_{di} + D_i\hat{\mathcal{D}}_i(\tau)C_{dyi} \end{bmatrix}^T$$
$$\tilde{\Lambda}_{45,i}(\tau) = \begin{bmatrix} F_i + D_i\hat{\mathcal{D}}_i(\tau)F_{yi} \end{bmatrix}^T$$
$$\tilde{\Lambda}_{66,i}(\tau) = -4e^{-\kappa h}\tilde{\mathcal{R}}_i(\tau) - \tilde{\mathcal{Q}}_{2,i}(\tau)e^{-\kappa h}$$
$$\tilde{\Lambda}_{67,i}(\tau) = -\tilde{\mathcal{P}}_{22,i}^T(\tau) + 6h^{-1}e^{-\kappa h}\tilde{\mathcal{R}}_i(\tau)$$
$$\tilde{\Lambda}_{77,i}(\tau) = -12h^{-2}e^{-\kappa h}\tilde{\mathcal{R}}_i(\tau) + \partial_\tau\tilde{\mathcal{P}}_{22,i}(\tau)$$
$$\tilde{\Lambda}_{99,i}(\tau) = -\mathrm{Sym}[\tilde{\mathcal{Z}}_i(\tau)] + h^2\tilde{\mathcal{R}}_i(\tau),$$

then the system Σ_p with (47) is uniformly asymptotically stable in the absence of disturbance w with $u \equiv 0$. Moreover, the $\mathcal{L}_2$-gain of the map $w \mapsto z$ is at most γ, that is, $J \leq \gamma$.

Moreover, once the unknown matrices satisfying the LMIs conditions are obtained, the clock-dependent output feedback control matrices can be computed by first obtaining two matrices $T_i \in \mathbb{R}^{n\times n}$ and $S_i \in \mathbb{R}^{n\times n}$ from the factorization problem

$$I_n - X_iW_i = T_iS_i^T, \tag{58}$$

and then by reversing the following transformation

$$\hat{\mathcal{A}}_i(\tau) = X_iA_iW_i + X_iB_i\mathcal{D}_i(\tau)C_{yi}W_i + T_i\mathcal{B}_i(\tau)C_{yi}W_i + X_iB_i\mathcal{C}_i(\tau)S_i^T + T_i\mathcal{A}_i(\tau)S_i^T$$
$$\hat{\mathcal{A}}_{di}(\tau) = X_iA_{di}W_i + X_iB_i\mathcal{D}_i(\tau)C_{dyi}W_i + T_i\mathcal{B}_i(\tau)C_{dyi}W_i + X_iB_i\mathcal{C}_{di}(\tau)S_i^T + T_i\mathcal{A}_{di}(\tau)S_i^T$$
$$\hat{\mathcal{B}}_i(\tau) = X_iB_i\mathcal{D}_i(\tau) + T_i\mathcal{B}_i(\tau)$$
$$\hat{\mathcal{C}}_i(\tau) = \mathcal{D}_i(\tau)C_{yi}W_i + \mathcal{C}_i(\tau)S_i^T$$
$$\hat{\mathcal{C}}_{di}(\tau) = \mathcal{D}_i(\tau)C_{dyi}W_i + \mathcal{C}_{di}(\tau)S_i^T$$
$$\hat{\mathcal{D}}_i(\tau) = \mathcal{D}_i(\tau).$$

Proof. Since $A(\rho), A_d(\rho), B(\rho), E(\rho), C(\rho), C_d(\rho), D(\rho), F(\rho), C_y(\rho), C_{dy}(\rho)$, and $F_y(\rho)$ have the form (47), without loss of generality, we can choose $P(\tau,\eta) = \sum_{i=1}^{\mathcal{S}} \delta_{i,\eta}P_i(\tau)$, $P_i(\tau) \succ 0$, $Q_1(\tau,\eta) = \sum_{i=1}^{\mathcal{S}} \delta_{i,\eta}Q_{1,i}(\tau)$, $Q_{1,i}(\tau) \succ 0$, $Q_2(\tau,\eta) = \sum_{i=1}^{\mathcal{S}} \delta_{i,\eta}Q_{2,i}(\tau)$, $Q_{2,i}(\tau) \succ 0$, $R(\tau,\eta) = \sum_{i=1}^{\mathcal{S}} \delta_{i,\eta}R_i(\tau)$, $R_i(\tau) \succ 0$, $M(\tau,\rho) = \sum_{i=1}^{\mathcal{S}} \delta_{i,\rho}M_i(\tau)$, $Z(\tau,\rho) = \sum_{i=1}^{\mathcal{S}} \delta_{i,\rho}Z_i(\tau)$, $X(\rho) = \sum_{i=1}^{\mathcal{S}} \delta_{i,\rho}X_i$, $W(\rho) = \sum_{i=1}^{\mathcal{S}} \delta_{i,\rho}W_i$, $T(\rho) = \sum_{i=1}^{\mathcal{S}} \delta_{i,\rho}T_i$, and $S(\rho) = \sum_{i=1}^{\mathcal{S}} \delta_{i,\rho}S_i$. The result then follows. ∎

