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The generic character table of $\mathbf{Spin}_8^+(\mathbf{q})$

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Abstract

Deligne–Lusztig theory allows the parametrization of generic character tables of finite groups of Lie type in terms of families of conjugacy classes and families of irreducible characters "independently" of q. Only in small cases the theory also gives all the values of the table.

For most of the groups the completion of the table must be carried out with ad-hoc methods. The aim of the present work is to describe one possible computation which avoids Lusztig's theory of "character sheaves". In particular, the theory of Gel'fand–Graev characters and Clifford theory is used to complete the generic character table of $G = \mathrm{Spin}_8^+(q)$ for q odd. As an example of the computations, we also determine the character table of $\mathrm{SL}_4(q)$, for q odd.

In the process of finding character values, the following tools are developed. By explicit use of the Bruhat decomposition of elements, the fusion of the unipotent classes of G is determined. Among others, this is used to compute the 2-parameter Green functions of every Levi subgroup with disconnected centre of G. Furthermore, thanks to a certain action of the centre $Z(G)$ on the characters of G, it is shown how, in principle, the values of any character depend on its values at the unipotent elements.

It is important to consider $\mathrm{Spin}_8^+(q)$ as it is one of the "smallest" interesting examples for which Deligne–Lusztig theory is not sufficient to construct the whole character table. The reasons is related to the structure of $\mathbf{G} = \mathrm{Spin}_8$, from which G is constructed. Firstly, $\mathbf{G}$ has disconnected centre. Secondly, $\mathbf{G}$ is the only simple algebraic group which has an outer group automorphism of order 3. And finally, G can be realized as a subgroup of bigger groups, like $E_6(q)$, $E_7(q)$ or $E_8(q)$. The computation on $\mathrm{Spin}_8^+(q)$ serves as preparation for those cases.

Zusammenfassung

Die Deligne–Lusztig Theorie ist ein wichtiges Konstrukt in der Darstellungstheorie, mit welcher die Parametrisierung generischer Charaktertafeln endlicher Gruppen vom Lietyp durchgeführt werden kann. Diese Parametrisierung erfolgt durch Familien von Konjugiertenklassen und Familien irreduzibler Charaktere, welche "unabhängig" von q sind. Allerdings ergeben sich aller Werte einer Charaktertafel nur in kleinen Gruppen durch diese Theorie.

Für die meisten Gruppen muss die Vervollständigung der Charaktertafel mithilfe von Adhoc-Methoden durchgeführt werden. Das Ziel dieser Arbeit ist es, eine mögliche Rechnung zu beschreiben, welche Lusztigs Theorie von "character sheaves" vermeidet. Insbesondere wird die generische Charaktertafel der Gruppe $G = \mathrm{Spin}_8^+(q)$ für ungerade Werte von q mithilfe von Gel'fand–Greav Charakteren und der Clifford Theorie vervollständigt. Wir bestimmen die Charaktertafel von $\mathrm{SL}_4(q)$, mit ungeradem q, um ein Beispiel für die Rechnungen zu geben.

Um die Charakterwerte zu berechnen, werden im Laufe der Arbeit verschiedene Werkzeuge entwickelt werden. So wird zum Beispiel durch die explizite Nutzung der Bruhat-Zerlegung von Gruppenelementen die Fusion unipotenter Klassen in G festgelegt. Dies wird unter anderem verwendet, um die 2-Parameter Green-Funktionen jeder Leviuntergruppe von G mit unzusammenhängendem Zentrum zu berechnen. Dank einer bestimmten Operation des Zentrums $Z(G)$ auf den Charakteren von G, kann weiterhin gezeigt werden, dass die Werte jedes Charakters im Prinzip nur von seinen Werten auf den unipotenten Elementen abhängen.

Die Gruppe $\mathrm{Spin}_8^+(q)$ ist hier von besonderem Interesse, da diese Gruppe eines der "kleinsten" interessanten Beispiele ist, für welches die Deligne–Lusztig Theorie nicht genügt um die ganze Charaktertafel zu berechnen. Dies lässt sich auf die Struktur der Gruppe $\mathbf{G} = \mathrm{Spin}_8$ zurückführen, von welcher G konstruiert wird. Zum einen hat $\mathbf{G}$ ein unzusammenhängendes Zentrum. Andererseits ist $\mathbf{G}$ die einzige einfache algebraische Gruppe, die einen Gruppenautomorphismus der Ordnung 3 besitzt. Schließlich kann G als eine Untergruppe größerer Gruppen wie $E_6(q)$, $E_7(q)$ oder $E_8(q)$ aufgefasst werden. Die Berechnung für $\mathrm{Spin}_8^+(q)$ in dieser Arbeit wird als Vorbereitung für diese Fälle dienen.

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Introduction

The representation theory of finite groups is a rich and still wide open field of mathematics. In this work, we are interested in the character theory of a particular class of finite groups called of "Lie Type". These are defined (at least in the present work) as being the rational points of connected reductive linear algebraic groups over a finite field. One interesting property of these groups is that they can be gathered in families, and each family can be treated more or less "uniformly" (for what concerns character theory). One example are the groups of $n \times n$ invertible matrices with entries in a finite field $\mathbb{F}_q$, $\{\mathrm{GL}_n(q) \mid q \text{ prime power}\}$. A goal of character theory is to study irreducible representations via their traces, called irreducible characters. For each finite group G this information is recorded in a (square) array, called character table, where rows are labelled by irreducible characters of G and columns by conjugacy classes of G. For finite groups of Lie type it is possible to write one "generic" character table for a whole family of groups (or a subset with some mild conditions on q). By "generic" we mean that the table has a fixed size for all considered groups and the values are given with q as parameter. Then, the evaluation of q at a certain prime power yields the character table for a particular group of the family.

Here we are interested in computing the generic character tables for

$$\{\mathrm{SL}_4(q) \mid q \text{ prime power}, q \equiv 1 \pmod 4\},$$

as a first "easy" example, and

$$\{\mathrm{Spin}_8^+(q) \mid q \text{ prime power}, q \equiv 1 \pmod 4\}.$$

Of great relevance in the representation theory of finite groups of Lie type is the work of Deligne and Lusztig, now known as Deligne–Lusztig theory. This gives not only a parametrization of the irreducible characters but also a theoretical way of explicitly computing values of certain class functions. These class functions are integer linear combinations of some irreducible characters, moreover each irreducible character appears as a constituent of at least one of them. In some cases (for example $\mathrm{GL}_n(q)$) there are enough of these class functions to actually compute the full character table. In general, the theory gives enough information if the considered connected reductive group has connected centre and is "of type A". Here we are interested in cases where the theory does not yield the full character table, for instance both SL_4 and Spin_8 have disconnected centre.

There are two main goals in this thesis that are of interest for the character theory of finite groups of Lie type.

On the one hand, we want to complete the generic character table for the spin groups $\mathrm{Spin}_8^+(q)$. We say "complete" since the partial table containing the information coming from Deligne–Lusztig theory has been computed and furnished by Frank Lübeck, and serves therefore as starting point for all the computations. These groups are of interest because they are constructed from a connected reductive group $\mathbf{G}$ "of type D_4" which is a simple algebraic group of simply connected type. This makes them the "smallest" (in some sense) interesting case where Deligne–Lusztig theory fails to give the full character table. Being of type D_4, the group $\mathbf{G}$ has an accidental outer automorphism of order three, called "triality", that does not exist in other simple algebraic groups. Furthermore, it can be embedded as a subgroup inside the much bigger exceptional simple groups of type E_6, E_7 and E_8. As a result, this computation can be seen as a preparation for the treatment of those cases. In the same way $\mathrm{SL}_4(q)$ can be embedded as a subgroup inside $\mathrm{Spin}_8^+(q)$. We treat it first as a smaller example for which we expose in details the computations. Then, we can use it as reference for the computations on $\mathrm{Spin}_8^+(q)$, which are analogous but would need much more text for basically the same procedure.

On the other hand, we want to develop a method for completing character tables that is as elementary as possible and at the same time applicable to other groups. We complete the tables thanks to the construction of class functions orthogonal to the space of “uniform class functions” (which arise from Deligne–Lusztig theory), or by direct computation of character values at some problematic conjugacy classes. To do this, we will use the existence of regular embeddings, which allow us to apply Clifford theory. We will also use Gel’fand–Graev characters (and a modified version) and Harish–Chandra/Lusztig induction/restriction. These are all more or less directly tied to the fusion of the unipotent conjugacy classes.

The Gel’fand–Graev characters are special class functions that by definition have distinct values on some of those afore mentioned problematic conjugacy classes. We define modified Gel’fand–Graev characters in order to cover all problematic cases.

Finally, we remark that the determination of the generic character table for $\mathrm{Spin}_8^+(q)$ was already started by Geck and Pfeiffer in [GePf92]. In their work, they explicitly computed the so-called unipotent characters for the group of rational points of a connected reductive group of type D_4 but with connected centre. This can be adapted to find the unipotent characters of $\mathrm{Spin}_8^+(q)$.

This thesis is divided in three parts. Part I contains a survey of the theory that we need to define all objects we use and an outline of the applied methods. Then, in Part II and Part III we explain the details of the computations made for the character tables of $\mathrm{SL}_4(q)$ and $\mathrm{Spin}_8^+(q)$, respectively. We will give only a quick overview of the main results for Spin_8, since the computations are mostly analogous to those made for SL_4. We will however put some emphasis on the passages where differences occur with the case of SL_4. For example the construction of the group $\mathrm{Spin}_8^+(q)$ and the computation of the 2-parameter Green functions need a special treatment, different from the one for $\mathrm{SL}_4(q)$.

Part I

Background theory

We introduce in this part of the theory all the tools needed for the computations in Parts II and III.

In Section 1 we give the definition and properties of the finite groups of Lie type. In Section 2 we recall some basics of representation theory of finite groups. In Section 3 we summarize the results from representation theory of finite groups of Lie type which are interesting for us. In Section 4 and Section 5 we introduce two essential objects for our computations. Respectively, the 2-parameter Green functions and the modified Gel'fand–Graev characters. Finally, in Section 6 we describe the method used to compute the character tables of $\mathrm{SL}_4(q)$ and $\mathrm{Spin}_8^+(q)$.

1 Finite groups of Lie type

The finite groups that we consider here are $\mathrm{SL}_4(q)$ (Part II) and $\mathrm{Spin}_8^+(q)$ (Part III), for odd prime powers q. Both belong to the family of finite groups of Lie type. In these cases they are constructed as fixed points of some simply connected algebraic groups under an algebraic group endomorphism. In this section we recall the relevant theory behind the construction of finite groups of Lie type. References for this section are, for example, [Ca85, Chapter 1 and 2], [GeMa20, Chapter 1], [Hu75] and [MaTe11].

We start in Section 1.1 by giving the definition of connected reductive/semisimple algebraic groups. In Section 1.2 we recall the definitions of root system and Weyl group of a connected reductive group. In Section 1.3 we discuss isogenies of semisimple groups and define simply connected groups. In Section 1.4 we give the definition of split BN-pair and its consequences (Bruhat decomposition, Chevalley relations, ...). Finally in Section 1.5 we introduce Frobenius endomorphisms and finite groups of Lie type.

Throughout this section we denote by K an algebraically closed field.

1.1 Connected reductive groups

As mentioned above, the objects considered in this work are (linear) algebraic groups, which are affine algebraic varieties with a group structure. For details on the definition see [Ca85, Chapter 1.1 an 1.2], [GeMa20, Chapter 1.1.1-1.1.3] or [MaTe11, Chapter 1].

It is easy to characterize algebraic groups thanks to the following crucial result.

Theorem 1.1 ([MaTe11, Theorem 1.7])**.** *Let* $\mathbf{G}$ *be a linear algebraic group over* K*. Then* $\mathbf{G}$ *can be embedded as a closed subgroup into* $\mathrm{GL}_n(K)$ *for some* $n \in \mathbb{N}$*.*

Two examples of linear algebraic groups are $K^+ := (K, +)$, the additive group of K, and $K^\times = (K \setminus \{0\}, *)$, the multiplicative group of K.

Notation 1.2. From now on we say in short “algebraic group” for “linear algebraic group”. For an algebraic group $\mathbf{G}$, we denote by $\mathbf{G}^\circ$ its connected component containing $1 \in \mathbf{G}$. We mention that $\mathbf{G}^\circ$ is a closed normal subgroup of finite index in $\mathbf{G}$ ([MaTe11, Proposition 1.13]). See [MaTe11, Chapter 1.3] for details about connectedness of algebraic groups.

It is important to distinguish two special types of elements in an algebraic group $\mathbf{G}$. By Jordan decomposition every element of $\mathrm{GL}(V)$ can be written as the product of two (commuting) elements, one semisimple (i.e. diagonalizable) and the other unipotent (from linear algebra). Thanks to Theorem 1.1, this property holds in any algebraic group.

Theorem 1.3 ([MaTe11, Theorem 2.5], Jordan Decomposition). *Let* $\mathbf{G}$ *be an algebraic group.*

(a) *For any embedding* ρ *of* $\mathbf{G}$ *into* $\mathrm{GL}(V)$ *(for some vector space* V *over* K*) and for any* $g \in \mathbf{G}$*, there exist unique* $g_s, g_u \in \mathbf{G}$ *such that* $g = g_s g_u = g_u g_s$*, where* $\rho(g_s)$ *is semisimple and* $\rho(g_u)$ *is unipotent.*

(b) *The decomposition* $g = g_s g_u = g_u g_s$ *is independent of the chosen embedding.*

(c) *Let* $\varphi : \mathbf{G}_1 \to \mathbf{G}_2$ *be a morphism of algebraic groups. Then* $\varphi(g_s) = \varphi(g)_s$ *and* $\varphi(g_u) = \varphi(g)_u$*.*

Definition 1.4. Let $\mathbf{G}$ be an algebraic group. The decomposition $g = g_s g_u = g_u g_s$ of Theorem 1.3 is called the *Jordan decomposition* of $g \in \mathbf{G}$. If $g = g_s$, it is called *semisimple*, while if $g = g_u$ it is said to be *unipotent*.

For any algebraic group $\mathbf{G}$, we will denote by $\mathbf{G}_{\mathrm{uni}}$ its subset of unipotent elements.

Definition 1.5. Let $\mathbf{G}$ be an algebraic group. The *radical* of $\mathbf{G}$, denoted by $R(\mathbf{G})$, is the maximal closed connected solvable normal subgroup of $\mathbf{G}$. The *unipotent radical* of $\mathbf{G}$, denoted by $R_u(\mathbf{G}) = R(\mathbf{G})_{\mathrm{uni}}$, is the maximal closed connected normal unipotent subgroup of $\mathbf{G}$.

Notation 1.6. Some important subgroups of a connected algebraic group $\mathbf{G}$ over K are the following:

- A Borel subgroup $\mathbf{B}$ is a maximal closed connected solvable subgroup.
- A torus $\mathbf{T}$ is a subgroup isomorphic to a product of copies of $K^\times$, $\mathbf{T} \cong K^\times \times \cdots \times K^\times$.
- A parabolic subgroup $\mathbf{P}$ is a subgroup of $\mathbf{G}$ that contains a Borel subgroup.
- A Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ is a Levi complement of a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, i.e. such that $\mathbf{P} = \mathbf{L} \ltimes R_u(\mathbf{P})$.

We list some important properties of Borel subgroups and tori that we will use without further mention. Let $\mathbf{G}$ be an algebraic group. Then:

- Any two Borel subgroups of $\mathbf{G}$ are conjugate ([MaTe11, Theorem 6.4 (a)]).
- Any two maximal tori of $\mathbf{G}$ are conjugate ([MaTe11, Corollary 6.5]).
- Any Borel subgroup $\mathbf{B}$ is the semidirect product $\mathbf{T} \ltimes \mathbf{B}_{\mathrm{uni}}$, where $\mathbf{T}$ is any maximal torus of $\mathbf{B}$ ([MaTe11, Theorem 4.4 (b)]).

Assume moreover that $\mathbf{G}$ is connected. Then:

- For any Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ we have $\mathbf{G} = \cup_{g \in \mathbf{G}} \mathbf{B}^g$ ([MaTe11, Theorem 6.10]).
- Every semisimple element of $\mathbf{G}$ lies in a maximal torus ([MaTe11, Corollary 6.11 (a)]).
- Every unipotent element of $\mathbf{G}$ lies in a closed connected unipotent subgroup ([MaTe11, Corollary 6.11 (b)]).
- The maximal closed connected unipotent subgroups of $\mathbf{G}$ are all conjugate and they are of the form $\mathbf{B}_{\mathrm{uni}}$ for some Borel subgroup $\mathbf{B}$ ([MaTe11, Corollary 6.11 (c)]).

A crucial consequence of these properties is that we can choose a reference Borel subgroup $\mathbf{B}_0$ of $\mathbf{G}$ and a maximal torus $\mathbf{T}_0 \leq \mathbf{B}_0$ of $\mathbf{G}$. Then, every unipotent conjugacy class of $\mathbf{G}$ has a representative in $\mathbf{U}_0 = (\mathbf{B}_0)_{\mathrm{uni}}$ and every semisimple conjugacy class of $\mathbf{G}$ has a representative in $\mathbf{T}_0$. Once this choice has been made, we will call $\mathbf{U}_0$ *the* unipotent subgroup of $\mathbf{G}$.

Definition 1.7. An algebraic group $\mathbf{G}$ is called *reductive* if $R_u(\mathbf{G}) = 1$. It is called *semisimple* if it is connected and $R(\mathbf{G}) = 1$. A semisimple algebraic group $\mathbf{G} \neq 1$ is said to be *simple* if it does not have non-trivial proper closed connected normal subgroups.

The groups that we consider in Part II ($\mathrm{SL}_4(K)$) and III ($\mathrm{Spin}_8(K)$) are both simple.

1.2 Root system, Weyl group and structure of connected reductive groups

There is a rather precise description of connected reductive groups if we consider their associated Lie algebras. The Lie algebra of a connected algebraic group is defined as being its tangent space at the identity. The description of the adjoint action of a connected reductive group on its Lie algebra says a lot on the structure of the group itself. It leads, among other results, to the classification of semisimple groups. However, because here we are only interested in two specific cases all this information would be overwhelming for the practical purpose of this work. Instead, we decide to follow the more economical introduction from [DiMi20, Chapter 1].

Before continuing, we recall the definition of an (abstract) root system, an object that will appear many times in what follows.

Definition 1.8. A subset Φ of a finite-dimensional real vector space E is called a *root system* in E if the following properties are satisfied:

- Φ is finite, $0 \notin \Phi$, $\langle \Phi \rangle_{\mathbb{R}} = E$;
- if $c \in \mathbb{R}$ is such that $\alpha, c\alpha \in \Phi$, then $c = \pm 1$;
- for each $\alpha \in \Phi$ there exists a reflection $s_\alpha \in \mathrm{GL}(E)$ along α stabilizing Φ;
- for $\alpha, \beta \in \Phi$, $s_\alpha(\beta) - \beta$ is an integral multiple of α.

The group $W = W(\Phi) := \langle s_\alpha \mid \alpha \in \Phi \rangle$ is called the *Weyl group* of Φ. The dimension of E is called the *rank* of Φ.

A subset $\Delta \subset \Phi$ is called a *base* of Φ if it is a vector space basis of E and any root $\beta \in \Phi$ is an integral linear combination $\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$ with either all $c_\alpha \geq 0$ or all $c_\alpha \leq 0$. The roots $\alpha \in \Delta$ are called *simple roots* of Φ. If Δ is a base of Φ, then the subset

$$\Phi^+ := \left\{ \sum_{\alpha \in \Delta} c_\alpha \alpha \mid c_\alpha \geq 0 \right\} \subset \Phi$$

is called the *system of positive roots* of Φ with respect to the base Δ, and its elements are called *positive roots*.

By definition, a root system Φ that generates a vector space E is finite and is stabilized by its Weyl group W. Then, it follows that also W is finite. Moreover, W stabilizes a positive definite W-invariant symmetric bilinear form of E, unique up to non-zero scalars on each irreducible W-submodule of E. We assume that such bilinear form is chosen once and for all. Then, root system Φ with base Δ is called *decomposable* if there exists a partition $\Delta = \Delta_1 \sqcup \Delta_2$, with Δ_1, Δ_2 non-empty and orthogonal to each other. If such a decomposition does not exist, then, if $\Phi \neq \emptyset$, the root system is said to be *indecomposable*.

We recall that indecomposable root systems are classified by type and have an associated Dynkin diagram.

Theorem 1.9 ([MaTe11, Theorem 9.6]). *Let Φ be an indecomposable root system in some real vector space isomorphic to $\mathbb{R}^n$. Then, up to isomorphism, Φ is one of the following types:*

$$A_n\ (n \geq 1),\quad B_n\ (n \geq 2),\quad C_n\ (n \geq 3),\quad D_n\ (n \geq 4),\quad E_6,\quad E_7,\quad E_8,\quad F_4,\quad G_2.$$

In Parts II and III we will encounter the types A_1, A_3 and D_4.

We introduce now the Weyl group of an algebraic group.

Definition 1.10. Let $\mathbf{T}$ be a torus.

Homomorphisms of algebraic groups $\chi : \mathbf{T} \to K^\times$ are called *characters* of $\mathbf{T}$. The abelian group of characters of $\mathbf{T}$ is denoted by $X(\mathbf{T})$.

Homomorphisms of algebraic groups $\gamma : K^\times \to \mathbf{T}$ are called *cocharacters* of $\mathbf{T}$. The abelian group of cocharacters of $\mathbf{T}$ is denoted by $Y(\mathbf{T})$.

Notice that, given $\chi \in X(\mathbf{T})$ and $\gamma \in Y(\mathbf{T})$, the composition $\chi \circ \gamma$ is a homomorphism from $K^\times$ to itself. The only homomorphisms of this type are given by $x \mapsto x^n$ for some integer n. Then, the map $\langle -, - \rangle : X(\mathbf{T}) \times Y(\mathbf{T}) \to \mathbb{Z}$, $(\chi, \gamma) \mapsto n$ is a perfect pairing between $X(\mathbf{T})$ and $Y(\mathbf{T})$ (which makes them dual to each other), see [MaTe11, Proposition 3.6].

From now on, $\mathbf{G}$ denotes a connected reductive group and $\mathbf{T}_0$ a fixed maximal torus of $\mathbf{G}$. Moreover, we write $X := X(\mathbf{T}_0)$ and $Y := Y(\mathbf{T}_0)$.

We recall next how, after choosing a reference maximal torus $\mathbf{T}_0$ of $\mathbf{G}$, it is possible to associate to $\mathbf{G}$ a root system Φ and a finite Weyl group W relative to $\mathbf{T}_0$. Note that these definitions are independent of the choice of $\mathbf{T}_0$.

