INTERFACE LAYERS AND COUPLING CONDITIONS ON NETWORKS FOR LINEARIZED KINETIC BGK EQUATION

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Abstract. We consider a linearized kinetic BGK equation and the associated acoustic system on a network. Coupling conditions for the macroscopic equations are derived from the kinetic conditions via an asymptotic analysis near the nodes of the network. This analysis leads to the consideration of a fixpoint problem involving the solutions of kinetic half-space problems. This work extends the procedure developed in [13], where coupling conditions for a simplified BGK model have been derived. Numerical comparisons between different coupling conditions confirm the accuracy of the proposed approximation.

Keywords. Kinetic layer, coupling condition, kinetic half-space problem, networks

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1. Introduction. Coupling conditions for macroscopic partial differential equations on networks including scalar hyperbolic equations and hyperbolic systems like the wave equation or Euler type models have been discussed in many works, see [4, 5, 10, 11, 14, 16–18, 22, 28]. In [17, 29] coupling conditions for scalar hyperbolic equations on networks are discussed and investigated. [2] treats the wave equation and general nonlinear hyperbolic systems are considered in [4, 5, 10, 14, 18]. We finally note, that, for example, for hyperbolic systems on networks there are still many unsolved problems, like finding suitable coupling conditions without restricting to subsonic situations.

On the other hand, coupling conditions for kinetic equations on networks have been discussed in a much smaller number of publications, see, for example, [12, 13, 30, 31]. In [12] a first attempt to derive a coupling condition for a macroscopic equation from the underlying kinetic model has been presented for the case of a kinetic equations for chemotaxis. In [13] a more general and more accurate procedure has been presented motivated by the classical procedure to find kinetic slip boundary conditions for macroscopic equations based on the analysis of the kinetic layer [3,7,8,23,24,33,36–38]. In this work coupling conditions for the wave equation on a network have been derived from a simplified BGK equation via an asymptotic analysis of the situation near the nodes. In the present paper we extend this approach to the full linear BGK problem [15, 27] with the linearized Euler equations as limit equations for small Knudsen numbers.

The paper is organized as follows. In section 2, we present the basic kinetic and macroscopic equations defined on a graph. Section 3 describes the half-moment approximation of the kinetic equation. Section 4 contains the general procedure to obtain coupling conditions for the macroscopic equations from the kinetic conditions. In section 5 the application of the half moment approximation to determine accurate approximate solutions of the kinetic half-space problem is discussed. Section 6 contains a derivation of coupling conditions for the macroscopic problem using the half-moment approximation. These conditions include an approximation of the kinetic quantities in the fluid dynamic limit at the nodes. Finally, in section 7, we give

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a numerical investigation of the kinetic and macroscopic equations on the network. The results show the very good approximation of the underlying kinetic model by the macroscopic model with the coupling conditions derived here.

2. Kinetic and macroscopic equations. For $f = f(x, v, t)$ with $x \in \mathbb{R}$ and $v \in \mathbb{R}$ at time $t \in [0, T]$ we consider the linear kinetic BGK model

$$\partial_t f + v\partial_x f = -\frac{1}{\epsilon}Q(f) = -\frac{1}{\epsilon}\left(f - \left(\rho + vq + \frac{1}{2}(v^2-1)(S-\rho)\right)M(v)\right), \tag{2.1}$$

where density, mean flux and total energy are given by

$$\rho = \int_{-\infty}^{\infty} f(v)dv, \quad q = \int_{-\infty}^{\infty} vf(v)dv, \quad S = \int_{-\infty}^{\infty} v^2 f(v)dv$$

and the standard Maxwellian is defined by

$$M(v) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{v^2}{2}).$$

The associated limit equation for $\epsilon \to 0$ is the acoustic system

$$\begin{aligned} \partial_t \rho + \partial_x q &= 0, \\ \partial_t q + \partial_x S &= 0, \\ \partial_t S + a^2 \partial_x q &= 0 \end{aligned} \tag{2.2}$$

with $a^2 = 3$. See [6,9,25] for the derivation of the acoustic limit from Boltzmann and BGK equation. (2.2) is the linearization of the compressible Euler equations around a state with velocity zero. This is a linear strictly hyperbolic system with eigenvalues $-a, 0$ and $+a$. On a bounded interval, here $[0, 1]$, we prescribe for the kinetic equation boundary values at $x = 0$ and $x = 1$. The values of f for all ingoing velocities have to be fixed, i.e.

$$f(0, v), v > 0 \quad \text{and} \quad f(1, v), v < 0.$$

For (2.2) boundary conditions should be formulated in characteristic variables. The corresponding Riemann invariants are

$$r_- = \ S - aq, \qquad r_0 = \ S - a^2\rho, \qquad r_+ = \ S + aq. \tag{2.3}$$

At the left boundary the data for the Riemann invariant associated to the positive eigenvalue $r_+ = S + aq$ have to be prescribed, on the right end the Riemann invariant for the negative eigenvalue $r_- = S - aq$. The Riemann invariant r_0 associated to the 0-characteristic is not specified at the boundaries.

3. Half-moment approximation. In this section we will construct a half-moment approximation for the kinetic equation (2.1). We refer, for example, to [21]. We will use this approximation in the following sections to obtain explicit and accurate coupling conditions for the equations in the fluid limit. We define

$$\begin{aligned} \rho^- &= \int_{-\infty}^{0} f(v)dv, \qquad q^- = \int_{-\infty}^{0} vf(v)dv, \qquad S^- = \int_{-\infty}^{0} v^2 f(v)dv, \\ \rho^+ &= \int_{0}^{\infty} f(v)dv, \qquad q^+ = \int_{0}^{\infty} vf(v)dv, \qquad S^+ = \int_{0}^{\infty} v^2 f(v)dv. \end{aligned} \tag{3.1}$$

As closure assumption, we use the following approximation of the distribution function f by quadratic functions in v to determine half-moment equations

$$f(v) = (a^+ + vb^+ + v^2c^+)M(v), v \geq 0, \tag{3.2}$$
$$f(v) = (a^- + vb^- + v^2c^-)M(v), v \leq 0.$$

Inserting (3.2) into (3.1) leads to

$$\begin{aligned}
\rho^{\mp} &= \frac{1}{2}a^{\mp} \mp \frac{1}{\sqrt{2\pi}}b^{\mp} + \frac{1}{2}c^{\mp}, \\
q^{\mp} &= \mp\frac{1}{\sqrt{2\pi}}a^{\mp} + \frac{1}{2}b^{\mp} \mp \frac{2}{\sqrt{2\pi}}c^{\mp}, \\
S^{\mp} &= \frac{1}{2}a^{\mp} \mp \frac{2}{\sqrt{2\pi}}b^{\mp} + \frac{3}{2}c^{\mp},
\end{aligned}$$

and inverting gives

$$\begin{aligned}
a^{\mp} &= \frac{1}{\pi-3}\left((3\pi-8)\rho^{\mp} \pm \sqrt{2\pi}q^{\mp} - (\pi-4)S^{\mp}\right), \\
b^{\mp} &= \frac{1}{\pi-3}\left(\pm\sqrt{2\pi}\rho^{\mp} + 2\pi q^{\mp} \pm \sqrt{2\pi}S^{\mp}\right), \\
c^{\mp} &= \frac{1}{\pi-3}\left(-(\pi-4)\rho^{\mp} \pm \sqrt{2\pi}q^{\mp} + (\pi-2)S^{\mp}\right).
\end{aligned}$$

Furthermore, we obtain the following approximation of the higher order half moments

$$\begin{aligned}
\int_0^\infty v^3 f(v)dv &= \frac{2}{\sqrt{2\pi}}a^+ + \frac{3}{2}b^+ + \frac{8}{\sqrt{2\pi}}c^+ \\
&= \frac{-(5\pi-16)\rho^+ + \sqrt{2\pi}(3\pi-10)q^+ + (3\pi-8)S^+}{\sqrt{2\pi}(\pi-3)}, \qquad (3.3) \\
\int_{-\infty}^0 v^3 f(v)dv &= -\frac{2}{\sqrt{2\pi}}a^- + \frac{3}{2}b^- - \frac{8}{\sqrt{2\pi}}c^- \\
&= \frac{(5\pi-16)\rho^- + \sqrt{2\pi}(3\pi-10)q^- - (3\pi-8)S^-}{\sqrt{2\pi}(\pi-3)}. \qquad (3.4)
\end{aligned}$$

