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The primitive equations in the scaling-invariant space $L^\infty(L^1)$

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Abstract. Consider the primitive equations on $\mathbb{R}^2 \times (z_0, z_1)$ with initial data a of the form $a = a_1 + a_2$, where $a_1 \in \mathrm{BUC}_\sigma(\mathbb{R}^2; L^1(z_0, z_1))$ and $a_2 \in L^\infty_\sigma(\mathbb{R}^2; L^1(z_0, z_1))$. These spaces are scaling-invariant and represent the anisotropic character of these equations. It is shown that for a_1 arbitrary large and a_2 sufficiently small, this set of equations admits a unique strong solution which extends to a global one and is thus strongly globally well posed for these data provided a is periodic in the horizontal variables. The approach presented depends crucially on mapping properties of the hydrostatic Stokes semigroup in the $L^\infty(L^1)$-setting. It can be seen as the counterpart of the classical iteration schemes for the Navier–Stokes equations, now for the primitive equations in the $L^\infty(L^1)$-setting.

1. Introduction

The primitive equations for ocean and atmospheric dynamics serve as a fundamental model for many geophysical flows. This set of equations describing the conservation of momentum and mass of a fluid, assuming hydrostatic balance of the pressure, is given in the isothermal setting by

$$\begin{cases} \partial_t v + u \cdot \nabla v - \Delta v + \nabla_H \pi = 0, & \text{in } \Omega \times (0, T), \\ \partial_z \pi = 0, & \text{in } \Omega \times (0, T), \\ \operatorname{div} u = 0, & \text{in } \Omega \times (0, T), \\ v(0) = a. \end{cases} \tag{1.1}$$

Here $\Omega := \mathbb{R}^2 \times J$, where $J = (z_0, z_1)$ is an interval. Denoting the horizontal coordinates by $x, y \in \mathbb{R}^2$ and the vertical one by $z \in (z_1, z_2)$, we use the notation $\nabla_H = \left(\partial_x, \partial_y\right)^{\mathrm{T}}$, whereas Δ denotes the three-dimensional Laplacian and ∇ and div the three-dimensional gradient and divergence operators. The velocity u of the fluid is

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described by $u = (v, w)$, where $v = (v_1, v_2)$ denotes the horizontal component and w the vertical one.

In the literature, various sets of boundary conditions are considered such as Neumann, Dirichlet and mixed boundary conditions. In this article, we choose Neumann boundary conditions for v, i.e.

$$\begin{cases} \frac{\partial}{\partial z} v = 0, & \text{on } \partial\Omega \times (0, T), \\ w = 0, & \text{on } \partial\Omega \times (0, T), \end{cases} \tag{1.2}$$

and mixed boundary conditions are discussed in [8].

The rigorous analysis of the primitive equations started with the pioneering work of Lions et al. [20–22], who proved the existence of a global weak solution for this set of equations for initial data $a \in L^2$. The uniqueness problem for weak solutions remains an open problem until today.

The existence of a local, strong solution for this equation with data $a \in H^1$ was proved by Guillén-González et al. in [9].

In 2007, Cao and Titi [3] proved a breakthrough result for this set of equation which says, roughly speaking, that there exists a unique, global strong solution to the primitive equations for *arbitrarily large* initial data $a \in H^1$. Their proof is based on *a priori* H^1-bounds for the solution, which in turn are obtained by $L^\infty(L^6)$ energy estimates. Kukavica and Ziane [16] proved global strong well-posedness of the primitive equations with respect to arbitrary large H^1-data for the case of mixed Dirichlet–Neumann boundary conditions. For a different approach, see also Kobelkov [14]. We also would like to draw the attention of the reader to the recent survey article by Li and Titi [19] on the primitive equations.

Recently, a new approach to the primitive equations based on the theory of evolution equations has been developed in [10,11]. This approach is also valid in the L^p-setting for all $1 < p < \infty$ and, using this approach, the authors proved global strong well-posedness of the primitive equations subject to mixed Dirichlet–Neumann boundary conditions for arbitrary large data in the Bessel potential space $H^{2/p,p}$. Taking formally the limit $p \to \infty$, it is now tempting to consider initial data $a \in L^\infty$ with no differentiability assumption on the initial data. This article aims to find a function space, as large as possible, for the initial data for which the primitive equations are strongly and globally well posed.

Recent regularity results for weak solutions by Li and Titi [18] and Kukavica et al. [15] are also pointing in this direction. More specifically, starting from a weak solution to the primitive equations, these authors investigated regularity criteria for weak solutions for the primitive equations, following hereby in a certain sense the spirit of Serrin's condition in the theory of the Navier–Stokes equations and methods of weak–strong uniqueness. Li and Titi proved in [18] that weak solutions are unique for initial values in C^0 or in $\{u \in L^6 \colon \partial_z u \in L^2\}$ including a small perturbation belonging to L^∞. By the weak–strong uniqueness property, it follows that these weak solutions regularize and become strong solutions.

Our approach to rough initial data results for the primitive equations is very different: It considers the primitive equation as an evolution equation in an anisotropic function space of the form $L^\infty(\mathbb{R}^2; L^1(J))$. This space is invariant under the scaling

$$v_\lambda(t, x_1, x_2, x_3) = \lambda v(\lambda^2 t, \lambda(x_1, x_2, x_3)), \quad \lambda > 0.$$

By this, we mean that $\|v_\lambda\|_{L^\infty(\mathbb{R}^2;L^1(\lambda^{-1}J))} = \|v\|_{L^\infty(\mathbb{R}^2;L^1(J))}$ for all $\lambda > 0$. Moreover, v_λ is a solution to the primitive equations whenever v has this property. For further information on the Navier–Stokes equations in critical spaces see [2,4,17].

Based on L^∞-type estimates for the underlying hydrostatic Stokes semigroup S on $L^\infty(L^1)$ and its gradient, we develop an iteration scheme yielding first the existence of a unique, local mild solution for initial data of the form $a = a_1 + a_2$ with

$$a_1 \in \mathrm{BUC}_\sigma(\mathbb{R}^2, L^1(J)) \text{ and } a_2 \text{ being a small perturbation in } L^\infty_\sigma(\mathbb{R}^2; L^1(J)).$$

The main idea of our approach may be described as follows: In a first step, we extend the hydrostatic Stokes semigroup S from the $L^p(L^p)$-setting to the $L^\infty(L^1)$-setting. Duhamel's formula leads us then to terms of the form $S(t)\mathbb{P}\mathrm{div}\,(u \otimes v)$. Observe that $u = (v, w)$ involves first derivatives through $w = w(v)$, and thus, second-order derivatives appear in the above term. This implies a singularity of order t^{-1} for $S(t)\mathbb{P}\mathrm{div}\,(u \otimes v)$ for $t > 0$, which is nonintegrable. In order to surpass this difficulty, we smoothen the horizontal derivatives by inserting fractional powers of the horizontal Laplacian and the vertical derivative by inserting fractional vertical derivatives and obtain

$$\begin{aligned} S(t)\mathbb{P}\nabla \cdot (u \otimes v) &= \underbrace{S(t)\mathbb{P}(-\Delta_H)^{(1-\alpha)/2}}_{\text{decay term}} \nabla_H \cdot (-\Delta_H)^{-(1-\alpha)/2}(v \otimes v) \\ &\quad + \underbrace{S(t)\partial_z I^\alpha_{z_0}}_{\text{decay term}}\, I^{1-\alpha}_{z_0}\partial_z(wv), \quad t > 0. \end{aligned}$$

This leads us on the one hand to estimates for the decay terms in the $L^\infty(L^1)$-norm and on the other hand to estimates for fractional derivatives of functions within this $L^\infty(L^1)$ framework. The iteration scheme developed yields sequences defined for $m \in \mathbb{N}$ by

$$K_m(t) := \sup_{0<\tau<t} \tau^{1/2}\|v_m(\tau)\|_{1,\infty,1}, \quad \|v_m(\tau)\|_{1,\infty,1} := \|v_m(\tau)\|_{\infty,1} + \|\nabla v_m(\tau)\|_{\infty,1}.$$

We show that this sequence and related sequences converge provided $K_0(t) = \sup_{0<\tau<t} \tau^{1/2}\,\|S(\tau)a\|_{1,\infty,1}$ is sufficiently small. The sequence (v_m) is then a Cauchy sequence and converges to the unique solution of (1.1). The smallness can be balanced by taking the time as well as the rough part a_2 sufficiently small.

Assuming that a_1, a_2 are periodic with respect to horizontal variables, we are able to prove that the solution regularizes sufficiently and thus, by an *a priori* estimate, can be extended to global, strong solution without any restriction on the size of a_1.

Comparing our assumptions on the initial data with the ones given by Li and Titi [18], let us note that our assumptions are slightly less restrictive for the case of continuous initial data, while our assumptions are not comparable to their second case.

Our approach may be viewed as the counterpart of the classical iteration schemes for the 3-D Navier–Stokes equations due to Giga [5] and Kato [13] which yield initial values in the scaling-invariant space L^3. Note that, in contrast to the case of the Navier–Stokes equations, our iteration scheme presented here combined with a suitable *a priori* estimate yields the existence of a unique, global strong solutions not only for small data as in the case of the Navier–Stokes equations, but for arbitrary large solenoidal data $a \in \mathrm{BUC}_\sigma(\mathbb{R}^2; L^1(J))$.

As written above, our approach depends crucially on $L^\infty(L^1)$-mapping properties of the underlying hydrostatic Stokes semigroup as well as on its gradient. These estimates are collected in Proposition 2.2 and are of independent interest.

This article is organized as follows. Section 2 presents the result of this article. Sections 3, 4 and 5 are devoted to anisotropic estimates for fractional derivatives, the heat semigroup as well as for the hydrostatic semigroup. In Sect. 6, we present a proof of our main results based on our iteration scheme.