Definition 1.11. The Weyl group W of a connected reductive group $\mathbf{G}$ is the (abstract) finite group isomorphic to $W(\mathbf{T}) := N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ for a maximal torus $\mathbf{T}$ of $\mathbf{G}$.

Notice that the definition makes sense since all maximal tori are conjugate.

Notation 1.12. For $w \in W$ we denote by $\dot{w}$ a representative in $N_{\mathbf{G}}(\mathbf{T}_0)$ with respect to the reference torus $\mathbf{T}_0$.

The Weyl group acts by definition as automorphisms of $\mathbf{T}_0$. This action can be extended to an action on both X and Y in the following way. Let $w \in W$, $\chi \in X$ and $\gamma \in Y$, then for $t \in \mathbf{T}_0$ and $\lambda \in K^\times$ we have $w(\chi)(t) := \chi(t^{\dot{w}})$ and $w(\gamma)(\lambda) := \gamma(\lambda)^{\dot{w}}$. These two actions are related by $\langle \chi,\, w(\gamma)\rangle = \langle w(\chi),\, \gamma\rangle$.

On the other side, it is possible to identify a root system Φ as a subset of X. Then, the Weyl group $W(\Phi)$ of this root system is isomorphic to the Weyl group $W(\mathbf{T}_0)$. This comes from the following theorem.

Theorem 1.13 ([DiMi91, 0.31 Theorem (i) and (ii)]). *Let $\mathbf{G}$ be connected reductive and $\mathbf{T}_0$ the reference torus.*

(a) *Non-trivial minimal closed unipotent subgroups of $\mathbf{G}$ normalized by $\mathbf{T}_0$ are isomorphic to K^+; the conjugation action of $t \in \mathbf{T}_0$ is mapped by this isomorphism to an action of $\mathbf{T}_0$ on K^+ of the form $x \mapsto \alpha(t)x$, where $\alpha \in X$.*

(b) *The elements $\alpha \in X$ obtained in* (i) *are all distinct, non-zero and finite in number. They form a root system Φ in the subspace of $X \otimes \mathbb{R}$ that they generate. The group $W(\mathbf{T}_0) = N_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0$ is isomorphic to the Weyl group of Φ.*

Definition 1.14. The elements $\alpha \in X$ from Theorem 1.13 (i) are called the *roots* of $\mathbf{G}$ relative to $\mathbf{T}_0$. The set Φ is called the *root system* of $\mathbf{G}$ (relative to $\mathbf{T}_0$).

For every root $\alpha \in \Phi \subset X$ there exists a coroot $\alpha^\vee \in Y$ such that $\langle \alpha,\, \alpha^\vee \rangle = 2$ and such that Φ is stable under the reflection $s_\alpha : X \otimes \mathbb{R} \to X \otimes \mathbb{R}, x \mapsto x - \langle x,\, \alpha^\vee \rangle \alpha$. These $\alpha^\vee$ are called the *coroots* of $\mathbf{G}$ (relative to $\mathbf{T}_0$), and they form a root system in $Y \otimes \mathbb{R}$ denoted by $\Phi^\vee$.

Definition 1.15. The one-dimensional unipotent subgroup of $\mathbf{G}$ corresponding to the root $\alpha \in \Phi$ as in Theorem 1.13 is denoted by $\mathbf{U}_\alpha$ and is called the *root subgroup* of $\mathbf{G}$ associated to α.

The isomorphism of Theorem 1.13 (a) is denoted by $u_\alpha : K^+ \to \mathbf{U}_\alpha$ and is called *root map.*

These root subgroups are essential for the description of the structure of connected reductive groups.

Theorem 1.16 ([DiMi20, Theorem 2.3.1 (iv)]). *Let $\mathbf{G}$ be connected reductive and $\mathbf{T}_0$ the reference torus. Any closed connected subgroup $\mathbf{H}$ of $\mathbf{G}$ normalized by $\mathbf{T}_0$ is generated by $(\mathbf{T}_0 \cap \mathbf{H})^0$ and the $\mathbf{U}_\alpha$ it contains; in particular, $\mathbf{G}$ is generated by $\mathbf{T}_0$ and the $\mathbf{U}_\alpha$.*

It follows that $\mathbf{U}_0 = \prod_{\alpha\in\Phi^+} \mathbf{U}_\alpha$ for a suitable choice of a base of the root system Φ relative to $\mathbf{T}_0$.

The multiplication law of a maximal unipotent group like $\mathbf{U}_0$ can be described thanks to the so-called commutation relations of its root subgroups and root maps. A proof of which can be found in [Hu75, Lemma 32.5].

Proposition 1.17 ([MaTe11, Theorem 11.8],Commutation relations). *Let Φ be the root system relative to a maximal torus of a connected reductive group such that Φ^+ has a fixed total order compatible with addition. Then, for $\alpha, \beta \in \Phi^+$ there exist integers $c^{nm}_{\alpha\beta}$ and a choice of root maps such that*

$$[u_\alpha(t),\, u_\beta(s)] = \prod u_\gamma(c^{nm}_{\alpha\beta} t^n s^m) \text{ for all } t, s \in K$$

where the product is over the positive roots of the form $\gamma = n\alpha + m\beta \in \Phi^+$ for integers $n, m > 0$ (in the chosen ordering). We use the convention $[x, y] = x^{-1}y^{-1}xy$.

Although Theorem 1.16 already gives the structure of connected reductive groups, we give a more explicit result that describes the structure of connected reductive/semisimple groups.

Theorem 1.18 ([MaTe11, Theorem 8.17(g),(h) and Theorem 8.21(a),(b)]). *Let $\mathbf{G}$ be a connected reductive group, $\mathbf{T}_0$ the reference maximal torus and Φ the root system (of $\mathbf{G}$ relative to $\mathbf{T}_0$). Then:*

(a) $\mathbf{G} = \langle \mathbf{T}_0, \mathbf{U}_\alpha \mid \alpha \in \Phi \rangle$.

(b) $Z(\mathbf{G}) = \bigcap_{\alpha\in\Phi} \ker \alpha$.

Furthermore, if $\mathbf{G}$ is semisimple, then:

(c) $\mathbf{G} = \langle \mathbf{U}_\alpha \mid \alpha \in \Phi \rangle$.

(d) $\mathbf{G} = [\mathbf{G}, \mathbf{G}]$.

Recall that the *rank* $\mathrm{rk}(\mathbf{G})$ of an algebraic group $\mathbf{G}$ is the dimension of any of its maximal tori. If $\mathbf{G}$ is reductive, its *semisimple rank* is $\mathrm{rk}_{\mathrm{ss}}(\mathbf{G}) := \mathrm{rk}(\mathbf{G}/R(\mathbf{G}))$.

Corollary 1.19 ([MaTe11, Corollary 8.22]). *Let $\mathbf{G}$ be connected reductive. Then*

$$\mathbf{G} = [\mathbf{G},\, \mathbf{G}]\, R(\mathbf{G}) = [\mathbf{G},\, \mathbf{G}]\, Z(\mathbf{G})^\circ;$$

in particular, $\mathrm{rk}_{\mathrm{ss}}(\mathbf{G}) := \mathrm{rk}([\mathbf{G},\, \mathbf{G}])$ *and* $\mathrm{rk}(\mathbf{G}) = \mathrm{rk}_{\mathrm{ss}}(\mathbf{G}) + \dim Z(\mathbf{G})$.

An important object in the representation theory of finite groups of Lie type is the centralizer of semisimple elements. We have again an explicit description of its structure.

Theorem 1.20 ([MaTe11, Theorem 14.2])**.** *Let* $\mathbf{G}$ *be connected reductive. Let* $s \in \mathbf{G}$ *be semisimple,* $\mathbf{T} \leq \mathbf{G}$ *a maximal torus containing* s *with corresponding root system* Φ*, and fix the set* $\Psi := \{\alpha \in \Phi \mid \alpha(s) = 1\}$*. Then:*

(a) $C_{\mathbf{G}}(s)^\circ = \langle \mathbf{T}, \mathbf{U}_\alpha \mid \alpha \in \Psi \rangle$.

(b) $C_{\mathbf{G}}(s) = \langle \mathbf{T}, \mathbf{U}_\alpha, \dot{w} \mid \alpha \in \Psi,\ w \in W \text{ with } s^{\dot{w}} = s \rangle$.

Moreover, $C_{\mathbf{G}}(s)^\circ$ *is reductive with root system* Ψ *and Weyl group* $W(s)^\circ := \langle s_\alpha \mid \alpha \in \Psi \rangle$*.*

Notation 1.21. We denote by $W(s) := \{w \in W \mid s^{\dot{w}} = s\}$ the "Weyl group"[1] of $C_{\mathbf{G}}(s)$. The group $W(s)^\circ$ is normal in $W(s)$ and $W(s)/W(s)^\circ \cong C_{\mathbf{G}}(s)/C_{\mathbf{G}}(s)^\circ$ ([DiMi20, Remark 3.5.2]).

From now on, we use the notation $A(g) := A_{\mathbf{G}}(g) := C_{\mathbf{G}}(g)/C_{\mathbf{G}}(g)^\circ$ for $g \in \mathbf{G}$.

Remark 1.22. Recall that the Weyl group is finite and there are only finitely many root subsystems $\Psi \subseteq \Phi$. Then, it follows from the proposition that there are only finitely many distinct centralizers of semisimple elements of a connected reductive group, up to conjugacy

1.3 Classification and isogenies of semisimple groups

Semisimple algebraic groups are classified thanks to a combinatorial tool called the root datum.

Definition 1.23. A quadruple $(X, \Phi, Y, \Phi^\vee)$ is called a *root datum* if

- $X \cong \mathbb{Z}^n \cong Y$, with a perfect pairing $\langle -, - \rangle : X \times Y \to \mathbb{Z}$, for some n;
- $\Phi \subset X$, $\Phi^\vee \subset Y$ are abstract root systems in $\mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathbb{Z}\Phi^\vee \otimes_{\mathbb{Z}} \mathbb{R}$, respectively;
- there exists a bijection $\Phi \to \Phi^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$ for all $\alpha \in \Phi$; and
- the reflections s_α of the root system Φ and $s_{\alpha^\vee}$ of $\Phi^\vee$ are given, respectively, by $s_\alpha(\chi) = \chi - \langle \chi, \alpha^\vee \rangle \alpha$ for all $\chi \in X$, and $s_{\alpha^\vee}(\gamma) = \gamma - \langle \alpha, \gamma \rangle \alpha^\vee$ for all $\gamma \in Y$.

If $(X, \Phi, Y, \Phi^\vee)$ is a root datum, we call $(Y, \Phi^\vee, X, \Phi)$ its *dual* root datum. More generally, we say that two root data are *dual* if one of the root data is isomorphic (see definition below) to the dual of the other root datum.

This definition is justified by the fact that for a connected reductive group the quadruple $(X(\mathbf{T}_0), \Phi, Y(\mathbf{T}_0), \Phi^\vee)$ is a root datum (see [MaTe11, Proposition 9.11]), where Φ is the root system relative to the reference maximal torus $\mathbf{T}_0$ and $\Phi^\vee$ is the set of coroots as in Definition 1.14.

Definition 1.24. Two root data $(X, \Phi, Y, \Phi^\vee)$ and $(X', \Phi', Y', \Phi'^\vee)$ are said to be *isomorphic* if there exist isomorphisms of abelian groups $\delta : X \to X'$ and $\varepsilon : Y \to Y'$ such that:

- $\langle \delta(\chi), \varepsilon(\gamma) \rangle = \langle \chi, \gamma \rangle$ for all $\chi \in X$ and $\gamma \in Y$.
- $\delta(\Phi) = \Phi'$ and $\varepsilon(\Phi^\vee) = \Phi'^\vee$.
- $\varepsilon(\alpha^\vee) = \delta(\alpha)^\vee$ for all $\alpha \in \Phi$.

Theorem 1.25 (Chevalley Classification Theorem, [MaTe11, Theorem 9.13])**.** *Two semisimple algebraic groups are isomorphic if and only if they have isomorphic root data. For each root datum there exists a semisimple algebraic group which realizes it. This group is simple if and only if its root system is indecomposable.*

---

[1] There is also a notion of Weyl group for disconnected groups, that we do not need explicitly here.

It is possible to give a more precise description of semisimple groups with the same root system by means of the fundamental group. First of all, define $\Omega := \mathrm{Hom}(\mathbb{Z}\Phi^\vee, \mathbb{Z})$. Second, notice that, thanks to the perfect pairing $\langle -, - \rangle$, we can identify $X \cong \mathrm{Hom}(Y, \mathbb{Z})$. And, by restriction, we get an injection $\mathrm{Hom}(Y, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}\Phi^\vee, \mathbb{Z})$. Then we have the inclusions $\mathbb{Z}\Phi \subseteq X \subseteq \Omega$ (each of finite index). These are the ingredients for the next definition.

Definition 1.26. Let $\mathbf{G}$ be a semisimple algebraic group with root datum $(X, \Phi, Y, \Phi^\vee)$ and Ω as above. Then the finite group $\Lambda(\mathbf{G}) := \Omega/X$ is called the *fundamental group* of $\mathbf{G}$.
If $X = \Omega$ ($\Lambda(\mathbf{G}) = 1$), then $\mathbf{G}$ is said to be *simply connected*, and is denoted by $\mathbf{G}_{\mathrm{sc}}$. If $X = \mathbb{Z}\Phi$, then $\mathbf{G}$ is said to be of *adjoint type*, and is denoted by $\mathbf{G}_{\mathrm{ad}}$.

A surjective homomorphism of algebraic groups with finite kernel is called an *isogeny*. Two groups with such a morphism between them are said to be *isogenous*.

Notice that for connected reductive groups the kernel of an isogeny lies in all maximal tori.

Proposition 1.27 ([MaTe11, Proposition 9.15]). *Let $\mathbf{G}$ be semisimple with root system Φ. Then there exist natural isogenies*

$$\mathbf{G}_{\mathrm{sc}} \xrightarrow{\pi_1} \mathbf{G} \xrightarrow{\pi_2} \mathbf{G}_{\mathrm{ad}}$$

from a simply connected group $\mathbf{G}_{\mathrm{sc}}$ and to an adjoint group $\mathbf{G}_{\mathrm{ad}}$, each with root system Φ, with $\ker(\pi_1) \cong \Lambda(\mathbf{G})_{p'}$, $\ker(\pi_2) \cong (\Lambda(\mathbf{G}_{\mathrm{ad}})/\Lambda(\mathbf{G}))_{p'}$, *where* $p = \mathrm{char}(K)$.

The various semisimple groups with the same root system Φ in between $\mathbf{G}_{\mathrm{sc}}$ and $\mathbf{G}_{\mathrm{ad}}$, according to Proposition 1.27, are called the *isogeny types* corresponding to Φ.

Example 1.28. In Parts II and III, we consider, respectively, $\mathrm{SL}_4(K)$ and $\mathrm{Spin}_8(K)$. They are both simple algebraic groups of simply connected type. $\mathrm{SL}_4(K)$ is of type A_3 and $\mathrm{Spin}_8(K)$ is of type D_4. See [MaTe11, Table 9.2].

1.4 BN-pair, Bruhat decomposition and Chevalley relations

An important property of finite groups of Lie type and of the connected reductive groups from which they are constructed is their (split) BN-pair. This pair encodes structural information about these groups, which is useful in practical computations. In what follows below, we recall the definition of split BN-pairs and we list the properties that are used in Parts II and III.

Definition 1.29. Let G be any group. Two subgroups B and N are said to form a *BN-pair* if the following axioms are satisfied.

- $G = \langle B, N \rangle$.
- $H := B \cap N$ is normal in N.
- $W := N/H$ is generated by a set of elements s_i, $i \in I$, with $s_i^2 = 1$.
- If $n_i \in N$ maps to $s_i \in W$ under the natural homomorphism, then $n_i B n_i \neq B$.
- For each $n \in N$ and each n_i we have $n_i B n \subseteq B n_i n B \cup B n B$.

The group W is called the *Weyl group* of the BN-pair.

The Weyl group of a BN-pair is a Coxeter group, see [Ca85, Proposition 2.1.7], but this fact is not used explicitly in the rest of this work and will not be discussed any further.

Let G be a group with a BN-pair. Then $G = BNB$ ([Ca85, Proposition 2.1.1]). This decomposition can be described more precisely. The double cosets $BnB, Bn'B \in BNB$ are equal if n and n' represent the same element of $W = N/H$ in N ([Ca85, Proposition 2.1.2]). Then, this implies that the group has a decomposition

$$G = \coprod_{w \in W} B\dot{w}B$$

where $\dot{w} \in N$ denotes a representative of $w \in W$. It is called the *Bruhat decomposition* of G.

It is possible to define a notion of parabolic subgroups of a group with a BN-pair by considering subgroups of the Weyl group .

Definition 1.30. Let J be a subset of the index set I. Let W_J be the subgroup of W generated by the elements s_i with $i \in J$ and let N_J be the subgroup of N satisfying $N_J/H = W_J$. Then the subgroup $P_J := BN_JB$ is called a *standard parabolic subgroup* of G. A *parabolic subgroup* of G is a subgroup conjugate to P_J for some $J \subseteq I$ (this is well-defined by [Ca85, Proposition 2.1.6]).

The definition of BN-pairs can be specialized to split BN-pairs for algebraic groups.

Definition 1.31. The algebraic group $\mathbf{G}$ has a *split BN-pair* if it satisfies the following axioms:

- $\mathbf{G}$ has closed subgroups $\mathbf{B}$ and $\mathbf{N}$ which form a BN-pair.
- $\mathbf{B} = \mathbf{H} \ltimes \mathbf{U}$, where $\mathbf{H} := \mathbf{B} \cap \mathbf{N}$ is a closed commutative subgroup of semisimple elements and $\mathbf{U}$ is a closed normal unipotent group.
- $\bigcap_{n \in \mathbf{N}} n\mathbf{B}n^{-1} = \mathbf{H}$.

Analogously, for finite groups we define split BN-pairs of characteristic p.

Definition 1.32. A finite group G is said to have a *split BN-pair of characteristic p* if the following conditions hold:

- G has subgroups B, N which form a BN-pair.
- $B = H \ltimes U$, where U is a normal p-subgroup of B and H is an abelian subgroup of order prime to p.
- $\bigcap_{n \in N} nBn^{-1} = H$.

Remark 1.33. Notice that any finite group can be seen as an algebraic group (see discussion at the end of Chapter 1.2 of [MaTe11]). It follows that a finite group with a split BN-pair of characteristic p can be seen as an algebraic group over an algebraically closed field of characteristic p with a split BN-pair (according to Definition 1.31). Then, Definition 1.31 includes both the case of connected reductive groups and the finite groups of Lie type that we construct in the next section.

We list now some properties of algebraic groups with BN-pairs that are useful for the computations in Parts II and III. But, first we fix some notations.

Notation 1.34. For the remainder of this section, $\mathbf{G}$ denotes an algebraic group with a split BN-pair formed by $\mathbf{B}$ and $\mathbf{N}$. It has Weyl group $W := \mathbf{N}/\mathbf{H}$ generated by elements s_i indexed by $i \in I$, where $\mathbf{H} := \mathbf{B} \cap \mathbf{N}$. We denote by w_0 the element of maximal reduced length of W (this exists and is unique by [Ca85, Proposition 2.2.11]) and for every $w \in W$ we denote a representative in $\mathbf{N}$ by $\dot{w}$.

We define the following subgroups of $\mathbf{G}$:

$$\mathbf{U}^- := \mathbf{U}^{\dot{w}_0},\ \mathbf{U}_i := \mathbf{U} \cap \mathbf{U}^{\dot{w}_0 \dot{s}_i},\ \mathbf{U}_{-i} := \mathbf{U}_i^{\dot{s}_i},\ \text{and } \mathbf{U}_w := \mathbf{U} \cap \mathbf{U}^{\dot{w}_0 \dot{w}},$$

for $i \in I$ and $w \in W$.

For a subset $J \subset I$, we denote by Φ_J the root subsystem of Φ with base $\Delta_J = \{\alpha_i \mid i \in J\}$ (and Weyl group W_J). We denote by $(w_0)_J$ the element of W_J of maximal reduced length.

Proposition 1.35 ([Ca85, Proposition 2.5.1])**.** $\mathbf{U}$ *is a maximal unipotent subgroup of* $\mathbf{G}$.

For split BN-pairs it is possible to refine the Bruhat decomposition by adding a "uniqueness of expression" statement.

Theorem 1.36 ([Ca85, Theorem 2.5.14], Sharp form of the Bruhat decomposition)**.** *Each element of* $\mathbf{G}$ *is uniquely expressible in the form*

$$uh\dot{w}u_w$$

where $u \in \mathbf{U}$*,* $h \in \mathbf{H}$*,* $w \in W$ *and* $u_w \in \mathbf{U}_w$*.*

Proposition 1.37 ([Ca85, Proposition 2.5.15])**.** *The set of subgroups* $\dot{w}\mathbf{U}_i\dot{w}^{-1}$ *for* $w \in W$ *and* $i \in I$ *is in bijective correspondence with the set* Φ *of roots. The root corresponding to the subgroup* $\dot{w}\mathbf{U}_i\dot{w}^{-1}$ *is* $w(\alpha_i)$*.*

If $w(\alpha_i) = \alpha$, we denote the root subgroup $\dot{w}\mathbf{U}_i\dot{w}^{-1}$ by $\mathbf{U}_\alpha$.

Proposition 1.38 ([Ca85, Corollary 2.5.17])**.** $\mathbf{U} = \prod_{\alpha \in \Phi^+} \mathbf{U}_\alpha$ *with uniqueness, if the positive roots are taken in any fixed order in the product.*

We turn now to properties of standard parabolic subgroups and standard Levi subgroups of groups with split BN-pairs.

For each subset $J \subseteq I$ define the subgroup $\mathbf{L}_J$ of $\mathbf{G}$ by $\mathbf{L}_J := \langle \mathbf{H}, \mathbf{U}_\alpha \mid \alpha \in \Phi_J \rangle$.

Proposition 1.39 ([Ca85, Proposition 2.6.3])**.** *The group* $\mathbf{L}_J$ *has a split* BN*-pair corresponding to subgroups* $\mathbf{B}_J := \mathbf{U}_{(w_0)_J}\mathbf{H}$ *and* $\mathbf{N}_J$*, where* $\mathbf{N}_J \leq \mathbf{N}$ *is such that* $\mathbf{N}_J/\mathbf{H} = W_J$*.*

Our aim is to apply the results on BN-pairs to connected reductive groups. For the remainder of this Section we choose a total ordering of Φ^+ such that the subgroups $\mathbf{U}_\alpha$, for $\alpha \in \Phi^+$, follow the commutation relations of Proposition 1.17. This is needed to discuss the Levi decomposition of standard parabolic subgroups.