Multiplying (2.1) with $1, v$ and v^2 and integrating over positive and negative velocity domains, we can obtain a closed system for the half-moment approximation of the kinetic equation. With the notation

$$\alpha := \frac{5\pi-16}{\sqrt{2\pi}(\pi-3)}, \quad \beta := \frac{3\pi-10}{(\pi-3)}, \quad \gamma := \frac{3\pi-8}{\sqrt{2\pi}(\pi-3)},$$

the system for the half-moments reads

$$
\begin{aligned}
&\partial_t \rho^+ + \partial_x q^+ &&= \frac{-1}{\epsilon}\left(\rho^+ - \left(\frac{1}{2}\rho + \frac{1}{\sqrt{2\pi}}q\right)\right)\\
&\partial_t q^+ + \partial_x S^+ &&= \frac{-1}{\epsilon}\left(q^+ - \left(\frac{\rho}{\sqrt{2\pi}} + \frac{q}{2} + \frac{1}{\sqrt{8\pi}}(S-\rho)\right)\right)\\
&\partial_t S^+ + \partial_x(-\alpha\rho^+ + \beta q^+ + \gamma S^+) &&= \frac{-1}{\epsilon}\left(S^+ - \left(\frac{1}{2}\rho + \frac{2}{\sqrt{2\pi}}q + \frac{1}{2}(S-\rho)\right)\right)\\
&\partial_t \rho^- + \partial_x q^- &&= \frac{-1}{\epsilon}\left(\rho^- - \left(\frac{1}{2}\rho - \frac{1}{\sqrt{2\pi}}q\right)\right)\\
&\partial_t q^- + \partial_x S^- &&= \frac{-1}{\epsilon}\left(q^- - \left(-\frac{\rho}{\sqrt{2\pi}} + \frac{q}{2} - \frac{1}{\sqrt{8\pi}}(S-\rho)\right)\right)\\
&\partial_t S^- + \partial_x(\alpha\rho^- + \beta q^- - \gamma S^-) &&= \frac{-1}{\epsilon}\left(S^- - \left(\frac{1}{2}\rho - \frac{2}{\sqrt{2\pi}}q + \frac{1}{2}(S-\rho)\right)\right).
\end{aligned}
\tag{3.5}
$$

On a finite interval this system requires boundary conditions on both sides. For the left boundary the positive quanitites ρ^+, q^+, S^+ and for the right boundary the negative quantities ρ^-, q^-, S^- have to be prescribed.

For later use we introduce the even-odd variables

$$
\begin{aligned}
\rho &= \rho^+ + \rho^-, & q &= q^+ + q^-, & S &= S^+ + S^-,\\
\hat{\rho} &= \rho^+ - \rho^-, & \hat{q} &= q^+ - q^-, & \hat{S} &= S^+ - S^-
\end{aligned}
$$

and rewrite the system as

$$
\begin{aligned}
&\partial_t \rho + \partial_x q &&= 0\\
&\partial_t q + \partial_x S &&= 0\\
&\partial_t S + \partial_x(-\alpha\hat{\rho} + \beta q + \gamma\hat{S}) &&= 0\\
&\partial_t \hat{\rho} + \partial_x \hat{q} &&= -\frac{1}{\epsilon}\left(\hat{\rho} - \frac{2}{\sqrt{2\pi}}q\right)\\
&\partial_t \hat{q} + \partial_x \hat{S} &&= -\frac{1}{\epsilon}\left(\hat{q} - \frac{2}{\sqrt{2\pi}}\rho - \frac{1}{\sqrt{2\pi}}(S-\rho)\right)\\
&\partial_t \hat{S} + \partial_x(-\alpha\rho + \beta\hat{q} + \gamma S) &&= -\frac{1}{\epsilon}\left(\hat{S} - \frac{4}{\sqrt{2\pi}}q\right).
\end{aligned}
\tag{3.6}
$$

Obviously, for $\epsilon \to 0$ one obtains the same limit problem (2.2) as before.

3.1. Coupling conditions for the kinetic model. To consider the above equations on a network coupling conditions have to be imposed at the nodes. In the following, we consider a node connecting n edges, which are oriented away from the node, as in Fig.3.1. Each edge i is parameterized by the interval $[0, \infty]$ and the kinetic and macroscopic quantities are denoted by f_i and ρ_i, q_i, S_i respectively. A possible choice of coupling conditions for the kinetic problem are given by

$$
f^i(0, v) = \sum_{j=1}^{n} c_{ij} f^j(0, -v), \quad v > 0, \tag{3.7}
$$

where the ingoing quantities are distributed according to given parameters $c_{i,j}$, $i, j = 1, \ldots, n$. Further properties of these coupling conditions can be found in [12,13]. The

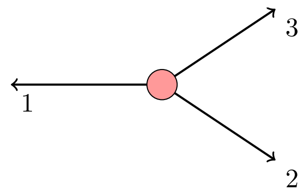

Fig. 3.1: *Node connecting three edges and orientation of the edges.*

total mass in the system is conserved, if

$$\sum_{i=1}^{n} c_{ij} = 1$$

holds for any $j = 1, \dots, n$. In the following, we use the vector notation

$$f^+ = C f^-, \quad v > 0,$$

where $C = (c_{i,j})_{i,j=1,\cdots,n}$ and $f^{\pm} = (f^1(0, \pm v), \dots, f^n(0, \pm v))^T$.

The coupling condition for the half-moment approximation is easily found from the kinetic coupling conditions by integration with respect to v. We obtain

$$\begin{aligned}\int_0^{\infty} \begin{pmatrix} 1 \\ v \\ v^2 \end{pmatrix} f(0,v) dv &= \sum_{j=1}^{n} \int_0^{\infty} \begin{pmatrix} 1 \\ v \\ v^2 \end{pmatrix} c_{ij} f^j(0,-v) dv \\ &= \sum_{j=1}^{n} \int_{-\infty}^{0} \begin{pmatrix} 1 \\ -v \\ v^2 \end{pmatrix} c_{ij} f^j(0,v) dv\end{aligned}$$

or

$$\begin{pmatrix} \rho^+ \\ q^+ \\ S^+ \end{pmatrix} = \sum_{j=1}^{n} c_{ij} \begin{pmatrix} \rho^- \\ -q^- \\ S^- \end{pmatrix}. \tag{3.8}$$

The coupling conditions for the macroscopic quantities are conditions on the characteristic variables

$$S_B^i + a q_B^i$$

using the given values of

$$S_B^i - a q_B^i,$$

where the subscript B denotes the value of the macroscopic solution at the boundary or node. In the following, we discuss how to determine macroscopic conditions from the kinetic coupling conditions (3.7) and perform an explicit procedure for the half-moment approximation with coupling conditions (3.8).

4. Derivation of coupling conditions for macroscopic equations via kinetic layer analysis. We consider first the classical procedure to derive boundary conditions for macroscopic equations from the underlying kinetic equation.