2. Preliminaries and main results

Let $z_0 \in \mathbb{R}$, $z_1 = z_0 + h$ for some $h > 0$, J be the interval $J = (z_0, z_1)$ and $\Omega := \mathbb{R}^2 \times J$. The incompressibility condition $\operatorname{div} u = 0$ in $\Omega \times (0, T)$ implies

$$w(x, y, z) = \int_z^{z_1} \operatorname{div}_H v(x, y, \xi) \mathrm{d}\xi,$$

where the boundary condition $w = 0$ on $\partial\Omega$ has been taken into account. Also, $w = 0$ on $\partial\Omega$ implies

$$\operatorname{div}_H \overline{v} = 0 \text{ in } \mathbb{R}^2,$$

where $\overline{v}$ denotes the vertical average of v, i.e.,

$$\overline{v}(x, y) := \frac{1}{z_1 - z_0} \int_{z_0}^{z_1} v(x, y, z) \mathrm{d}z.$$

The linearization of Eq. (1.1) are the *hydrostatic Stokes equations*, which are given by

$$\begin{cases} \partial_t v - \Delta v + \nabla_H \pi = f, & \text{in } \Omega \times (0, T), \\ \operatorname{div}_H \overline{v} = 0, & \text{in } \Omega \times (0, T), \\ v(0) = a & \text{in } \Omega. \end{cases} \tag{2.1}$$

The name ‘hydrostatic Stokes equations’ is motivated by the assumption of the hydrostatic balance when deriving the full primitive equations. Equation (2.1) are supplemented by Neumann boundary conditions (1.2) for v.

For a function $f : \mathbb{R}^2 \times J \to \mathbb{C}$, we define for $1 \le p, q < \infty$ the $L^q(\mathbb{R}^2, L^p(J))$-norm of f by

$$\|f\|_{L^q(\mathbb{R}^2;L^p(J))} := \Big(\int_{\mathbb{R}^2}\Big(\int_J \big|f(x',x_3)\big|^q \,\mathrm{d}x_3\Big)^{q/p}\mathrm{d}x'\Big)^{1/q},$$

where we use the shorthand notation $L^q(L^p)$ for the spaces and $\|\cdot\|_{q,p}$ for the norms. The usual modifications hold for the cases $p = \infty$ or $q = \infty$. The space $L^p(\mathbb{R}^2; L^q(J))$ consisting of all measurable functions f with $\|f\|_{p,q} < \infty$ and equipped with the above norm becomes a Banach space.

Following [6,11], we introduce the hydrostatic Helmholtz projection as follows. For a function $f : \mathbb{R}^2 \times J \to \mathbb{C}^2$, we define the hydrostatic Helmholtz projection by

$$\mathbb{P}f := f + \nabla_H(-\Delta)^{-1}\mathrm{div}_H\, \overline{f}.$$

The solenoidal subspace $L^\infty_\sigma(\mathbb{R}^2; L^p(J))$ is defined for $1 \le p \le \infty$ as the closed subspace of $L^\infty(\mathbb{R}^2; L^p(J))$ given by

$$L^\infty_\sigma(\mathbb{R}^2; L^p(J)) := \{v \in L^\infty(\mathbb{R}^2; L^p(J)) : \int_{\mathbb{R}^2} \overline{v}\, \nabla_H\varphi \,\mathrm{d}x = 0 \text{ for all } \varphi \in \widehat{W}^{1,1}(\mathbb{R}^2)\}. \tag{2.2}$$

Here $\widehat{W}^{1,1}(\mathbb{R}^2)$ denotes the homogeneous Sobolev space of the form $\widehat{W}^{1,1}(\mathbb{R}^2) = \{\varphi \in L^1_{\mathrm{loc}}(\mathbb{R}^2) : \nabla_H\varphi \in L^1(\mathbb{R}^2)\}$, that is, the condition $\mathrm{div}_H\, \overline{v} = 0$ is understood in the sense of distributions.

If $a \in L^\infty_\sigma(\mathbb{R}^2; L^p(J))$ for some $1 \le p \le \infty$ and $f \equiv 0$, then the solution of Eq. (2.1) can be represented as $v(t) = S(t)a$ for $t \ge 0$, where S denotes the *hydrostatic Stokes semigroup* on $L^\infty_\sigma(\mathbb{R}^2; L^p(J))$. The latter semigroup may be represented using the heat semigroup as follows: Consider the one-dimensional heat equation in $J \times (0,\infty)$,

$$u_t - u_{zz} = 0, \quad u(0) = u_0,$$

subject to the boundary conditions

$$u_z(z_1) = 0, \quad u_z(z_0) = 0. \tag{2.3}$$

For $p \in [1,\infty]$ and $u_0 \in L^p(J)$, its solution u is given by $u(t) = e^{t\Delta_N}u_0$. Here $e^{t\Delta_N}$ denotes the analytic semigroup on $L^p(J)$ generated by the Laplacian subject to Neumann boundary conditions. The corresponding heat semigroup on $L^\infty(\mathbb{R}^2; L^p(J))$ for all $p \in [1,\infty]$ denoted by S_∞ is thus given by

$$S_\infty(t) := e^{t\Delta_H} \otimes e^{t\Delta_N}, \quad t \ge 0,$$

where $e^{t\Delta_H}$ denotes the heat semigroup on $L^\infty(\mathbb{R}^2)$. The hydrostatic Stokes semigroup S is then given as the restriction of S_∞ to the space of solenoidal functions

$L^\infty_\sigma(\mathbb{R}^2; L^p(J))$. Here, one uses the fact that the solenoidal space $L^\infty_\sigma(\mathbb{R}^2; L^p(J))$ is left invariant by the heat semigroup. This follows from (2.2) since

$$\begin{aligned}\int_{\mathbb{R}^2} \overline{S_\infty(t)v}\ \nabla_H\varphi\ \mathrm{d}x &= \frac{1}{z_1-z_0}\int_{\mathbb{R}^2}\int_J e^{t\Delta_N}v\ e^{t\Delta_H}\nabla_H\varphi\ \mathrm{d}z\mathrm{d}x\\ &= \int_{\mathbb{R}^2}\overline{v}\ \nabla_H e^{t\Delta_H}\varphi\ \mathrm{d}x = 0\end{aligned}$$

for $v \in L^\infty_\sigma(\mathbb{R}^2; L^p(J))$, $\varphi \in \widehat{W}^{1,1}(\mathbb{R}^2)$ and $t \geq 0$, where one uses that S_∞ acts as identity on functions constant with respect to z, and that $e^{t\Delta_H}\varphi \in \widehat{W}^{1,1}(\mathbb{R}^2)$.

So, for $a \in L^\infty_\sigma(\mathbb{R}^2; L^p(J))$ and $f \equiv 0$, the solution of (2.1) is thus given as the restriction of the heat semigroup to solenoidal functions, that is,

$$v(t) = S(t)a, \quad \text{where } S(t)a = S_\infty(t)a, \quad t \geq 0.$$

The semigroup S is not strongly continuous on $L^\infty_\sigma(\mathbb{R}^2; L^p(J))$; however, its restriction to

$$\mathrm{BUC}_\sigma(L^p) := \mathrm{BUC}(\mathbb{R}^2; L^p(J)) \cap L^\infty_\sigma(L^p)$$

defines for $1 \leq p < \infty$ an analytic C_0-semigroup on this space satisfying $\|S(t)\|_{\infty,p} \leq M$ for all $t > 0$ and for some $M > 0$, where $\|\cdot\|_{\infty,p}$ denotes the $L^\infty(\mathbb{R}^2; L^p(J))$-norm. It is well known that the Helmholtz projection is not bounded on L^∞-spaces, and this carries over to the hydrostatic Helmholtz projection considered here. However, for sufficiently smooth a one has $S(t)a = S_\infty(t)\mathbb{P}a$ for $t \geq 0$.

The first part of our main result concerns the existence of a unique, local mild solution to the primitive equations with initial value a. By this, we mean a function v satisfying the integral equation

$$v(t) = S(t)a - \int_0^t S(t-s)\mathbb{P}\nabla\cdot(u(s)\otimes v(s))\mathrm{d}s, \quad 0 < t < T, \tag{2.4}$$

for some $T > 0$, where $w(s) = \int_z^{z_1} \mathrm{div}_H\, v(s)\ \mathrm{d}x_3$ and $u(s) = (v(s), w(s))$ for all $s \in [0,t]$ and $\nabla\cdot(u\otimes v) = u\cdot\nabla v$ since $\mathrm{div}\, u = 0$. In the second part of the main result, we state that this solution regularizes in space and time and extends to a unique, global, strong solution. Applying [7], this solution is in fact a classical one, i.e., $v \in C^\infty((0,T)\times\overline{\Omega})$, and it is even real analytic in time and space.

In order to formulate our result precisely, we introduce for $v \in L^\infty(L^1)$ the norm

$$\|v\|_{1,\infty,1} := \|v\|_{\infty,1} + \|\nabla v\|_{\infty,1}.$$

Theorem 2.1. (Local and global strong well-posedness) *Let $a \in L^\infty_\sigma(L^1)$.*

(a) *Then there exists a constant $\varepsilon_0 > 0$ such that if $\overline{\lim}_{t\to 0} t^{1/2}\|S(t)a\|_{1,\infty,1} \leq \varepsilon_0$, there exists $T > 0$ such that (1.1) subject to (1.2) admits a unique, local mild solution*

$$v \in C\big((0,T); \mathrm{BUC}_\sigma\left(L^1(J)\right)\big) \cap L^\infty\big((0,T); \mathrm{BUC}_\sigma\left(L^1(J)\right)\big)$$

with

$$t^{1/2}\nabla v \in L^\infty\left((0,T); L^\infty\left(L^1(J)\right)\right) \quad \textit{and} \quad \overline{\lim}_{t\to 0} t^{1/2}\|v(t)\|_{1,\infty,1} \le \varepsilon_0.$$

(b) *The above condition* $\limsup_{t\to 0} t^{1/2}\|S(t)a\|_{1,\infty,1} \le \varepsilon_0$ *is in particular satisfied for* a *of the form* $a = a_1 + a_2$ *with* $a_1 \in \mathrm{BUC}_\sigma(L^1)$ *and* $a_2 \in L^\infty_\sigma(L^1)$ *provided* $\|a_2\|_{L^\infty_\sigma(L^1)}$ *is sufficiently small. In this case,*

$$\overline{\lim_{t\to 0}}\, t^{1/2}\|\nabla v(t)\|_{L^\infty(L^1)} \le C\|a_2\|_{L^\infty_\sigma(L^1)}$$

for some $C > 0$*, independent of* a.

(c) *If* $a_2 = 0$*, then* $v \in C\left([0,T); \mathrm{BUC}_\sigma\left(L^1(J)\right)\right)$.