Proposition 1.40 ([Ca85, Proposition 2.6.4])**.** *Let* $\mathbf{G}$ *be an algebraic group with a split* BN*-pair which satisfies the commutator relations. Let* $\mathbf{U}_J := \mathbf{U} \cap \mathbf{U}^{(\dot{w}_0)_J}$*. Then* $\mathbf{U}_J$ *is a normal subgroup of the standard parabolic subgroup* $\mathbf{P}_J$*,* $\mathbf{P}_J = \mathbf{U}_J\mathbf{L}_J$ *and* $\mathbf{U}_J \cap \mathbf{L}_J = 1$*.*

Definition 1.41. The decomposition $\mathbf{U}_J\mathbf{L}_J$ of the standard parabolic subgroup $\mathbf{P}_J$ is called its *Levi decomposition* and $\mathbf{L}_J$ is called a *standard Levi subgroup* of $\mathbf{G}$. A *Levi subgroup* of $\mathbf{P}_J$ is a $\mathbf{P}_J$-conjugate of $\mathbf{L}_J$.

The parabolic/Levi subgroup structure of $\mathbf{G}$ is well behaved with respect to inclusion. Since any Levi subgroup of an algebraic group with a split BN-pair is itself an algebraic group with a split BN-pair, it is possible to relate parabolic/Levi subgroups of a Levi subgroup to those of $\mathbf{G}$.

Proposition 1.42 ([Ca85, Proposition 2.6.6]). *The standard parabolic subgroup of $\mathbf{L}_J$ corresponding to the subset $J' \subseteq J$ is $\mathbf{P}_{J'} \cap \mathbf{L}_J$.*

Proposition 1.43 ([Ca85, Proposition 2.6.7]). *The maximal normal unipotent subgroup of $\mathbf{P}_{J'} \cap \mathbf{L}_J$ is $\mathbf{U}_{J'} \cap \mathbf{L}_J$. The standard Levi subgroup of $\mathbf{P}_{J'} \cap \mathbf{L}_J$ is $\mathbf{L}_{J'}$.*

Notation 1.44. The Weyl group of $\mathbf{G}$ is generated by a subset of involutions S indexed by a set I and corresponding to the simple roots Δ. We use interchangeably the notations for $\mathbf{P}_J$, $\mathbf{L}_J$ and W_J with J a subset of the index set I, a subset of S or a subset of Δ.

Every result in this section is valid for any connected reductive (and therefore also any semisimple) group.

Proposition 1.45 ([MaTe11, Theorem 11.16]). *Let $\mathbf{G}$ be a connected reductive algebraic group with Borel subgroup $\mathbf{B}$ and $\mathbf{N} := N_{\mathbf{G}}(\mathbf{T})$ for some maximal torus $\mathbf{T} \leq \mathbf{B}$. Then $\mathbf{B}$ and $\mathbf{N}$ form a BN-pair in $\mathbf{G}$ whose Weyl group is equal to that of $\mathbf{G}$.*

It is important to notice that Levi subgroups of a connected reductive group are themselves connected and reductive, see [MaTe11, Proposition 12.6].

Corollary 1.46. *Connected reductive groups have a split BN-pair (the same pair as in Proposition 1.45).*

Proof. For connected reductive groups, $\mathbf{B} = \mathbf{T} \ltimes \mathbf{B}_{\text{uni}}$ holds. Thus we only need to prove that $\bigcap\limits_{n \in \mathbf{N}} n\mathbf{B}n^{-1} = \mathbf{T}$.

We fix a base of the root system Φ of $\mathbf{G}$ relative to $\mathbf{T}$ and we denote the subsets of positive roots and negative roots by Φ^+ and Φ^-, respectively. Then we have two Borel subgroups $\mathbf{B}^+ = \mathbf{T} \cdot \prod_{\alpha \in \Phi^+} \mathbf{U}_\alpha$ and $\mathbf{B}^- = \mathbf{T} \cdot \prod_{\alpha \in \Phi^-} \mathbf{U}_\alpha$. It easy to see that $\mathbf{B}^+ \cap \mathbf{B}^- = \mathbf{T}$. Moreover, $\mathbf{B}^+$ and $\mathbf{B}^-$ are conjugate by $\dot{w}_0 \in \mathbf{N}$, since $w_0(\Phi^+) = \Phi^-$ (see [MaTe11, Corollary A.23]) and because for all $\alpha \in \Phi$ we have that $\dot{w}\mathbf{U}_\alpha \dot{w}^{-1} = \mathbf{U}_{w(\alpha)}$ for all $w \in W$. This last statement can be seen explicitly on the root maps:

Fix $t \in K$, $w \in W$ and $h \in \mathbf{T}$. Then for all $\alpha \in \Phi$ we have

$$h\dot{w}u_\alpha(t)\dot{w}^{-1}h^{-1} = \dot{w}h'u_\alpha(t)h'^{-1}\dot{w}^{-1} = \dot{w}u_\alpha(\alpha(h')t)\dot{w}^{-1}$$

where $h' = h^{\dot{w}} \in \mathbf{T}$.

Now, by [Ca85, Proposition 3.1.2(ii)] the group $Y(\mathbf{T}) \otimes K^\times$ is isomorphic to the torus $\mathbf{T}$. This means that we can find a cocharacter $\gamma \in Y(\mathbf{T})$ and $c \in K^\times$ such that $h = \gamma(c)$. Then we obtain $\alpha(h') = \alpha\left(\gamma(c)^{\dot{w}}\right) = \alpha\left(w(\gamma)(c)\right) = c^{\langle \alpha, w(\gamma) \rangle} = c^{\langle w(\alpha), \gamma \rangle} = w(\alpha)(\gamma(c)) = w(\alpha)(h)$. Plugging this back in, we see that ${}^{\dot{w}}\mathbf{U}_\alpha = \mathbf{U}_{w(\alpha)}$. $\square$

We are now interested in describing more precisely the elements of semisimple groups. One important information used in the computations of Parts II and III is the fusion of unipotent classes of a semisimple group. By fusion of unipotent classes we mean identifying for each unipotent conjugacy class of a maximal unipotent subgroup $\mathbf{U}$ of the group $\mathbf{G}$ to which conjugacy class of $\mathbf{G}$ they belong, i.e. for any unipotent element $u \in \mathbf{G}$ find those unipotent conjugacy classes $u_i^{\mathbf{U}}$ of $\mathbf{U}$ such that $\bigcup u_i^{\mathbf{U}} = u^{\mathbf{G}} \cap \mathbf{U}$.

This implies that we need to be able to conjugate unipotent elements by arbitrary elements $g \in \mathbf{G}$. Due to the sharp form of the Bruhat decomposition (Theorem 1.36), we can write

$$g = ut\dot{w}u_w$$

for uniquely determined $u \in \mathbf{U}_0 = R_u(\mathbf{B}_0)$, $t \in \mathbf{T}_0$, $w \in W = N_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0$ and $u_w \in \mathbf{U}_w = \mathbf{U}_0^{\dot{w}_0\dot{w}}$ (for our choice of a reference BN-pair, $\mathbf{B}_0, \mathbf{N}_0$ and maximal torus $\mathbf{T}_0 \leq \mathbf{B}_0$).

In other words, to effectively describe all the elements of $\mathbf{G}$ we need to explicitly express the elements of $\mathbf{U}_0$, of the torus $\mathbf{T}_0$ and representatives of the simple reflections of W. Also, we need to know how they act (by conjugation) on each other.

Fortunately, semisimple groups are generated by their unipotent elements (Theorem 1.18 (c)). Thus, it is possible to write down elements that generate the BN-pair. These are called Chevalley generators of $\mathbf{G}$, and the relations between them are called the Chevalley relations.

These generators and relations are summarized in [GLS98, Theorem 1.12.1] and they are treated in detail in [Ca72, chapter 12.1].

Notation 1.47 (Chevalley generators)**.** Let $\mathbf{G}$ be a semisimple group with split BN-pair $\mathbf{B}$, $\mathbf{N}$ and root system Φ. Denote by $\mathbf{T}$ the maximal torus $\mathbf{B} \cap \mathbf{N}$, by $\mathbf{U}$ the maximal unipotent subgroup of $\mathbf{B}$ and by $W = \mathbf{N}/\mathbf{T}$ the Weyl group.

For every root $\alpha \in \Phi$ we define the root maps $u_\alpha : K^+ \to \mathbf{U}_\alpha$ as in Definition 1.15. Then, we define maps $n_\alpha : K^\times \to \mathbf{N}$ by $n_\alpha(t) := u_\alpha(t)u_{-\alpha}(-t^{-1})u_\alpha(t)$ and define maps $h_\alpha : K^\times \to \mathbf{T}$ by $h_\alpha(t) := n_\alpha(t)n_\alpha(-1)$ for all $t \in K^\times$ (which are cocharacters). Moreover, we define elements $n_\alpha := n_\alpha(1)$.

Then, as shown in the proof of [Ca72, Theorem 12.1.1] we have $\mathbf{T} = \langle h_\alpha(t) \mid \alpha \in \Phi,\, t \in K^\times\rangle$ and $\mathbf{N} = \langle \mathbf{T},\, n_\alpha \mid \alpha \in \Phi\rangle$. Furthermore, for each root $\alpha \in \Phi$ we have $n_\alpha\mathbf{T} = s_\alpha \in W$ where s_α is the reflection relative to the root α.

Remark 1.48. A semisimple group $\mathbf{G} = \langle \mathbf{U}_\alpha \mid \alpha \in \Phi\rangle$, with root system Φ and generated by its root subgroups, is actually generated just by the subgroups corresponding to a particular base Δ of Φ, i.e. $\mathbf{G} = \langle \mathbf{U}_\alpha \mid \alpha \in \pm\Delta\rangle$.

This follows from the facts listed below:

- for all roots $\alpha \in \Phi$ and $w \in W$ we have ${}^{\dot{w}}\mathbf{U}_\alpha = \mathbf{U}_{w(\alpha)}$ (seen above in the proof of Corollary 1.46);
- for every root $\beta \in \Phi$ there exist $\alpha \in \Delta$ and $w \in W$ such that $\beta = w(\alpha)$ (see [MaTe11, Proposition A.11]); and
- every element of W is a product of the simple reflections s_α, $\alpha \in \Delta$.

Then, we have

$$\begin{gathered}\mathbf{G} = \langle u_\alpha(t) \mid \alpha \in \pm\Delta,\, t \in K\rangle,\\ \mathbf{T} = \langle h_\alpha(t) \mid \alpha \in \Delta,\, t \in K^\times\rangle,\\ \mathbf{N} = \langle \mathbf{T},\, n_\alpha \mid \alpha \in \Delta\rangle.\end{gathered}$$

Now, the Chevalley relations that we use are given by

- the commutation relations $[u_\alpha(t),\, u_\beta(s)] = \prod u_\gamma(c_{\alpha\beta}^{nm}t^n s^m)$ as in Proposition 1.17;
- $u_\alpha(t)^{h_\beta(s)} = u_\alpha(s^{\langle\alpha,\beta^\vee\rangle}t)$ for $\alpha, \beta \in \Phi$ and $t \in K$, $s \in K^\times$;
- $u_\alpha(t)^{n_\beta} = u_{s_\beta(\alpha)}(c_{\alpha\beta}t)$ for $\alpha, \beta \in \Phi$ and $t \in K$, where $c_{\alpha\beta}$ are signs independent of t;

- $h_\alpha(s)^{n_\beta} = h_{s_\beta(\alpha)}(s)$ for $\alpha, \beta \in \Phi$ and $s \in K^\times$.

Notice moreover that for a simply connected group we have $\prod_{\alpha\in\Delta} h_\alpha(t_\alpha) = 1$ if and only if $t_\alpha = 1$ for all $\alpha \in \Delta$.

It is clear that knowing all the values of $c^{nm}_{\alpha\beta}$ and $c_{\alpha\beta}$ is equivalent to knowing the multiplication law of the whole group. These constants must be computed case by case by finding a suitable (faithful) representation of the group. Once the Chevalley relations have been explicitly computed with the chosen representation, it is possible to carry out any computation in **G** with the use of the Chevalley generators and relations in the form of u_α, h_α and n_α without having to use the underlying representation. The Chevalley generators together with the Chevalley relations are called the *Steinberg presentation* of **G** (compare this with [DiMi20, Theorem 2.4.11]).

Every computation made in Parts II and III is expressed in the Steinberg presentation. The use of a Steinberg presentation makes it easier to write results of computations in a clearer way. On one side it avoids having matrices laying around in the text, and on the other side it underlines the "structure" of the elements (unipotent part, semisimple part, ...).

1.5 Steinberg maps and finite groups of Lie type

We can now discuss the finite groups of Lie type. We construct these here as fixed points of a connected reductive group under a certain map called Steinberg endomorphism. For this section we use [GeMa20, Chapter 1.4] and [MaTe11, Chapter 21] as references.

We first need to introduce some basic definitions (which are valid more generally for affine varieties).

Definition 1.49. Let $\mathbf{X}$ be an affine variety over $K = \bar{\mathbb{F}}_q$, for some prime power q. We say that $\mathbf{X}$ has an *$\mathbb{F}_q$-rational structure* (or that $\mathbf{X}$ is *defined over* $\mathbb{F}_q$) if there exists some $n > 1$ and an isomorphism of affine varieties $i : \mathbf{X} \to \mathbf{X}'$ where $\mathbf{X}' \subseteq K^n$ is Zariski closed and stable under the *standard Frobenius map*

$$F_q : K^n \to K^n,\ (\xi_1, \dots, \xi_n) \mapsto (\xi_1^q, \dots, \xi_n^q).$$

In this case, there is a unique morphism of affine varieties $F : \mathbf{X} \to \mathbf{X}$ such that $i \circ F = F_q \circ i$; it is called the *Frobenius map* corresponding to the $\mathbb{F}_q$-rational structure of $\mathbf{X}$.

Notice that F_q is a bijective morphism with a finite number of fixed point (by elementary Galois theory). It follows that F is a bijective morphism such that $\mathbf{X}^F := \{x \in \mathbf{X} \mid F(x) = x\}$ is finite.

These definitions are easily adapted to algebraic groups. We take an algebraic group $\mathbf{G}$ such that, as an affine variety, it is defined over $\mathbb{F}_q$ with corresponding Frobenius map F. Then, we say that $\mathbf{G}$ (as an algebraic group) is *defined over* $\mathbb{F}_q$ if F is a group homomorphism. In this case, the set of fixed points $\mathbf{G}^F$ is a finite group (since group homomorphisms commute with inversion and multiplication).

Thanks to Theorem 1.1 it is possible to give a more concrete description in terms of matrices, for algebraic groups. The morphism

$$F_q : \mathrm{GL}_n(K) \to \mathrm{GL}_n(K),\ (a_{ij}) \mapsto (a_{ij}^q)$$

is called the *standard Frobenius map* of $\mathrm{GL}_n(K)$ for $n \in \mathbb{N}$. An algebraic group $\mathbf{G}$ is defined over $\mathbb{F}_q$ if and only if for some n there is a homomorphism of algebraic groups $i : \mathbf{G} \to \mathrm{GL}_n(K)$ which is an isomorphism onto its image and such that $i(\mathbf{G})$ is F_q-stable. Then, a corresponding Frobenius map $F : \mathbf{G} \to \mathbf{G}$ is again defined by $i \circ F = F_q \circ i$, and $\mathbf{G}^F$ is a finite group.

This construction can be generalized to account for some finite groups (called the Ree and Suzuki groups) that are excluded when considering only Frobenius maps. Although these groups are not treated here, the majority of results that we state in these pages are true in the more general case. Then, we discuss the general case for completeness.

Definition 1.50. Let $F : \mathbf{G} \to \mathbf{G}$ be an endomorphism of algebraic groups. Then F is called a *Steinberg map* if some power of F is a Frobenius map of $\mathbf{G}$. Note that, in this case, F is a bijective homomorphism of algebraic groups and $\mathbf{G}^F$ is a finite group. If $\mathbf{G}$ is connected and reductive, then $\mathbf{G}^F$ is called a *finite group of Lie type* or a *finite reductive group.*

We will see that for a connected reductive group $\mathbf{G}$ with a split BN-pair and a Steinberg endomorphism F related to its $\mathbb{F}_q$-structure, the finite group $\mathbf{G}^F$ inherits some of the structural properties of $\mathbf{G}$. For instance, not only both $\mathbf{G}$ and $\mathbf{G}^F$ have split BN-pairs but these are tightly related by F.

A crucial tool that we need to understand the relation between the structures of $\mathbf{G}$ and $\mathbf{G}^F$ is given in the following theorem.

Theorem 1.51 ([MaTe11, Theorem 21.7], Lang–Steinberg)**.** *Let* $\mathbf{G}$ *be a connected algebraic group over* $\bar{\mathbb{F}}_q$ *with a Steinberg endomorphism* F*. Then the morphism*

$$\mathscr{L} : \mathbf{G} \to \mathbf{G},\ g \mapsto g^{-1}F(g)$$

is surjective.

The morphism $\mathscr{L}$ of the theorem is called the *Lang map.*

Before discussing the BN-pairs in finite groups of Lie type we list some important consequences of the Lang–Steinberg theorem.

Definition 1.52. Let G be a group and σ a group automorphism of G. Two elements $g_1, g_2 \in G$ are said to be *σ-conjugate* if there is an element $x \in G$ such that $g_2 = \sigma(x)g_1x^{-1}$. The equivalence classes for this relation are called *σ-conjugacy classes* of G (or just σ-classes of G). The set of σ-conjugacy classes of G is denoted by $H^1(\sigma, G)$.

Proposition 1.53 ([GeMa20, Proposition 1.4.9] and [MaTe11, Theorem 21.11])**.** *Let* $\mathbf{G}$ *be a connected algebraic group with Steinberg map* F*. Let* $X \neq \varnothing$ *be an abstract set on which* $\mathbf{G}$ *acts transitively; let* $F' : X \to X$ *be a map such that* $F'(g.x) = F(g).F'(x)$ *for all* $g \in \mathbf{G}$ *and* $x \in X$*.*

(a) *There exists some* $x_0 \in X$ *such that* $F'(x_0) = x_0$*.*

(b) *If* x_0 *is as in* (a) *and if* $\mathrm{Stab}_{\mathbf{G}}(x_0) \subseteq \mathbf{G}$ *is closed and connected, then* $\{x \in X \mid F'(x) = x\}$ *is a single* $\mathbf{G}^F$*-orbit.*

More generally we have:

(c) *If* $\mathrm{Stab}_{\mathbf{G}}(x) \subseteq \mathbf{G}$ *is closed for some* $x \in X$*, then for any* $x \in X^{F'}$ *there is a natural 1-1 correspondence:*

$$\{\mathbf{G}^F\textit{-orbits on } X^{F'}\} \leftrightarrow \{F\textit{-classes in } \mathrm{Stab}_{\mathbf{G}}(x)/\mathrm{Stab}_{\mathbf{G}}(x)^\circ\}.$$

If we set $\mathbf{X} = \mathbf{G}$ (with the action given by conjugation and $F' = F$) in this Proposition we gain important information on the F-stable conjugacy classes of $\mathbf{G}$. Let us see this explicitly.

Let $\mathbf{G}$ be a connected algebraic group with Steinberg map F and let C be an F-stable conjugacy class of $\mathbf{G}$. Then $\mathbf{G}$ acts transitively on C by conjugation and we can apply the proposition above with $X = C$ and $F' = F$. In this case, the stabilizer of an element is its centralizer. Notice that centralizers in algebraic groups are always closed subgroups (see [Hu75, 8.2 Proposition (b)]). It follows that there is a representative $c \in C$ such that $F(c) = c$ and if $C_{\mathbf{G}}(c)$ is connected, then C^F is a single $\mathbf{G}^F$-class. Otherwise, if the centralizer is not connected, the number of $\mathbf{G}^F$-classes contained in C is $|H^1\left(F, C_{\mathbf{G}}(c)/C_{\mathbf{G}}(c)^\circ\right)|$.

In what follows we say that the class C *splits* into $|H^1\left(F, C_{\mathbf{G}}(c)/C_{\mathbf{G}}(c)^\circ\right)|$ classes of $\mathbf{G}^F$.

Another important consequence of Proposition 1.53 for a connected reductive group $\mathbf{G}$ is that all pairs $(\mathbf{T}, \mathbf{B})$ are $\mathbf{G}^F$-conjugate, where $\mathbf{T}$ and $\mathbf{B}$ are respectively an F-stable torus and an F-stable Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$ (see [GeMa20, Proposition 1.4.12]). An F-stable torus contained in an F-stable Borel is called *maximally split.* Notice that maximally split tori exist by Proposition 1.53 (a).

Usually at this point, in the literature, one restricts the action of the Steinberg map to a maximal torus which in turn defines an action on the character and cocharacter groups of the torus. This is used to classify the finite groups of Lie type. Since neither this classification nor the methods to get it are explicitly used in the rest of this work, we just give a quick description of how to obtain it, in order to fix the notation used later. Details can be found in [MaTe11, Chapter 22] and [Ca85, Chapters 1.18 and 1.19].

Let $\mathbf{G}$ be a connected reductive group with a Steinberg map F and $\mathbf{T}$ a maximally split torus contained in an F-stable Borel subgroup $\mathbf{B}$. The action of F on the characters and cocharacters of $\mathbf{T}$ is given by $F(\chi)(t) := \chi(F(t))$ and $F(\gamma)(c) := F(\gamma(c))$ for $\chi \in X(\mathbf{T})$, $\gamma \in Y(\mathbf{T})$, $t \in \mathbf{T}$ and $c \in K^\times$. At the same time, F induces a permutation ρ of the positive roots of $\mathbf{G}$ relative to $\mathbf{T}$ due to its action on the root subgroups (these are subgroups of an F-stable Borel subgroup). We have $F(\mathbf{U}_\alpha) = \mathbf{U}_{\rho(\alpha)}$ for every positive root α. Recall that roots are characters. Then, these two actions are related by the fact that $F(\rho(\alpha))$ is a positive multiple of $\alpha \in \Phi^+$. Whenever $\mathbf{G}$ is simple, and not of type B_2, F_4 or G_2 with $\mathrm{char}(K) = 2, 2, 3$, respectively[2], the action of F is given more explicitly on root maps by $F(u_\alpha(c)) = u_{\rho(\alpha)}(a_\alpha c^q)$ with $c, a_\alpha \in K^\times$ and on $X(\mathbf{T}) \otimes_\mathbb{Z} \mathbb{R}$ by $F = q\phi$, where q is a power of $\mathrm{char}(K)$, and $\phi \in \mathrm{Aut}(X(\mathbf{T}) \otimes_\mathbb{Z} \mathbb{R})$ induces ρ^{-1} on Φ^+.