4 | ILLUSTRATIONS

In this section, we provide an application and an illustrative example to demonstrate our result's efficacy.

4.1 | Computational aspects

Polynomials functions of an arbitrary order can be used to approximate the continuous matrix-valued functions appearing in LMIs of Theorem 1 and Corollary 1 with arbitrary precision. To this aim, we employ clock- and parameter-dependent polynomials of order 1 for the sake of simplicity. Moreover, due to the continuous-time parameters τ and ρ, the inequalities in Theorem 1 and Corollary 1 yield intractable infinite-dimensional semidefinite programs, therefore, we consider robust optimization module[33] provided in YALMIP[42] that approximates infinite-dimensional semidefinite programs which can then be solved using standard solvers, for instance, SeDuMi.[41] The robust optimization module of YALMIP can robustify semidefinite (SDP) constraints with affine dependence on the uncertainty, under the restriction that the uncertainty is constrained to a polytopic set. In this case, YALMIP derives a robust counterpart by enumerating the vertices of the uncertainty set and define an SDP constraint for every vertex.[33]

4.2 | Application to VTOL helicopter model

In this section, we illustrate the efficacy of our result by applying it to VTOL helicopter model introduced in References 20,25,43 . The dynamics of the system can be given as

$$\begin{aligned}\dot{x}(t) &= A(\rho)x(t) + A_d x(t-d(t)) + B(\rho)u(t) + Ew(t)\\ z(t) &= Cx(t) + C_d x(t-d(t)) + Du(t) + Fw(t)\\ y(t) &= C_y x(t) + C_{dy} x(t-d(t)) + F_y w(t)\\ x(\theta) &= \phi(\theta),\ \forall \theta \in [-h, 0],\end{aligned} \tag{59}$$

where ρ is a time-varying parameter and the state variables $x(t) = [x_1\ x_2\ x_3\ x_4]^T$ are horizontal velocity, the vertical velocity, the pitch rate, and the pitch angle, respectively. The delay is taken as $d(t) = 1.5 + 0.5\ \sin^2(0.45t)$, and the upper bound on its magnitude and time-derivative are taken as $h = 2$ and $\mu = 0.9$, respectively. The matrices in (59) are given as

$$A(\rho) = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555\\ 0.0482 & -1.01 & 0.0024 & -4.0208\\ 0.1002 & a_{32}(\rho) & -0.7071 & a_{34}(\rho)\\ 0 & 0 & 1 & 0 \end{bmatrix},\ B(\rho) = \begin{bmatrix} 0.4422 & 0.1761\\ b_{21}(\rho) & -7.5922\\ -5.52 & 4.49\\ 0 & 0 \end{bmatrix}, E = \begin{bmatrix} I_4 & 0_{4\times1} \end{bmatrix},\ A_d = \mathcal{O}_4,$$

$$C = \begin{bmatrix} I_4\\ 0_{2\times4} \end{bmatrix},\ C_d = C,\ D = \begin{bmatrix} 0_{4\times2}\\ I_2 \end{bmatrix},\ F = 0_{6\times5},\ C_y = [\,0\quad 1\quad 0\quad 0\,],\ C_{dy} = 0_{1\times4},\ F_y = [\,0\quad 0\quad 0\quad 0\quad 1\,],$$

where

$$a_{32}(\rho) = 0.0670 + 0.4390\rho,\ a_{34}(\rho) = 0.0479 + 2.3440\rho,\ b_{21}(\rho) = 0.9263 + 4.0760\rho,$$

and the parameter ρ is defined as $\rho = 0.0091(\theta_t - 60)$, where $\theta_t \in [60,170]$ is airspeed in knots. It is easy to see that the parameter space $\mathcal{P} = [0, 1]$ is a bounded closed set. We choose the minimum dwell time $T_D = 0.01$, and the tuning parameter $\kappa = 0.1$ to solve the inequalities in Theorem 1. Polynomials of order 1 have been employed to approximate the unknown matrices in Theorem 1. The controller matrices obtained after solving the LMIs are given in the Appendix A. The stabilizing state trajectories in Figure 1 (top) for an arbitrary initial condition subject to a piecewise constant parameter trajectory