4.1. Boundary conditions. Consider the left boundary of the interval $[0,1]$. A rescaling of the spatial variable in equation (2.1) with $x \to \frac{x}{\epsilon}$, gives

$$\partial_t f + \frac{1}{\epsilon} v \partial_x f = \frac{1}{\epsilon} Q(f).$$

This yields to first order in ϵ the stationary kinetic half-space problem for the layer function $f_L = f_L(x)$ with $x \in [0, \infty)$

$$v \partial_x f_L = Q(f_L).$$

Such half-space problems have been investigated in many papers, see, for example, [1,20,26]. One obtains a well-posed problem, if on the one hand, at $x = 0$, a boundary condition for the half space problem is prescribed

$$f_L(0, v) = k(v) = f(0, v), \quad v > 0.$$

Moreover, an additional condition prescribing the value of the Riemann Invariant (2.3) $r_- = S - aq$ of the macroscopic system (2.2) is required

$$\int_{-\infty}^{\infty} (v^2 - av) f_L(\infty, v) dv = \int_{-\infty}^{\infty} (v^2 - av) f_L(x, v) dv = S_B - aq_B.$$

Note that $\int_{-\infty}^{\infty} v^k f_L(x, v) dv, k = 1, 2$ are invariants for the half-space problem.

The boundary condition for (2.2) is obtained by determining r_+ from the asymptotic solution of the half space problem by setting $r_+ = S_\infty + aq_\infty$, where the values ρ_∞, q_∞ and S_∞ are the macroscopic quantities associated to the solution of the half-space problem at infinity.

Then, the solution of the half space problem can be used to determine the outgoing distribution

$$f(0, v) = f_L(0, v), \quad v < 0$$

in the limit $\epsilon \to 0$.

4.2. Coupling conditions for the macroscopic model. We use the above procedure to determine the coupling conditions for the macroscopic equations. Starting from the kinetic coupling conditions (3.7) we determine the coupling conditions for the macroscopic equations in the following way. From the kinetic coupling conditions we obtain conditions on the in- and outgoing solutions of the half space problems on the different arcs. That means the layer functions satisfy

$$f_L^i(0, v) = \sum_{j=1}^{n} c_{ij} f_L^j(0, -v), \quad v > 0.$$

If the outgoing values $f_L^i(0, v)$ are determined via the solution of the half-space problems on arc then, this is a fix point equation for $f_L(0, v)) = (f_L^1, \ldots, f_L^n)(0, v), v > 0$. Note that, to solve the half-space problems, we need the additional conditions

$$\int_{-\infty}^{\infty} (v^2 - av) f_L^i(0, v) dv = S_B^i - aq_B^i,$$

which connect the values of the layer to the states in the domain.

Then, the outgoing characteristics on the different arcs are determined by

$$r_+^i(0) = S_\infty^i[k^i] + aq_\infty^i[k^i],$$

which gives the coupling values for the acoustic system.

In the next section, we use the half moment approximation to find numerically tractable expressions for these coupling conditions. This approximation is numerically compared to the solution of the full kinetic equation in the last section.

5. Half-moment half-space problem. Proceeding as for the original kinetic problem and rescaling the spatial variable at the node in the half-moment problem with ϵ and neglecting lower order terms we obtain the half-space problem of (3.6) for $x \in [0,\infty)$

$$\begin{aligned}
\partial_x q &= 0 \\
\partial_x S &= 0 \\
\partial_x(-\alpha\hat{\rho} + \beta q + \gamma\hat{S}) &= 0 \\
\partial_x \hat{q} &= -\left(\hat{\rho} - \frac{2}{\sqrt{2\pi}} q\right) \\
\partial_x \hat{S} &= -\left(\hat{q} - \frac{2}{\sqrt{2\pi}}\rho - \frac{1}{\sqrt{2\pi}}(S-\rho)\right) \\
\partial_x(-\alpha\rho + \beta\hat{q} + \gamma S) &= -\left(\hat{S} - \frac{4}{\sqrt{2\pi}} q\right) .
\end{aligned} \tag{5.1}$$

Boundary conditions for $\rho^+(0), q^+(0), S^+(0)$ are needed, as well as a condition at infinity

$$S_\infty - aq_\infty = r_-(0) = C.$$

With these data at hand, the half-space problem can be solved explicitly. We determine a bounded solution up to four constants, which will be fixed with the above four conditions. First, we observe that we have three invariants

$$\begin{aligned}
q &= C_1, \\
S &= C_2, \\
-\alpha\hat{\rho} + \beta q + \gamma\hat{S} &= C_3.
\end{aligned} \tag{5.2}$$

So, we can find $\hat{\rho}$ from (5.2) as

$$\hat{\rho} = \frac{\gamma}{\alpha}\hat{S} + \tilde{C}_3,$$

where $\tilde{C}_3 = \beta q - C_3 = \beta C_1 - C_3$. Plugging these invariants into (5.1) yields

$$\begin{aligned}
\partial_x \hat{q} &= -\left(\frac{\gamma}{\alpha}\hat{S} + \tilde{C}_3 - \frac{2}{\sqrt{2\pi}} C_1\right) \\
\partial_x \hat{S} &= -\left(\hat{q} - \frac{1}{\sqrt{2\pi}}(C_2 + \rho)\right) \\
\partial_x(-\alpha\rho + \beta\hat{q}) &= -\left(\hat{S} - \frac{4}{\sqrt{2\pi}} C_1\right) .
\end{aligned} \tag{5.3}$$

Combining the first and last equation of (5.3) gives

$$\partial_x \left(\alpha\rho - \beta\hat{q} + \frac{\alpha}{\gamma}\hat{q} \right) = -\frac{\alpha}{\gamma}\tilde{C}_3 + \frac{2\alpha}{\sqrt{2\pi}\gamma}C_1 - \frac{4}{\sqrt{2\pi}}C_1.$$

If

$$-\frac{\alpha}{\gamma}\tilde{C}_3 + \frac{2\alpha}{\sqrt{2\pi}\gamma}C_1 - \frac{4}{\sqrt{2\pi}}C_1 \neq 0$$

then, the last differential equation has no bounded solution. This requires

$$-\frac{\alpha}{\gamma}\tilde{C}_3 + \frac{2\alpha}{\sqrt{2\pi}\gamma}C_1 - \frac{4}{\sqrt{2\pi}}C_1 = 0 \qquad \Rightarrow \qquad \tilde{C}_3 = \frac{2}{\sqrt{2\pi}}C_1 \left(1 - \frac{2\gamma}{\alpha}\right).$$

We obtain

$$\hat{\rho} = \frac{\gamma}{\alpha}\hat{S} + \frac{2}{\sqrt{2\pi}}C_1 \left(1 - \frac{2\gamma}{\alpha}\right)$$

and invariance of

$$\alpha\rho - \beta\hat{q} + \frac{\alpha}{\gamma}\hat{q} \qquad \Rightarrow \qquad \rho = \frac{\beta}{\alpha}\hat{q} - \frac{1}{\gamma}\hat{q} + \hat{C}_3.$$

Inserting this into (5.3) gives

$$\begin{aligned} \partial_x \hat{q} &= -\left(\frac{\gamma}{\alpha}\hat{S} - \frac{4\gamma}{\sqrt{2\pi}\alpha}C_1 \right) \\ \partial_x \hat{S} &= -\left(\hat{q} - \frac{1}{\sqrt{2\pi}}(C_2 + \frac{\beta}{\alpha}\hat{q} - \frac{1}{\gamma}\hat{q} + \hat{C}_3) \right) \end{aligned}$$

or finally

$$\begin{aligned} \partial_x \hat{q} &= -\left(\frac{\gamma}{\alpha}\hat{S} - \frac{4\gamma}{\sqrt{2\pi}\alpha}C_1 \right) \\ \partial_x \hat{S} &= -\left(\frac{\hat{q}}{\sqrt{2\pi}} \left(\frac{2}{\alpha} + \frac{1}{\gamma} \right) - \frac{1}{\sqrt{2\pi}}C_4 \right) \end{aligned} \tag{5.4}$$

with $C_4 = C_2 + \hat{C}_3$.