It turns out that the Steinberg maps can be classified for simple algebraic groups in terms of q and ρ, see [MaTe11, Theorem 22.5, Proposition 22.7 and Example 22.8].

We fix some definitions/notation used later, related to the discussion above.

Definition 1.54. A Steinberg endomorphism F of a connected reductive group $\mathbf{G}$ that acts as $q\,\mathrm{id}$ on $X(\mathbf{T})$ is said to be $\mathbb{F}_q$*-split*. Otherwise it is called *twisted*. In this case, we also say that the finite group of Lie type $\mathbf{G}^F$ is *twisted*.

Notation 1.55. Let $\mathbf{G}$ be a simple algebraic group with a Steinberg endomorphism F inducing a graph automorphism of order δ on the Dynkin diagram (such that $F^\delta = q^\delta \mathrm{id}$ on $X(\mathbf{T}) \otimes_\mathbb{Z} \mathbb{R}$). If the root system of $\mathbf{G}$ is of type R, we denote $\mathbf{G}^F$ also by ${}^\delta R(q)$ (if $\delta = 1$ we simply write $R(q)$).

More generally, we use a similar notation for connected reductive groups. If the root system has connected components $R_1, \dots, R_n$ on which F acts trivially, we denote the group $\mathbf{G}^F$ by $R_1(q) \cdot \ldots \cdot R_n(q) \cdot |Z^{\circ F}|$. If F doesn't act trivially on the root system, the notation is easily adapted, for example if F interchanges the roots in $A_1 \times A_1$ we denote $\mathbf{G}^F$ by $A_1(q^2)$.

In what follows we encounter (and explicitly construct), for example, $D_4(q)$, $A_3(q)(q-1)$, ${}^2A_3(q)(q+1)$ and $A_1(q)^2(q^2-1)$ (as Levi subgroups of $D_4(q)$). Of course, this notation can be misleading since the isogeny type is not evident. However, here, we are only interested in explicit computations with simple groups of simply connected type. Therefore, it will always be clear what finite group we are talking about.

The various possibilities for finite groups of Lie type constructed from a simple algebraic group are described in [MaTe11, Chapter 22.2] and in [Ca85, Chapter 1.19].

Before continuing the discussion about BN-pairs in finite groups of Lie type, we introduce some signs depending on the rational structure of $\mathbf{G}$. These signs appear frequently in the character theory of finite reductive groups.

Definition 1.56. Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$. Let $q\phi$ be the map of $X(\mathbf{T}) \otimes_\mathbb{Z} \mathbb{R}$ induced by F, with $\phi \in \mathrm{Aut}(X(\mathbf{T}) \otimes_\mathbb{Z} \mathbb{R})$. Then, the *relative F-rank* of $\mathbf{T}$ is defined as the dimension of the q-eigenspace of $q\phi$ on $X(\mathbf{T}) \otimes_\mathbb{Z} \mathbb{R}$. We set

$$\varepsilon_\mathbf{T} := (-1)^{\text{relative } F\text{-rank of } \mathbf{T}}.$$

If $\mathbf{T}_0$ is our reference maximally split torus of $\mathbf{G}$ we call the *relative F-rank of* $\mathbf{G}$ the relative F-rank of $\mathbf{T}_0$, and we set $\varepsilon_\mathbf{G} := \varepsilon_{\mathbf{T}_0}$. Analogously, we call *semisimple F-rank of* $\mathbf{G}$ the relative F-rank of $[\mathbf{G}, \mathbf{G}]$ and we set

$$\eta_\mathbf{G} := (-1)^{\text{semisimple } F\text{-rank of } \mathbf{G}}.$$

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[2] We are not interested in these cases here.

Finally, we describe the BN-pair in finite groups of Lie type and their consequences.

Proposition 1.57. *Let* $\mathbf{G}$ *be a connected reductive group with a Steinberg map* F *and Weyl group* W*. There exist* F*-stable subgroups* $\mathbf{B}$, $\mathbf{N}$ *of* $\mathbf{G}$ *that form a split* BN*-pair. Furthermore,* $\mathbf{B}^F$, $\mathbf{N}^F$ *is a split* BN*-pair of the finite reductive group* $\mathbf{G}^F$ *with Weyl group* W^F*.*

Proof. By Proposition 1.53 there exists a maximally split torus $\mathbf{T}$ contained in an F-stable Borel subgroup $\mathbf{B}$. The subgroup $\mathbf{N} = N_{\mathbf{G}}(\mathbf{T})$ is also F-stable since for all $n \in \mathbf{N}$ we have ${}^{F(n)}\mathbf{T} = F({}^{n}\mathbf{T}) = F(\mathbf{T}) = \mathbf{T}$. It follows that $\mathbf{B}$ and $\mathbf{N}$ form an F-stable split BN-pair of $\mathbf{G}$ by Proposition 1.45 and Corollary 1.46.

By [MaTe11, Theorem 24.1], $\mathbf{G}^F$ inherits the Bruhat decomposition (sharp form) of the split BN-pair of $\mathbf{G}$.

Then, it is proved in [MaTe11, Theorem 24.10] (with the help of the Bruhat decomposition) that $\mathbf{B}^F$ and $\mathbf{N}^F$ form a BN-pair of $\mathbf{G}^F$ with Weyl group W^F.

Next we have, by [MaTe11, Corollary 24.11], that $\mathbf{B}^F = \mathbf{T}^F \ltimes \mathbf{U}^F$ (where $\mathbf{U} = R_u(\mathbf{B})$).

To conclude, we need to show that $\bigcap_{n\in\mathbf{N}^F} n\mathbf{B}^F n^{-1} = \mathbf{T}^F$. First, notice that $\mathbf{T}^F$ is normal in $\mathbf{N}^F$, so we have the inclusion $\mathbf{T}^F \subseteq \bigcap_{n\in\mathbf{N}^F} n\mathbf{B}^F n^{-1}$.

We set $\mathbf{B}^- := \mathbf{B}^{w_0}$ and we denote by Φ the root system of $\mathbf{G}$ relative to $\mathbf{T}$. The Borel subgroup $\mathbf{B}^-$ is F-stable since $\mathbf{B} = \mathbf{T} \cdot \prod_{\alpha\in\Phi^+} \mathbf{U}_\alpha$ and $\mathbf{B}^- = \mathbf{T} \cdot \prod_{\alpha\in\Phi^-} \mathbf{U}_\alpha$ and we have seen that F permutes the positive roots, then analogously F permutes the negatives roots.

We saw (in the discussion after Proposition 1.53) that pairs $(\mathbf{T}', \mathbf{B}')$ of F-stable tori and Borel subgroups are $\mathbf{G}^F$-conjugate. In particular, $(\mathbf{T}, \mathbf{B})$ is $\mathbf{G}^F$-conjugate to $(\mathbf{T}, \mathbf{B}^-)$. Clearly they are actually $\mathbf{N}^F$-conjugate. Finally, we get

$$\bigcap_{n\in\mathbf{N}^F} n\mathbf{B}^F n^{-1} \subseteq \mathbf{B}^F \cap (\mathbf{B}^-)^F = (\mathbf{B} \cap \mathbf{B}^-)^F = \mathbf{T}^F.$$

□

This result shows that finite reductive groups have a natural structure of parabolic/Levi subgroups. The precise way Levi subgroups are described makes them invaluable in the representation theory of finite reductive groups. We will see in Section 3 how we can "induce" irreducible representations of Levi subgroups and relate them to irreducible representations of the whole group.

An important consequence of Proposition 1.57 is that the sharp form of the Bruhat decomposition gives a practical way of counting elements of finite groups of Lie type. Therefore, it is possible to compute the order of finite groups of Lie type (see discussion in [MaTe11, Chapter 24.1]). A list of orders of finite groups of Lie type can be found in [MaTe11, Table 24.1].

A question that arises naturally is how the Chevalley generators/relations change from the connected reductive group $\mathbf{G}$ to the finite reductive group $\mathbf{G}^F$. By identifying F-stable subsets I of the generating set $S \subset W$, it is possible to explicitly write the structure of the root subgroups of $\mathbf{G}^F$ in terms of those of $\mathbf{G}$. The theory behind these structures can be found in [MaTe11, Chapter 23] and is not used explicitly in what follows, so we do not expand on it here.

A list of consequences/properties of the BN-pair of finite groups of Lie type (constructed from simple algebraic groups) can be found in [GLS98, Chapter 2.4]. In particular Theorems 2.4.1, 2.4.5, 2.4.7 and 2.4.8 give Chevalley generators for the finite group and their relations. However, we are mainly interested in the case where F is a Frobenius morphism that has trivial action on the root system. Then $\mathbf{G}^F$ inherits the Chevalley generators/relations of $\mathbf{G}$, by simply restricting the field of definition to $\mathbb{F}_q$.

We end this section with some remarks about F-stable subgroups (Levi subgroups and tori) and the computation of fixed points of connected algebraic groups under the action of a Steinberg endomorphism F.

Proposition 1.58 ([GeMa20, Lemma 1.4.14])**.** *Assume that* $\mathbf{G}$ *is a connected algebraic group and let* $F : \mathbf{G} \to \mathbf{G}$ *be a Steinberg map. Let* $y \in \mathbf{G}$ *and define* $F' : \mathbf{G} \to \mathbf{G}$ *by* $F'(g) = yF(g)y^{-1}$ *for all* $g \in \mathbf{G}$*. Then* F' *is a Steinberg map and we have* $G^F \cong G^{F'}$*. Furthermore, if* F *is a Frobenius map corresponding to an* $\mathbb{F}_q$*-rational structure, then so is* F' *(with the same* q*).*

This result can be useful when doing explicit computations with elements of finite groups of Lie type. It might be easier in some cases to find fixed points under a certain F' as in the proposition than with F. For example, the finite general unitary groups are classically defined as being the fixed points of a general linear group under an endomorphism F which is the composition of a standard Frobenius map, the transposition map and the inverse map. In this case, the subgroup of upper triangular matrices is not F-stable. However, it is F'-stable if F' is the composition of F and the conjugation by a matrix which has ones on the anti-diagonal. Then, Proposition 1.58 allows one to choose how to realize the finite general unitary group. This fact will be used later in Part III since the special unitary group $\mathrm{SU}_4(q)$ can be realized as a subgroup of the finite spin group $\mathrm{Spin}_8^+(q)$ (to be more precise it is isomorphic to the derived subgroup of a Levi subgroup of $\mathrm{Spin}_8^+(q)$ of type ${}^2A_3(q)$).

Next, we consider a crucial remark on the construction of Levi subgroups and tori of finite groups of Lie type.

Let $\mathbf{G}$ be a connected reductive group with Weyl group W, generated by the set S of simple reflections, and let F be a Steinberg endomorphism of $\mathbf{G}$. We choose a reference maximally split torus $\mathbf{T}_0$ contained in a reference F-stable Borel subgroup $\mathbf{B}_0$ of $\mathbf{G}$.

We know that every maximal torus $\mathbf{T}$ of $\mathbf{G}$ is conjugate to $\mathbf{T}_0$ by a certain element $g \in \mathbf{G}$, i.e. $\mathbf{T} = {}^g\mathbf{T}_0$. It is easy to prove that this torus is also F-stable if and only if $g^{-1}F(g) \in N_{\mathbf{G}}(\mathbf{T}_0)$ ([Ca85, Proposition 3.3.1]). In this case, there is an element $w \in W$ such that $g^{-1}F(g) = \dot{w}$ and we say that $\mathbf{T}$ is *obtained from* $\mathbf{T}_0$ *by twisting with* w or that $\mathbf{T}$ is *of type* w. To make this explicit we will denote it by $\mathbf{T}_w$.

A general description exists for Levi subgroups (tori are a special case of Levi subgroups). We first make some basic remarks about parabolic and Levi subgroups.

By definition, for any parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ there exist an element $g \in \mathbf{G}$ and a subset $I \subset \Delta$ such that $\mathbf{P} = {}^g\mathbf{P}_I$ for the standard parabolic subgroup $\mathbf{P}_I$ of $\mathbf{G}$ (containing $\mathbf{B}_0$). Conversely, for any $I \subset \Delta$ and $g \in \mathbf{G}$ the subgroup ${}^g\mathbf{P}_I$ is parabolic (it is closed and contains the Borel subgroup ${}^g\mathbf{B}_0$).

We have the Levi decomposition $\mathbf{P}_I = \mathbf{L}_I \ltimes \mathbf{U}_I$ where $\mathbf{L}_I$ is the standard Levi subgroup of $\mathbf{G}$ associate with I and $\mathbf{U}_I = R_u(\mathbf{P}_I)$. Therefore, we also have the Levi decomposition $\mathbf{P} = {}^g\mathbf{P}_I = {}^g\mathbf{L}_I \ltimes {}^g\mathbf{U}_I$ and $\mathbf{L} := {}^g\mathbf{L}_I$ is a Levi subgroup of $\mathbf{G}$ containing the maximal torus ${}^g\mathbf{T}_0$.

We can now classify the F-stable Levi subgroups of $\mathbf{G}$ in terms of combinatorial data. For every F-stable Levi subgroup $\mathbf{L}$ there are $g \in \mathbf{G}$ and $I \subset S$ such that $\mathbf{L} = {}^g\mathbf{L}_I$. Moreover, $\mathbf{T} = {}^g\mathbf{T}_0$ is a maximal torus of $\mathbf{L}$ obtained by twisting with w ($\dot{w} = g^{-1}F(g)$). In particular $\mathbf{T}$ is F-stable. We have ${}^{F(g)}F(\mathbf{L}_I) = F(\mathbf{L}) = \mathbf{L} = {}^g\mathbf{L}_I$. Since $\mathbf{L}_I$ is generated by the F-stable torus $\mathbf{T}_0$ and the root subgroups $\mathbf{U}_\alpha$ for $\alpha \in I$, it follows that $F(\mathbf{L}_I) = \mathbf{L}_{F(I)}$, and for any subset $J \subseteq S$ we have ${}^{\dot{w}}\mathbf{L}_J = \mathbf{L}_{w(J)}$. Then, gathering these properties we see that a Levi subgroup $\mathbf{L}$ is F-stable if and only if $\mathbf{L}_{wF(I)} = \mathbf{L}_I$ (for I and w defined as above).

Every F-stable Levi subgroup determines a pair (I, w) with $I \subseteq \Delta$ and $w \in W$ such that $wF(I) = I$. We say that $\mathbf{L}$ is a *Levi of type* (I, w).

It follows from this discussion that there is a bijection between $\mathbf{G}^F$-conjugacy classes of F-stable Levi subgroups and equivalence classes of pairs (I, w). Two pairs (I_1, w_1), (I_2, w_2) are equivalent if and only if there is $x \in W$ such that $xI_1 = I_2$ and $xw_1 = w_2F(x)$.

It follows that for an F-stable Levi of type (I, w) we have $\mathbf{L}^F \cong \mathbf{L}_I^{F'}$ where $F' = w \circ F$ and the isomorphism is given by the conjugation with $g \in \mathbf{G}$ such that $g^{-1}F(g) = \dot{w}$. In fact, we know that $\mathbf{L} = {}^g\mathbf{L}_I$ which implies that for any $l' \in \mathbf{L}^F$ there exists an $l \in \mathbf{L}_I$ with $l' = {}^gl$. So,

${}^{g}l = l' = F(l') = F({}^{g}l) = {}^{F(g)}F(l)$ which is equivalent to $l = \dot{w}F(l)\dot{w}^{-1}$. In particular, for any F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$ we have that $\mathbf{T}^F \cong \mathbf{T}_0^{F'}$.

This is crucial when determining elements explicitly in a twisted Levi subgroup (or twisted torus). Normally, it is much easier to describe $\mathbf{L}_I$ than $\mathbf{L}$ and the element $w \in W$ can easily be computed thanks to the GAP part of CHEVIE [MiChv] (with the commands `Twistings` and `TwistingElement`).

Definition 1.59. If an F-stable Levi subgroup is contained in an F-stable parabolic subgroup of $\mathbf{G}$ we say that it is *F-split*, otherwise it is called a *twisted* Levi subgroup.

We end this section by stating an important result on the centre of finite groups of Lie type.

Proposition 1.60 ([Ca85, Proposition 3.6.8])**.** *Let $\mathbf{G}$ be a connected reductive group and F a Steinberg map. Then*

$$Z(\mathbf{G}^F) = Z(\mathbf{G})^F.$$

1.6 Regular embeddings

We describe in Section 3 the character theory of finite groups of Lie type. It turns out that the theory is much richer for connected reductive groups with connected centre.

We are only interested in the case of simply connected groups with disconnected centre. Regular embeddings provide a way to associate any connected reductive group with disconnected centre to another connected reductive group but with connected centre, and to relate the representation theory of the two groups.

The reference for this section is mainly [GeMa20, Chapter 1.7].

Definition 1.61. Let $\mathbf{G}$, $\tilde{\mathbf{G}}$ be connected reductive algebraic groups over $K = \bar{\mathbb{F}}_p$ and $F : \mathbf{G} \to \mathbf{G}$, $\tilde{F} : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ be Steinberg maps. Let $i : \mathbf{G} \to \tilde{\mathbf{G}}$ be a homomorphism of algebraic groups such that $i \circ F = \tilde{F} \circ i$. We say that i is a *regular embedding* if $\tilde{\mathbf{G}}$ has connected centre, i is an isomorphism of $\mathbf{G}$ with a closed subgroup of $\tilde{\mathbf{G}}$ and if the derived subgroups of $\tilde{\mathbf{G}}$ and $i(\mathbf{G})$ are the same.

Remark 1.62. By definition $[\tilde{\mathbf{G}}, \tilde{\mathbf{G}}] \subseteq i(\mathbf{G})$ and therefore $i(\mathbf{G})$ is normal in $\tilde{\mathbf{G}}$ and the quotient $\tilde{\mathbf{G}}/i(\mathbf{G})$ is abelian.

It follows that the finite group $i(\mathbf{G}^F) = i(\mathbf{G})^{\tilde{F}}$ also contains the derived subgroup of $\tilde{\mathbf{G}}^{\tilde{F}}$. Again $i(\mathbf{G}^F)$ is normal in $\tilde{\mathbf{G}}^{\tilde{F}}$ with abelian quotient $\tilde{\mathbf{G}}^{\tilde{F}}/i(\mathbf{G}^F)$. This means that we can apply Clifford theory to relate characters of $\mathbf{G}^F$ and $\tilde{\mathbf{G}}^{\tilde{F}}$.

There is an explicit description of how to build regular embeddings (which also proves their existence), that is given in [GeMa20, Lemma 1.7.3].

Example 1.63. We are interested in two cases.

When $\mathbf{G} = \mathrm{SL}_4(K)$ we can choose $\tilde{\mathbf{G}} = \mathrm{GL}_4(K)$. Although it is easy to embed $\mathrm{SL}_4(K)$ as a subgroup of $\mathrm{GL}_4(K)$, via their matrix representation, an explicit regular embedding is given in [GeMa20, Example 1.7.2]. For F such that $\mathbf{G}^F = \mathrm{SL}_4(q)$ we have $\tilde{\mathbf{G}}^F = \mathrm{GL}_4(q)$ and $\mathrm{GL}_4(q)/\mathrm{SL}_4(q) \cong \mathbb{F}_q^\times$ (this is clear from the explicit form of the regular embedding from [GeMa20, Example 1.7.2 (b)]).

For $\mathbf{G} = \mathrm{Spin}_8(K)$, the group $\tilde{\mathbf{G}}$ is explicitly constructed in [GePf92]. In this case, if F is a Frobenius map such that $\mathbf{G}^F = \mathrm{Spin}_8^+(q)$, then by [GeMa20, Proposition 1.7.5 (a)] we have a surjective map $\tilde{\mathbf{G}}^{\tilde{F}}/i(\mathbf{G}^F) \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ if q is odd.

In both cases, we are not using explicitly the group $\tilde{\mathbf{G}}$ but just the fact that it exists for the computations in Parts II and III.

We will need to know how $\tilde{\mathbf{G}}^{\tilde{F}}$ acts (by conjugation) on $\mathbf{G}^F$.

Remark 1.64. Clearly, the centre $\tilde{\mathbf{Z}}^{\tilde{F}}$ of $\tilde{\mathbf{G}}^{\tilde{F}}$ acts trivially on $\mathbf{G}^F$. By [GeMa20, Remark 1.7.6] we have

$$\tilde{\mathbf{G}}^{\tilde{F}}/\mathbf{G}^F.\tilde{\mathbf{Z}}^{\tilde{F}} \cong H^1(F, Z(\mathbf{G})).$$

Recall that $H^1(F, Z(\mathbf{G}))$ denotes the F-classes of $Z(\mathbf{G})$.

It is possible to define an action of $H^1(F, Z(\mathbf{G}))$ on $\mathbf{G}^F$. We follow [Bo00, Section 1.8]. For every $z \in H^1(F, Z(\mathbf{G}))$ we choose an element $g_z \in \mathbf{G}$ such that $g_z^{-1}F(g_z) \in Z(\mathbf{G})$, representing z. It is easy to see that g_z normalizes $\mathbf{G}^F$. Therefore we can define the action of z on $\mathbf{G}^F$ by the conjugation with g_z. This is well defined up to an inner automorphism of $\mathbf{G}^F$. To prove this statement we compare two different elements $g_a, m_a \in \mathbf{G}$ such that both $m_a^{-1}F(m_a)$ and $g_a^{-1}F(g_a)$ represent $a \in H^1(F, Z(\mathbf{G}))$ in $Z(\mathbf{G})$. Then, by definition there exists $z \in Z(\mathbf{G})$ such that

$$m_a^{-1}F(m_a) = F(z)g_a^{-1}F(g_a)z^{-1} \Leftrightarrow m_a^{-1}F(m_a) = (g_a z)^{-1}F(g_a z).$$

Finally, it follows the existence of an element $g \in \mathbf{G}^F$ for which $m_a = g g_a z$.

One of the most important results, for the present work, on regular embeddings concerns the restriction of characters (here stated with modules), and it is due to Lusztig.

Theorem 1.65 (Lusztig, [GeMa20, Theorem 1.7.15], Multiplicity-Freeness Theorem)**.** *Let $i : \mathbf{G} \to \tilde{\mathbf{G}}$ be a regular embedding and K be any algebraically closed field. Then the restriction of every simple $K\tilde{\mathbf{G}}^{\tilde{F}}$-module to $\mathbf{G}^F$ (via i) is multiplicity-free.*

1.7 Dual group and geometric conjugacy

For this section, $\mathbf{G}$ is a connected reductive group and F a Steinberg map.