(d) *Let* $p, q \in (1,\infty)$ *with* $1/p + 1/q \le 1$ *and assume in addition to* (a) *or* (b) *that* a *is periodic with respect to the horizontal variables. Then, for any* $T^* > 0$ *the solution* v *extends to a unique, strong solution of* (1.1) *on* $(0, T^*)$*, i.e., for any* $\delta > 0$

$$v \in H^q((\delta, T^*); L^p(\Omega)^2) \cap L^q((\delta, T^*); H^{2,p}(\Omega)^2).$$

Estimates for the linearized problem applied to fractional derivatives are essential in our proof of the local well-posedness. Here, we denote for $\alpha > 0$ by $I^\alpha_{z_0}$ the Riemann–Liouville operator of the form

$$\left(I^\alpha_{z_0} f\right)(z) := \frac{1}{\Gamma(\alpha)} \int_{z_0}^{z} (z-\zeta)^{\alpha-1} f(\zeta)\mathrm{d}\zeta, \quad z \in \overline{J}, \tag{2.5}$$

where Γ denotes the usual Gamma function, cf. [12, Section 23.16] or [1, Section 3.9] for basic facts about Riemann–Liouville operators.

Proposition 2.2. (Linear estimates) *Let* $p \in [1,\infty]$*. Then the following assertions hold:*

(i) *There exists a constant* $C > 0$ *such that for all* $f \in L^\infty(L^p)$

$$\|\nabla S_\infty(t) f\|_{\infty,p} \le Ct^{-1/2}\|f\|_{\infty,p} \quad \textit{and} \quad \|S_\infty(t)\nabla_H \cdot f\|_{\infty,p} \le Ct^{-1/2}\|f\|_{\infty,p}, \quad t > 0.$$

(ii) *For* $\alpha \in [0,1)$*, there exists a constant* $C > 0$ *such that for all* $f \in L^\infty(L^p)$ *satisfying* $I^\alpha_{z_0} f(z_1) = 0$

$$\left\|S_\infty(t)\partial_z I^\alpha_{z_0} f\right\|_{\infty,p} \le Ct^{-(1-\alpha)/2}\|f\|_{\infty,p}, \quad t > 0.$$

(iii) *For* $\alpha \in (0,2]$*, there exists a constant* $C > 0$ *such that for all* $f \in L^\infty(L^p)$

$$\left\|S_\infty(t)\mathbb{P}(-\Delta_H)^{\alpha/2} f\right\|_{\infty,p} \le Ct^{-\alpha/2}\|f\|_{\infty,p}, \quad t > 0.$$

(iv) *There exists a constant* $C > 0$ *such that for all* $f \in L^\infty(L^p)$

$$\|S_\infty(t)\mathbb{P}\nabla_H \cdot f\|_{\infty,p} \le Ct^{-1/2}\|f\|_{\infty,p}, \quad t > 0.$$

(v) *There exists a constant* $C > 0$ *such that for all* $f \in L^\infty(L^p)$

$$\|S_\infty(t)f\|_{\infty,p} \le Ct^{-(1-1/p)}\|f\|_{\infty,1}, \quad t > 0.$$

Remark 2.3. In the case where $\alpha = 0$ in assertion (ii), the operator $I_{z_0}^0$ is interpreted as the identity operator and there is no restriction for f other than $f \in L^\infty(L^p)$.

In order to establish the global well-posedness of Eq. (1.1), we use the smoothing effect of the local solution described in Proposition 2.4 below. Roughly speaking, the solution regularizes into the well understood L^q-L^p-setting and extends hence to the global smooth solution, see [7]. In order to apply this strategy, we use first the $L^\infty(L^p)$-$L^\infty(L^1)$ smoothing properties of S_∞ and assuming then initial data in $L^\infty(L^p)$ for $p > 1$, we control the $L^\infty(L^p)$-norms of the pair $(v(t), \nabla v(t))$ by the corresponding $L^\infty(L^1)$-norms. In turn, the $L^\infty(L^p)$-norms of $v(t)$ and $\nabla v(t)$ for $p \ge 2$ give rise to control of the H^1-norms assuming periodicity. This strategy is formulated precisely in the following proposition.

Proposition 2.4. (Local existence for $p > 1$) *Let a and* $T > 0$ *be as in Theorem* 2.1. *If in addition to the assumptions of Theorem* 2.1 *the initial data a satisfies*

(i) $a \in L^\infty_\sigma(L^p)$ *for some* $p \in (1, \infty]$, *then* $t^{1/2-1/2p}v, t^{1-1/2p}\nabla v \in L^\infty\big(0, T; L^\infty_\sigma(L^p)\big)$;
(ii) $a \in \mathrm{BUC}_\sigma(L^p)$ *for some* $p \in (1, \infty]$, *then* $t^{1/2-1/2p}v, t^{1-1/2p}\nabla v \in C\left([0, T), \mathrm{BUC}_\sigma(L^p)\right)$;
(iii) $a \in \mathrm{BUC}_\sigma(\mathrm{BUC})$, *then* $t^{1/2}v, t\nabla v \in C\left([0, T), \mathrm{BUC}_\sigma(\mathrm{BUC})\right)$.

The local mild solution constructed in Theorem 2.1 exists at least on the interval $[0, T)$, where $T > 0$ depends on a. Instead of using smoothing properties to obtain a global strong solution, we may also estimate the existence time $T > 0$ explicitly from below in terms of the $|||\cdot|||$-norm, defined for $a \in L^\infty_\sigma(\mathbb{R}^2; L^1(J))$ and for $\mu \in [0, 1/2)$ by

$$|||a||| := [a]_\mu + \|a\|_{\infty,1}, \text{ where } [a]_\mu := \sup_{0<t<1} t^\mu \|\nabla S(t)a\|_{\infty,1}.$$

Proposition 2.5. (Estimate on the life span) *Let a and* $T > 0$ *be as in Theorem* 2.1. *Assume in addition that* $[a]_\mu < \infty$ *for some* $\mu \in [0, 1/2)$. *Then there exists* $C > 0$, *depending on* μ *only, such that*

$$1/T \le \min\left(C|||a|||, 1\right)^{2/(1/2-\mu)}.$$

3. Interpolation inequalities for fractional derivatives

In this section, we consider fractional derivatives and prove anisotropic interpolation inequalities. To this end, let $f \in L^\infty(J;\mathbb{C})$, and consider the zero extension of f to $\mathbb{R}$, denoted by f_+, and the zero extension of z from $(0,h)$ to $\mathbb{R}$, denoted by z_+. The Riemann–Liouville operator introduced as in (2.5) is given by

$$I_{z_0}^\alpha f = \frac{z_+^{\alpha-1}}{\Gamma(\alpha)} * f_+, \quad f \in L^\infty(J).$$

Then $I_{z_0}^\alpha f$ is called the α-times integral of f from z_0 whenever $\alpha > 0$ and we have $I_{z_0}^{\alpha_1+\alpha_2} = I_{z_0}^{\alpha_1} I_{z_0}^{\alpha_2}$ for all $\alpha_1, \alpha_2 > 0$, cf. [12, Section 23.16]. We also set $I_{z_0}^0 f = f$.

The Caputo derivative ∂_z^α for $\alpha \in (0,1)$ is defined by

$$\left(\partial_z^\alpha f\right)(z) := \left(I_{z_0}^{1-\alpha}(\partial_z f)\right)(z), \ z \in \overline{J},$$

where $\partial_z f = \partial f/\partial z$. This formula is well defined provided $f \in W^{1,p}(J)$. Indeed, the Hausdorff–Young inequality for convolutions yields

$$\left\|\partial_z^\alpha f\right\|_{L^p(z_0,z_0+\mu)} = \left(\int_{z_0}^{z_0+\mu} \left|\partial_z^\alpha f(z)\right|^p \mathrm{d}z\right)^{1/p} \leq \frac{\mu^{1-\alpha}}{\Gamma(2-\alpha)} \|\partial_z f\|_{L^p(z_0,z_0+\mu)} \tag{3.1}$$

for $\mu \in (0,h)$, since $\int_0^\mu z^{-\alpha}\mathrm{d}z = \mu^{1-\alpha}/(1-\alpha)$. Here we identified $\partial_z f$ with $(\partial_z f)\cdot \chi_{(z_0,z_0+\mu)}$ denoting by $\chi_{(z_0,z_0+\mu)}$ the characteristic function.

We next state an interpolation inequality for $\left\|\partial_z^\alpha f\right\|_p = \left\|\partial_z^\alpha f\right\|_{L^p(J)}$.

Lemma 3.1. (Interpolation inequality for the Caputo derivative) *Let $\alpha \in (0,1)$ and $p \in [1,\infty]$. Then the estimate*

$$\left\|\partial_z^\alpha f\right\|_p \leq \frac{2}{\Gamma(1-\alpha)} \|f\|_p^{1-\alpha} \|\partial_z f\|_p^\alpha \tag{3.2}$$

holds true for all $f \in W^{1,p}(J)$ satisfying $f(z_0) = 0$.

Proof. We may assume that $\|\partial_z f\|_p \neq 0$ and $\|f\|_p \neq 0$. Given $\mu \in (0,h)$ and $z \in (z_0+\mu, z_1]$, we subdivide the integral into two parts and integrate by parts to obtain

$$\begin{aligned}
(\partial_z^\alpha f)(z) =& \frac{1}{\Gamma(1-\alpha)}\left(\int_{z_0}^{z-\mu} + \int_{z-\mu}^{z}\right)(z-\zeta)^{-\alpha}\partial_\zeta f(\zeta)\mathrm{d}\zeta \\
=& \frac{1}{\Gamma(1-\alpha)}\left(\int_{z-\mu}^{z} (z-\zeta)^{-\alpha}\partial_\zeta f(\zeta)\mathrm{d}\zeta\right. \\
&\left. + \alpha\int_{z_0}^{z-\mu} (z-\zeta)^{-\alpha-1} f(\zeta)\mathrm{d}\zeta + \mu^{-\alpha} f(z-\mu) - (z-z_0)^{-\alpha} f(z_0)\right).
\end{aligned}$$

Since $f(z_0) = 0$, applying the Hausdorff–Young inequality yields

$$\Big(\int_{z_0+\mu}^{z_1} \big|\partial_z^\alpha f(z)\big|^p \,\mathrm{d}z \Big)^{1/p} \le \frac{\mu^{1-\alpha}}{\Gamma(2-\alpha)} \|\partial_z f\|_p + \frac{1}{\Gamma(1-\alpha)} \mu^{-\alpha} \|f\|_p + \frac{\mu^{-\alpha}}{\Gamma(1-\alpha)} \|f\|_p . \quad (3.3)$$