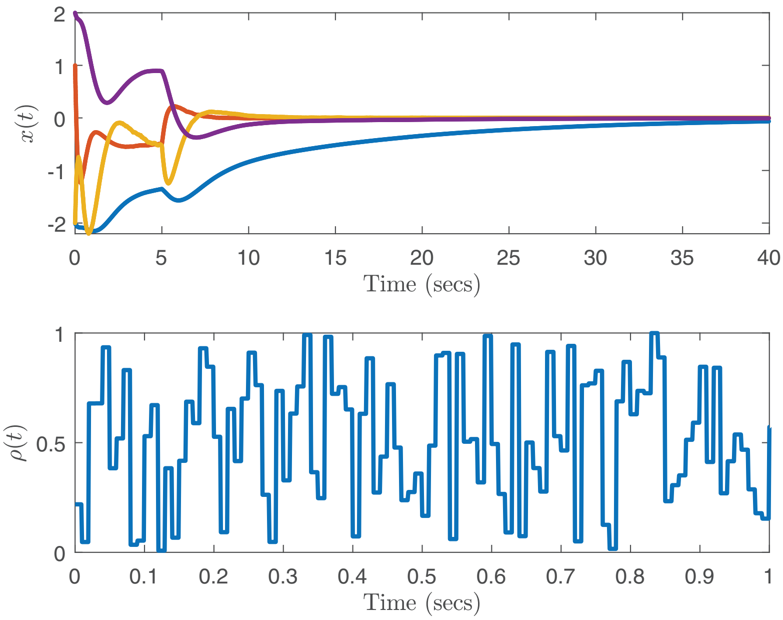

FIGURE 1 Evolution of the states of the closed-loop with disturbance (top) subject to a typical parameter trajectory with $T_D = 0.01$s [Colour figure can be viewed at wileyonlinelibrary.com]

shown in Figure 1 (bottom) under the disturbance $w(t) = [v(t)\ v(t)\ v(t)\ v(t)\ v(t)]^T$, where $v(t) = 0.5(H(t) - H(t-5))$, support the applicability of the approach.

4.3 | Application to delayed switched systems

Consider the following delayed switched system. The system comprising of two subsystems:

$$A_1 = \begin{bmatrix} -0.9 & 0.2 & -0.2 \\ 0.2 & -0.6 & 0.3 \\ -0.3 & 0.1 & -0.1 \end{bmatrix},\ A_{d1} = \begin{bmatrix} 0.2 & 0 & 0.1 \\ 0.1 & 0.3 & 0.1 \\ 0.3 & 0.1 & 0.2 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 0.5 \\ 2.0 \end{bmatrix},\ E_1 = \begin{bmatrix} 0.3 \\ 0.5 \\ 0.2 \end{bmatrix},$$
$$C_1 = [\,0.8 \quad 1.0 \quad 0.5\,],\ C_{d1} = [\,0.2 \quad 0.3 \quad 0.1\,],\ C_{y1} = [\,-1.2 \quad 1.5 \quad 0.9\,],$$
$$C_{dy1} = [\,0.3 \quad 0.1 \quad 0.2\,],\ F_{y1} = 0.2,\ D_1 = F_1 = 0. \tag{60}$$

$$A_2 = \begin{bmatrix} -0.8 & -0.1 & -0.2 \\ 0.2 & -0.7 & 0.3 \\ 0.2 & -0.1 & 0.1 \end{bmatrix},\ A_{d2} = \begin{bmatrix} 0.2 & 0.1 & 0 \\ 0.1 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.3 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.5 \\ 0.7 \\ 1.5; \end{bmatrix},\ E_2 = \begin{bmatrix} 0.4 \\ 0.2 \\ 0.3 \end{bmatrix},$$
$$C_2 = [\,-1.0 \quad 1.2 \quad 0.5\,],\ C_{d2} = [\,0.1 \quad 0.3 \quad 0.4\,],\ C_{y2} = [\,-0.1 \quad 1.2 \quad 0.5\,],$$
$$C_{dy2} = [\,0.1 \quad 0.3 \quad 0.4\,],\ F_{y2} = 0.1,\ D_2 = F_2 = 0, \tag{61}$$

and $d(t) = 0.9 + 0.3\sin(t)$. A straightforward calculation gives $h = 1.2$ and $\mu = 0.3$. It can be verified that the switched system (60) and (61) with $u(t) = 0$ is unstable for an arbitrary switching signal. To support the latter claim, a typical open-loop system response is provided in Figure 2 with the initial condition $x(t) = [-1.0\ 0.5\ 1.0]^T$, $t \in [-1.2, 0]$.