An important property of $\mathrm{Irr}(\mathbf{G}^F)$[3] is to be partitioned in a way that the irreducible characters of $\mathbf{G}^F$ are classified in terms of semisimple conjugacy classes of another group, called the dual group of $\mathbf{G}$.

We define here the dual of $\mathbf{G}$ and its relations with $\mathbf{G}$. The reference for this section is [Ca85, Chapter 4].

Definition 1.66. Two connected reductive groups are said to be *dual* to one another if their root data are dual.

Notice that, up to isomorphism, each connected reductive group $\mathbf{G}$ has a dual group (by Theorem 1.25). It is denoted by $\mathbf{G}^*$.

We are interested in defining duality of groups in relation to a certain rational structure.

Definition 1.67. Let $\mathbf{G}$ and $\mathbf{G}^*$ be connected reductive groups with respective Steinberg maps F and F^*. We say that the pairs $(\mathbf{G}, F)$ and $(\mathbf{G}^*, F^*)$ are in *duality* if there are maximally split tori $\mathbf{T}_0 \subseteq \mathbf{G}$ and $\mathbf{T}_0^* \subseteq \mathbf{G}^*$ such that the root data $(X(\mathbf{T}_0), \Phi, Y(\mathbf{T}_0), \Phi^\vee)$ of $\mathbf{G}$ and $(X(\mathbf{T}_0^*), \Phi^*, Y(\mathbf{T}_0^*), \Phi^{*\vee})$ of $\mathbf{G}^*$ are dual and, additionally, if the isomorphism $\delta : X(\mathbf{T}_0) \to Y(\mathbf{T}_0^*)$ given by the isomorphism of root data (see Definition 1.24) is such that $\delta(F(\chi)) = F^*(\delta(\chi))$ for all $\chi \in X(\mathbf{T}_0)$.

The duality relation allows us to study many important properties of $\mathbf{G}$ and $\mathbf{G}^F$ by considering related structures in the dual group. In the present work, we are mainly interested in using duality to study geometric conjugacy which is the object of the next definition.

For some $n > 0$, we denote the norm map of an F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$ by $N_{F^n/F}$, it is defined by

$$N_{F^n/F} : \mathbf{T} \to \mathbf{T},\ t \mapsto tF(t)F^2(t)\dots F^{n-1}(t).$$

---

[3] We recall the definition of $\mathrm{Irr}(G)$, for a finite group G, in the next section.

Definition 1.68. Let $\mathbf{T}$, $\mathbf{T}'$ be F-stable maximal tori of $\mathbf{G}$ and $\theta \in \operatorname{Irr}(\mathbf{T}^F)$, $\theta' \in \operatorname{Irr}(\mathbf{T}'^F)$. We say that the pairs $(\mathbf{T}, \theta)$, $(\mathbf{T}', \theta')$ are *geometrically conjugate* if for some $n > 0$ there is an element $g \in \mathbf{G}^{F^n}$ such that $\mathbf{T}' = g\mathbf{T}g^{-1}$ and g conjugates $\theta \circ N_{F^n/F} \in \operatorname{Irr}(\mathbf{T}^{F^n})$ to $\theta' \circ N_{F^n/F} \in \operatorname{Irr}(\mathbf{T}'^{F^n})$ (via the isomorphism $\mathbf{T}^{F^n} \to \mathbf{T}'^{F^n}$, $t \mapsto gtg^{-1}$).

Geometric conjugacy is an equivalence relation.

In the case of $n = 1$, the study of geometric conjugacy is made easier by considering the dual group. This will give a powerful tool for the classification of irreducible characters of finite groups of Lie type in Section 3.

Proposition 1.69 ([DiMi20, Proposition 11.1.16])**.** *The $\mathbf{G}^F$-conjugacy classes of pairs $(\mathbf{T}, \theta)$ where $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$ and $\theta \in \operatorname{Irr}(\mathbf{T}^F)$ are in one-to-one correspondence with the $\mathbf{G}^{*F^*}$-conjugacy classes of pairs $(\mathbf{T}^*, s)$ where s is a semisimple element of $\mathbf{G}^{*F^*}$ and $\mathbf{T}^*$ is an F^*-stable maximal torus of $\mathbf{G}^*$ containing s.*

2 Basic ordinary representation theory of finite groups

We recall here some basic facts about ordinary representation theory of finite groups (for a basic introduction see [JaLi01]). The main reference for this section is [Is76].

Even though most of the definitions and results that we show here are true in a more general setting we restrict ourselves to representations over $\mathbb{C}$.

For this (and only this) section G denotes any finite group.

We call a (*ordinary*) *representation* of G a homomorphism of groups

$$\rho : G \to \mathrm{GL}_n(\mathbb{C})$$

for some n, which is called the *degree* of the representation.

Although it is practical to work with matrices to describe elements of the group, it is sometimes more convenient to use the analogous description of modules of G. A (complex) vector space V is called a $\mathbb{C}G$-*module* if there is an action of G on V such that for any elements $g, h \in G$, $v, u \in V$ and $\lambda \in \mathbb{C}$ we have $gv \in V$, $h(gv) = hg(v)$, $1v = v$, $g(\lambda v) = \lambda gv$ and $g(v + u) = gv + gu$.

It is easy to see that a $\mathbb{C}G$-module V determines a representation of degree $\dim(V)$ (since G acts as endomorphisms of V). Conversely, a representation of degree n determines a $\mathbb{C}G$-module of dimension n (G acts, through the representation, on the vector space $\mathbb{C}^n$).

Two representations ρ_1 and ρ_2 of degree n are said to be *similar* if there exists an invertible matrix $T \in \mathrm{GL}_n(\mathbb{C})$ such that for all $g \in G$ we have $T\rho_1(g)T^{-1} = \rho_2(g)$. On the other hand, two modules of G are said to be *isomorphic* if there is an isomorphism of vector spaces between them compatible with the action of the group.

Up to similarity, the representations of G are in bijection with the modules of G, up to isomorphism. Thus, we will use these two notions interchangeably without further comments.

A $\mathbb{C}G$-*submodule* W of a $\mathbb{C}G$-module V is a subspace of V which is also a $\mathbb{C}G$-module. A $\mathbb{C}G$-module $V \neq 0$ is called *simple* or *irreducible* if it has no non-trivial proper submodules, else it is called *reducible*. A $\mathbb{C}G$-module which is the direct sum of irreducible $\mathbb{C}G$-submodules is called *semisimple* or *completely reducible*. The representation corresponding to an irreducible $\mathbb{C}G$-module is also called *irreducible*.

It is crucial that every $\mathbb{C}G$-module is semisimple by Maschke's Theorem (see [JaLi01, Chapter 8]). This means that, to understand all the modules of a finite group (which are infinitely many), we only need to study the irreducible ones. It turns out that for any finite group, up to isomorphism, there are only a finite number of irreducible modules (see [JaLi01, Theorem 10.5 and Corollary 10.7]).

It follows that we want to study irreducible representations and in more generality we want to study representations up to similarity. For a representation of degree n, we associate an $n \times n$ matrix to each element of the group. Clearly, this is a redundancy of information. This is why we study representations thanks to characters, which are the traces of representations. We denote by $\mathrm{tr}(A)$ the trace of the square matrix A.

Definition 2.1. Let G be a finite group and ρ a representation of G of degree n. Then, the *character* χ of G *afforded* by ρ is the function

$$\chi : G \to \mathbb{C},\ g \mapsto \chi(g) := \mathrm{tr}(\rho(g)).$$

We call $n = \chi(1)$ the *degree* of χ. We say that χ is *irreducible* if ρ is irreducible. A character of degree 1 is called a *linear character*. The character $1_G : g \in G \mapsto 1$ afforded by the representation $g \mapsto 1$ is called the *trivial character*.

It is clear from the definition that linear characters are irreducible.

Recall that the trace has the property that if A and B are two square matrices of the same size then $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. It follows immediately that characters are class functions (meaning that they are constant on conjugacy classes) and, more importantly, that similar representations afford the same character.

Notation 2.2. For a finite group G the set of irreducible characters is denoted by $\mathrm{Irr}(G)$ and the set of conjugacy classes is denoted by $\mathrm{Cl}(G)$. The vector space of complex valued class functions over G is denoted by $\mathrm{CF}(G)$.

We list now some important properties of characters of finite groups.

Proposition 2.3. *Let G be a finite group.*

(a) *The group G is abelian if and only if every irreducible character is linear.* [Is76, (2.6) Corollary]

(b) $|\mathrm{Irr}(G)| = |\mathrm{Cl}(G)|$ *and* $\sum_{\chi\in\mathrm{Irr}(G)} \chi(1)^2 = |G|$. [Is76, (2.7) Corollary]

(c) *The set* $\mathrm{Irr}(G)$ *is a basis of* $\mathrm{CF}(G)$ *and a class function $\varphi \neq 0$ is a character if and only if it is a nonnegative integer linear combination of* $\mathrm{Irr}(G)$. [Is76, (2.8) Theorem]

(d) *Two representations are similar if and only if the characters they afford are equal.* [Is76, (2.9) Corollary]

Remark 2.4. By Proposition 2.3 (a) and (b) it is clear that any cyclic group $C_n = \langle g \rangle$ (where g is an element of order n) has n irreducible (linear) characters. Because linear characters are representations of degree 1 and representations are homomorphisms of finite groups, the n linear characters $\chi_k^{(n)}$ of C_n are defined by

$$\chi_k^{(n)}(g^j) = \mathrm{e}^{\frac{2\pi i}{n}jk}$$

for all $j = 1, ..., n$.

It is known that finite abelian groups are isomorphic to direct products of cyclic groups. Therefore, it is possible to give the complete character table for any finite abelian group.

Lemma 2.5. *Let G be the finite abelian group*

$$C_{n_1} \times \cdots \times C_{n_k}$$

with $n_1, ..., n_k$ integers.

Then

$$\mathrm{Irr}(G) = \left\{ \chi_{i_1}^{(n_1)} \cdots \chi_{i_k}^{(n_k)} \mid i_j = 1, ..., n_j, \text{ for } j = 1, ..., k \right\}.$$

Definition 2.6. For every character χ of G we write $\chi = \sum_{\varphi\in\mathrm{Irr}(G)} n_\varphi \varphi$ with n_φ non-negative integers (according to Proposition 2.3 (c)). Then we call $\varphi \in \mathrm{Irr}(G)$ an *irreducible constituent* of χ when $n_\varphi \neq 0$.

In general, we call a (non-zero) class function in $\mathbb{Z}\mathrm{Irr}(G)$ a *virtual character* of G.

Definition 2.7. The square table obtained with the values of the irreducible characters at representatives of the conjugacy classes with rows labelled by $\mathrm{Irr}(G)$ and columns labelled by $\mathrm{Cl}(G)$ is called the *character table* of G.

The rows and columns of the character table of G respect orthogonality relations.

Proposition 2.8 ([Is76, (2.14) Corollary and (2.18) Theorem], Orthogonality relations)**.** *The* first orthogonality relation *(between rows of the character table) is given for* $\chi, \varphi \in \mathrm{Irr}(G)$ *by*

$$\frac{1}{|G|} \sum_{g \in G} \chi(g)\varphi(g^{-1}) = \delta_{\chi\varphi}.$$

Analogously, we have the second orthogonality relation *(between columns of the character table). For two classes* $C_1, C_2 \in \mathrm{Cl}(G)$ *choose representatives* $g_1 \in C_1$ *and* $g_2 \in C_2$*, then*

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(g_1)\chi(g_2^{-1}) = \delta_{C_1C_2}|C_G(g_1)|.$$

It follows from the first orthogonality relation that $\mathrm{CF}(G)$ is a Hilbert space endowed with the inner product

$$\langle \chi, \varphi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\varphi(g^{-1})$$

with orthonormal basis $\mathrm{Irr}(G)$.

Remark 2.9. It is clear that an irreducible character $\chi \in \mathrm{Irr}(G)$ is a constituent of a class function φ if and only if $\langle \chi, \varphi \rangle \neq 0$ (> 0 if φ is a character).

Definition 2.10. Let χ be a character of G. Then $\ker\chi := \{g \in G \mid \chi(g) = \chi(1)\}$.

Definition 2.11. Let $N \lhd G$. Then it is possible to *inflate* characters of G/N to G. Let χ be a character of G/N. We define its *inflation* $\chi_G \in \mathrm{CF}(G)$ by $\chi_G(g) := \chi(gN)$.

Analogously, if χ is a character of G such that $N \subseteq \ker\chi$, then we can define $\tilde{\chi} \in \mathrm{CF}(G/N)$ by $\tilde{\chi}(gN) := \chi(g)$.

By working with representations it is easy to see that these constructions always results in characters.

Proposition 2.12 ([Is76, (2.22) Lemma])**.** *Let* $N \lhd G$*.*

(a) *If* χ *is a character of* G *and* $N \subseteq \ker\chi$*, then* χ *is constant on cosets of* N *in* G *and the function* $\tilde{\chi}$ *on* G/N *defined by* $\tilde{\chi}(gN) = \chi(g)$ *is a character of* G/N*.*

(b) *If* $\tilde{\chi}$ *is a character of* G/N*, then the function* χ *defined by* $\chi(g) = \tilde{\chi}(gN)$ *is a character of* G

In both (a) *and* (b)*,* $\chi \in \mathrm{Irr}(G)$ *if and only if* $\tilde{\chi} \in \mathrm{Irr}(G/N)$*.*

Remark 2.13. Notice that linear characters are homomorphisms of groups into the abelian group $\mathbb{C}^\times$. This means that, for any linear character $\lambda \in \mathrm{Irr}(G)$, we have $[G, G] \subseteq \ker\lambda$. Therefore, the linear characters are precisely the inflations of the (linear) irreducible characters of $G/[G, G]$.

Proposition 2.14 ([Is76, (3.6) Corollary])**.** *Let* χ *be a character of* G*. Then* $\chi(g)$ *is an algebraic integer for all* $g \in G$*.*

Apart from some small groups, the information given until now is not enough to compute the whole character table of finite groups. It is possible to gain further information on the character table of a finite group G by using restriction of characters to a subgroup $H \leq G$ or induction of characters from a subgroup $H \leq G$.

Definition 2.15. Let H be a subgroup of G and $\chi \in \mathrm{CF}(G)$. Then its *restriction* to H is the class function $\chi|_H$ defined by

$$\chi|_H(h) := \chi(h)$$

for all $h \in H$.

For $\varphi \in \mathrm{CF}(H)$ we call its *induction* to G the class function $\mathrm{Ind}_H^G(\varphi)$ defined by

$$\mathrm{Ind}_H^G(\varphi)(g) := \frac{1}{|H|} \sum_{\substack{x \in G \\ xgx^{-1} \in H}} \varphi(xgx^{-1})$$

for all $g \in G$.

An important result that relates induction and restriction is the Frobenius reciprocity.

Proposition 2.16 ([Is76, (5.2) Lemma], Frobenius reciprocity)**.** *Let $H \leq G$, $\chi \in \mathrm{CF}(G)$ and $\varphi \in \mathrm{CF}(H)$. Then*

$$\langle \chi, \mathrm{Ind}_H^G \varphi \rangle_G = \langle \chi|_H, \varphi \rangle_H.$$

In other words, the induction and restriction functors are (hermitian) adjoint to each other.

Remark 2.17. It is easy to see that the restriction of a character is a character. Due to the Frobenius reciprocity the induction of characters are also characters (see [Is76, (5.3) Corollary]).

For our computations in Parts II and III it is more useful to rewrite the induction formula in the following way.

Lemma 2.18. *Let $H \leq G$ and $\varphi \in \mathrm{CF}(H)$ then*

$$\mathrm{Ind}_H^G(\varphi)(g) = \sum_x \frac{|C_G(g)|}{|C_H(x)|} \varphi(x)$$

for all $g \in G$, where the sum is over representatives x of the conjugacy classes of H that are G-conjugate to g.

Proof. Let us assume that the conjugacy class g^G intersects m classes of H with representatives x_i, for $i = 1, \dots, m$, i.e. $g^G \cap H = \bigsqcup_{i=1}^m x_i^H$. Then we can rewrite the formula

$$\begin{aligned}
\mathrm{Ind}_H^G(\varphi)(g) &= \frac{1}{|H|} \sum_{\substack{x \in G \\ xgx^{-1} \in H}} \varphi(xgx^{-1}) = \frac{1}{|H|} \sum_{i=1}^m \sum_{\substack{x \in G \\ xgx^{-1} \in x_i^H}} \varphi(xgx^{-1}) \\
&= \frac{1}{|H|} \sum_{i=1}^m \varphi(x_i) \left| \{ x \in G \mid xgx^{-1} \in x_i^H \} \right| = \frac{1}{|H|} \sum_{i=1}^m \varphi(x_i) |x_i^H| |C_G(g)|
\end{aligned}$$

and the claim follows. ☐

A priori not much can be said for a general subgroup H of G on the restriction/induction of characters to/from H. However, if we consider normal subgroups, the theory provides us with a variety of useful results. The study of restriction/induction to/from normal subgroups is known as Clifford theory.

First we need to introduce the conjugates of a character.

Definition 2.19. Let $N \lhd G$ be a normal subgroup of G. For $\theta \in \mathrm{CF}(N)$ and $g \in G$ we define the class function θ^g by $\theta^g(h) := \theta(ghg^{-1})$ for all $h \in N$ (analogously ${}^g\theta(h) := \theta(g^{-1}hg)$). We say that θ^g is *conjugate* to θ in G.

Remark 2.20. With the same notation of the definition, if moreover θ is a character so is θ^g.

It is straightforward that for $\theta_1, \theta_2 \in \mathrm{CF}(N)$ we have $\langle \theta_1^g, \theta_2^g \rangle = \langle \theta_1, \theta_2 \rangle$. It follows that conjugation with elements of G permutes $\mathrm{Irr}(N)$. Moreover, since N acts trivially on $\mathrm{Irr}(N)$, G/N also permutes $\mathrm{Irr}(N)$.

We state now the main theorem of Clifford theory.

Theorem 2.21 ([Is76, (6.2) Theorem], Clifford)**.** *Let $N \lhd G$ and let $\chi \in \mathrm{Irr}(G)$. Let θ be an irreducible constituent of $\chi|_N$ and suppose $\theta = \theta_1, \theta_2, \ldots, \theta_n$ are the distinct conjugates of θ in G. Then*

$$\chi|_N = e \sum_{i=1}^{n} \theta_i$$

where $e = \langle \chi|_N, \theta \rangle$.

This theorem has many important consequences. However, we will not use explicitly more than the theorem itself. We end our general discussion on ordinary characters with a last remark on another method of obtaining characters from subgroups, taken from [DiMi20, Chapter 5.1].

Remark 2.22. Let $H \leq G$ and let M be a $\mathbb{C}G$-module-$\mathbb{C}H$ (i.e. M is a bimodule with a left $\mathbb{C}G$-action and a right $\mathbb{C}H$-action). We define a functor R_H^G from the category of left $\mathbb{C}H$-modules to that of left $\mathbb{C}G$-modules by

$$R_H^G : E \mapsto M \otimes_{\mathbb{C}H} E.$$

Analogously, taking the tensor product with the dual module $M^* = \mathrm{Hom}(M, \mathbb{C})$ defines the adjoint functor ${}^*R_H^G$.

Composition of these functors is transitive [DiMi20, Proposition 5.1.4].

The analogous description for characters can be found by taking traces, see [DiMi20, Proposition 5.1.5],

$$\mathrm{Trace}(g \mid R_H^G E) = \frac{1}{|H|} \sum_{h \in H} \mathrm{Trace}\left((g, h^{-1}) \mid M\right) \mathrm{Trace}(h \mid E)$$

for every $g \in G$, where E is a $\mathbb{C}H$-module. We use the same notation for characters as for the modules. For a $\mathbb{C}H$-module (respectively $\mathbb{C}G$-module) E affording the character χ we write $R_H^G(\chi)$ (respectively ${}^*R_H^G(\chi)$) for the character afforded by $R_H^G(E)$ (respectively ${}^*R_H^G(E)$). Notice that these functors always take character to character.

In the next section we apply these functors to the special case where G is a finite group of Lie type and H is a Levi subgroup. This gives a much more powerful method of constructing characters of the group G than the induction functor Ind_H^G (it is easier to identify the irreducible constituents, also because usually their number is smaller).

3 Character theory of finite groups of Lie type

We are interested in this section in the representation theory of finite groups of Lie type. Because of their rich structure (in particular the existence of a BN-pair) and the precise description of their elements/subgroups (see Section 1) much can be added to the ordinary representation theory of finite groups.

We discuss in this section how to gain more information on the character table of finite groups of Lie type first by introducing Harish–Chandra induction/restriction and then Deligne–Lusztig induction/restriction. This makes it possible to classify irreducible characters into families, called Lusztig series.

The theory behind the results listed in this section is known as Deligne–Lusztig theory. It is a far reaching theory based on representations built from some so-called ℓ-adic cohomology groups with compact support, where ℓ is a prime different from the characteristic of the group. We will not discuss this machinery here but only its consequences.

This section's references are [DiMi20, Chapters 5, 9, 10 and 11] and [GeMa20, Chapters 2 and 3].

In Remark 2.22 we described a way of building characters of a group from characters of a subgroup. In the case where $G = \mathbf{G}^F$ is a finite group of Lie type and the subgroup $H = \mathbf{L}^F$ is a split Levi subgroup, this construction gives rise to what is known as Harish–Chandra theory.

For this section, $\mathbf{G}$ is a connected reductive group defined over $\mathbb{F}_q$ and F a Steinberg map. Furthermore, let $(\mathbf{G}^*, F^*)$ be the dual of the pair $(\mathbf{G}, F)$.

Definition 3.1. Let $\mathbf{P}$ be an F-stable parabolic subgroup of $\mathbf{G}$ with Levi decomposition $\mathbf{P} = \mathbf{LU}$ where $\mathbf{L}$ is an F-stable Levi subgroup of $\mathbf{P}$ and $\mathbf{U}$ is the unipotent radical of $\mathbf{P}$.

Then $\mathbb{C}[\mathbf{G}^F/\mathbf{U}^F]$ is a $\mathbf{G}^F$-module-$\mathbf{L}^F$ ($\mathbf{G}^F$ acts by left translation and $\mathbf{L}^F$ by right translations). The functor $R_{\mathbf{L}}^{\mathbf{G}}$ obtained according to Remark 2.22 is called *Harish–Chandra induction.*

Analogously, $\mathbb{C}[\mathbf{U}^F \setminus \mathbf{G}^F]$ is an $\mathbf{L}^F$-module-$\mathbf{G}^F$ ($\mathbf{G}^F$ acts by right translation and $\mathbf{L}^F$ by left translations). This gives rise to the adjoint functor ${}^*R_{\mathbf{L}}^{\mathbf{G}}$ called *Harish–Chandra restriction.*

Notice that by construction (see Remark 2.22) Harish–Chandra induction/restriction map characters to characters.