Combining (3.1) with (3.3), we obtain

$$\big\|\partial_z^\alpha f\big\|_p \le \frac{2\mu^{1-\alpha}}{\Gamma(2-\alpha)} \|\partial_z f\|_p + \frac{2\mu^{-\alpha}}{\Gamma(1-\alpha)} \|f\|_p . \quad (3.4)$$

We obtain the desired estimate by setting $\mu = \|f\|_p / \|\partial_z f\|_p$ in (3.4). $\square$

We next derive an interpolation inequality for the horizontal Laplace operator in the space $L^\infty(L^p)$. Denote by G_t the two-dimensional Gauss kernel, i.e., $G_t(x) = (4\pi t)^{-1} \exp\big(-|x|^2/4t\big)$ for $x \in \mathbb{R}^2$ and $t > 0$ and let $e^{t\Delta_H} f = G_t *_H f$, where $*_H$ denotes convolution in the horizontal variables, only. Then the negative fractional powers of $-\Delta_H$ are defined as

$$(-\Delta_H)^{-\alpha/2} f = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty s^{\frac{\alpha}{2}-1} e^{s\Delta_H} f \,\mathrm{d}s, \quad \alpha \in (0, 2). \quad (3.5)$$

Lemma 3.2. (Interpolation inequality for horizontal derivatives) *Let* $\alpha \in (0, 1)$ *an* $p \in [1, \infty]$. *Then there exists a constant* $C > 0$ *such that*

$$\Big\|\nabla_H(-\Delta_H)^{-\alpha/2} f\Big\|_{\infty,p} \le C \|f\|_{\infty,p}^\alpha \, \|\nabla_H f\|_{\infty,p}^{1-\alpha} \quad (3.6)$$

for all $f \in L^\infty(\mathbb{R}^2; L^p(J))$ *with* $\nabla_H f \in L^\infty(\mathbb{R}^2; L^p(J))$.

Proof. We only give a detailed proof for the case $p = 1$. The proof of the remaining cases can be then adapted to this case. Without loss of generality, we may assume that $\|\nabla_H f\|_{\infty,1} \ne 0$ and $\|f\|_{\infty,1} \ne 0$. Writing for $\mu \in (0, \infty)$

$$|\nabla_H(-\Delta_H)^{-\alpha/2} f| \le \frac{1}{\Gamma(\alpha/2)} \Big(\int_\mu^\infty s^{\frac{\alpha}{2}-1} |\nabla_H e^{s\Delta_H} f| \mathrm{d}s + \int_0^\mu s^{\frac{\alpha}{2}-1} |e^{s\Delta_H} \nabla_H f| \mathrm{d}s \Big).$$

and employing the estimates

$$|\nabla_H e^{s\Delta_H} f| = |(\nabla_H G_s) *_H f| \le |\nabla_H G_s| *_H |f|, \quad |e^{s\Delta_H} \nabla_H f| \le |G_s| *_H |\nabla_H f|.$$

as well as

$$|\partial_i G_t(x)| \le C t^{-1/2} G_{2t}(x), \quad x \in \mathbb{R}^2, t > 0, i = 1, 2, \quad (3.7)$$

an application of Fubini's theorem yields

$$\int_J |\nabla_H(-\Delta_H)^{-\alpha/2} f|(\cdot, z)\mathrm{d}z \le C \int_\mu^\infty s^{\frac{\alpha}{2}-1-\frac{1}{2}} |G_{2s}| *_H \Big(\int_J |f(\cdot, z)|\mathrm{d}z \Big) \mathrm{d}s + C \int_0^\mu s^{\frac{\alpha}{2}-1} |G_s| *_H \Big(\int_J |\nabla_H f(\cdot, z)|\mathrm{d}z \Big) \mathrm{d}s.$$

There exists thus a constant $C > 0$ such that

$$\|\nabla_H(-\Delta_H)^{-\alpha/2} f\|_{\infty,1} = C\mu^{\frac{\alpha}{2}-\frac{1}{2}}\|f\|_{\infty,1} + C\mu^{\frac{\alpha}{2}}\|\nabla_H f\|_{\infty,1}.$$

Choosing $\mu = \left(\|f\|_{\infty,1}/\|\nabla f\|_{\infty,1}\right)^2$, we obtain the desired estimate. □

4. Pointwise and L^∞ bounds for the heat semigroup, Riesz transforms and fractional powers of the Laplacian

In this section, we derive estimates on time and space fractional derivatives for the semigroups $e^{t\Delta_N}$ and $e^{t\Delta_H}$.

Lemma 4.1. (Decay estimates for the heat semigroup acting on fractional derivatives on an interval) *Given $\alpha \in [0, 1]$ and $p \in [1, \infty]$, there exists a constant $C > 0$ such that*

$$\left\|e^{t\Delta_N}\partial_z I^\alpha_{z_0} f\right\|_p \le Ct^{-(1-\alpha)/2}\|f\|_p, \quad t > 0$$

for all $f \in L^p(J)$ satisfying $I^\alpha_{z_0} f(z_1) = 0$.

Recall that for $\alpha = 0$ the operator $I^0_{z_0}$ is interpreted as identity.

Proof. We start by observing that due to duality

$$\left\|e^{t\Delta_N}\partial_z I^\alpha_{z_0} f\right\|_p = \sup\left\{\left\langle e^{t\Delta_N}\partial_z I^\alpha_{z_0} f, \varphi\right\rangle \mid \varphi \in C^\infty_c(J), \|\varphi\|_{p'} \le 1\right\},$$

where $\langle\varphi, \psi\rangle = \int_J \varphi\psi \mathrm{d}z$ and $\frac{1}{p} + \frac{1}{p'} = 1$ for $p \in [1, \infty)$. Note further that

$$\left\langle e^{t\Delta_N}\partial_z I^\alpha_{z_0} f, \varphi\right\rangle = \left\langle \partial_z I^\alpha_{z_0} f, e^{t\Delta_N}\varphi\right\rangle = -\left\langle I^\alpha_{z_0} f, \partial_z e^{t\Delta_N}\varphi\right\rangle,$$

where in the last identity we used the fact that $(I^\alpha_{z_0} f)(z_1) = 0$ and $(I^\alpha_{z_0} f)(z_0) = 0$. Since

$$\left\langle I^\alpha_{z_0} f, \psi\right\rangle = \left\langle f, \overline{I}^\alpha_{z_1}\psi\right\rangle$$

with

$$\overline{I}^\alpha_{z_1}\psi(z) = \frac{1}{\Gamma(\alpha)}\int_z^{z_1}(\xi - z)^{\alpha-1}\psi(\xi)\mathrm{d}\xi,$$

we conclude that

$$\left\langle e^{t\Delta_N}\partial_z I^\alpha_{z_0} f, \varphi\right\rangle = -\left\langle f, \overline{I}^\alpha_{z_1}\partial_z e^{t\Delta_N}\varphi\right\rangle.$$

Since $\overline{I}^\alpha_{z_1}\partial_z$ resembles the Caputo derivative and $\partial_z e^{t\Delta}\varphi(z_1, t) = 0$ by (2.3), we are able to adapt Lemma 3.1 to obtain

$$\left\|\overline{I}^\alpha_{z_1}\partial_z e^{t\Delta_N}\varphi\right\|_{p'} \le \frac{2}{\Gamma(\alpha)}\left\|e^{t\Delta_N}\varphi\right\|^\alpha_{p'}\left\|\partial_z e^{t\Delta_N}\varphi\right\|^{1-\alpha}_{p'}.$$

Notice that $\left\|e^{t\Delta_N}\varphi\right\|_{p'} \leq \|\varphi\|_{p'}$ for all $t > 0$ and there is a $C > 0$ independent of φ such that

$$\left\|\partial_z e^{t\Delta_N}\varphi\right\|_{p'} \leq Ct^{-1/2}\|\varphi\|_{p'} \quad \text{for all} \quad t > 0.$$

This can be seen by extending the problem to the whole space problem, cf. Lemma 5.3 below. To this end, extend φ periodically in a suitable way to $\mathbb{R}$ to obtain $e^{t\Delta_N}\varphi = G_t * \tilde{\varphi}$, where $\tilde{\varphi}$ denotes the extension of φ. Thus,

$$\left\|\overline{I}^{\alpha}_{z_1}\partial_z e^{t\Delta_N}\varphi\right\|_{p'} \leq Ct^{-(1-\alpha)/2}\|\varphi\|_{p'}, \quad t > 0,$$

with $C > 0$ depending on α, only. We thus conclude that

$$\left|\left\langle e^{t\Delta_N}\varphi_z I^{\alpha}_{z_0} f, \varphi\right\rangle\right| \leq \|f\|_p \left\|\overline{I}^{\alpha}_{z_1}\partial_z e^{t\Delta_N}\varphi\right\|_{p'} \leq Ct^{-(1-\alpha)/2}\|f\|_p\|\varphi\|_{p'}, \quad t > 0.$$

The case $p = \infty$ follows by duality from the case $p = 1$. For $\alpha = 1$, the assertion remains true since the $\partial_z I^{\alpha}_{z_0} f = f$ if $f(z_0) = 0$. □

We proceed with pointwise estimates for the heat semigroup $e^{t\Delta}$ on $L^\infty(\mathbb{R}^d)$ combined with Riesz transforms and fractional powers of the Laplacian. The heat semigroup $e^{t\Delta}$ on $L^\infty(\mathbb{R}^d)$ is given by

$$e^{t\Delta}f := G_t * f, \quad \text{where } G_t(x) = (4\pi t)^{-d/2}\exp\left(-|x|^2/4t\right)$$
$$\text{for } x \in \mathbb{R}^d, t > 0 \text{ and } f \in L^\infty(\mathbb{R}^d).$$

We use the Bochner representation formula for the fractional powers of the Laplacian given by

$$(-\Delta)^{-\alpha/2}f = \frac{1}{\Gamma(\alpha/2)}\int_0^\infty s^{\alpha/2-1}(G_s * f)\,\mathrm{d}s, \quad \alpha > 0. \tag{4.1}$$

Using the smoothing effect of $e^{t\Delta}$ for $t > 0$, we obtain

$$e^{t\Delta}(-\Delta)^{\alpha/2}f = (-\Delta)^{-(1-\alpha/2)}(-\Delta)e^{t\Delta}f$$

and the representation (4.1) yields

$$e^{t\Delta}(-\Delta)^{\alpha/2}f = \frac{1}{\Gamma(1-\alpha/2)}\int_0^\infty s^{-\alpha/2}(-\Delta G_{s+t}) * f\,\mathrm{d}s,$$

interpreting $(-\Delta)^0$ hereby as the identity operator. The ith Riesz transform is denoted by

$$R_i = \partial_i(-\Delta)^{-1/2}, \quad \text{where } \partial_i = \partial/\partial x_i \quad \text{for all } 1 \leq i \leq d.$$

Lemma 4.2. (Pointwise bounds for $e^{t\Delta}(-\Delta)^{\alpha/2}$ and $e^{t\Delta}R_iR_j(-\Delta)^{\alpha/2}$) *Let* $d \in \mathbb{N}$.