We choose $T_D = 0.1$ and $\kappa = 0.1$, and solve the LMIs given in Corollary 1 via robust optimization module of YALMIP. Once LMIs are feasible, we compute controller matrices and form a closed-loop system. The computed controller matrices are given in Appendix B. To show the effectiveness of the designed controller through simulation, we consider the exogenous disturbance input as $w(t) = \exp(-t)\sin(t)$. Figure 3 (top) demonstrates the evolution of the closed-loop states subject to a typical periodic switching signal with $T_D = 0.1$ provided at the bottom of the same figure (here, 1 and 2

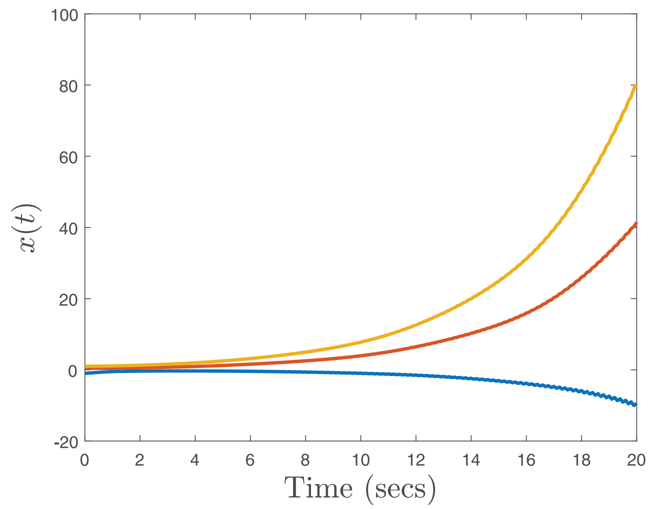

FIGURE 2 Unstable behavior of states for the open-loop system [Colour figure can be viewed at wileyonlinelibrary.com]

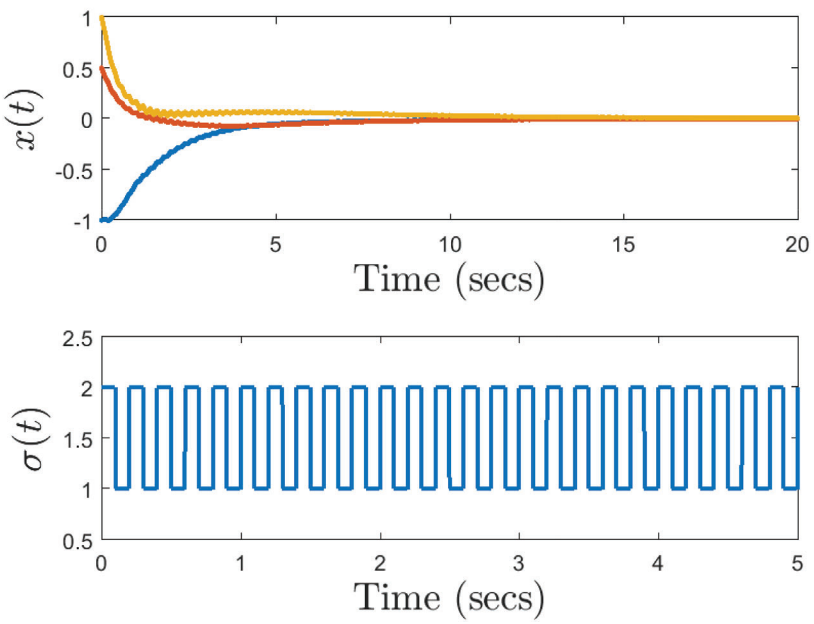

FIGURE 3 Evolution of the states of the closed-loop with disturbance (top) subject to a typical switching signal with $T_D = 0.1$s [Colour figure can be viewed at wileyonlinelibrary.com]

represent the first and the second subsystems, respectively). Since our simulation depicts stability of the closed-loop system, it helps illustrate our general theory, in the special case of the system (60) and (61).