The eigenvalues of the system matrix of the linear system (5.4) are $\pm\lambda$, with $\lambda = \frac{\sqrt{\alpha+2\gamma}}{\sqrt[4]{2\pi}\alpha} < 0$. The solutions of (5.4) can be computed explicitly as

$$\begin{aligned} \hat{q} &= \delta_1 \exp(\lambda x) + \delta_2 \exp(-\lambda x) + \frac{\alpha\gamma C_4}{\alpha + 2\gamma}, \\ \hat{S} &= -\frac{\lambda\alpha\delta_1}{\gamma}\exp(\lambda x) + \frac{\lambda\alpha\delta_2}{\gamma}\exp(-\lambda x) + \frac{4}{\sqrt{2\pi}}C_1, \end{aligned}$$

where δ_1, δ_2 are positive constants. As for layers only the bounded solutions of (5.4) can be used, we look at

$$\begin{aligned} \hat{q} &= \delta_1 \exp(\lambda x) + \frac{\alpha\gamma C_4}{\alpha + 2\gamma}, \\ \hat{S} &= -\frac{\lambda\alpha\delta_1}{\gamma}\exp(\lambda x) + \frac{4}{\sqrt{2\pi}}C_1 \end{aligned}$$

which gives for the other variables

$$\begin{aligned} \rho &= \frac{\beta}{\alpha}\hat{q} - \frac{1}{\gamma}\hat{q} + C_4 - C_2, \\ \hat{\rho} &= \frac{\gamma}{\alpha}\hat{S} + \frac{2}{\sqrt{2\pi}}C_1\left(1 - \frac{2\gamma}{\alpha}\right), \\ q &= C_1, \\ S &= C_2. \end{aligned}$$

The four parameters are fixed with the four conditions

$$\begin{aligned} &\frac{1}{2}(\rho(0) + \hat{\rho}(0)) = \rho_+(0), \\ &\frac{1}{2}(q(0) + \hat{q}(0)) = q_+(0), \\ &\frac{1}{2}(S(0) + \hat{S}(0)) = S_+(0) \end{aligned}$$

and the condition at infinity

$$S_\infty - aq_\infty = r_-(0) = C. \tag{5.5}$$

Altogether we obtain

$$\begin{aligned} \rho_+(0) &= \frac{1}{2}\left(\left(\frac{\gamma\beta - \alpha}{\alpha\gamma}\right)\left(\delta_1 + \frac{\alpha\gamma C_4}{\alpha + 2\gamma}\right) + C_4 - C_2 - \lambda\delta_1 + \frac{2}{\sqrt{2\pi}}C_1\right), \\ q_+(0) &= \frac{1}{2}\left(C_1 + \delta_1 + \frac{\alpha\gamma C_4}{\alpha + 2\gamma}\right), \\ S_+(0) &= \frac{1}{2}\left(C_2 - \frac{\lambda\alpha\delta_1}{\gamma} + \frac{4}{\sqrt{2\pi}}C_1\right). \end{aligned}$$

With help of the asymptotic values

$$C_1 = q_\infty, \quad C_2 = S_\infty, \quad C_4 = \frac{\alpha + 2\gamma}{\sqrt{2\pi}\alpha\gamma}(\rho_{\infty + S_\infty})$$

this can be rewritten as

$$\begin{aligned} \rho_+(0) &= \frac{1}{2}\left(\left(\frac{\gamma\beta - \alpha}{\alpha\gamma} - \lambda\right)\delta_1 + \rho_\infty + \frac{2}{\sqrt{2\pi}}q_\infty\right), \\ q_+(0) &= \frac{1}{2}\left(q_\infty + \delta_1 + \frac{\rho_\infty + S_\infty}{\sqrt{2\pi}}\right), \\ S_+(0) &= \frac{1}{2}\left(S_\infty - \frac{\lambda\alpha\delta_1}{\gamma} + \frac{4}{\sqrt{2\pi}}q_\infty\right). \end{aligned} \tag{5.6}$$

Together with the condition at infinity, this determines the asymptotic values ρ_∞, q_∞, S_∞ and δ_1. The outgoing quantities $\rho_-(0), q_-(0), S_-(0)$ are then determined by

$$\begin{aligned} \rho_-(0) &= \frac{1}{2}\left(\left(\frac{\gamma\beta - \alpha}{\alpha\gamma} + \lambda\right)\delta_1 + \rho_\infty - \frac{2}{\sqrt{2\pi}}q_\infty\right), \\ q_-(0) &= \frac{1}{2}\left(q_\infty - \delta_1 - \frac{\rho_\infty + S_\infty}{\sqrt{2\pi}}\right), \\ S_-(0) &= \frac{1}{2}\left(S_\infty + \frac{\lambda\alpha\delta_1}{\gamma} - \frac{4}{\sqrt{2\pi}}q_\infty\right). \end{aligned}$$

Remark 1 (Maxwell approximation for the half-moment problem [34, 35]). *A straightforward approximation of the asymptotic states is obtained by equalizing the first two half fluxes at $x = 0$ with those at $x = \infty$, i.e.*

$$\int_0^\infty \begin{pmatrix} v \\ v^2 \end{pmatrix} f(0,v)dv = \int_0^\infty \begin{pmatrix} v \\ v^2 \end{pmatrix} \left(\rho + vq + \frac{1}{2}(v^2-1)(S-\rho)\right) M(v)dv.$$

This gives

$$\begin{aligned} q_+(0) = q_+(\infty) &= \frac{1}{2}(q_\infty + \hat{q}_\infty) = \frac{q_\infty}{2} + \frac{\rho_{\infty + S_\infty}}{2\sqrt{2\pi}}, \\ S_+(0) = S_+(\infty) &= \frac{1}{2}(S_\infty + \hat{S}_\infty) = \frac{S_\infty}{2} + \frac{2q_\infty}{\sqrt{2\pi}} \end{aligned}$$

and the condition at infinity (5.5)*. This leads to an approximation of the outgoing quantities by*

$$\begin{aligned} \rho_-(0) = \rho_-(\infty) &= \frac{\rho_\infty}{2} - \frac{q_\infty}{\sqrt{2\pi}}, \\ q_-(0) = q_-(\infty) &= \frac{q_\infty}{2} - \frac{\rho_\infty + S_\infty}{2\sqrt{2\pi}}, \\ S_-(0) = S_-(\infty) &= \frac{S_\infty}{2} - \frac{2q_\infty}{\sqrt{2\pi}}. \end{aligned}$$

5.1. Comparison with a direct computation. To estimate the accuracy of our approximation method, we consider a boundary value problem related to the determination of the slip boundary coefficient. For $x \in \mathbb{R}^+, v \in \mathbb{R}$, we consider the half-space equation

$$v\partial_x f = -\left(f - \left(\rho + vq + \frac{1}{2}(v^2-1)(S-\rho)\right) M(v)\right) \tag{5.7}$$

with $\int_{\mathbb{R}} vf dv = q = 0$ and $f(0,v) = v(v^2 - a^2)M(v), v > 0$ where $a^2 = 3$. We compute the asymptotic values by a direct numerical solver for the equation (5.7), compare [19] and [32]. Choosing a velocity discretization with $N_v = 350$ grid points on $[-5, 5]$ and a spatial discretization with $\Delta x = \frac{L}{N_x}$ with $N_x = 500$ and $L = 5$, we obtain the asymptotic values as $\rho_\infty = -2.41561$ and $S_\infty = 1.71757$.