(i) *Let* $\alpha \in [0, 2]$. *Then there exists* $H_t \in L^1(\mathbb{R}^d)$ *satisfying* $\|H_t\|_1 \leq C$ *for some* $C > 0$ *independent of* $t > 0$ *such that all* $f \in L^\infty(\mathbb{R}^d)$

$$\left|e^{t\Delta}(-\Delta)^{\alpha/2} f(x)\right| \leq t^{-\alpha/2}(H_t * |f|)(x), \quad x \in \mathbb{R}^d,\ t > 0.$$

In particular,

$$\left\|e^{t\Delta}(-\Delta)^{\alpha/2} f\right\|_\infty \leq C t^{-\alpha/2}\|f\|_\infty, \quad t > 0.$$

(ii) *Let* $\alpha \in (0, 2]$. *Then there exists* $\tilde{H}_t \in L^1(\mathbb{R}^d)$ *satisfying* $\|\tilde{H}_t\|_1 \leq C$ *for some* $C > 0$ *independent of* $t > 0$ *such that for all* $f \in L^\infty(\mathbb{R}^d)$

$$\left|e^{t\Delta} R_i R_j (-\Delta)^{\alpha/2} f(x)\right| \leq t^{-\alpha/2}(\tilde{H}_t * |f|)(x), \quad x \in \mathbb{R}^d,\ t > 0.$$

In particular,

$$\left\|e^{t\Delta} R_i R_j (-\Delta)^{\alpha/2} f\right\|_\infty \leq C t^{-\alpha/2}\|f\|_\infty, \quad t > 0.$$

(iii) *There exists* $\breve{H}_t \in L^1(\mathbb{R}^d)$ *satisfying* $\|\breve{H}_t\|_1 \leq C$ *for some* $C > 0$ *independent of* $t > 0$ *such that for all* $f \in L^\infty(\mathbb{R}^d)$

$$\left|e^{t\Delta} R_i R_j \partial_k f(x)\right| \leq t^{-1/2}(\breve{H}_t * |f|)(x), \quad x \in \mathbb{R}^d,\ t > 0.$$

In particular,

$$\left\|e^{t\Delta} R_i R_j \partial_k f\right\|_\infty \leq C t^{-1/2}\|f\|_\infty, \quad t > 0.$$

Remark 4.3. Note that although the Riesz transforms are unbounded operators on $L^\infty(\mathbb{R}^d)$, the compositions of the operators $e^{t\Delta} R_i R_j (-\Delta)^{\alpha/2}$ and $\partial_k e^{t\Delta} R_i R_j$ define nevertheless bounded operators on $L^\infty(\mathbb{R}^d)$ for all $t > 0$.

Proof of Lemma 4.2. Let $\beta \in \mathbb{N}^d$. Then there exists a constant $C = C_{d,\beta} > 0$ such that for all $t > 0$

$$|\partial^\beta G_t| \leq C t^{-|\beta|/2} G_{2t}. \tag{4.2}$$

It follows from (4.2) that

$$\begin{aligned}
|e^{t\Delta}(-\Delta)^{\alpha/2} f| &\leq \frac{C}{\Gamma(1-\alpha/2)} \int_0^\infty s^{-\alpha/2}(s+t)^{-1} G_{2(s+t)} * |f| \,\mathrm{d}s \\
&= \frac{C}{\Gamma(1-\alpha/2)} t^{-\alpha/2} \int_0^\infty u^{-\alpha/2}(u+1)^{-1} G_{2t(u+1)} * |f| \,\mathrm{d}u.
\end{aligned}$$

Setting

$$H_t := \frac{C}{\Gamma(1-\alpha/2)} \int_0^\infty u^{-\alpha/2}(u+1)^{-1} G_{2t(u+1)} \,\mathrm{d}u$$

and observing that $\|H_t\|_1 \le C < \infty$ for all $t > 0$ provided $\alpha \in (0, 2)$ yields estimate (i) for those values of α. For $\alpha = 0$ and $\alpha = 2$, we set $H_t := G_t$ and $H_t := G_{2t}$ respectively, and apply (4.2).

In order to prove estimate (ii), we observe that

$$e^{t\Delta} R_i R_j (-\Delta)^{\alpha/2} f = (-\Delta)^{-(1-\alpha/2)} \partial_i \partial_j e^{t\Delta} f, \quad 1 \le i, j \le d.$$

The case $\alpha = 2$ then follows from (4.2) by setting $\tilde{H}_t := G_{2t}$, whereas for $\alpha \in (0, 2)$ we have

$$e^{t\Delta} R_i R_j (-\Delta)^{\alpha/2} f = \frac{1}{\Gamma(1-\alpha/2)} \int_0^\infty s^{-\alpha/2} (\partial_i \partial_j G_{s+t}) * f \,\mathrm{d}s$$

and thus the same argument used to derive (i) applies.

For (iii) we write

$$e^{t\Delta} R_i R_j \partial_k f = (-\Delta)^{-1} \partial_i \partial_j \partial_k e^{t\Delta} f = \int_0^\infty \partial_i \partial_j \partial_k G_{s+t} * f \,\mathrm{d}s$$

and since by (4.2) we have $|\partial_i \partial_j \partial_k G_{s+t}| \le C(s+t)^{-3/2} G_{2(s+t)}$ for $s, t > 0$ we may set

$$\breve{H}_t := \int_0^\infty (u+1)^{-3/2} G_{2t(u+1)} \,\mathrm{d}u.$$

Then $\|\breve{H}_t\|_1 \le \int_0^\infty (u+1)^{-3/2} \,\mathrm{d}u < \infty$, which yields estimate (iii). The corresponding norm estimates then follow from estimates (i)–(iii) and the Hausdorff–Young inequality. $\square$

5. Anisotropic estimates for the hydrostatic Stokes semigoup

We recall from Sect. 2 that the hydrostatic Stokes semigroup S on $L^\infty_\sigma(L^p(J))$ for $p \in [1, \infty]$ is given by

$$S(t) = e^{t\Delta_H} \otimes e^{t\Delta_N}, \quad t > 0,$$

and that its extension to the larger space $L^\infty(L^p)$ for $p \in [1, \infty]$ is denoted by S_∞. In this section, we give a proof of Proposition 2.2. For this, it is helpful to investigate first the periodic heat semigroup on $L^p(J)$.

Lemma 5.1. (Estimate for the periodic heat semigroup) *Let $\mathbb{T} = \mathbb{R}/\omega_0\mathbb{Z}$ for some $\omega_0 > 0$, $p \in [1, \infty]$ and $f \in L^p(\mathbb{T})$. Then*

$$(G_t * f)(z) = \int_0^{\omega_0} E_t(z-y) f(y) \mathrm{d}y, \quad z \in \mathbb{T}, \quad t > 0,$$

where $E_t(z) = \sum_{k=-\infty}^{\infty} G_t(z - k\omega_0)$ for $z \in \mathbb{T}$. In particular,

$$\|G_t * f\|_{L^p(\mathbb{T})} \le \|f\|_{L^p(\mathbb{T})}, \quad t > 0.$$

Proof. The above representation for $G_t * f$ follows by noting that

$$(G_t * f)(z) = \sum_{k=-\infty}^{\infty} \int_{k\omega_0}^{(k+1)\omega_0} G_t(z-y)f(y)\mathrm{d}y, \quad z \in \mathbb{T}, \quad t > 0,$$

and

$$\int_{k\omega_0}^{(k+1)\omega_0} G_t(z-y)f(y)\mathrm{d}y = \int_0^{\omega_0} G_t(z-y-k\omega_0)f(y+k\omega_0)\mathrm{d}y, \quad t > 0,$$

where $f(y+k\omega_0) = f(y)$ for all $k \in \mathbb{Z}$ by periodicity. The estimate claimed follows by Young's inequality since $\int_0^{\omega_0} E_t(z-y)\mathrm{d}z = \int_{-\infty}^{\infty} G_t(z-y)\mathrm{d}z = 1$ for all $t > 0$ and since $E_t \geq 0$ for all $t > 0$. $\square$

Lemma 5.2. (Derivative estimate for the periodic heat semigroup) *Given the assumptions of Lemma 5.1, there exists a constant $C > 0$, independent of ω_0, such that*

$$|\partial_z(G_t * f)(z)| \leq Ct^{-1/2} \int_0^{\omega_0} E_{2t}(z-y)\,|f(y)|\,\mathrm{d}y, \quad z \in \mathbb{T}, \quad t > 0.$$

In particular,

$$\|\partial_z(G_t * f)\|_{L^p(\mathbb{T})} \leq Ct^{-1/2}\|f\|_{L^p(\mathbb{T})}, \quad t > 0.$$

Proof. By (3.7)

$$|\partial_z G_t(z)| \leq Ct^{-1/2} G_{2t}(z), \quad z \in \mathbb{T}, \quad t > 0,$$

which implies the first assertion. The second one follows by Young's inequality. $\square$

Lemma 5.3. (Periodization) *Given $p \in [1, \infty]$, then there exists a constant $C > 0$ such that for all $f \in L^p(J)$*

$$\left\|e^{t\Delta_N} f\right\|_{L^p(J)} \leq \|f\|_{L^p(J)} \quad \text{and} \quad \left\|\partial_z e^{t\Delta_N} f\right\|_{L^p(J)} \leq Ct^{-1/2}\|f\|_{L^p(J)}, \quad t > 0.$$

Proof. We first extend $f \in L^p(J)$ to $(z_0 - h, z_0)$ by even extension, i.e., by setting $f(z_0 - z) = f(z - z_0)$ for $z \in (z_0 - h, z_0)$ and extend then f to a periodic function f_{per} with period $\omega_0 = 2h$ by $f_{per}(z) = f(z - k\omega_0)$ for $z \in (k\omega_0, (k+1)\omega_0)$ and $k \in \mathbb{Z}$. It then follows that

$$e^{t\Delta_N} f = e^{t\Delta} f_{per}\big|_J,$$

and $\|f\|_{L^p(J)} = \frac{1}{2}\|f_{per}\|_{L^p(-h,h)}$. The desired estimates follow then from Lemma 5.1 and Lemma 5.2. $\square$

Proof of Proposition 2.2. (i) These assertions follow from Lemma 5.3, and from the pointwise estimates

$$\left|\nabla_H e^{t\Delta_H} f\right| \leq Ct^{-1/2} G_{2t} * |f|, \quad \left|e^{t\Delta_H} f\right| \leq G_t * |f|,$$

compare (3.7), as well as $e^{t\Delta_H}\partial_{x_i} f = \partial_{x_i} e^{t\Delta_H} f$ for $i = 1, 2$.