5 | CONCLUDING REMARKS

We provided sufficient conditions for the existence of a full-order clock-dependent gain-scheduled output-feedback controller for delayed LPV systems with piecewise constant parameters. These systems generalize the framework of switched systems with finite/infinite countable sets to uncountable bounded sets. We formulated sufficient conditions as clock-dependent convex infinite-dimensional LMIs and employed standard approximations to relax the semidefinite program. Incorporating parameter variation during the analysis reduces the conservatism of the existing methods and makes them more efficient.

One of the future perspectives of this work includes designing a memory-resilient dynamic $\mathcal{L}_2$ output-feedback compensator, where the delay will be approximated to be used for the controller.[31] Controller synthesis for other dwell-time notions, such as range, average, and maximum dwell-time, also seems an appealing extension of this work. Another intriguing future research direction is to employ the decoupling technique proposed in References 44-46 to tackle nonconvex matrix inequality (nonlinear coupling) conditions. We also plan to employ reciprocally convex inequality[38] to achieve new stability results. Controller synthesis for delayed LPV systems with piecewise constant parameters subject to uncertainties also seems an interesting future research direction, which will be a topic for future exploration. These results will be reported elsewhere.

ACKNOWLEDGMENT

The authors would like to acknowledge fruitful discussions with Dr. Corentin Briat and Prof. Hitay Özbay. Open access funding enabled and organized by Projekt DEAL.

ORCID

Saeed Ahmed https://orcid.org/0000-0001-9545-9313

REFERENCES

1. Shamma JS. *Analysis and Design of Gain Scheduled Control Systems* [PhD thesis]. Cambridge, MA: Massachusetts Institute of Technology; 1988.
2. Shamma JS, Athans M. Gain scheduling: potential hazards and possible remedies. *IEEE Control Syst*. 1992;12(3):101-107.
3. Savaresi SM, Poussot-Vassal C, Spelta C, Sename O, Dugard L. *Semi-Active Suspension Control Design for Vehicles*. Oxford, England: Butterworth-Heinemann; 2011.
4. Sename O, Gaspar P, Bokor J, eds. *Robust Control and Linear Parameter Varying Approaches: Application to Vehicle Dynamics*. Berlin, Germany: Springer; 2013.
5. Gilbert W, Henrion D, Bernussou J, Boyer D. Polynomial LPV synthesis applied to turbofan engines. *Control Eng Pract*. 2010;18(9):1077-1083.
6. Robert D, Sename O, Simon D. An H_∞ LPV design for sampling varying controllers: experimentation with a T-inverted pendulum. *IEEE Trans Control Syst Technol*. 2010;18(3):741-749.
7. Kajiwara H, Apkarian P, Gahinet P. LPV techniques for control of an inverted pendulum. *IEEE Control Syst*. 1999;19(1):44-54.
8. Briat C. *Linear Parameter-Varying and Time-Delay Systems: Analysis, Observation, Filtering & Control*. Berlin, Germany: Springer; 2015.
9. Mohammadpour J, Scherer CW, eds. *Control of Linear Parameter Varying Systems with Applications*. New York, NY: Springer; 2012.
10. Apkarian P, Gahinet P, Becker G. Self-scheduled H_∞ control of linear parameter-varying systems: a design example. *Automatica*. 1995;31(9):1251-1261.
11. Briat C. Stability analysis and control of a class of LPV systems with piecewise constant parameters. *Syst Control Lett*. 2015;82:10-17.
12. Briat C. Stability analysis and stabilization of LPV systems with jumps and (piecewise) differentiable parameters using continuous and sampled-data controllers; 2017. arXiv [mathOC]. http://arxiv.org/abs/1705.00056.
13. Briat C, Khammash M. Stability analysis of LPV systems with piecewise differentiable parameters. *IFAC-PapersOnLine*. 2017;50(1):7554-7559.
14. Packard A. Gain scheduling via linear fractional transformations. *Syst Control Lett*. 1994;22(2):79-92.
15. Scherer CW. LPV control and full block multipliers. *Automatica*. 2001;37(3):361-375.
16. Briat C. Stability analysis and state-feedback control of LPV systems with piecewise constant parameters subject to spontaneous Poissonian jumps. *IEEE Control Syst Lett*. 2018;2(2):230-235.
17. Zakwan M, Ahmed S, Bajcinca N. On output feedback stabilization of time-varying decomposable systems with switching topology and delay. Paper presented at: Proceedings of the 21st IFAC World Congress; July 11-17, 2020; Berlin, Germany. http://arxiv.org/abs/2001.01593.
18. Joo H, Kim SH. H_∞ LPV control with pole placement constraints for synchronous buck converters with piecewise-constant loads. *Math Probl Eng*. 2015;2015:1-8.
19. Tan K, Grigoriadis KM, Wu F. Output-feedback control of LPV sampled-data systems. *Int J Control*. 2002;75(4):252-264.
20. Zakwan M. Dynamic L_2 output feedback stabilization of LPV systems with piecewise constant parameters subject to spontaneous Poissonian jumps. *IEEE Control Syst Lett*. 2020;4(2):408-413.
21. Briat C. Convex conditions for robust stabilization of uncertain switched systems with guaranteed minimum and mode-dependent dwell-time. *Syst Control Lett*. 2015;78:63-72.
22. Ahmed S, Mazenc F, Özbay H. Dynamic output feedback stabilization of switched linear systems with delay via a trajectory based approach. *Automatica*. 2018;93:92-97.
23. Niculescu S-I. *Delay Effects on Stability: A Robust Control Approach*. London, England: Springer; 2003.
24. Chiasson J, Loiseau JJ, eds. *Applications of Time Delay Systems*. Berlin, Germany: Springer; 2007.
25. Zakwan M, Ahmed S. Distributed output feedback control of decomposable LPV systems with delay and switching topology: application to consensus problem in multi-agent systems. *Int J Control*. 2020;1-12. https://doi.org/10.1080/00207179.2019.1710257.
26. Wu F, Grigoriadis KM. LPV systems with parameter-varying time delays. Paper presented at: Proceedings of the 36th IEEE Conference on Decision and Control; December 12-12, 1997; San Diego, CA
27. Wu F, Grigoriadis KM. LPV Systems with parameter-varying time delays: analysis and control. *Automatica*. 2001;37(2):221-229.
28. Zhang X, Tsiotras P, Knospe C. Stability analysis of LPV time-delayed systems. *Int J Control*. 2002;75(7):538-558.
29. Zhang F, Grigoriadis KM. Delay-dependent stability analysis and $\mathcal{H}_\infty$ control for state-delayed LPV system. Paper presented at: Proceedings of the 2005 IEEE International Symposium on Mediterrean Conference on Control and Automation Intelligent Control; June 27-29, 2005; Limassol, Cyprus.