The corresponding values by the Maxwell approximation from Remark 1 are

$$\rho_\infty = -\frac{4}{\sqrt{2\pi}} \approx -1.59617, \qquad S_\infty = \frac{4}{\sqrt{2\pi}} \approx 1.59617.$$

The above half-moment approximation gives

$$\begin{aligned} \frac{1}{2}\left(\left(\frac{\gamma\beta - \alpha}{\alpha\gamma} - \lambda\right)\delta_1 + \rho_\infty\right) &= -\frac{1}{\sqrt{2\pi}}, \\ \frac{1}{2}\left(\delta_1 + \frac{\rho_\infty + S_\infty}{\sqrt{2\pi}}\right) &= 0, \\ \frac{1}{2}\left(S_\infty - \frac{\lambda\alpha\delta_1}{\gamma}\right) &= \frac{2}{\sqrt{2\pi}}, \end{aligned}$$

which yields

$$\left(\frac{\gamma\beta-\alpha}{\alpha\gamma}-\lambda\right)\frac{\rho_\infty+S_\infty}{\sqrt{2\pi}}-\rho_\infty=\frac{2}{\sqrt{2\pi}},$$
$$S_\infty+\frac{\lambda\alpha}{\sqrt{2\pi}\gamma}(\rho_\infty+S_\infty)=\frac{4}{\sqrt{2\pi}}.$$

The solution of these equations is

$$\rho_\infty=\frac{2(\lambda\alpha+2)(2\gamma+\alpha)}{\sqrt{2\pi}(2\gamma+\alpha+\lambda\alpha(\alpha+\gamma))}\approx-2.23425,$$
$$S_\infty=\frac{2(4\gamma+2\alpha-\sqrt{2\pi}\alpha\gamma+\lambda\alpha(2\gamma+\alpha))}{\sqrt{2\pi}(2\gamma+\alpha+\lambda\alpha(\alpha+\gamma))}\approx 1.68775.$$

Thus the error for the above half-moment method is approximately 7.5% for ρ_∞ and 1.7% for S_∞. For the Maxwell method the errors are 33.9% and 7.1% respectively.

	Numerical solver	Maxwell	Half-moment	Maxw./Num.	Half/Num.
ρ_∞	-2.41561	-1.59617	-2.23425	33.9%	7.5%
S_∞	1.71757	1.59617	1.68775	7.1%	1.7%

Table 5.1: *Comparison of results with the different approximations.*

6. Half-moment coupling conditions. In this section, we determine the coupling conditions on the basis of the half-moment approximation of the half-space problem and compare them to coupling conditions based on the Maxwell approximation and to a condition given by the invariance of the second moment S.

Multiplying first the kinetic coupling conditions (3.7) by v and integrating over positive velocities gives for $i=1,\dots,n$

$$q_+^i(0)=-\sum_{j=1}^n c_{ij}q_-^j(0).$$

Inserting (5.6) yields for $i=1,\dots,n$

$$\frac{1}{2}\left(q_\infty^i+\delta_1^i+\frac{\rho_\infty^i+S_\infty^i}{\sqrt{2\pi}}\right)=-\sum_{j=1}^n c_{ij}\frac{1}{2}\left(q_\infty^j-\delta_1^j-\frac{\rho_\infty^j+S_\infty^j}{\sqrt{2\pi}}\right). \tag{6.1}$$

By summing these equations one obtains directly the balance of fluxes

$$\sum_{j=1}^n q_\infty^j=0. \tag{6.2}$$

6.1. Invariants at the nodes. For a uniform node with equal distribution $c_{ij}=\frac{1}{n-1}, i\neq j$ and 0 otherwise, we have

$$-\sum_{j=1}^n c_{ij}\frac{q_\infty^j}{2}=\sum_{j=1}^n(-c_{ij})\frac{q_\infty^j}{2}+\frac{1}{n}\sum_{j=1}^n q_\infty^j=\sum_{j\neq i}\left(-\frac{1}{2(n-1)}\right)q_\infty^j+\sum_{j\neq i}\frac{1}{n}q_\infty^j+\frac{1}{n}q_\infty^i$$
$$=\sum_{j\neq i}\left(\frac{1}{n}-\frac{1}{2(n-1)}\right)q_\infty^j+\frac{1}{n}q_\infty^i=\sum_{j\neq i}\frac{1}{n-1}\frac{n-2}{2n}q_\infty^j+\frac{1}{n}q_\infty^i.$$

Hence, (6.1) can be written as

$$\frac{1}{2}\left(q_\infty^i + \delta_1^i + \frac{\rho_\infty^i + S_\infty^i}{\sqrt{2\pi}}\right) = \sum_{j\neq i} \frac{1}{n-1}\frac{1}{2}\left(\frac{n-2}{n}q_\infty^j + \delta_1^j + \frac{\rho_\infty^j + S_\infty^j}{\sqrt{2\pi}}\right) + \frac{q_\infty^i}{n}.$$

This gives for each $i = 1 \ldots, n$

$$\frac{n-2}{n}q_\infty^i + \delta_1^i + \frac{\rho_\infty^i + S_\infty^i}{\sqrt{2\pi}} = \sum_{j\neq i} \frac{1}{n-1}\left(\frac{n-2}{n}q_\infty^j + \delta_1^j + \frac{\rho_\infty^j + S_\infty^j}{\sqrt{2\pi}}\right).$$

These equations can be rearranged to

$$\frac{n-2}{2n}q_\infty^i + \frac{\delta_1^i}{2} + \frac{\rho_\infty^i + S_\infty^i}{2\sqrt{2\pi}} = \frac{n-2}{2n}q_\infty^j + \frac{\delta_1^j}{2} + \frac{\rho_\infty^j + S_\infty^j}{2\sqrt{2\pi}},$$

which can be equivalently formulated as the invariance of the quantity

$$\frac{n-2}{n}q_\infty + \delta_1 + \frac{\rho_\infty + S_\infty}{\sqrt{2\pi}}.$$

Integrating the kinetic coupling conditions (3.7) for positive v additionally gives

$$\rho_+^i(0) = \sum_{j=1}^n c_{ij}\rho_-^j(0).$$

This yields

$$\frac{1}{2}\left(\left(\frac{\gamma\beta-\alpha}{\alpha\gamma} - \lambda\right)\delta_1^i + \rho_\infty^i + \frac{2}{\sqrt{2\pi}}q_\infty^i\right) =$$
$$= \sum_j c_{ij}\frac{1}{2}\left(\left(\frac{\gamma\beta-\alpha}{\alpha\gamma} + \lambda\right)\delta_1^j + \rho_\infty^j - \frac{2}{\sqrt{2\pi}}q_\infty^j\right).$$

Using again $\sum_{j=1}^n q_\infty^j = 0$, we have

$$\sum_{i=1}^n \delta_1^i = 0.$$

For a uniform node, one obtains using the same arguments as before

$$\frac{1}{2}\left(\frac{\gamma\beta-\alpha}{\alpha\gamma} - \frac{\lambda(n-2)}{n}\right)\delta_1^i + \frac{\rho_\infty^i}{2} + \frac{n-2}{n\sqrt{2\pi}}q_\infty^i =$$
$$= \frac{1}{n-1}\sum_{j=1}^n \frac{1}{2}\left(\frac{\gamma\beta-\alpha}{\alpha\gamma} - \frac{\lambda(n-2)}{n}\right)\delta_1^j + \frac{\rho_\infty^j}{2} + \frac{n-2}{n\sqrt{2\pi}}q_\infty^j.$$

This gives for any i and j

$$\frac{1}{2}\left(\frac{\gamma\beta-\alpha}{\alpha\gamma} - \frac{\lambda(n-2)}{n}\right)\delta_1^i + \frac{\rho_\infty^i}{2} + \frac{n-2}{n\sqrt{2\pi}}q_\infty^i =$$
$$= \frac{1}{2}\left(\frac{\gamma\beta-\alpha}{\alpha\gamma} - \frac{\lambda(n-2)}{n}\right)\delta_1^j + \frac{\rho_\infty^j}{2} + \frac{n-2}{n\sqrt{2\pi}}q_\infty^j$$

and therefore invariance of

$$\frac{1}{2}\left(\frac{\gamma\beta-\alpha}{\alpha\gamma}-\frac{\lambda(n-2)}{n}\right)\delta_1+\frac{\rho_\infty}{2}+\frac{n-2}{n\sqrt{2\pi}}q_\infty$$

at the node. Finally, one obtains from the kinetic coupling conditions

$$S_+^i(0)=\sum_{j=1}^n c_{ij}S_-^j(0)$$

which yields

$$\frac{1}{2}\left(S_\infty^i-\frac{\lambda\alpha\delta_1^i}{\gamma}+\frac{4}{\sqrt{2\pi}}q_\infty^i\right)=\sum_{j=1}^n c_{ij}\frac{1}{2}\left(S_\infty^j+\frac{\lambda\alpha\delta_1^j}{\gamma}-\frac{4}{\sqrt{2\pi}}q_\infty^j\right)$$

and, considering again the case of a uniform node, invariance of

$$S_\infty-\frac{\lambda\alpha(n-2)}{\gamma n}\delta_1+\frac{4(n-2)}{\sqrt{2\pi}n}q_\infty.$$