We first prove that $\|\partial_z S_\infty(t) f\|_{\infty,p} \le Ct^{-1/2}\|f\|_{\infty,p}$ for all $t > 0$. By Lemma 5.3

$$\left\|\partial_z S_\infty(t) f(x', \cdot)\right\|_{L^p(J)} \le Ct^{-1/2}\|e^{t\Delta_H} f(x', \cdot)\|_{L^p(J)}$$

for almost all $x' \in \mathbb{R}^2$. By Minkowski's inequality and due to the positivity of $e^{t\Delta_H}$,

$$\|e^{t\Delta_H} f(x', \cdot)\|_{L^p(J)} \le e^{t\Delta_H}\|f(x', \cdot)\|_{L^p(J)},$$

and thus

$$\begin{aligned}\|\partial_z S_\infty(t) f\|_{\infty,p} &\le Ct^{-1/2}\operatorname{ess\,sup}_{x'}\left(e^{t\Delta_H}\|f(x', \cdot)\|_{L^p(J)}\right)\\ &\le Ct^{-1/2}\|f\|_{\infty,p}, \quad t > 0.\end{aligned}$$

We next prove that $\|\nabla_H S_\infty(t) f\|_{\infty,p} \le Ct^{-1/2}\|f\|_{\infty,p}$ for all $t > 0$. To this end, we estimate

$$\|e^{t\Delta_N}\nabla_H e^{t\Delta_H} f(x', \cdot)\|_{L^p(J)} \le \|\nabla_H e^{t\Delta_H} f(x', \cdot)\|_{L^p(J)}.$$

As in the proof of Lemma 5.2, we observe that

$$|\nabla_H e^{t\Delta_H} f(x', z)| \le Ct^{-1/2}\left(G_{2t} *_H |f|\right)(x', z),$$

and applying Minkowski's inequality yields

$$\|\nabla_H e^{t\Delta_H} f(x', \cdot)|_{L^p(J)} \le Ct^{-1/2}(G_{2t} *_H \|f(x', \cdot)\|_{L^p(J)}).$$

We thus conclude that

$$\|\nabla_H S_\infty(t) f\|_{\infty,p} \le Ct^{-1/2}\|f\|_{\infty,p}, \quad t > 0.$$

(ii) Since

$$\left|e^{t\Delta_H} g\right|(x') \le (G_t * |g|)(x'), \quad t > 0,$$

Fubini's theorem implies

$$\left\|e^{t\Delta_H} e^{t\Delta_N}\partial_z I^\alpha_{z_0} f(x', \cdot)\right\|_{L^p(J)} \le G_t * \left\|e^{t\Delta_N}\partial_z I^\alpha_{z_0} f(x', \cdot)\right\|_{L^p(J)}, \quad t > 0,$$

for almost all $x' \in \mathbb{R}^2$. By Lemma 4.1

$$\left\|e^{t\Delta_N}\partial_z I^\alpha_{z_0} f(x', \cdot)\right\|_{L^p(J)} \le Ct^{-(1-\alpha)/2}\|f(x', \cdot)\|_{L^p(J)}, \quad t > 0,$$

which allows us to conclude that

$$\left\|S_\infty(t)\partial_z I^\alpha_{z_0} f\right\|_{\infty,p} \le Ct^{-(1-\alpha)/2}\|G_t\|_1\|f\|_{\infty,p} = Ct^{-(1-\alpha)/2}\|f\|_{\infty,p}, \quad t > 0.$$

The proof is also valid for the case $\alpha = 0$ yielding $\|S_\infty(t)\partial_z f\|_{\infty,p} \le Ct^{-1/2}\|f\|_{\infty,p}$ for all $t > 0$.

(iii) We verify by Lemma 4.2 (i) and (ii) that

$$\begin{aligned}\left\|S_\infty(t)\mathbb{P}(-\Delta_H)^{\alpha/2}f(x',\cdot)\right\|_{L^p(J)} &\leq \left\|e^{t\Delta_H}e^{t\Delta_N}(-\Delta_H)^{\alpha/2}f(x',\cdot)\right\|_{L^p(J)} \\ &\quad+\sum_{1\leq i,j\leq 2}\left\|e^{t\Delta_H}e^{t\Delta_N}R_iR_j(-\Delta_H)^{\alpha/2}\overline{f}\right\|_{L^p(J)} \\ &\leq t^{-\alpha/2}\Big(\left\|H_t *_H |f|(x',\cdot)\right\|_{L^p(J)} \\ &\quad+h\;(\tilde{H}_t *_H \overline{f})(x')\Big),\quad t>0,\end{aligned}$$

for almost all $x'\in\mathbb{R}^2$ since $\overline{f}$ is independent of z. By Fubini's theorem,

$$\int_J |H_t *_H |f|(x',z)|\,\mathrm{d}z = \Big(H_t *_H \int_J |f(\cdot,z)|\,\mathrm{d}z\Big)(x'),\quad \text{for a.a. } x'\in\mathbb{R}^2,$$

which allows us to conclude that

$$\begin{aligned}\left\|S_\infty(t)\mathbb{P}(-\Delta_H)^{\alpha/2}f\right\|_{\infty,p} &\leq t^{-\alpha/2}\Big(\|H_t\|_{L^1(\mathbb{R}^2)}+\|\tilde{H}_t\|_{L^1(\mathbb{R}^2)}\Big)\|f\|_{\infty,p}. \\ &\leq 2Ct^{-\alpha/2}\|f\|_{\infty,p},\quad t>0.\end{aligned}$$

(iv) As above, we have

$$\|S_\infty(t)\mathbb{P}\nabla_H\cdot f\|_{L^p(J)} \leq \left\|\nabla_H e^{t\Delta_H}f\right\|_{L^p(J)}+\sum_{1\leq i,j\leq 2}\left\|e^{t\Delta_H}R_iR_j\nabla_H\cdot\overline{f}\right\|_{L^p(J)}.$$

The first term was already estimated and the second one is treated in the same way as in (iii). □

6. Proof of the main results

In this section, we construct a solution to the integral equation (2.4). We start by estimating the integral term for functions with vanishing vertical average.

Lemma 6.1. *For $\alpha\in[0,1)$, there exists a constant $C>0$ such that*

$$\begin{aligned}\|S(t)\mathbb{P}\nabla\cdot(\tilde{u}\otimes v)\|_{\infty,1} &\leq Ct^{-(1-\alpha)/2}\Big(\|\nabla\tilde{v}\|_{\infty,1}\|v\|_{\infty,1}+\|\tilde{v}\|_{\infty,1}\|\nabla v\|_{\infty,1}\Big)^{1-\alpha} \\ &\quad\times\Big(\|\nabla v\|_{\infty,1}\|\nabla\tilde{v}\|_{\infty,1}\Big)^{\alpha},\quad t>0,\end{aligned}$$

for all $v\in L^\infty_\sigma(L^1)$ satisfying $\overline{v}=0$ and all $\tilde{u}=(\tilde{v},\tilde{w})$ with $\tilde{v}\in L^\infty_\sigma(L^1)$ satisfying $\overline{\tilde{v}}=0$ as well as $\tilde{w}=\int_z^{z_1}\operatorname{div}_H\tilde{v}\,\mathrm{d}x_3$.

Proof. We first note that

$$\nabla\cdot(\tilde{u}\otimes v)=\nabla_H\cdot(\tilde{v}\otimes v)+\partial_z(\tilde{w}v).$$

Since $\mathrm{div}_H \overline{\tilde{v}} = 0$ we obtain $\tilde{w} = 0$ at $z = z_0$ and since $\tilde{w} = 0$ at $z = z_1$ by definition, we see that $\overline{\partial_z(\tilde{w}v)} = 0$. Hence,

$$\mathbb{P}\nabla \cdot (\tilde{u} \otimes v) = \mathbb{P}\nabla_H \cdot (\tilde{v} \otimes v) + \partial_z(\tilde{w}v).$$

The case $\alpha = 0$ is now straightforward using Proposition 2.2 (ii), (iv). Consider now the case $\alpha \in (0, 1)$.

Noting that $(-\Delta_H)^{(1-\alpha)/2}$,$(-\Delta_H)^{-(1-\alpha)/2}$ and ∇_H commute, we write

$$\begin{aligned} S(t)\mathbb{P}\nabla \cdot (\tilde{u} \otimes v) &= S(t)\mathbb{P}(-\Delta_H)^{(1-\alpha)/2}\nabla_H \cdot (-\Delta_H)^{-(1-\alpha)/2}(\tilde{v} \otimes v) \\ &\quad + S(t)\partial_z I_{z_0}^{\alpha} I_{z_0}^{1-\alpha}\partial_z(\tilde{w}v), \quad t > 0, \\ &=: I + II. \end{aligned}$$

Applying Proposition 2.2 (iii) and Lemma 3.2 yields

$$\begin{aligned} \|I\|_{\infty,1} &\le Ct^{-(1-\alpha)/2} \left\|\nabla_H \cdot (-\Delta_H)^{-(1-\alpha)/2}\tilde{v} \otimes v\right\|_{\infty,1} \\ &\le Ct^{-(1-\alpha)/2} \|\nabla_H(\tilde{v} \otimes v)\|_{\infty,1}^{\alpha} \|\tilde{v} \otimes v\|_{\infty,1}^{1-\alpha}, \quad t > 0. \end{aligned}$$

Since $\overline{v} = 0$ and $\overline{\tilde{v}} = 0$, we obtain the estimates

$$\begin{aligned} \|\nabla(\tilde{v} \otimes v)\|_{\infty,1} &\le \|\nabla\tilde{v}\|_{\infty,1} \|v\|_{\infty,\infty} + \|\tilde{v}\|_{\infty,\infty}\|\nabla v\|_{\infty,1}, \\ \|\tilde{v} \otimes v\|_{\infty,1} &\le \|\tilde{v}\|_{\infty,1}\|v\|_{\infty,\infty} + \|\tilde{v}\|_{\infty,\infty}\|v\|_{\infty,1}, \\ \|v\|_{\infty,\infty} &\le \|\partial_z v\|_{\infty,1}, \\ \|\tilde{v}\|_{\infty,\infty} &\le \|\partial_z\tilde{v}\|_{\infty,1} \end{aligned}$$

and the term $\|I\|_{\infty,1}$ can be thus estimated as claimed.