30. Briat C, Sename O, Lafay J-F. Parameter dependent state-feedback control of LPV time delay systems with time varying delays using a projection approach. *IFAC Proc*. 2008;41(2):4946-4951.
31. Briat C, Sename O, Lafay J-F. Memory-resilient gain-scheduled state-feedback control of uncertain LTI/LPV systems with time-varying delays. *Syst Control Lett*. 2010;59(8):451-459.
32. Briat C, Sename O, Lafay J-F. A LFT/$\mathcal{H}_\infty$ state feedback design for linear parameter varying time delay systems. Paper presented at: Proceedings of the 2007 European Control Conference (ECC); July 2-5, 2007; Kos, Greece.
33. Löfberg J. Automatic robust convex programming. *Optim Methods Softw*. 2012;27(1):115-129.
34. Wu F, Prajna S. SOS-based solution approach to polynomial LPV system analysis and synthesis problems. *Int J Control*. 2005;78(8): 600-611.
35. Scherer CW, Hol CWJ. Matrix sum-of-squares relaxations for robust semi-definite programs. *Math Program*. 2006;107(1-2): 189-211.
36. Wu F, Yang XH, Packard A, Becker G. Induced L_2-norm control for LPV systems with bounded parameter variation rates. *Int J Robust Nonlinear Control*. 1996;6(9-10):983-998.
37. Zakwan M, Ahmed S. Dwell-time based stability analysis and $\mathcal{L}_2$ control of LPV systems with piecewise constant parameters and delay. *Syst Control Lett*. 2020;145(104805):1-9.
38. Seuret A, Gouaisbaut F. Wirtinger-based integral inequality: application to time-delay systems. *Automatica*. 2013;49(9): 2860-2866.
39. Gu K, Kharitonov VL, Chen J. *Stability of Time-Delay Systems*. Boston, MA: Birkhäuser; 2003.
40. Gahinet P, Apkarian P. A linear matrix inequality approach to H_∞ control. *Int J Robust Nonlinear Control*. 1994;4(4):421-448.
41. Sturm JF. Using SeDuMi 1.02, A Matlab toolbox for optimization over symmetric cones. *Optim Methods Softw*. 1999;11(1-4): 625-653.
42. Löfberg J. YALMIP: a toolbox for modeling and optimization in MATLAB. Paper presented at: Proceedings of the 2004 IEEE International Conference on Robotics and Automation (IEEE Cat. No.04CH37508); September 2-4, 2004; New Orleans, LA.
43. De Farias DP, Geromel JC, Do Val JBR, Costa OLV. Output feedback control of Markov jump linear systems in continuous-time. *IEEE Trans Automat Control*. 2000;45(5):944-949.
44. Chang X-H, Xiong J, Li Z-M, Park JH. Quantized static output feedback control for discrete-time systems. *IEEE Trans Ind Inform*. 2017;14(8):3426-3435.
45. Chang X-H, Xiong J, Park JH. Fuzzy robust dynamic output feedback control of nonlinear systems with linear fractional parametric uncertainties. *Appl Math Comput*. 2016;291:213-225.
46. Chang X-H, Yang G-H. New results on output feedback H_∞ control for linear discrete-time systems. *IEEE Trans Automat Control*. 2013;59(5):1355-1359.