This construction was generalised by Deligne and Lusztig to include the case of twisted Levi subgroups.

Definition 3.2. Let $\mathbf{L}$ be an F-stable Levi subgroup of a parabolic subgroup $\mathbf{P} = \mathbf{LU}$ (not necessarily F-stable) of $\mathbf{G}$. The *Lusztig induction* $R_{\mathbf{L}}^{\mathbf{G}}$ is the generalized induction functor associated with the $\mathbf{G}^F$-module-$\mathbf{L}^F$ afforded by $H_c^*(\mathscr{L}^{-1}(\mathbf{U})) := \sum_i (-1)^i H_c^i(\mathscr{L}^{-1}(\mathbf{U}))$ (for some details on ℓ-adic cohomology groups with compact support, see [DiMi20, Chapter 8]), where $\mathscr{L}$ is the Lang map. The module structure is induced by the action on $\mathscr{L}^{-1}(\mathbf{U})$, namely $x \mapsto gxl$ for $g \in \mathbf{G}^F$ and $l \in \mathbf{L}^F$.

The adjoint functor ${}^*R_{\mathbf{L}}^{\mathbf{G}}$ is called *Lusztig restriction.*

In character notation we have, for characters χ of $\mathbf{L}^F$ and ψ of $\mathbf{G}^F$,

$$\left(R_{\mathbf{L}}^{\mathbf{G}}\chi\right)(g) = \frac{1}{|\mathbf{L}^F|} \sum_{l \in \mathbf{L}^F} \operatorname{Trace}\left((g, l^{-1}) \mid H_c^*(\mathscr{L}^{-1}(\mathbf{U}))\right) \chi(l)$$

and

$$\left({}^*R_{\mathbf{L}}^{\mathbf{G}}\psi\right)(l) = \frac{1}{|\mathbf{G}^F|} \sum_{g \in \mathbf{G}^F} \operatorname{Trace}\left((g^{-1}, l) \mid H_c^*(\mathscr{L}^{-1}(\mathbf{U}))\right) \psi(g)$$

for $g \in \mathbf{G}^F$ and $l \in \mathbf{L}^F$ (see [DiMi20, Proposition 9.1.6 and Lemma 9.1.5]).

Two preliminary remarks about the notation are in order here.

Remark 3.3. We use the same notation for Harish–Chandra and Lusztig functors. This makes sense since the Harish–Chandra induction/restriction is actually a particular case of Lusztig induction/restriction, see discussion after [DiMi20, Proposition 9.1.4].

Remark 3.4. In the notation $R_{\mathbf{L}}^{\mathbf{G}}$, ${}^*R_{\mathbf{L}}^{\mathbf{G}}$ it is not explicit which parabolic subgroup $\mathbf{P}$ we use. In fact these functors are independent of the choice of $\mathbf{P}$. This is a consequence of the Mackey formula (see [GeMa20, Theorem 3.3.8]) which we state in the next theorem (for the case that interests us).

Theorem 3.5 ([GeMa20, Theorem 3.3.7 (4)], Mackey formula)**.** *Let* $\mathbf{G}$ *be of classical type (A, B, C or D) and* F *a Frobenius map. Let* $\mathbf{P}$*,* $\mathbf{Q}$ *be parabolic subgroups of* $\mathbf{G}$ *with* F*-stable Levi complements* $\mathbf{L}$*,* $\mathbf{M}$ *respectively. For any character* $\psi \in \operatorname{Irr}(\mathbf{M}^F)$ *we have*

$${}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} \circ R_{\mathbf{M}\subset\mathbf{Q}}^{\mathbf{G}}(\psi) = \sum_{w \in \mathbf{L}^F \backslash \mathcal{S}(\mathbf{L},\mathbf{M})^F/\mathbf{M}^F} R_{\mathbf{L}\cap{}^w\mathbf{M}\subset\mathbf{L}\cap{}^w\mathbf{Q}}^{\mathbf{L}} \circ \operatorname{ad}(w) \circ {}^*R_{\mathbf{L}^w\cap\mathbf{M}\subset\mathbf{P}^w\cap\mathbf{M}}^{\mathbf{M}}(\psi),$$

where w *runs over a system of* $\mathbf{L}^F$*-*$\mathbf{M}^F$ *double coset representatives in*

$$\mathcal{S}(\mathbf{L},\mathbf{M})^F := \{g \in \mathbf{G}^F \mid \mathbf{L} \cap {}^g\mathbf{M} \text{ contains a maximal torus of } \mathbf{G}\}.$$

Notice that Lusztig induction takes characters to virtual characters ($H_c^*(\mathscr{L}^{-1}(\mathbf{U}))$ is a virtual vector space). We consider Lusztig induction because there is a certain family of virtual characters (obtained by Lusztig induction) that contains, in some sense, all the characters of $\operatorname{Irr}(\mathbf{G}^F)$. We introduce these virtual characters now.

Definition 3.6. If $\mathbf{L} = \mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$ and $\theta \in \operatorname{Irr}(\mathbf{T}^F)$, we call $R_{\mathbf{T}}^{\mathbf{G}}\theta$ a *Deligne–Lusztig character.*

We list now some important properties of Deligne–Lusztig characters that we will use without further mention.

Let $\mathbf{G}$ be a connected reductive group with Steinberg map F. Let $\mathbf{T}$, $\mathbf{T}'$ be F-stable maximal tori of $\mathbf{G}$ and $\theta \in \operatorname{Irr}(\mathbf{T}^F)$, $\theta' \in \operatorname{Irr}(\mathbf{T}'^F)$. Then:

- We have the scalar product formula ([GeMa20, Theorem 2.2.8]):

$$\langle R_{\mathbf{T}}^{\mathbf{G}}\theta, R_{\mathbf{T}'}^{\mathbf{G}}\theta'\rangle = \frac{|\{g \in \mathbf{G}^F \mid {}^g\mathbf{T} = \mathbf{T}',\ {}^g\theta = \theta'\}|}{|\mathbf{T}^F|}.$$

- Two Deligne–Lusztig characters $R_{\mathbf{T}}^{\mathbf{G}}\theta$ and $R_{\mathbf{T}'}^{\mathbf{G}}\theta'$ are either equal or orthogonal to each other. We have $R_{\mathbf{T}}^{\mathbf{G}}\theta = R_{\mathbf{T}'}^{\mathbf{G}}\theta'$ if and only if there exists some $g \in \mathbf{G}^F$ such that ${}^g\mathbf{T} = \mathbf{T}'$ and ${}^g\theta = \theta'$ ([GeMa20, Corollary 2.2.10]).

- For any irreducible character $\rho \in \operatorname{Irr}(\mathbf{G}^F)$, there is a pair $(\mathbf{T}, \theta)$ such that $\langle R_{\mathbf{T}}^{\mathbf{G}}\theta, \rho\rangle \neq 0$ ([GeMa20, Corollary 2.2.19]).

Definition 3.7. We call *uniform functions* the class functions on $\mathbf{G}^F$ that are linear combinations of Deligne–Lusztig characters.

Clearly, the Deligne–Lusztig characters $R_{\mathbf{T}}^{\mathbf{G}}\theta$ form an orthogonal basis of the space of uniform functions, up to $\mathbf{G}^F$-conjugacy.

Notation 3.8. By Proposition 1.69, we may write $R_{\mathbf{T}^*}^{\mathbf{G}}(s)$ instead of $R_{\mathbf{T}}^{\mathbf{G}}\theta$ where the $\mathbf{G}^F$-conjugacy class of $(\mathbf{T}, \theta)$ corresponds to the $\mathbf{G}^{*F^*}$-conjugacy class of $(\mathbf{T}^*, s)$.

This notation gives a convenient way to write a partition of the irreducible characters of a finite group of Lie type.

Definition 3.9. Let $s \in \mathbf{G}^{*F^*}$ be semisimple. A *rational series of characters* of $\mathbf{G}^F$ (or also *Lusztig series of characters*) is a set $\mathscr{E}(\mathbf{G}^F, s)$ of all $\rho \in \operatorname{Irr}(\mathbf{G}^F)$ such that $\langle \rho, R_{\mathbf{T}^*}^{\mathbf{G}}(s)\rangle \neq 0$ for some F^*-stable maximal torus $\mathbf{T}^*$ of $\mathbf{G}^*$ containing s.

One of the rational series of characters is particularly important for the representation theory of finite groups of Lie type.

Definition 3.10. The characters of $\mathscr{E}(\mathbf{G}^F, 1)$ are called the *unipotent characters* of $\mathbf{G}^F$. The set of unipotent characters is denoted by

$$\operatorname{Uch}(\mathbf{G}^F) := \mathscr{E}(\mathbf{G}^F, 1).$$

The following is a crucial result on unipotent characters.

Proposition 3.11 ([DiMi20, Proposition 11.3.8])**.** *Let $(\mathbf{G}, F)$ and $(\mathbf{G}_1, F_1)$ be two connected reductive groups with Steinberg maps and let $f : \mathbf{G} \to \mathbf{G}_1$ be a morphism of algebraic groups with a central kernel such that $f \circ F = F_1 \circ f$ and such that $f(\mathbf{G})$ contains $[\mathbf{G}_1, \mathbf{G}_1]$; then the unipotent characters of $\mathbf{G}^F$ are the $\chi \circ f$, where χ runs over the unipotent characters of $\mathbf{G}_1^{F_1}$.*

The reason why we introduced the Lusztig series of characters is clear from the next theorem, also due to Lusztig.

Theorem 3.12 (Lusztig, [GeMa20, Theorem 2.6.2])**.** *If $s_1, s_2 \in \mathbf{G}^{*F^*}$ are semisimple and conjugate in $\mathbf{G}^{*F^*}$, then $\mathscr{E}(\mathbf{G}^F, s_1) = \mathscr{E}(\mathbf{G}^F, s_2)$. We have a partition*

$$\operatorname{Irr}(\mathbf{G}^F) = \bigsqcup_{s} \mathscr{E}(\mathbf{G}^F, s)$$

where s runs over representatives of the conjugacy classes of semisimple elements in $\mathbf{G}^{*F^*}$

This partition is a great theoretical tool. However for practical computations (like in this work) it is not enough to determine $\operatorname{Irr}(\mathbf{G}^F)$ since the Deligne–Lusztig characters are hardly ever irreducible or in number equal to $|\operatorname{Irr}(\mathbf{G}^F)|$. We need more information regarding and complementary to the Deligne–Lusztig characters.

The representation theory of finite groups of Lie type is richer if $\mathbf{G}$ has connected centre (for example every character of $\operatorname{Irr}(\mathrm{GL}_n(q))$ is a uniform function, see [DiMi20, Theorem 11.7.3]). For this reason, one usually gathers all the available informations for a regular embedding. Hopefully, this is useful to gain information on the representation theory of $\mathbf{G}^F$.

We fix a regular embedding $i : \mathbf{G} \to \tilde{\mathbf{G}}$ (and identify $\mathbf{G}$ with $i(\mathbf{G}) \subset \tilde{\mathbf{G}}$) and we denote by $(\tilde{\mathbf{G}}^*, \tilde{F}^*)$ the dual of $(\tilde{\mathbf{G}}, \tilde{F})$. Let $i^* : \tilde{\mathbf{G}}^* \to \mathbf{G}^*$ be the corresponding central isotypy (see [GeMa20, Section 1.7.11]) such that $i^* \circ \tilde{F}^* = F^* \circ i^*$.

Proposition 3.13 ([GeMa20, Proposition 2.5.22])**.** *Let $\mathbf{T}^* \subseteq \mathbf{G}^*$ be an F^*-stable maximal torus and $s \in \mathbf{T}^{*F^*}$. Let $\tilde{\mathbf{T}}^* := {i^*}^{-1}(\mathbf{T}^*) \subseteq \tilde{\mathbf{G}}^*$. Then there exists a semisimple element $\tilde{s} \in (\tilde{\mathbf{T}}^*)^{\tilde{F}^*}$ such that $i^*(\tilde{s}) = s$. For any such $\tilde{s}$ we have $R_{\mathbf{T}^*}^{\mathbf{G}}(s) = R_{\tilde{\mathbf{T}}^*}^{\tilde{\mathbf{G}}}(\tilde{s})|_{\mathbf{G}^F}$.*

Proposition 3.14. *Let $s \in \mathbf{G}^{*F^*}$ be semisimple and $\tilde{s} \in \tilde{\mathbf{G}}^{*\tilde{F}^*}$ be any semisimple element such that $i^*(\tilde{s}) = s$. Then*

$$\mathscr{E}(\mathbf{G}^F, s) = \{\rho \in \operatorname{Irr}(\mathbf{G}^F) \mid \langle \tilde{\rho}|_{\mathbf{G}^F}, \rho\rangle \neq 0 \textit{ for some } \tilde{\rho} \in \mathscr{E}(\tilde{\mathbf{G}}^{\tilde{F}}, \tilde{s})\}.$$

Remark 3.15. By Proposition 3.13 we can see the Deligne–Lusztig characters of $\mathbf{G}^F$ as the restriction of those of $\tilde{\mathbf{G}}^{\tilde{F}}$. As a consequence, we can apply Clifford theory to these groups (see Theorem 2.21 and Theorem 1.65).

Let $\tilde{s} \in \tilde{\mathbf{G}}^{*\tilde{F}^*}$ be semisimple and $\tilde{\rho} \in \mathscr{E}(\tilde{\mathbf{G}}^{\tilde{F}}, \tilde{s})$. By the multiplicity freeness theorem (Theorem 1.65), we can write

$$\tilde{\rho}|_{\mathbf{G}^F} = \rho_1 + \cdots + \rho_r$$

where $\rho_1, \ldots, \rho_r \in \mathscr{E}(\mathbf{G}^F, s)$ (where $i^*(\tilde{s}) = s$) are the distinct $\tilde{\mathbf{G}}^{\tilde{F}}$-conjugates of an irreducible constituent of $\tilde{\rho}|_{\mathbf{G}^F}$, by Clifford's theorem (Theorem 2.21).

This will be one of the key ingredients in the explicit decomposition of the Deligne–Lusztig characters performed in Parts II and III.

We define now an important functor for the character theory of finite groups of Lie type.

Definition 3.16. The *Alvis–Curtis–Kawanaka–Lusztig duality operator* on the space of class functions $\mathrm{CF}(\mathbf{G}^F)$ is defined as

$$D_{\mathbf{G}} := \sum_{I \subseteq S} (-1)^{|I|} R^{\mathbf{G}}_{\mathbf{L}_I} \circ {}^*R^{\mathbf{G}}_{\mathbf{L}_I}$$

where S is the set of simple reflections of W.

We list some important properties of this duality functor that we need in later sections.

Proposition 3.17 ([GeMa20, Theorem 3.4.4])**.** *Let* $\mathbf{L}$ *be an* F*-stable Levi subgroup of* $\mathbf{G}$ *such that the Mackey formula holds for* $R^{\mathbf{G}}_{\mathbf{L}}$*. Then*

$$\varepsilon_{\mathbf{G}} D_{\mathbf{G}} \circ R^{\mathbf{G}}_{\mathbf{L}} = \varepsilon_{\mathbf{L}} R^{\mathbf{G}}_{\mathbf{L}} \circ D_{\mathbf{L}}$$

and

$$\varepsilon_{\mathbf{L}} D_{\mathbf{L}} \circ {}^*R^{\mathbf{G}}_{\mathbf{L}} = \varepsilon_{\mathbf{G}} {}^*R^{\mathbf{G}}_{\mathbf{L}} \circ D_{\mathbf{G}}.$$

Recall that the signs $\varepsilon_{\mathbf{G}}$ were defined in Definition 1.56.

Proposition 3.18 ([GeMa20, Proposition 3.4.2 and Corollary 3.4.5])**.** *The functor* $D_{\mathbf{G}}$ *is self-adjoint and* $D_{\mathbf{G}} \circ D_{\mathbf{G}}$ *is the identity on* $\mathrm{CF}(\mathbf{G}^F)$.

Even more importantly, the duality functor sends irreducible characters to irreducible characters (up to sign). Explicitly, we have that it permutes (up to sign) the irreducible characters of all the Lusztig series:

Corollary 3.19 ([GeMa20, Corollary 3.4.6])**.** *If* $\rho \in \mathscr{E}(\mathbf{G}^F, s)$ *then* $\pm D_{\mathbf{G}}(\rho) \in \mathscr{E}(\mathbf{G}^F, s)$.

With the duality functor we can introduce an irreducible character of finite groups of Lie type that we will encounter in Part III.

Definition 3.20. The irreducible character $\mathrm{St}_{\mathbf{G}} := D_{\mathbf{G}}(1_{\mathbf{G}^F})$ is the *Steinberg character* of $\mathbf{G}^F$.

The values of the Steinberg characters are known, and given in the following proposition.

Proposition 3.21 ([GeMa20, Proposition 3.4.10])**.** *Let* $\mathbf{G}^F$ *be a finite group of Lie type and* $g \in \mathbf{G}^F$*. Then*

$$\mathrm{St}_{\mathbf{G}}(g) = \begin{cases} \varepsilon_{\mathbf{G}} \varepsilon_{C_{\mathbf{G}}(g)^\circ} \, |C_{\mathbf{G}}(g)^{\circ F}|_p & \textit{if } g \textit{ is semisimple,} \\ 0 & \textit{else.} \end{cases}$$

We introduce now another partition of $\mathrm{Irr}(\mathbf{G}^F)$ in terms of Harish–Chandra induction.

The set $\{(\mathbf{L}, \lambda)\}$ of pairs of F-stable split Levi subgroups $\mathbf{L}$ of $\mathbf{G}$ and characters $\lambda \in \mathrm{Irr}(\mathbf{L}^F)$ can be endowed with a partial ordering $\leqslant$. The ordering is defined by $(\mathbf{L}', \lambda') \leqslant (\mathbf{L}, \lambda)$ if $\mathbf{L}' \subseteq \mathbf{L}$ and $\langle \lambda, R_{\mathbf{L}'}^{\mathbf{L}} \lambda' \rangle \neq 0$ (this is well defined by the transitivity of the induction functor).

Definition 3.22. A pair $(\mathbf{L}, \lambda)$ is called a *cuspidal pair* of $\mathbf{G}^F$ if it is minimal with respect to the ordering $\leqslant$, or equivalently, if for any proper split Levi subgroup $\mathbf{L}'$ of $\mathbf{L}$ we have ${}^*R_{\mathbf{L}'}^{\mathbf{L}} \lambda = 0$.

When $(\mathbf{L}, \lambda)$ is a cuspidal pair of $\mathbf{G}^F$ we say that λ is a *cuspidal character* of $\mathbf{L}^F$.

Let us state some properties of cuspidal pairs.

Proposition 3.23 ([GeMa20, Proposition 3.2.2], Uniform criterion for cuspidality)**.** *The character $\rho \in \mathrm{Irr}(\mathbf{G}^F)$ is cuspidal if and only if ${}^*R_{\mathbf{T}}^{\mathbf{G}} \rho = 0$ for all F-stable maximal tori $\mathbf{T}$ contained in some proper split Levi subgroup of $\mathbf{G}$.*

Proposition 3.24 ([DiMi20, Remark 5.3.10])**.** *Let $(\mathbf{L}, \lambda)$ and $(\mathbf{L}', \lambda')$ be cuspidal pairs of $\mathbf{G}^F$. Then*

$$\langle R_{\mathbf{L}}^{\mathbf{G}} \lambda, R_{\mathbf{L}'}^{\mathbf{G}} \lambda' \rangle = |\{x \in \mathbf{G}^F \mid {}^x\mathbf{L} = \mathbf{L}' \textit{ and } {}^x\lambda = \lambda'\}/\mathbf{L}^F|.$$

In particular, if we set

$$N_{\mathbf{G}^F}(\mathbf{L}, \lambda) := \{n \in N_{\mathbf{G}^F}(\mathbf{L}) \mid {}^n\lambda = \lambda\} \text{ and } W_{\mathbf{G}^F}(\mathbf{L}, \lambda) := N_{\mathbf{G}^F}(\mathbf{L}, \lambda)/\mathbf{L}^F,$$

it directly follows that:

Corollary 3.25. *Let $(\mathbf{L}, \lambda)$ be a cuspidal pair of $\mathbf{G}^F$. Then*

$$\langle R_{\mathbf{L}}^{\mathbf{G}} \lambda, R_{\mathbf{L}}^{\mathbf{G}} \lambda \rangle = |W_{\mathbf{G}^F}(\mathbf{L}, \lambda)|.$$

The group $W_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ is called the *relative Weyl group of the cuspidal pair* $(\mathbf{L}, \lambda)$.

The importance of cuspidal pairs for the representation theory of finite groups of Lie type comes from the next result.

Proposition 3.26 ([DiMi91, 6.4 Theorem])**.** *Let $\chi \in \mathrm{Irr}(\mathbf{G}^F)$; then, up to $\mathbf{G}^F$-conjugacy, there exists a unique minimal pair $(\mathbf{L}, \lambda)$ such that $(\mathbf{L}, \lambda) \leqslant (\mathbf{G}, \chi)$.*

In other words, we have the partition

$$\mathrm{Irr}(\mathbf{G}^F) = \bigsqcup_{(\mathbf{L},\lambda)/\mathbf{G}^F} \mathscr{E}(\mathbf{G}^F, (\mathbf{L}, \lambda))$$

where we define the *Harish–Chandra series* of the cuspidal pair $(\mathbf{L}, \lambda)$ by

$$\mathscr{E}(\mathbf{G}^F, (\mathbf{L}, \lambda)) := \left\{\rho \in \mathrm{Irr}(\mathbf{G}^F) \mid \mathbf{L} \text{ is minimal such that } {}^*R_{\mathbf{L}}^{\mathbf{G}} \rho \neq 0 \text{ and } \langle \rho, R_{\mathbf{L}}^{\mathbf{G}} \lambda \rangle \neq 0\right\}.$$

The cuspidal characters can be identified by their degrees. To give the next result we need to set some notation first.

Definition 3.27. Let $\mathbf{L}$ be a Levi subgroup of $\mathbf{G}$ of type (I, w). Then, its *relative rank* is defined to be $r(\mathbf{L}) := |I|$.

If we denote by r_F the relative F-rank, we have that the relative rank is $r(\mathbf{L}^F) = r_F([\mathbf{L}, \mathbf{L}])$ for an F-stable Levi subgroup $\mathbf{L}$.