In order to estimate $\|II\|_{\infty,1}$, we observe that Proposition 2.2 (ii) and Lemma 3.1 yield

$$\|II\|_{\infty,1} \le Ct^{-(1-\alpha)/2} \left\|\partial_z^{\alpha}(\tilde{w}v)\right\|_{\infty,1} \le Ct^{-(1-\alpha)/2}\|\tilde{w}v\|_{\infty,1}^{1-\alpha} \|\partial_z(\tilde{w}v)\|_{\infty,1}^{\alpha}, \quad t > 0.$$

Here we invoked the fact that

$$I_{z_0}^{\alpha}\left(I_{z_0}^{1-\alpha}\partial_z(\tilde{w}v)\right)(z_1) = (\tilde{w}v)(z_1) = 0.$$

Since

$$\|\tilde{w}\|_{\infty,\infty} \le C\|\partial_z\tilde{w}\|_{\infty,1} \le C\|\nabla_H\tilde{v}\|_{\infty,1}$$

we are able to estimate $\|II\|_{\infty,1}$ in the same way as I. This completes the proof. $\square$

Our next step consists of proving a similar estimate for the above integral term, however, without assuming that the vertical average of the functions involved is vanishing. To this end, we set

$$\|v\|_{1,\infty,1} := \|v\|_{\infty,1} + \|\nabla v\|_{\infty,1}.$$

Proposition 6.2. (Estimate for the nonlinear term) *Let $\alpha \in [0, 1)$, then there exists a constant $C > 0$ such that*

$$\|S(t)\mathbb{P}\nabla \cdot (\tilde{u} \otimes v)\|_{\infty,1} \leq Ct^{-(1-\alpha)/2}\Big(\|\tilde{v}\|_{1,\infty,1}\|v\|_{\infty,1} + \|v\|_{1,\infty,1}\|\tilde{v}\|_{\infty,1}\Big)^{1-\alpha} \times \Big(\|\tilde{v}\|_{1,\infty,1}\|v\|_{1,\infty,1}\Big)^{\alpha}, \quad t > 0,$$

for all $\tilde{u} = (\tilde{v}, \tilde{w})$ with $\tilde{v} \in L^\infty_\sigma(L^1)$, $\nabla\tilde{v} \in L^\infty(L^1)$ where $\tilde{w} = \int_z^{z_1} \operatorname{div}_H \tilde{v}\, \mathrm{d}x_3$, and $v \in L^\infty_\sigma(L^1)$ satisfying $\nabla v \in L^\infty(L^1)$.

Proof. We argue similarly as in the proof of Lemma 6.1. In order to estimate $\|v\|_{\infty,\infty}$, we write

$$\|v\|_{\infty,\infty} \leq \|v - \overline{v}\|_{\infty,\infty} + \|\overline{v}\|_{\infty,\infty}.$$

Observing that

$$\|v - \overline{v}\|_{\infty,\infty} \leq \|\partial_z v\|_{\infty,1}, \quad \|\overline{v}\|_{\infty,\infty} \leq \|v\|_{\infty,1}$$

we conclude that

$$\|v\|_{\infty,\infty} \leq \|\partial_z v\|_{\infty,1} + \|v\|_{\infty,1}.$$

Thus,

$$\begin{aligned} \|\nabla(\tilde{v} \otimes v)\|_{\infty,1} &\leq \|\tilde{v}\|_{1,\infty,1}\|v\|_{1,\infty,1} \\ \|\tilde{v} \otimes v\|_{\infty,1} &\leq \|\tilde{v}\|_{\infty,1}\|v\|_{1,\infty,1} + \|v\|_{\infty,1}\|\tilde{v}\|_{1,\infty,1}, \end{aligned}$$

and the desired estimate follows as in the proof of Lemma 6.1. □

We now give a proof of our main results.

Proof of Theorem 2.1. Step 1 Consider the sequence (v_m) recursively defined for $t \geq 0$ by

$$\begin{aligned} v_{m+1}(t) &:= S(t)a - \int_0^t S(t-s)\mathbb{P}\nabla \cdot (u_m(s) \otimes v_m(s))\mathrm{d}s, \quad m \in \mathbb{N} \\ v_0(t) &:= S(t)a. \end{aligned}$$

Applying Proposition 2.2 (i), (ii) with $\alpha = 0$, there exists $C > 0$ such that

$$\begin{aligned} \|v_{m+1}(t)\|_{\infty,1} &\leq \|S(t)a\|_{\infty,1} + C\int_0^t (t-s)^{-1/2}\|u_m(s) \otimes v_m(s)\|_{\infty,1}\mathrm{d}s \\ &\leq \|S(t)a\|_{\infty,1} + C\int_0^t (t-s)^{-1/2}\|u_m(s)\|_{\infty,\infty}\|v_m(s)\|_{\infty,1}\mathrm{d}s \\ &\leq \|S(t)a\|_{\infty,1} + C\int_0^t (t-s)^{-1/2}\|v_m(s)\|_{1,\infty,1}\|v_m(s)\|_{\infty,1}\mathrm{d}s. \end{aligned} \tag{6.1}$$

Note that constants $C > 0$ here and below are independent of v_m, u_m and t. We now estimate $\|\nabla v_{m+1}(t)\|_{\infty,1}$ by Proposition 6.2. Since

$$\nabla S(t-s) = \nabla S\big(\tfrac{t-s}{2}\big) S\big(\tfrac{t-s}{2}\big)$$

Proposition 2.2 (i) and Proposition 6.2 with $\alpha = 1/2$ yield

$$\begin{aligned}\|\nabla v_{m+1}(t)\|_{\infty,1} \le{}& \|\nabla S(t)a\|_{\infty,1} \\ &+C\int_0^t (t-s)^{-1/2}(t-s)^{-1/4}\|v_m(s)\|_{1,\infty,1}^{3/2}\|v_m(s)\|_{\infty,1}^{1/2}\mathrm{d}s, \quad t>0.\end{aligned} \tag{6.2}$$

Note that in the above estimate we may also take any $\alpha \in (0, 1)$. For $m \in \mathbb{N} \cup \{0\}$ and $t > 0$ we now set

$$\begin{aligned}K_m(t) &:= \sup_{0<\tau<t} \tau^{1/2}\|v_m(\tau)\|_{1,\infty,1},\\ H_m(t) &:= \sup_{0<\tau<t} \|v_m(\tau)\|_{\infty,1},\\ M_m(t) &:= \sup_{0<\tau<t} \tau^{1/2}\|v_m(\tau)\|_{\infty,1}.\end{aligned}$$

Estimate (6.1) combined with $\|S(t)a\|_{\infty,1} \le \|a\|_{\infty,1}$ for all $t > 0$ yields

$$H_{m+1}(t) \le \|a\|_{\infty,1} + CK_m(t)H_m(t), \quad t > 0. \tag{6.3}$$

Multiplying (6.2) by $t^{1/2}$ yields

$$\sup_{0<\tau<t} \tau^{1/2}\|\nabla v_{m+1}(\tau)\|_{\infty,1} \le \sup_{0<\tau<t} \tau^{1/2}\,\|\nabla S(\tau)a\|_{\infty,1} + CK_m(t)^{3/2}H_m(t)^{1/2}, \quad t > 0, \tag{6.4}$$

and by multiplying (6.1) with $t^{1/2}$, we obtain

$$M_{m+1}(t) \le \sup_{0<\tau<t} \tau^{1/2}\,\|S(\tau)a\|_{\infty,1} + Ct^{1/2}K_m(t)H_m(t), \tag{6.5}$$

provided $t \le T$ for some $T \le 1$. Adding (6.4) and (6.5) yields

$$K_{m+1}(t) \le K_0(t) + CK_m(t)^{3/2}H_m(t)^{1/2} + Ct^{1/2}K_m(t)H_m(t), \quad m \ge 0 \tag{6.6}$$

with

$$K_0(t) = \sup_{0<\tau<t} \tau^{1/2}\,\|S(\tau)a\|_{1,\infty,1}\,.$$

By assumption $K_0(t) \le \varepsilon_0$ and by the following Lemma 6.3, the sequences (H_m) and (K_m) are thus uniformly bounded provided ε_0 is sufficiently small.

It is not difficult to prove that (v_m) is a Cauchy sequence in $L^\infty\big((0, t_0), L^\infty_\sigma(L^1)\big)$ and that $(t^{1/2}\nabla(v_m))$ is a Cauchy sequence in $L^\infty\big((0, t_0), L^\infty(L^1)\big)$. Hence, v as the limit of (v_m), satisfies the desired estimate. Moreover, $v_m, \nabla v_m \in C\left((0,t_0); L^\infty(L^1)\right)$ and therefore, $v \in C\left((0,t_0); \mathrm{BUC}_\sigma(L^1)\right)$. This proves assertion (a).

In order to prove (b) let $a_1 \in \mathrm{BUC}_\sigma(L^1)$, then

$$\tau^{1/2}\, \|\nabla S(\tau)a_1\|_{\infty,1} \to 0 \quad \text{as} \quad \tau \searrow 0, \quad \text{and} \quad \tau^{1/2}\, \|\nabla S(\tau)a_2\|_{\infty,1} \leq C\, \|a_2\|_{\infty,1}$$

by Proposition 2.2 (i).

Thus, by (6.3) and (6.6), the sequences (H_m) and (K_m) fulfill the assumptions of the following Lemma 6.3 provided t is small enough, say $t \leq t_0$, since $K_0(t) \to 0$ as $t \searrow 0$, and $\|a_2\|_{\infty,1}$ is sufficiently small. The sequences (H_m) and (K_m) are thus uniformly bounded.