How to cite this article: Zakwan M, Ahmed S, Bajcinca N. Dynamic $\mathcal{L}_2$ output feedback control of delayed linear parameter varying systems with piecewise constant parameters: A clock-dependent L–K approach. *Int J Robust Nonlinear Control*. 2021;31:1145–1167. https://doi.org/10.1002/rnc.5343

APPENDIX A. CONTROLLER MATRICES FOR VTOL HELICOPTER MODEL

The clock- and parameter-dependent controller matrices appearing in the expression of Σ_c for the specific example of VTOL aircraft model given in (59) are computed as

$$\mathcal{A}(\tau,\rho) = \begin{bmatrix} 1.1693\rho + 0.11303\tau - 3.0872 & 3.4396\rho + 0.82665\tau - 7.8907 & 5.9491 - 0.49595\tau - 2.6331\rho & 6.6653 - 0.58384\tau - 3.0373\rho \\ -5.8544\rho - 3.3759\tau - 3.0099 & -18.989\rho - 24.072\tau - 28.81 & 14.365\rho + 14.46\tau + 14.928 & 16.464\rho + 17.027\tau + 18.075 \\ 2.122\tau - 8.9525\rho + 29.363 & 14.457\tau - 24.358\rho + 93.343 & 18.643\rho - 8.7041\tau - 66.592 & 21.525\rho - 10.254\tau - 75.613 \\ -0.44292\rho - 6.0309\tau - 27.131 & -5.1879\rho - 41.451\tau - 103.04 & 3.7461\rho + 24.945\tau + 67.74 & 4.1706\rho + 29.384\tau + 78.001 \end{bmatrix}, \tag{A1}$$

$$\mathcal{A}_d(\tau,\rho)=\begin{bmatrix} 0.0001177\rho-0.0027107\tau-0.00048673 & -0.0017939\tau-0.0002946\rho+0.000047637 \\ 0.079411\tau-0.0015755\rho+0.0087851 & 0.0021794\rho+0.052009\tau+0.01844 \\ 0.00014834\rho-0.048915\tau-0.0036706 & 0.0014933\rho-0.032038\tau-0.017342 \\ 0.13951\tau-0.0021222\rho+0.014086 & 0.0016073\rho+0.091356\tau+0.037035 \end{bmatrix.$$
$$\begin{matrix} 0.00025001\rho+0.00052855\tau-0.00013593 & 0.00025598\rho+0.0012053\tau-0.00010276 \\ -0.015099\tau-0.0028309\rho-0.0098077 & -0.03492\tau-0.0022891\rho-0.012987 \\ 0.0092991\tau-0.00023452\rho+0.010217 & 0.021511\tau-0.00088194\rho+0.012834 \\ -0.026514\tau-0.0034232\rho-0.020467 & -0.061336\tau-0.0022261\rho-0.026564 \end{bmatrix},$$