Proposition 3.28 ([GeMa20, Corollary 3.2.21]). *Let $\rho \in \operatorname{Irr}(\mathbf{G}^F)$ lie in the Harish–Chandra series of the cuspidal pair $(\mathbf{L}, \lambda)$. Then, the degree polynomial* [4], *in $\mathbb{R}[\mathbf{q}]$, of ρ has the form*

$$\mathbb{D}_\rho = (\mathbf{q} - 1)^{r(\mathbf{L}^F)} f(\mathbf{q}),$$

where $f \in \mathbb{Q}[\mathbf{q}]$ is not divisible by $\mathbf{q} - 1$.

In particular, ρ is cuspidal if and only if $(\mathbf{q} - 1)^{r(\mathbf{G}^F)}$ is the precise power of $\mathbf{q} - 1$ dividing its degree polynomial.

We will need to explicitly perform Lusztig/Harish–Chandra induction/restriction. For this purpose, we introduce the character formula that we will use.

Definition 3.29. Let $\mathbf{L}$ be an F-stable Levi subgroup of a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ with Levi decomposition $\mathbf{P} = \mathbf{UL}$. Then

$$Q_{\mathbf{L}}^{\mathbf{G}} : \mathbf{G}_{\text{uni}}^F \times \mathbf{L}_{\text{uni}}^F \to \mathbb{Q},\ (u, v) \mapsto \frac{1}{|\mathbf{L}^F|} \operatorname{Trace}\big((u, v) \mid H_c^*(\mathscr{L}^{-1}(\mathbf{U}))\big),$$

is called the associated *2-parameter Green function.*

As in the case of $R_{\mathbf{L}}^{\mathbf{G}}$, we hide $\mathbf{P}$ from the notation, since we will only work with cases for which the Mackey formula holds.

Proposition 3.30 ([DiMi20, Proposition 10.1.2], Character formula). *Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$ and let $\psi \in \operatorname{Irr}(\mathbf{G}^F)$ and $\chi \in \operatorname{Irr}(\mathbf{L}^F)$.*

(a) *If $g = su$ is the Jordan decomposition of $g \in \mathbf{G}^F$ (s semisimple and u unipotent) we have*

$$(R_{\mathbf{L}}^{\mathbf{G}}\chi)(g) = \frac{1}{|\mathbf{L}^F||C_{\mathbf{G}}(s)^{\circ F}|} \sum_{\{h \in \mathbf{G}^F | s \in {}^h\mathbf{L}\}} |C_{{}^h\mathbf{L}}(s)^{\circ F}| \sum_{v \in C_{{}^h\mathbf{L}}(s)^{\circ F}_{\text{uni}}} Q_{C_{{}^h\mathbf{L}}(s)^\circ}^{C_{\mathbf{G}}(s)^\circ}(u, v^{-1})\, {}^h\chi(sv).$$

(b) *If $l = tv$ is the Jordan decomposition of $l \in \mathbf{L}^F$ (t semisimple and v unipotent) we have*

$$({}^*R_{\mathbf{L}}^{\mathbf{G}}\psi)(l) = \frac{|C_{\mathbf{L}}(t)^{\circ F}|}{|C_{\mathbf{G}}(t)^{\circ F}|} \sum_{u \in C_{\mathbf{G}}(t)^{\circ F}_{\text{uni}}} Q_{C_{\mathbf{L}}(t)^\circ}^{C_{\mathbf{G}}(t)^\circ}(u, v^{-1})\, \psi(tu).$$

We call the functions $Q_{\mathbf{L}}^{\mathbf{G}}(-,-)$ "2-parameter" Green functions to distinguish them from the ordinary (1-parameter) Green functions, given in the next definition.

Definition 3.31. For $\mathbf{T} \subseteq \mathbf{G}$ an F-stable maximal torus define its *Green function* by

$$Q_{\mathbf{T}}^{\mathbf{G}} : \mathbf{G}_{\text{uni}}^F \to \mathbb{Z},\ u \mapsto Q_{\mathbf{T}}^{\mathbf{G}}(u) := \left(R_{\mathbf{T}}^{\mathbf{G}} 1_{\mathbf{T}}\right)(u).$$

Remark 3.32. The Green functions $Q_{\mathbf{T}}^{\mathbf{G}}(u)$ can be seen as a special case of the 2-parameter Green functions $Q_{\mathbf{L}}^{\mathbf{G}}(u, v)$ when the Levi $\mathbf{L}$ is a maximal torus and v is forced to be 1 (the only unipotent element of any torus).

The ordinary Green functions obey the following orthogonality relations, when extended by zero on non-unipotent elements of $\mathbf{G}^F$.

Proposition 3.33 ([DiMi20, Proposition 13.2.2], Orthogonality relations for Green functions). *For F-stable maximal tori $\mathbf{T}, \mathbf{T}'$ of $\mathbf{G}$ we have*

$$\langle Q_{\mathbf{T}}^{\mathbf{G}}, Q_{\mathbf{T}'}^{\mathbf{G}} \rangle_{\mathbf{G}^F} = \begin{cases} \frac{|W(\mathbf{T})^F|}{|\mathbf{T}^F|} & \textit{if } \mathbf{T} \textit{ and } \mathbf{T}' \textit{ are } \mathbf{G}^F\textit{-conjugate,} \\ 0 & \textit{otherwise.} \end{cases}$$

---

[4] See [GeMa20, Definition 2.3.25] for a definition of the degree polynomial of an irreducible character.

The idea behind the decomposition of the Deligne–Lusztig characters performed in this work comes from the following remark.

Remark 3.34. The (ordinary) Green functions (and therefore the Deligne–Lusztig characters) do not distinguish between splitting classes, see the discussion in Section 4, in particular Remark 4.14, in the special case where $\mathbf{L}$ is a maximal torus.

To decompose them, we need to consider class functions that have different values on splitting classes.

It is clear that the unipotent splitting conjugacy classes have a representative in the unipotent group $\mathbf{U}^F$. Some of them even have representatives in $\mathbf{U}^F/[\mathbf{U}^F, \mathbf{U}^F]$, meaning that these are distinguished by some linear characters of $\mathbf{U}^F$. The (usual) induction to $\mathbf{G}^F$ of these linear characters produces characters of $\mathbf{G}^F$ that distinguish those particular unipotent splitting classes. These characters have nice properties and are the main objects used in the decomposition of Deligne–Lusztig characters in Parts II and III. These characters are called Gel'fand–Graev characters and are studied in the disconnected centre case in [DLM92]. We give a summary of their definition and relevant properties in Section 5.

4 About 2-parameter Green functions

The 2-parameter Green functions are a central ingredient in the character formula for Lusztig induction and restriction in finite groups of Lie type (Proposition 3.30). In this section, we discuss the elementary method used in [MaRo20] for the explicit determination of their values. We start by restating the general observations in [MaRo20, Sections 2 and 3.1] and continue by giving details on the explicit computation for the split cases (this is just mentioned but not explained in [MaRo20]). The explicit computations for $\mathrm{SL}_4(q)$ and $\mathrm{Spin}_8^+(q)$ will be detailed later respectively in Section 10 and Section 16.

In this section, we denote by $\mathbf{G}$ a connected reductive group (over a field of characteristic p) with Steinberg map F and by $\mathbf{L}$ is F-stable Levi subgroup of a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ with Levi decomposition $\mathbf{P} = \mathbf{UL}$.

4.1 General observations and known facts

We begin with a remark on the arguments of the 2-parameter Green functions.

Remark 4.1. For any F-stable Levi subgroup $\mathbf{L}$ the functors $R_{\mathbf{L}}^{\mathbf{G}}$ and ${}^*R_{\mathbf{L}}^{\mathbf{G}}$ send class functions to class functions. Applying the character formulas (Proposition 3.30) to characters $\psi \in \mathrm{Irr}(\mathbf{G}^F)$ and $\chi \in \mathrm{Irr}(\mathbf{L}^F)$, we have for unipotent elements $u \in \mathbf{G}^F$ and $v \in \mathbf{L}^F$

$$\big({}^*R_{\mathbf{L}}^{\mathbf{G}}\psi\big)(v) = \frac{|\mathbf{L}^F|}{|\mathbf{G}^F|} \sum_{u\in \mathbf{G}^F_{\mathrm{uni}}} Q_{\mathbf{L}}^{\mathbf{G}}(u, v^{-1})\psi(u) \text{ and } \big(R_{\mathbf{L}}^{\mathbf{G}}\chi\big)(u) = \sum_{v\in \mathbf{L}^F_{\mathrm{uni}}} Q_{\mathbf{L}}^{\mathbf{G}}(u, v^{-1})\chi(v).$$

Therefore, for any $u \in \mathbf{G}^F_{\mathrm{uni}}$ the function $Q_{\mathbf{L}}^{\mathbf{G}}(u, -)$ is constant on unipotent classes of $\mathbf{L}^F$ and, analogously, for any $v \in \mathbf{L}^F_{\mathrm{uni}}$ the function $Q_{\mathbf{L}}^{\mathbf{G}}(-, v)$ is constant on unipotent classes of $\mathbf{G}^F$.

Notice that, if $\mathbf{L} = \mathbf{T}$ is a maximal torus, and $\mathbf{P}$ is a Borel subgroup of $\mathbf{G}$, then $\mathbf{L}^F_{\mathrm{uni}} = \{1\}$ and the defining formula shows that $Q_{\mathbf{T}}^{\mathbf{G}}(u, 1) = \big(R_{\mathbf{T}}^{\mathbf{G}}1\big)(u)$ $(u \in \mathbf{G}^F_{\mathrm{uni}})$, which is the usual (1-parameter) Green function.

The values of $Q_{\mathbf{L}}^{\mathbf{G}}(u, v)$ at $u = 1$ are known for any $\mathbf{L}$, see [DiMi20, p. 157]:

$$Q_{\mathbf{L}}^{\mathbf{G}}(1, v) = \begin{cases} \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{L}}|\mathbf{G}^F : \mathbf{L}^F|_{p'} & \text{if } v = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $|\cdot|_{p'}$ indicates the part of $|\cdot|$ coprime with p.

From the definition of Harish–Chandra induction (Definition 3.1), it is easy to get the following formula for the 2-parameter Green functions in the split case:

Proposition 4.2. *If* $\mathbf{L}$ *is a split Levi subgroup of* $\mathbf{G}$*, then*

$$Q_{\mathbf{L}}^{\mathbf{G}}(u, v) = \frac{1}{|\mathbf{L}^F|}|\{g\mathbf{U}^F \mid g \in \mathbf{G}^F,\ u^g \in v\mathbf{U}^F\}|.$$

Thus, in the split case the 2-parameter Green functions can be computed in an elementary way (we give details on this in Section 4.3).

Remark 4.3. It follows directly from the formula above that for $\mathbf{L} = \mathbf{G}$ we have

$$Q_{\mathbf{G}}^{\mathbf{G}}(u, v) = \begin{cases} |v^{\mathbf{G}^F}|^{-1} & \text{if } v \text{ and } u \text{ are } \mathbf{G}^F\text{-conjugate,} \\ 0 & \text{otherwise,} \end{cases}$$

since, in this case, $\mathbf{U} = \{1\}$.

Proposition 4.2 is used to show:

Proposition 4.4. *Assume that* $\mathbf{P}$ *is an* F*-stable parabolic subgroup of* $\mathbf{G}$ *with Levi decomposition* $\mathbf{P} = \mathbf{UL}$*. Then for* $v \in \mathbf{L}^F_{\mathrm{uni}}$*,* $u \in \mathbf{G}^F_{\mathrm{uni}}$ *we have:*

(a) $|v^{\mathbf{L}^F}|\, Q^{\mathbf{G}}_{\mathbf{L}}(u, v) \in \mathbb{Z}_{\geq 0}$.

(b) *If* $Q^{\mathbf{G}}_{\mathbf{L}}(u, v^{-1}) \neq 0$ *then* $v^{\mathbf{G}} \subseteq \overline{u^{\mathbf{G}}} \subseteq \overline{\mathrm{Ind}^{\mathbf{G}}_{\mathbf{L}}(v^{\mathbf{L}})}$*.*

(c) *If* u *is regular unipotent (we give the definition of regular unipotent elements later in Section 5.1), then there is a unique* $\mathbf{L}^F$*-class* C *of regular unipotent elements of* $\mathbf{L}^F$ *such that*
$$Q^{\mathbf{G}}_{\mathbf{L}}(u, v) = \begin{cases} |v^{\mathbf{L}^F}|^{-1} & \text{if } v \in C, \\ 0 & \text{otherwise.} \end{cases}$$

Here, $\overline{\mathbf{C}}$ denotes the closure of a class $\mathbf{C}$ and $\mathrm{Ind}^{\mathbf{G}}_{\mathbf{L}}\mathbf{C}$ is the induced class in the sense of Lusztig–Spaltenstein [LuSp79].

Proof. Observe that if $g \in \mathbf{G}^F$ is such that $u^g \in v\mathbf{U}^F$ then for any $c \in C_{\mathbf{L}}(v)^F$, the element gc has the same property, so $|\{g\mathbf{U}^F \mid u^g \in v\mathbf{U}^F\}|$ is divisible by $|C_{\mathbf{L}}(v)^F|$, whence Proposition 4.2 shows that $|v^{\mathbf{L}^F}| Q^{\mathbf{G}}_{\mathbf{L}}(u, v)$ is an integer.

If $Q^{\mathbf{G}}_{\mathbf{L}}(u, v^{-1}) \neq 0$ there is $g \in \mathbf{G}^F$ with $u^g \in v\mathbf{U}^F$, thus we have $u \in v\mathbf{U}^F$ up to replacing u by a conjugate. Now by definition the induced class $\mathbf{C} := \mathrm{Ind}^{\mathbf{G}}_{\mathbf{L}}(v^{\mathbf{L}})$ has the property that $\mathbf{C} \cap v^{\mathbf{L}}\mathbf{U}$ is dense in $v^{\mathbf{L}}\mathbf{U}$. Hence, $u \in \overline{\mathbf{C}}$. Moreover, we have $u = vx$ for some $x \in \mathbf{U}^F$. Then $X := \{vx^c \mid c \in Z(\mathbf{L})^\circ\} \subseteq u^{\mathbf{G}}$. Now, $Z(\mathbf{L})^\circ$ acts non-trivially on all root subgroups of $\mathbf{U}$ as $\mathbf{L} = C_{\mathbf{G}}(Z^\circ(\mathbf{L}))$. Thus the closure of X contains v and so $v \in \overline{u^{\mathbf{G}}}$.

For (c) note that the centraliser dimension of v in $\mathbf{L}$ and of $v' \in \mathrm{Ind}^{\mathbf{G}}_{\mathbf{L}}(v^{\mathbf{L}})$ in $\mathbf{G}$ agree (see [LuSp79, Theorem 1.3(a)]). It follows that only the regular unipotent class of $\mathbf{L}$ induces to the regular unipotent class of $\mathbf{G}$, and thus $Q^{\mathbf{G}}_{\mathbf{L}}(u, v) = 0$ unless v is regular. Now, assume that $u \in v\mathbf{U}^F$, and so $Q^{\mathbf{G}}_{\mathbf{L}}(u, v) \neq 0$. Since u is regular, it lies in a unique Borel subgroup $\mathbf{B} \leq \mathbf{P}$ of $\mathbf{G}$. Thus, if $g \in \mathbf{G}^F$ with $g^{-1}ug \in \mathbf{P}$ then $g \in \mathbf{P}^F$. In particular, $g^{-1}ug \in v'\mathbf{U}^F$ for some $v' \in \mathbf{L}^F_{\mathrm{uni}}$ implies that v, v' are $\mathbf{L}^F$-conjugate. It is clear that there are exactly $|C_{\mathbf{L}}(v)^F|$ cosets $g\mathbf{U}^F$ with $g^{-1}ug \in v\mathbf{U}^F$. □

Notice that, alternatively, point (c) can be proven thanks to Conjecture 5.32 (from Section 5 later), which is valid in the split case.

Notation 4.5. From now on we write $\tilde{Q}^{\mathbf{G}}_{\mathbf{L}}$ for the matrix $\big(|v^{\mathbf{L}^F}| Q^{\mathbf{G}}_{\mathbf{L}}(u, v^{-1})\big)_{v,u}$ with rows and columns indexed by the unipotent conjugacy classes of $\mathbf{L}^F$, $\mathbf{G}^F$ respectively. We call it *modified* 2-parameter Green functions.

Lemma 4.6. *Let* $\mathbf{M} \leq \mathbf{G}$ *be an* F*-stable Levi subgroup containing* $\mathbf{L}$*. Then*
$$\tilde{Q}^{\mathbf{G}}_{\mathbf{L}} = \tilde{Q}^{\mathbf{M}}_{\mathbf{L}} \cdot \tilde{Q}^{\mathbf{G}}_{\mathbf{M}}.$$

Proof. By transitivity of Lusztig induction [DiMi20, Proposition 9.1.8] we have $R^{\mathbf{G}}_{\mathbf{L}} = R^{\mathbf{G}}_{\mathbf{M}} \circ R^{\mathbf{M}}_{\mathbf{L}}$. Therefore, by applying twice the formula in Remark 4.1 we get
$$\big(R^{\mathbf{G}}_{\mathbf{L}}\psi\big)(u) = \sum_{v \in \mathbf{M}^F_{\mathrm{uni}}} Q^{\mathbf{G}}_{\mathbf{M}}(u, v^{-1}) \sum_{x \in \mathbf{L}^F_{\mathrm{uni}}} Q^{\mathbf{M}}_{\mathbf{L}}(v, x^{-1})\psi(x)$$
for all $u \in \mathbf{G}^F_{\mathrm{uni}}$ and all class functions ψ on $\mathbf{L}^F$. The claim follows. □

Thus, for an inductive determination of the 2-parameter Green functions it is sufficient to consider the case when $\mathbf{L} < \mathbf{G}$ is maximal among F-stable Levi subgroups. Now let $\mathbf{T} \leq \mathbf{L}$ be an F-stable maximal torus, then Lemma 4.6 gives $\tilde{R}_{\mathbf{T}}^{\mathbf{G}} = \tilde{R}_{\mathbf{T}}^{\mathbf{L}} \cdot \tilde{Q}_{\mathbf{L}}^{\mathbf{G}}$ where we have set $\tilde{R}_{\mathbf{T}}^{\mathbf{G}} := \big((R_{\mathbf{T}}^{\mathbf{G}}1)(u)\big)_u$. The $\mathbf{L}^F$-conjugacy classes of F-stable maximal tori of $\mathbf{L}$ are parametrised by F-conjugacy classes in the Weyl group W_L of $\mathbf{L}$ (see discussion before Definition 1.59). Thus the above formula yields a linear system of equations

$$\big(R_{\mathbf{T}_w}^{\mathbf{G}}1\big)(u) = \sum_{v \in \mathbf{L}_{\mathrm{uni}}^F} \big(R_{\mathbf{T}_w}^{\mathbf{L}}1\big)(v) Q_{\mathbf{L}}^{\mathbf{G}}(u, v^{-1}) \qquad (w \in W_L) \tag{$*$}$$

for $\tilde{Q}_{\mathbf{L}}^{\mathbf{G}}$ with coefficient matrix $\big((R_{\mathbf{T}_w}^{\mathbf{L}}1)(v)\big)_{w,v}$.

Proposition 4.7. *Assume that the matrix $\big((R_{\mathbf{T}_w}^{\mathbf{L}}1)(v)\big)_{w,v}$ is square. Then the 2-parameter Green functions $Q_{\mathbf{L}}^{\mathbf{G}}$ are uniquely determined by the ordinary Green functions of $\mathbf{G}$ and of $\mathbf{L}$.*

Proof. This follows from the above considerations and the fact that the (by assumption) square matrix of values of Green functions on unipotent classes of $\mathbf{L}^F$ is invertible due to the orthogonality relations for the ordinary Green functions, see Proposition 3.33. □

This result can be used to give restrictions on the values of the 2-parameter Green functions. The assumption of Proposition 4.7 is satisfied, for example, for $\tilde{\mathbf{G}}$ and a proper F-stable Levi $\tilde{\mathbf{L}} \leq \tilde{\mathbf{G}}$ where $i : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ is a regular embedding and $\mathbf{G}$ is either SL_4 or Spin_8 (in these cases the assumption will be easily checked).

In practice, we use the computer program CHEVIE to determine the linear system ($*$). On one hand, we use the GAP part ([MiChv]) to identify the corresponding tori of the group and of the Levi subgroups (using the commands `CoxeterGroup`, `Twistings` and `Torus`). On the other hand, we use the MAPLE part ([GHLMP]) to actually get the ordinary Green functions (with the command `GreenFunTab`) and to solve the system ($*$) (with the command `solve`).

We give two examples on the application of Proposition 4.7 that we will use later for the computation of the 2-parameter Green functions of $\mathrm{SL}_4(q)$ and $\mathrm{Spin}_8^+(q)$. For readability reasons we replace "0" by "." and the polynomials in q are factored and written in terms of the cyclotomic polynomials $\Phi_1 = q-1$, $\Phi_2 = q+1$, $\Phi_3 = q^2+q+1$, $\Phi_4 = q^2+1$ and $\Phi_6 = q^2-q+1$.

We give now some examples (that we use later) of 2-parameter Green functions, computed following the discussion above. These are all cases in which Proposition 4.7 can be applied.

Example 4.8. Let $\mathbf{G} = \mathrm{GL}_4$, with the unipotent classes parametrized by partitions of 4 (labelling the Jordan blocks) ordered as $1^4, 21^2, 2^2, 31, 4$. First, let $\mathbf{L}_1$ be a standard Levi subgroup of type A_2. Its unipotent classes are labelled by the partitions $1^3, 21, 3$ of 3, and we obtain the matrix of (modified) 2-parameter Green functions

$$\tilde{Q}_{\mathbf{L}_1}^{\mathbf{G}} = \begin{pmatrix} \Phi_2\Phi_4 & 1 & . & . & . \\ . & q\Phi_2 & \Phi_2 & 1 & . \\ . & . & . & q & 1 \end{pmatrix}.$$

Next, let $\mathbf{L}_2$ be the standard Levi subgroup of type A_1^2. The resulting matrix is

$$\tilde{Q}_{\mathbf{L}_2}^{\mathbf{G}} = \begin{pmatrix} \Phi_3\Phi_4 & \Phi_2 & 1 & . & . \\ . & q^2 & . & 1 & . \\ . & q^2 & . & 1 & . \\ . & . & q\Phi_2 & \Phi_1 & 1 \end{pmatrix}$$

where the rows are labelled by the unipotent conjugacy classes of $\mathbf{L}_2$ parametrised by the pairs $(1^2, 1^2), (2, 1^2), (1^2, 2), (2, 2)$ of partitions of 2. Note that the second and third row agree, as the second and third unipotent class of $\mathbf{L}_2^F$ are conjugate in $N_{\mathbf{G}}(\mathbf{L}_2)^F$.