To prove (c), assume that $a_2 = 0$. Then one can use strong continuity of the semigroup to prove that (v_m) is a Cauchy sequence in $C\left([0,t_0], \mathrm{BUC}_\sigma(L^1)\right)$ and $(t^{1/2}\nabla v_m)$ is a Cauchy sequence in $C\left([0,t_0], L^\infty(L^1)\right)$. Hence, v as the limit of (v_m), has the desired regularity.

The proof of the uniqueness follows in both cases a similar line of arguments. We only give a detailed proof for (b). Let v, $\tilde{v}$ be two solutions, then

$$(v-\tilde{v})(t) = \int_0^t S(t-s)\mathbb{P}\nabla\cdot(u(s)\otimes(v-\tilde{v})(s) + (u-\tilde{u})\otimes\tilde{v}(s))\,\mathrm{d}s, \quad t>0,$$

and one obtains as above using Propositions2.2 (i) and 6.2 with $\alpha = 1/2$, and setting

$$K(v)(t) := \sup_{0<\tau<t} \tau^{1/2}\|v(\tau)\|_{1,\infty,1} \quad \text{and} \quad H(v)(t) := \sup_{0<\tau<t} \|v(\tau)\|_{\infty,1}$$

that for $N(v)(t) := \max\{K(v)(t), H(v)(t)\}$ one has

$$\begin{aligned} N(v) &\leq C(K(v)H(v-\tilde{v}) + K(v-\tilde{v})H(v))^{1/2}(K(v)K(v-\tilde{v}))^{1/2} \\ &\quad + C(K(\tilde{v})H(v-\tilde{v}) + K(v-\tilde{v})H(\tilde{v}))^{1/2}(K(\tilde{v})K(v-\tilde{v}))^{1/2}. \end{aligned}$$

Hence, one obtains

$$N(v-\tilde{v}) \leq N(v-\tilde{v})C\left\{(K(v)+H(v))^{1/2}K(v)^{1/2} + (K(\tilde{v})+H(\tilde{v}))^{1/2}K(\tilde{v})^{1/2}\right\}. \tag{6.7}$$

By assumption, if t is small we have

$$K(\tilde{v})(t), K(v)(t) \leq C\,\|a_2\|_{\infty,1}, \quad \text{and} \quad H(\tilde{v}), H(v)(t) \leq C\,\|a\|_{\infty,1}.$$

Thus supposing that t and $\|a_2\|_{\infty,1}\cdot\|a\|_{\infty,1}$ are small enough, one has

$$C\,\{(K(v)+H(v))K(v) + (K(\tilde{v})+H(\tilde{v}))K(\tilde{v})\} < 1,$$

and therefore by (6.7) one has $K(v-\tilde{v}) = 0$ on $(0,T_0)$ and $H(v-\tilde{v}) = 0$ on $[0,T_0]$ for some $0 < T_0 \leq T$. Iterating this argument it follows that the solutions are unique on $[0,T]$.

Step 2 Since $e^{t\Delta_H}$ as well as $\mathbb{P}$ and the nonlinearity leave horizontal periodicity-invariant, we see that if a is in addition to the assumption of Theorem 2.1 periodic with respect to the horizontal variables, then this is also true for $v(t)$ for any $t > 0$.

In order to extend the local solutions to a global one, we make use of the regularization of the solution v. By assertion (b), $v(t_0)$ and $\nabla v(t_0) \in \mathrm{BUC}(L^1)$ for some $t_0 > 0$ and in particular we have $v(t_0) \in \mathrm{BUC}(W^{1,1})$. Since $W^{1,1}(J) \hookrightarrow L^p(J)$ for all $p \in [1, \infty]$, we obtain $v(t_0) \in \mathrm{BUC}(L^p)$. Proposition 2.4 yields that $v(t_1)$ and $\nabla v(t_1) \in \mathrm{BUC}(L^p)$ for $t_1 > t_0$, and in particular the restriction $v(t_1)|_{[0,1]^2 \times J}$ satisfies

$$v(t_1)|_{[0,1]^2\times J} \in \{v \in W^{1,p}([0,1]^2 \times J) \mid v \text{ periodic in } x, y, \operatorname{div}_H \overline{v} = 0\} \quad 0 < t_1 < T.$$

Using $v(t_1)$ for $p \geq 2$ as new initial value, it follows from [7] or [19] that v extends to a global strong solution proving assertion (d). □

It remains to prove the uniform boundedness of the sequences (H_m) and (K_m) defined in the above proof.

Lemma 6.3. *Let $A, \varepsilon > 0$ be constants and assume that $(H_m) \subset \mathbb{R}$ and $(K_m) \subset \mathbb{R}$ are sequences satisfying*

$$\begin{aligned} &H_0 \leq A, \quad H_{m+1} \leq A + C H_m K_m, \\ &K_0 \leq \varepsilon, \quad K_{m+1} \leq \varepsilon + C K_m^{3/2} H_m^{1/2} + (4A)^{-1} K_m H_m, \end{aligned}$$

for all $m \geq 0$ and a constant $C > 0$ independent of m. Then there exists $\varepsilon_0 = \varepsilon_0(C, A) > 0$ such that (K_m) and (H_m) are bounded sequences provided that $\varepsilon \leq \varepsilon_0$.

Proof. Note first that if $K_m \leq 1/(2C)$ for $m \leq m_0$, then $H_m \leq 2A$ for all $m \leq m_0+1$. Next, we choose ε small enough so that the graphs of $y = x$ and
$y = \varepsilon + \sqrt{2A}Cx^{3/2} + x/2$ have an intersection. Denote by $x_0(\varepsilon)$ the abscissa of the intersection point closest to $x = 0$. Clearly $x_0(\varepsilon) \searrow 0$ as $\varepsilon \to 0$.

Choose now ε_0 so small that $x_0(\varepsilon_0) < 1/(2C)$. Then, $K_m \leq x_0(\varepsilon)$ and $H_m \leq 2A$ for all $m \geq 1$ provided $\varepsilon \leq \varepsilon_0$. Indeed, we proved this by induction. The estimate is trivial for $m = 1$. Assume that $K_m \leq x_0(\varepsilon)$, $H_m \leq 2A$ for all $m \leq m_0$. Since $x_0(\varepsilon) < 1/(2C)$, the inequality for H_m implies $H_{m+1} \leq 2A$ and the inequality for K_m implies $K_{m+1} \leq x_0(\varepsilon)$ by the choice of $x_0(\varepsilon)$ since $H_m \leq 2A$. We thus conclude that $K_m \leq x_0(\varepsilon)$ and $H_m \leq 2A$. □

We finally are able to prove Propositions 2.4 and 2.5. The solution v constructed in Theorem 2.1 exists at least for some nontrivial time interval $[0, T)$, where $T > 0$ depends on a. Given $a \in \mathrm{BUC}_\sigma(L^p)$ for some $p > 1$ we are unfortunately unable to estimate T from below by terms involving the norm of a, only. However, in Proposition 2.4 we claim that $v \in C([0, T), \mathrm{BUC}_\sigma(L^p))$ for all $p \in (1, \infty)$ in the same time interval.

Proof of Proposition 2.4. We estimate the integral equation (2.4) by writing $S(t) = S(\frac{t}{2})S(\frac{t}{2})$ and using the L^p-L^1-estimate from Proposition 2.2 (v) and Proposition 6.2 with $\alpha = 0$ to obtain

$$\begin{aligned}\|v(t)\|_{\infty,p} &\leq \|S(t)a\|_{\infty,p} + C\int_0^t (t-s)^{-(1-1/p)/2}(t-s)^{-1/2}\|v(s)\|_{1,\infty,1}\|v(s)\|_{\infty,1}\mathrm{d}s,\\ &\leq \|S(t)a\|_{\infty,p} + C\int_0^t (t-s)^{-(1-1/2p)}s^{-1/2}s^{1/2}\|v(s)\|_{1,\infty,1}\|v(s)\|_{\infty,1}\mathrm{d}s, \quad t>0.\end{aligned}$$

Since $a \in L^\infty_\sigma(L^p)$ and $v \in L^\infty\left(0,T; L^\infty_\sigma(L^1)\right)$ by Theorem 2.1, we see that $t^{1/2-1/2p}v$ is in $L^\infty\left(0,T; L^\infty_\sigma(L^p)\right)$.

Note that

$$\nabla S(t) = \nabla S\left(\tfrac{t}{3}\right)S\left(\tfrac{t}{3}\right)S\left(\tfrac{t}{3}\right), \quad t>0.$$

Differentiating (2.4), applying Proposition 2.2 (i) and the L^p-L^1-estimate from Proposition 2.2 (v) as well as Proposition 6.2 with $\alpha \in (0,1)$ yields

$$\begin{aligned}&\|\nabla v(t)\|_{\infty,p} \leq \|\nabla S(t)a\|_{\infty,p}\\ &+C\int_0^t (t-s)^{-(1-1/2p+(1-\alpha)/2)}s^{-(1+\alpha)/2}s^{(1+\alpha)/2}\|v(s)\|^{1+\alpha}_{1,\infty,1}\|v(s)\|^{1-\alpha}_{\infty,1}\mathrm{d}s.\end{aligned}$$

This gives the desired bound for $t^{1-2/p}\nabla v$ and the continuity of v follows from strong continuity of S. □

Proof of Proposition 2.5. We argue similarly as in the proof of Theorem 2.1. Setting

$$L_m(t) := \sup_{0<\tau<t} \tau^\mu \|v_m(\tau)\|_{1,\infty,1}, \quad 0<t<T,$$

we obtain by (6.1) for $m \geq 0$ and $t \in (0,T)$

$$H_{m+1}(t) \leq \|a\|_{\infty,1} + Ct^{1/2-\mu}L_m(t)H_m(t)$$

instead of (6.3). Similarly, instead of (6.6), we obtain now

$$L_{m+1}(t) \leq \|a\|_{\infty,1} + [a]_\mu + Ct^{(1/2-\mu)/2}L_m^{3/2}(t)H_m(t)^{1/2} + Ct^{1/2}L_m(t)H_m(t).$$

It follows that if T fulfills $1/T \geq \min\left(C|||a|||, 1\right)^{2/(1/2-\mu)}$ for some $C>0$ independent of a, then (L_m) and (H_m) are bounded sequences for $t \in [0,T]$. Moreover, (v_m) is a Cauchy sequence in $C\left([0,T], \mathrm{BUC}_\sigma(L^1)\right)$, which is proved as before. □

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