$$\mathcal{B}(\tau,\rho)=\begin{bmatrix} 3.2766\tau-110.78\rho+152.6 \\ 517.54\rho-81.865\tau+290.12 \\ 820.95\rho+34.104\tau-2255.9 \\ 15.23\rho-106.26\tau+2454.3 \end{bmatrix},\ \mathcal{C}(\tau)=\begin{bmatrix} 0.015642 & 0.0059962\rho-0.0023789\tau-0.0039152 \\ 0.043709 & 0.014969\rho-0.016101\tau-0.021909 \\ -0.031731 & 0.0096972\tau-0.010653\rho+0.012585 \\ -0.035727 & 0.011425\tau-0.011887\rho+0.01599 \end{bmatrix}^T,$$
$$\mathcal{C}_d(\tau)=0_{2\times 4},\ \mathcal{D}(\tau)=\begin{bmatrix} -0.83887 \\ 0.59394-0.0355\tau-0.24502\rho \end{bmatrix},$$

where $\rho \in [0,1]$, $\tau=\min\{t-t_k, T_D\}$, and $T_D=0.2$.

APPENDIX B. CONTROLLER MATRICES FOR THE DELAYED SWITCHED SYSTEM

The clock-dependent controller matrices for the specific example of the switched systems in Section 4.3 are given by

$$\mathcal{A}_1(\tau)=\begin{bmatrix} 0.078177\tau-1.5148 & 0.79289\tau-2.4233 & -0.12236\tau-1.1054 \\ 0.43546\tau+1.1966 & 0.5442\tau-4.1594 & -0.38428\tau-1.2285 \\ 0.43787\tau+0.44807 & 0.98199\tau-4.7281 & -0.42109\tau-2.508 \end{bmatrix},$$
$$\mathcal{A}_{d1}(\tau)=\begin{bmatrix} 0.13762-0.032851\tau & 0.085054\tau-0.039768 & 0.02302\tau+0.15243 \\ -0.0083713\tau-0.10522 & 0.18785-0.036128\tau & 0.00076656-0.043791\tau \\ 0.073338-0.021852\tau & 0.014718\tau-0.15337 & 0.015872-0.02795\tau \end{bmatrix},$$
$$\mathcal{B}_1(\tau)=\begin{bmatrix} 2.8052-0.25627\tau \\ -1.0899\tau-0.48908 \\ 1.0462-1.1022\tau \end{bmatrix},\ \mathcal{C}_1(\tau)=\begin{bmatrix} 0.0034713\tau+0.073795 \\ 0.46754-0.16181\tau \\ 0.16385-0.0001203\tau \end{bmatrix}^T,\ \mathcal{C}_{d1}(\tau)=\begin{bmatrix} 0.0018302\tau-0.00041468 \\ 0.018746-0.010073\tau \\ 0.0080545-0.0048618\tau \end{bmatrix}^T,$$
$$\mathcal{D}_1(\tau)=0.022468\tau-0.46118. \tag{B1}$$

$$\mathcal{A}_2(\tau)=\begin{bmatrix} 0.40232\tau+2.4086 & -0.79151\tau-3.7947 & -0.60792\tau-3.8863 \\ 1.174\tau+5.9596 & -1.8513\tau-8.6271 & -1.5819\tau-4.5217 \\ 0.84959\tau+5.0248 & -1.4384\tau-6.6277 & -1.1101\tau-5.0075 \end{bmatrix},$$
$$\mathcal{A}_{d2}(\tau)=\begin{bmatrix} 0.29368-0.28757\tau & 0.57125\tau+0.1688 & 0.41457\tau+0.01541 \\ 0.19985-0.23148\tau & 0.36635\tau+0.058319 & 0.24644\tau-0.082486 \\ 0.11925-0.28786\tau & 0.5497\tau+0.010571 & 0.35771\tau+0.1091 \end{bmatrix},$$

$$\mathcal{B}_2(\tau)=\begin{bmatrix}0.45203\tau+3.652\\0.34862\tau+1.851\\0.37272\tau+1.9117\end{bmatrix},\ \mathcal{C}_2(\tau)=\begin{bmatrix}-0.13512\tau-0.86276\\0.22604\tau+1.0378\\0.20417\tau+0.90664\end{bmatrix}^T,\ \mathcal{C}_{d2}(\tau)=\begin{bmatrix}0.040237\tau-0.0053372\\-0.080743\tau-0.017755\\-0.056259\tau-0.014578\end{bmatrix}^T,$$
$$\mathcal{D}_2(\tau)=-0.17336\tau-0.87176, \tag{B2}$$

where $\tau=\min\{t-t_k, T_D\}$ and $T_D=0.1$.