Note now that the combination of the invariants allows to eliminate δ_1 and using suitable linear combinations one obtains the invariants

$$S_\infty+\frac{n-2}{n}\frac{4n(2\gamma+\alpha)+5\sqrt{2\pi}\alpha\lambda(n-2)}{\sqrt{2\pi}n(2\gamma+\alpha)+8\alpha\lambda(n-2)}q_\infty \tag{6.3}$$

and

$$\rho_\infty+\frac{n-2}{n}\frac{(2\pi-4)n(\alpha-\gamma\beta)+2\sqrt{2\pi}\alpha\gamma n+3\sqrt{2\pi}\alpha\lambda(n-2)}{\sqrt{2\pi}n(2\gamma+\alpha)+4\alpha\lambda(n-2)}q_\infty. \tag{6.4}$$

Each of these invariants gives $n-1$ conditions for the quantities $\rho_\infty^i, q_\infty^i$ and S_∞^i.

Remark 2. *The Maxwell approximation for the half-space problem gives the conditions*

$$\frac{1}{2}\left(q_\infty^i+\frac{\rho_\infty^i+S_\infty^i}{\sqrt{2\pi}}\right)=-\sum_{j=1}^n c_{ij}\frac{1}{2}\left(q_\infty^j-\frac{\rho_\infty^j+S_\infty^j}{\sqrt{2\pi}}\right)$$

and

$$\frac{1}{2}\left(S_\infty^i+\frac{4}{\sqrt{2\pi}}q_\infty^i\right)=\sum_{j=1}^n c_{ij}\frac{1}{2}\left(S_\infty^j-\frac{4}{\sqrt{2\pi}}q_\infty^j\right),$$

which gives the invariants

$$\frac{n-2}{n}q_\infty+\frac{\rho_\infty+S_\infty}{\sqrt{2\pi}} \qquad \textit{and} \qquad S_\infty+\frac{4(n-2)}{\sqrt{2\pi}n}q_\infty.$$

Combining this implies the invariance of

$$\frac{n-2}{n}q_\infty+\frac{\rho_\infty+S_\infty}{\sqrt{2\pi}}-\frac{1}{\sqrt{2\pi}}\left(S_\infty+\frac{4(n-2)}{\sqrt{2\pi}n}q_\infty\right)=\frac{(n-2)(\pi-2)}{\pi n}q_\infty+\frac{\rho_\infty}{\sqrt{2\pi}}$$

or respectively

$$\rho_\infty+\frac{\sqrt{2\pi}(\pi-2)}{\pi}\frac{n-2}{n}q_\infty.$$

6.2. Coupling conditions for the macroscopic equation. The coupling conditions are up to now given by the invariance of the two quantities

$$\rho_\infty + C_1 q_\infty, \qquad \text{and} \qquad S_\infty + C_2 q_\infty \tag{6.5}$$

together with the balance of fluxes and the n conditions at infinity

$$S_\infty - a q_\infty = r_-(0) = S_B - a q_B. \tag{6.6}$$

Altogether, we have $1+2(n-1)+n = 3n-1$ conditions at a node for the $3n$ quantities $\rho^i_\infty, q^i_\infty$ and S^i_∞.

The conditions obtained by different approximations differ in the factors C_1 and C_2. With $a^2 = 3$ the half moment approximation gives the values

$$C_1 = \frac{n-2}{n} \frac{(2\pi - 4)n(\alpha - \gamma\beta) + 2\sqrt{2\pi}\alpha\gamma n + 3\sqrt{2\pi}\alpha\lambda(n-2)}{\sqrt{2\pi}n(2\gamma + \alpha) + 4\alpha\lambda(n-2)},$$

$$C_2 = \frac{n-2}{n} \frac{4n(2\gamma + \alpha) + 5\sqrt{2\pi}\alpha\lambda(n-2)}{\sqrt{2\pi}n(2\gamma + \alpha) + 8\alpha\lambda(n-2)}$$

compared to $C_1 = \frac{\sqrt{2\pi}(n-2)(\pi-2)}{\pi n}$, $C_2 = \frac{4(n-2)}{\sqrt{2\pi}n}$ for Maxwell.

For the case $n = 2$ we have simply $C_1 = C_2 = 0$, i.e. the continuity of density and second moment. From the balance of fluxes the continuity of the flux follows in this case.

For the case $n = 3$, this yields $C_1 \approx 0.37883$, $C_2 \approx 0.53006$ compared to $C_1 \approx 0.30327$, $C_2 \approx 0.532058$ for Maxwell.

We note that we already have a sufficient number of conditions for the coupling of the macroscopic equations: due to the 0-characteristic only $2n$ conditions are required. They are given by the balance of fluxes $\sum_j q^j_\infty = 0$, the invariance of $S_\infty + C_2 q_\infty$ and the n conditions at infinity. This fixes the values of q^i_∞ and S^i_∞ at the node, which is sufficient to couple the macroscopic equations.

Remark 3. *A simplified coupling condition corresponding to the widely used equality of pressure condition is given by choosing $C_2 = 0$ in the above invariant requiring the equality of the second moment S.*

Remark 4. *Considering again the situation with $n = 3$ we observe from the above consideration that due to the very similar numerical values for C_2 we do not expect to see a significant difference between the numerical results using the Maxwell conditions and those obtained from using the half moment conditions. For larger values of n the differences become slightly larger.*

6.3. Determination of the kinetic values at the node. To fix all values including ρ^i_∞ at the nodes and to obtain a complete approximation of the kinetic solution in the small ϵ limit at the node, we need additionally to the $3n-1$ conditions above a last condition at the node. To obtain such a condition we use a higher order approximation. We remark that the kinetic coupling conditions yields additionally to the balance of fluxes

$$\sum_{i=1}^n \int v f^i(v) dv = \sum_{i=1}^n q^i = 0 \tag{6.7}$$

also the balance of higher order odd moments, for example

$$\sum_{i=1}^{n} \int v^3 f^i(v) dv = 0. \tag{6.8}$$

We approximate this condition to first order via a Chapman-Enskog type approximation using equations (3.3), (3.4) and (3.6). This leads to

$$\int v^3 f(v) dv \;=\; -\alpha(\rho_+ - \rho_-) + \beta(q_+ + q_-) + \gamma(S_+ - S-) \;=\; -\alpha\hat{\rho} + \beta q + \gamma\hat{S}.$$

Due to (3.6), this is equal to

$$\int v^3 f(v) dv =$$
$$-\alpha\left(\frac{2}{\sqrt{2\pi}}q - \epsilon\partial_t\hat{\rho} - \epsilon\partial_x\hat{q}\right) + \beta q + \gamma\left(\frac{4}{\sqrt{2\pi}}q - \epsilon\partial_t\hat{S} - \epsilon\partial_x(-\alpha\rho + \beta\hat{q} + \gamma S)\right).$$

Using a Chapman-Enskog type procedure up to order $\mathcal{O}(\epsilon^2)$ this is approximated by

$$\begin{aligned}\int v^3 f(v) dv = &-\alpha\left(\frac{2}{\sqrt{2\pi}}q - \epsilon\partial_t\frac{2}{\sqrt{2\pi}}q - \epsilon\partial_x\frac{(\rho_\infty + S_\infty)}{\sqrt{2\pi}}\right) + \beta q + \\ &+\gamma\left(\frac{4}{\sqrt{2\pi}}q - \epsilon\partial_t\frac{4}{\sqrt{2\pi}}q - \epsilon\partial_x(-\alpha\rho + \beta\frac{(\rho_\infty + S_\infty)}{\sqrt{2\pi}} + \gamma S)\right).\end{aligned}$$

This gives

$$\begin{aligned}\int v^3 f(v) dv = &\left(-\frac{2\alpha}{\sqrt{2\pi}} + \beta + \frac{4\gamma}{\sqrt{2\pi}}\right)q + \epsilon\left(\frac{\alpha}{\sqrt{2\pi}} + \alpha\gamma - \frac{\beta\gamma}{\sqrt{2\pi}}\right)\partial_x\rho + \\ &+\epsilon\left(\frac{3\alpha}{\sqrt{2\pi}} - \frac{4\gamma}{\sqrt{2\pi}} - \frac{\beta\gamma}{\sqrt{2\pi}} - \gamma^2\right)\partial_x S.\end{aligned}$$

Thus, due to the balance of first (6.7) and third order fluxes (6.8), we have

$$\sum_{i=1}^{n}\left((\alpha + \sqrt{2\pi}\alpha\gamma - \beta\gamma)\partial_x\rho^i + (3\alpha - 4\gamma - \beta\gamma - \sqrt{2\pi}\gamma^2)\partial_x S^i\right) = 0$$

or

$$\sum_{i=1}^{n}\partial_x\rho^i = -\frac{(3\alpha - 4\gamma - \beta\gamma - \sqrt{2\pi}\gamma^2)}{(\alpha + \sqrt{2\pi}\alpha\gamma - \beta\gamma)}\sum_i \partial_x S^i.$$

Approximating the spatial derivatives yields the final form of the missing condition

$$\sum_{i=1}^{n}(\rho^i_\infty - \rho^i_B) = -\frac{(3\alpha - 10\gamma)}{\alpha + 2\gamma}\sum_i (S^i_\infty - S^i_B). \tag{6.9}$$

6.4. Summary of coupling conditions. In summary, the coupling condition (6.2) and (6.9) together with the conditions at infinity (6.6) and the conditions given by the invariants (6.3) and (6.4) yield a system of $3n$ linear equations for the $3n$ unknowns $\rho^i_\infty, q^i_\infty, S^i_\infty$.

Define $C_3 = -\frac{(3\alpha-10\gamma)}{\alpha+2\gamma}$, then, for example in the case $n = 3$, the system of equations for the 9 unknowns

$$(\rho^1_\infty, \rho^2_\infty, \rho^3_\infty, q^1_\infty, q^2_\infty, q^3_\infty, S^1_\infty, S^2_\infty, S^3_\infty)^T$$

is given by a linear system with the system matrix

$$\begin{pmatrix}
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & C_2 & -C_2 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & C_2 & -C_2 \\
0 & 0 & 0 & -a & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -a & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -a & 0 & 0 & 1 \\
1 & -1 & 0 & C_1 & -C_1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & C_1 & -C_1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & -C_3 & -C_3 & -C_3
\end{pmatrix}$$

and right hand side

$$\begin{pmatrix} 0 & 0 & 0 & S^1_B - aq^1_B & S^2_B - aq^2_B & S^3_B - aq^3_B & 0 & 0 & \sum_{i=1}^3 (\rho^i_B - C_3 S^i_B) \end{pmatrix}^T.$$

The determinant of the matrix C is equal to

$$9(1 + aC_2)^2.$$

Since, a as well as C_2 are positive we have a unique solution to our coupling problem for the Maxwell approximation as well as for the half-moment approximation.

7. Numerical results. In this section, we compare the numerical results of the different models. The solutions of the kinetic equation (2.1) serve as reference solutions for the half-moment approximation (3.5) and the macroscopic wave equation (2.2). For the wave equation we use the coupling conditions (6.5) with the different coefficients C_1 and C_2 derived from the half-moment approach. Moreover, a classical coupling condition, the equality of the second moment S on all edges, i.e. the above conditions with $C_2 = 0$ is included for comparison.

The networks are composed of coupled edges, each arc is given by an interval $x \in [0, 1]$, which is discretized with 500 spatial cells if not otherwise stated. In the kinetic model the velocity domain is discretized with 500 cells and we choose different values of ϵ.

For the advective part of the equations we use an upwind scheme. The source term in the kinetic and half moment equations is approximated with the implicit Euler method.

In general, at the outer boundaries of the network, boundary conditions have to be imposed. For the kinetic problem and the half-moment approximation, we use the conditions described above in the respective sections or free boundary conditions. For the linearized Euler equation, we use the approximations from the kinetic problem discussed above to determine boundary values or again free boundary conditions.

7.1. Tripod network. In the first example, we consider a tripod network with initial conditions $f^i(x, v) = (\rho^i + vq^i + \frac{1}{2}(v^2 - 1)(S^i - \rho^i))M(v)$ where $M(v)$ is the standard Maxwellian. The macroscopic states are $(\rho^1, q^1, S^1) = (1, 0, 2)$, $(\rho^2, q^2, S^2) = (0, 0, 0)$ and $(\rho^3, q^3, S^3) = (2, 0, 4)$. We use free boundary conditions at the exterior boundaries. The final time is chosen such that the waves generated at the node do not reach the exterior boundaries.

First we investigate the accuracy of the half-moment approximation of the full kinetic problem. In Figure 7.1 we compare the kinetic solution and the half-moment approach for a uniform node with symmetric coupling conditions at final time $T = 0.1$ and different values of ϵ between 0.0001 and 1. We note that kinetic and half-moment approach give almost identical results for times $T > \epsilon$. For $T \leq \epsilon$ one observes the influence of the hyperbolic characteristics of the half-moment approach, whereas the value at the node is still captured accurately.

Second, the different coupling conditions are compared for the tripod. In Figure 7.2 we compare the kinetic solution for $\epsilon = 0.01$ and the wave equation with coupling conditions given either by the the half-moment half-space approach or by the assumption of equal second moment S at time $T = 1$. We note that the Maxwell conditions and the half-moment approach give almost identical results for the macroscopic solution.

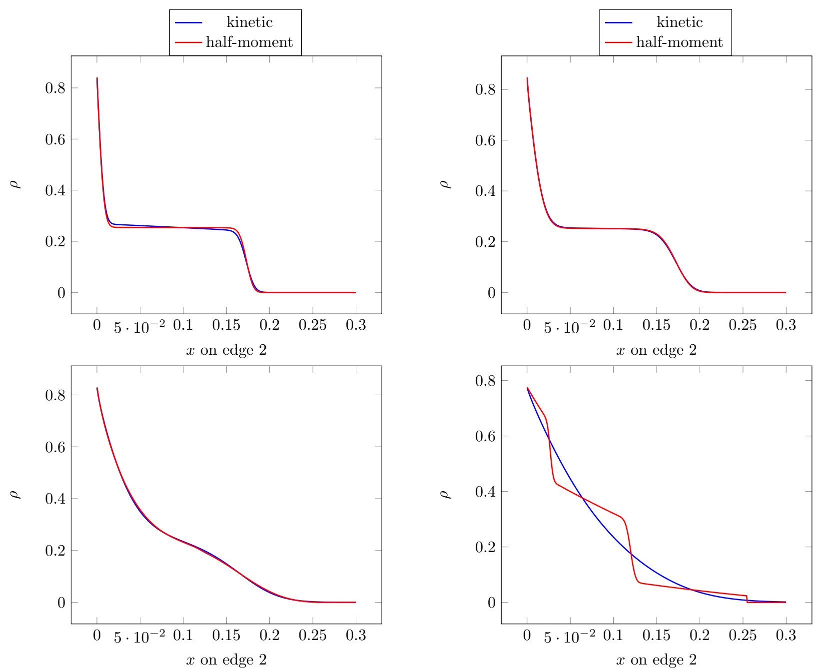

Fig. 7.1: Kinetic equation and half-moment approximation for different relations of $\Delta = \frac{\epsilon}{T}$. ρ on edge 2 of the tripod. Upper row: $\Delta = 0.001$ (left) and $\Delta = 0.01$ (right). Lower row: $\Delta = 0.1$ (left) and $\Delta = 1.0$ (right).

We observe that the interior state is approximated very accurately by the half-moment half-space coupling conditions. Moreover, also the kinetic state at the boundary, in particular the value of the density ρ, is very well approximated by the Chapman Enskog type procedure described in subsection 6.3, which is in Figure 7.2 denoted by a cross. This is obvious for the values of q and S which are transported into the domain, but less obvious for the values of ρ, which coincide very well with the actual value of the kinetic solution at the boundary.

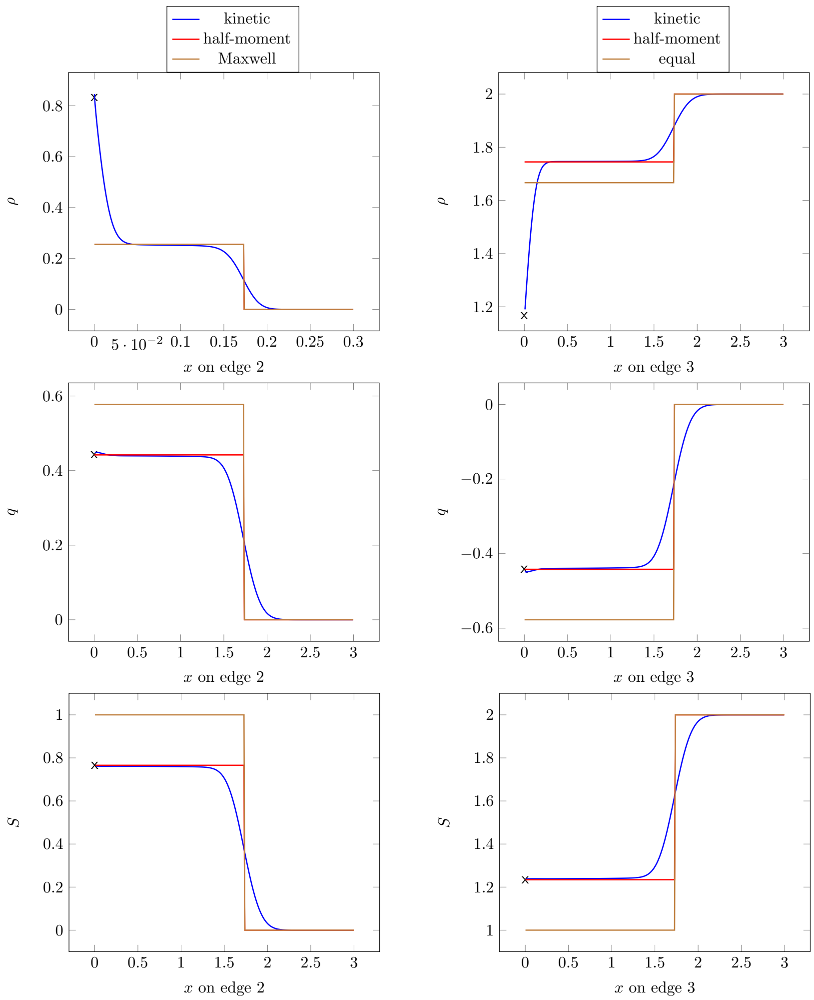

Fig. 7.2: Kinetic equation and wave equation with coupling conditions given by the Maxwell/half-moment approach and by the equality of S. ρ, q, S on edge 2 (left) and edge 3 (right) at time $T = 1$. The values at the nodes obtained from the coupling conditions are denoted by a cross

We note that the Maxwell approximation gives a similarly good approximation in the present case.

7.2. Diamond network. As a second example, we consider a network topology given in Figure 7.3.

As initial conditions for the kinetic equation, we have chosen $f^1(x,v) = 4M(v)$, $f^2(x,v) = 3M(v), f^3(x,v) = \frac{7}{3}M(v)$ and $f^j(x,v) = 2M(v)$ for $j = 4,\dots,7$ which correspond to macroscopic quantities $\rho^1 = 4, \rho^2 = 3, \rho^3 = \frac{7}{3}$ and $\rho^j = 2$ for $j = 4,\dots,7$, $q^j = 0$ for $j = 1,\dots,7$ and finally $S^1 = 4, S^2 = 3, S^3 = \frac{7}{3}$ and $S^j = 2$, for $j = 4,\dots,7$. These data are also prescribed at the two outer boundaries, i.e. $f^1(0,v) = 4M(v)$, $v \in [0,1]$ and $f^7(1,v) = \frac{7}{3}M(v)$, $v \in [-1,0]$. Boundary conditions

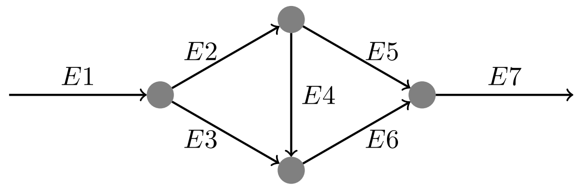

Fig. 7.3: Diamond network.

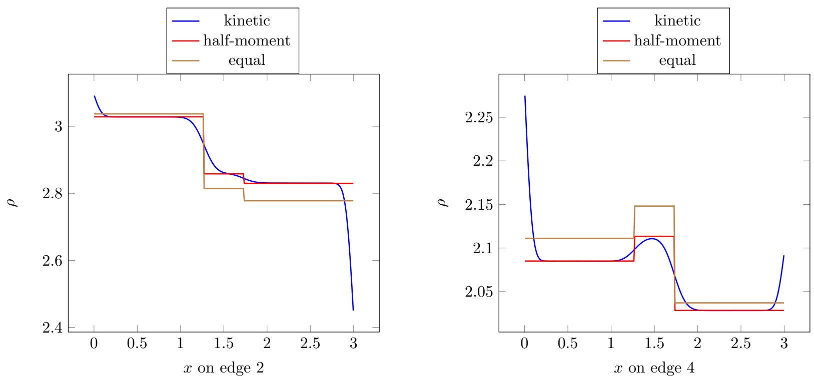

Fig. 7.4: Kinetic equation and wave equation with coupling conditions given by the Maxwell/half-moment approach and by the equality of S. ρ on edge 2 (left) and edge 4 (right) at time $T = 1$.

for the wave equation with full moment, Maxwell and half moment conditions are derived as detailed above.

In Figure 7.4 the density ρ^4 on edge 4 is displayed at time $t = 3$ and $t = 10$. As before, we observe a good agreement of the half moment coupling with the kinetic and half moment model. The Maxwell approximation gives again a result, which is very similarly to the one obtained from the half-moment approach.

Remark 5. *The above procedure can be extended to the linear BGK model with* $x \in \mathbb{R}, v = (v_1, v_2, v_3) \in \mathbb{R}^3$, *i.e.*

$$\partial_t f + v_1 \partial_x f = -\frac{1}{\epsilon} Q(f) = -\frac{1}{\epsilon}\left(f - \left(\rho + v_1 q + \frac{1}{2}(|v|^2 - 3)(S - \rho)\right) M(v)\right)$$

with

$$\rho = \int_{\mathbb{R}^3} f(v) dv, q = \int_{\mathbb{R}^3} v_1 f(v) dv, S = \frac{1}{3} \int_{\mathbb{R}^3} |v|^2 f(v) dv.$$

and standard Maxwellian in 3D given by

$$M(v) = \frac{1}{(2\pi)^{3/2}} \exp\left(\frac{-|v|^2}{2}\right).$$

The associated limit equation for $\epsilon \to 0$ *is again the acoustic system but in this case with* $a^2 = \frac{5}{3}$.

Conclusion. In this work, we have extended previous work in [13] to the case of the linearized one-dimensional BGK equation on a network with the acoustic system as the limit model. We have derived explicit coupling conditions for the acoustic system on a network. They are based on coupling conditions for the underlying kinetic BGK model, a half-moment approximation of the kinetic problem and a layer analysis near the nodes. It turns out that in this case approximate coupling conditions given by the equality of fluxes give a very good approximation. Due to the wave with zero speed an additional expansion of the coupling conditions is needed to determine all asymptotic states at the coupling points.

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