For the twisted Levi subgroup $\mathbf{L}_3$ of type $A_1(q^2).(q^2-1)$ we find

$$\tilde{Q}^{\mathbf{G}}_{\mathbf{L}_3} = \begin{pmatrix} \Phi_1^2\Phi_3 & -\Phi_1 & 1 & . & . \\ . & . & q\Phi_1 & -\Phi_1 & 1 \end{pmatrix}.$$

Finally, the split Levi subgroup of type A_1 is not maximal and we may apply Lemma 4.6, while for its twisted version $\mathbf{L}_4$ of type $A_1(q).(q^2-1)(q-1)$ we obtain

$$\tilde{Q}^{\mathbf{G}}_{\mathbf{L}_4} = \begin{pmatrix} -\Phi_1\Phi_3\Phi_4 & 1 & -\Phi_1 & 1 & . \\ . & -q^2\Phi_1 & q\Phi_2 & . & 1 \end{pmatrix}.$$

The assumptions of Proposition 4.7 are also satisfied for groups with connected centre of type D_4.

Example 4.9. Let $\mathbf{G}$ be of type D_4 with connected centre, F a split Frobenius map and $\mathbf{L}$ a split Levi subgroup of type A_3. We order the 13 unipotent classes of $\mathbf{G}^F$ by their Jordan normal forms (when projecting the elements in SO_8, they are given by partitions of 8)

$$1^8,\ 2^21^4,\ 2^4+,\ 2^4-,\ 31^5,\ 32^21,\ 3^21^2 \text{ (two classes)},\ 4^2+,\ 4^2-,\ 51^3,\ 53,\ 71$$

where the signs $+, -$ distinguish between classes with the same Jordan form that are swapped by the graph automorphism of order two of D_n ($n \geq 4$). In $\mathbf{L}^F$ we also order the unipotent classes by their Jordan form (partitions of 4) $1^4, 21^2, 2^2, 31, 4$. We find

$$\begin{pmatrix} \Phi_2^2\Phi_4\Phi_6 & \Phi_2 & . & . & 1 & . & . & . & . & . & . & . & . \\ . & q^2\Phi_2^2 & \Phi_2\Phi_4 & \Phi_2\Phi_4 & . & 1 & 2 & . & . & . & . & . & . \\ . & . & . & . & q\Phi_2\Phi_4 & q\Phi_2 & \Phi_1 & \Phi_2 & 1 & . & . & . & . \\ . & . & . & . & . & . & 2q^2 & . & . & \Phi_2 & \Phi_2 & 1 & . \\ . & . & . & . & . & . & . & . & q\Phi_2 & . & . & q & 1 \end{pmatrix}$$

for one of the three possible embeddings of the Levi, and a suitable permutation of the columns $(3,4,5)(9,10,11)$ for the other two. The Green functions for a twisted Levi subgroup of type ${}^2A_3(q).(q+1)$ are related to those of $A_3(q).(q-1)$ as follows. The ordinary Green functions of ${}^2A_3(q)$ are obtained from those of $A_3(q)$ by replacing q with $-q$ by Ennola duality [Ka85]. This also entails a permutation of maximal tori in the linear system $(*)$. Therefore, the Green functions of ${}^2A_3(q).(q+1)$ are obtained from those of $A_3(q).(q-1)$ by replacing q with $-q$ and swapping the classes with Jordan normal form 3^21^2 (they have centraliser orders $q^8(q\pm1)^2$, which are sent to one another by the transformation $q \mapsto -q$).

Next consider a split Levi subgroup $\mathbf{L}$ of type A_1^3. Here, $\tilde{Q}^{\mathbf{G}}_{\mathbf{L}}$ equals

$$\begin{pmatrix} \Phi_2\Phi_3\Phi_4^2\Phi_6 & * & \Phi_2\Phi_4 & \Phi_2\Phi_4 & \Phi_2\Phi_4 & \Phi_2 & 1 & 1 & . & . & . & . & . \\ . & q^4\Phi_2 & . & . & q^2\Phi_4 & . & 2q & . & . & . & 1 & . & . \\ . & q^4\Phi_2 & . & q^2\Phi_4 & . & . & 2q & . & . & 1 & . & . & . \\ . & q^4\Phi_2 & q^2\Phi_4 & . & . & . & 2q & . & 1 & . & . & . & . \\ . & . & q^2\Phi_2\Phi_4 & . & . & q^2 & 2q\Phi_1 & . & \Phi_2 & . & . & 1 & . \\ . & . & . & q^2\Phi_2\Phi_4 & . & q^2 & 2q\Phi_1 & . & . & \Phi_2 & . & 1 & . \\ . & . & . & . & q^2\Phi_2\Phi_4 & q^2 & 2q\Phi_1 & . & . & . & \Phi_2 & 1 & . \\ . & . & . & . & . & q^3\Phi_2 & q\Phi_1^2 & q\Phi_2^2 & \Phi_1\Phi_2 & \Phi_1\Phi_2 & \Phi_1\Phi_2 & q-2 & 1 \end{pmatrix}$$

where $* = q^4+3q^3+3q^2+q+1$, and the order of the unipotent classes of $\mathbf{L}^F$ is given in terms of triples of partitions of 2

$$(1^2,1^2,1^2), (2,1^2,1^2), (1^2,2,1^2), (1^2,1^2,2), (2,2,1^2), (2,1^2,2), (1^2,2,2), (2,2,2).$$

The symmetry from triality (one of the permutations of order 3 of the external nodes of D_4), cyclically permuting the classes 3,4,5 and 9,10,11 of $\mathbf{G}^F$, as well as the classes 2,3,4 and 5,6,7 of $\mathbf{L}^F$, is clearly visible. Again, the Green functions for a twisted Levi subgroup of type $A_1(q)^3.(q+1)$ are obtained by replacing q with $-q$ and interchanging the two classes with Jordan normal form 3^21^2. For a twisted Levi subgroup of type $A_1(q^2).(q^2+1)$ we find (the first row corresponds to the identity of $\mathbf{L}^F$)

$$\begin{pmatrix} -\Phi_1^3\Phi_2^3\Phi_3\Phi_6 & \Phi_1^2\Phi_2^2 & \Phi_1^2\Phi_2^2 & \Phi_1^2\Phi_2^2 & -\Phi_1\Phi_2 & -\Phi_1\Phi_2 & -\Phi_1 & \Phi_2 & . & . & 1 & . & . \\ . & . & . & . & -q^2\Phi_1^2\Phi_2^2 & q^2\Phi_1\Phi_2 & q\Phi_1^2 & -q\Phi_2^2 & -\Phi_1\Phi_2 & -\Phi_1\Phi_2 & . & 1 & 1 \end{pmatrix}$$

It will be extremely useful to compare the 2-parameter Green functions of groups of the same type (for example to use Proposition 4.7). In this regard, we list some results from [Bo00].

Remark 4.10. It is important to notice that our definition of the 2-parameter Green functions and the one of Bonnafé [Bo00] differ by a factor $|\mathbf{L}^F|$. The results taken from [Bo00], that we list below, have been rewritten with this extra factor included.

Let $\tilde{\mathbf{G}}$ be another connected reductive group defined over $\mathbb{F}_q$, with Frobenius endomorphism also denoted by $F : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$. We assume that there is a morphism of algebraic groups $i : \mathbf{G} \to \tilde{\mathbf{G}}$ defined over $\mathbb{F}_q$ and satisfying the following conditions:

(a) $\ker(i)$ is central in $\mathbf{G}$,

(b) $i(\mathbf{G})$ contains the derived group of $\tilde{\mathbf{G}}$.

In this setting, let $\mathbf{L} := i^{-1}(\tilde{\mathbf{L}})$ for an F-stable Levi subgroup $\tilde{\mathbf{L}}$ of $\tilde{\mathbf{G}}$. Then, $\mathbf{L}$ is an F-stable Levi subgroup of $\mathbf{G}$.

Proposition 4.11 ([Bo00, Proposition 2.2.1 (b)]). *Let $u \in \mathbf{G}^F$ and $v \in \mathbf{L}^F$ be unipotent elements. If i is injective, then*

$$Q_{\tilde{\mathbf{L}}}^{\tilde{\mathbf{G}}}(u,v) = \frac{|\mathbf{L}^F|}{|\tilde{\mathbf{L}}^F|} \sum_{g \in \tilde{\mathbf{G}}^F/\mathbf{G}^F} Q_{\mathbf{L}}^{\mathbf{G}}({}^g u, v) = \frac{|\mathbf{L}^F|}{|\tilde{\mathbf{L}}^F|} \sum_{l \in \tilde{\mathbf{L}}^F/\mathbf{L}^F} Q_{\mathbf{L}}^{\mathbf{G}}(u, {}^l v)$$

where $\mathbf{G}^F_{\mathrm{uni}}$ and $\tilde{\mathbf{G}}^F_{\mathrm{uni}}$ are identified via i.

By the hypothesis on i, the inclusion $\ker(i) \subseteq Z(\mathbf{G})$ translates into a morphism of groups $H^1(F, \ker(i)) \to H^1(F, Z(\mathbf{G}))$. Recall that by Remark 1.64, we have an action of $H^1(F, Z(\mathbf{G}))$ on the conjugacy classes of $\mathbf{G}^F$. Thus, for all $z \in H^1(F, \ker(i))$ we denote by $\hat{z}$ the action of the image of z in $H^1(F, Z(\mathbf{G}))$.

Proposition 4.12 ([Bo00, Proposition 2.2.2]). *Assume that i is surjective. Let $u \in \mathbf{G}^F$ and $v \in \mathbf{L}^F$ be unipotent elements. Then*

$$Q_{\tilde{\mathbf{L}}}^{\tilde{\mathbf{G}}}(i(u), i(v)) = \frac{|\mathbf{L}^F|}{|(\ker(i)^F)||\tilde{\mathbf{L}}^F|} \sum_{z \in H^1(F, \ker(i))} Q_{\mathbf{L}}^{\mathbf{G}}({}^{\hat{z}}u, v) = \frac{|\mathbf{L}^F|}{|(\ker(i)^F)||\tilde{\mathbf{L}}^F|} \sum_{z \in H^1(F, \ker(i))} Q_{\mathbf{L}}^{\mathbf{G}}(u, {}^{\hat{z}}v).$$

Something more precise can be said for the surjective case.

Proposition 4.13 ([Bo00, Corollary 2.2.3]). *If the morphism i is surjective and satisfies $\ker(i) \subset \{z^{-1}F(z) \mid z \in Z(\mathbf{L})\}$, then*

$$Q_{\tilde{\mathbf{L}}}^{\tilde{\mathbf{G}}}(i(u), i(v)) = \frac{|\mathbf{L}^F|}{|\ker(i)^{\circ F}||\tilde{\mathbf{L}}^F|} Q_{\mathbf{L}}^{\mathbf{G}}(u,v)$$

for all unipotent elements u and v in $\mathbf{G}^F$ and $\mathbf{L}^F$ respectively. In particular, this equality holds whenever $\ker(i) \subset Z(\mathbf{L})^\circ$.

4.2 Groups with non-connected centre

For groups $\mathbf{G}$ with non-connected centre, the number of unipotent classes is bigger than the number of irreducible characters of the Weyl group, so that Proposition 4.7 cannot be applied. However, additional considerations can lead to a solution.

To this end, let $\mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ be an F-equivariant regular embedding, such that $\tilde{\mathbf{G}} = \mathbf{G}Z(\tilde{\mathbf{G}})$ has connected centre and derived subgroup $[\tilde{\mathbf{G}}, \tilde{\mathbf{G}}] = \mathbf{G}$. Then, for any Levi subgroup $\mathbf{L} \leq \mathbf{G}$, $\tilde{\mathbf{L}} := \mathbf{L}Z(\tilde{\mathbf{G}})$ is a Levi subgroup of $\tilde{\mathbf{G}}$, F-stable if $\mathbf{L}$ is. Here, we are interested in cases where $\mathbf{G}$ is of type A or D_4. This implies that all proper Levi subgroups of $\mathbf{G}$, and hence of $\tilde{\mathbf{G}}$, are of type A. Therefore, all 2-parameter Green functions for $\tilde{\mathbf{G}}$ can be computed with Proposition 4.7 (see Example 4.8 and Example 4.9). Then, we just need to descend to the simply connected group $\mathbf{G}$.

Let us write $\mathscr{L}_{\mathbf{G}}$ for the Lang map on $\mathbf{G}$. While $\mathscr{L}_{\mathbf{G}}^{-1}(\mathbf{U})$ is not necessarily invariant under the action of $\tilde{\mathbf{G}} \times \tilde{\mathbf{L}}$, it is so under the action of the diagonally embedded subgroup $\Delta(\tilde{\mathbf{L}}^F) \cong \tilde{\mathbf{L}}^F$ of $\tilde{\mathbf{G}}^F \times \tilde{\mathbf{L}}^F$ (with $\Delta(l) = (l, l^{-1})$ for $l \in \tilde{\mathbf{L}}^F$). So $\Delta(\tilde{\mathbf{L}}^F)$ acts on the ℓ-adic cohomology groups $H_c^i(\mathscr{L}_{\mathbf{G}}^{-1}(\mathbf{U}))$. In particular, the 2-parameter Green functions $Q_{\mathbf{L}}^{\mathbf{G}}$ are invariant under the diagonal action of $\tilde{\mathbf{L}}^F$:

$$Q_{\mathbf{L}}^{\mathbf{G}}(u, v) = Q_{\mathbf{L}}^{\mathbf{G}}(u^l, v^l) \qquad \text{for all } l \in \tilde{\mathbf{L}}^F.$$

Furthermore, the 2-parameter Green functions $Q_{\tilde{\mathbf{L}}}^{\tilde{\mathbf{G}}}$ are "induced" from those of $\mathbf{L}$ inside $\mathbf{G}$ by Proposition 4.11. This implies relations between the 2-parameter Green functions of $\tilde{\mathbf{G}}$ and $\mathbf{G}$.

Remark 4.14. Suppose that the unipotent class of $u \in \tilde{\mathbf{G}}^F$ splits into n classes of $\mathbf{G}^F$ and that the unipotent class of $v \in \tilde{\mathbf{L}}^F$ splits into m classes of $\mathbf{L}^F$, then by Proposition 4.11

$$\tilde{Q}_{\tilde{\mathbf{L}}}^{\tilde{\mathbf{G}}}(u, v) = \sum_{i=1}^{m} \tilde{Q}_{\mathbf{L}}^{\mathbf{G}}(u, v_i) = \frac{m}{n} \sum_{i=1}^{n} \tilde{Q}_{\mathbf{L}}^{\mathbf{G}}(u_i, v) \tag{$**$}$$

where $u_i \in \mathbf{G}^F$ are the representatives of the $\mathbf{G}^F$-classes in $(u)^{\tilde{\mathbf{G}}^F}$, and $v_i \in \mathbf{L}^F$ are the representatives of the $\mathbf{L}^F$-classes in $(v)^{\tilde{\mathbf{L}}^F}$.

For example, in Sections 10 and 16 we need the following cases:

- If $n = 1$ then $(**)$ and the $\tilde{\mathbf{L}}^F$-invariance directly give
$$\tilde{Q}_{\mathbf{L}}^{\mathbf{G}}(u, v_i) = \frac{1}{m} \tilde{Q}_{\tilde{\mathbf{L}}}^{\tilde{\mathbf{G}}}(u, v) \qquad \text{for } i = 1, \ldots, m.$$
- If $m = 1$ then $(**)$ and the $\tilde{\mathbf{L}}^F$-invariance directly give
$$\tilde{Q}_{\mathbf{L}}^{\mathbf{G}}(u_i, v) = \tilde{Q}_{\tilde{\mathbf{L}}}^{\tilde{\mathbf{G}}}(u, v) \qquad \text{for } i = 1, \ldots, n.$$
- If $n = m = 2$ then the value $Q := \tilde{Q}_{\tilde{\mathbf{L}}}^{\tilde{\mathbf{G}}}(u, v)$ is replaced in the matrix of $\tilde{Q}_{\mathbf{L}}^{\mathbf{G}}$ by the submatrix
$$\begin{array}{c|cc} & u_1 & u_2 \\ \hline v_1 & a & Q-a \\ v_2 & Q-a & a \end{array}$$
where $a := \tilde{Q}_{\mathbf{L}}^{\mathbf{G}}(u_1, v_1)$.
- If $n = 4$, $m = 2$ then the value $Q := \tilde{Q}_{\tilde{\mathbf{L}}}^{\tilde{\mathbf{G}}}(u, v)$ is replaced in the matrix of $\tilde{Q}_{\mathbf{L}}^{\mathbf{G}}$ by the submatrix
$$\begin{array}{c|cccc} & u_1 & u_2 & u_3 & u_4 \\ \hline v_1 & a_1 & a_2 & a_3 & 2Q - a_1 - a_2 - a_3 \\ v_2 & Q - a_1 & Q - a_2 & Q - a_3 & a_1 + a_2 + a_3 - Q \end{array}$$
where $a_i := \tilde{Q}_{\mathbf{L}}^{\mathbf{G}}(u_i, v_1)$ for $i = 1, 2, 3$.

- If $n = 2$, $m = 4$ then the value $Q := \tilde{Q}_{\tilde{\mathbf{L}}}^{\tilde{\mathbf{G}}}(u, v)$ is replaced in the matrix of $\tilde{Q}_{\mathbf{L}}^{\mathbf{G}}$ by the submatrix

	u_1	u_2
v_1	a_1	$Q/2 - a_1$
v_2	a_2	$Q/2 - a_2$
v_3	a_3	$Q/2 - a_3$
v_4	$Q - a_1 - a_2 - a_3$	$a_1 + a_2 + a_3 - Q/2$

  where $a_i := \tilde{Q}_{\mathbf{L}}^{\mathbf{G}}(u_1, v_i)$ for $i = 1, 2, 3$.

In other words, every splitting class introduces some unknown entries in $\tilde{Q}_{\mathbf{L}}^{\mathbf{G}}$. These unknowns can be related to each other by studying the diagonal action of $\tilde{\mathbf{L}}^F$ on each unipotent class of $\mathbf{G}^F$ and $\mathbf{L}^F$.

We get the explicit action of $\tilde{\mathbf{L}}^F$ thanks to Remark 1.64. There, we associate an element $l_z \in \mathbf{L}$ to each $z \in H^1(F, Z(\mathbf{L}))$ such that $l_z^{-1}F(l_z) \in Z(\mathbf{L})$ represents z. Then, the action of $\tilde{\mathbf{L}}^F$ is given by conjugating with those elements l_z, up to $\mathbf{L}^F$-conjugacy. Since there is a canonical surjection $H^1(F, Z(\mathbf{G})) \twoheadrightarrow H^1(F, Z(\mathbf{L}))$, we choose, instead, for all $z \in H^1(F, Z(\mathbf{G}))$ an element $g \in \mathbf{G}$ such that $g^{-1}F(g)$ represents z. This gives us the desired action for all split Levi subgroups. For the non-split ones we replace F by the associated twisted Frobenius map F' (see discussion before Definition 1.59). Therefore, for all $z \in H^1(F, Z(\mathbf{G}))$ we have $g, g' \in \mathbf{G}$ such that both $g^{-1}F(g)$ and $g'^{-1}F'(g')$ represent z. Then, the action of $\tilde{\mathbf{L}}^F$ is given by $(u, v) \mapsto (u^g, v^{g'})$ for $u \in \mathbf{G}^F$ and $v \in \mathbf{L}^{F'}$.

One of the key tools for reducing the number of unknowns further is given by the next remark. We say that a class function (not only a character!) f of $\mathbf{G}^F$ is *absolutely cuspidal* if for every F-stable proper Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ we have ${}^*R_{\mathbf{L}}^{\mathbf{G}} f = 0$.

Remark 4.15. For any F-stable minimal Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ with disconnected centre (minimal with respect to having disconnected centre) a class function f of $\mathbf{L}^F$ that takes values 1, -1 on a pair of splitting unipotent classes (say with representatives u_1 and u_2) and zero elsewhere is absolutely cuspidal. This follows by the discussion above and minimality, since for any proper Levi subgroup $\mathbf{L}'$ of $\mathbf{L}$ we have that $\tilde{Q}_{\mathbf{L}'}^{\mathbf{L}}(u_1, v) = \tilde{Q}_{\mathbf{L}'}^{\mathbf{L}}(u_2, v)$ for all $v \in \mathbf{L}'^F_{\text{uni}}$. Then, $({}^*R_{\mathbf{L}'}^{\mathbf{L}} f) = 0$.

4.3 The split case

In this section, we show that the 2-parameter Green functions for split Levi subgroups can be computed by knowing the fusion of unipotent classes from the unipotent subgroup $\mathbf{U}_0^F$ to $\mathbf{G}^F$.

Proposition 4.16. *Let* $\mathbf{P}$ *be an* F*-stable parabolic subgroup containing the* F*-stable Levi subgroup* $\mathbf{L}$*. We denote by* $\mathbf{U}$ *the unipotent radical of* $\mathbf{P}$*, such that* $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$*. Assume that for* $v \in \mathbf{L}_{\text{uni}}^F$ *there exist subgroups* $H_i \leqslant \mathbf{G}^F$ *such that* $v\mathbf{U}^F = \dot{\bigcup} u_i^{H_i}$ *for class representatives* u_i *of* $\mathbf{G}^F$ *such that* $u_i^{\mathbf{G}^F} \cap v\mathbf{U}^F \neq \emptyset$*. Then*

$$\tilde{Q}_{\mathbf{L}}^{\mathbf{G}}(u,v) = \frac{|C_{\mathbf{G}^F}(u)|}{|\mathbf{U}^F||C_{\mathbf{L}^F}(v)|} \sum_{i \in I_u} [H_i : C_{H_i}(u_i)]$$

where I_u *is the set of indices of the elements* u_i *that are* $\mathbf{G}^F$*-conjugate to* u*.*

Proof. By Proposition 4.2 and by the fact that $\mathbf{L}^F$ normalizes $\mathbf{U}^F$ we have

$$\tilde{Q}_{\mathbf{L}}^{\mathbf{G}}(u,v) = \frac{1}{|C_{\mathbf{L}^F}(v)|}|\{g\mathbf{U}^F \mid g \in \mathbf{G}^F,\, u^g \in v\mathbf{U}^F\}| = \frac{1}{|\mathbf{U}^F||C_{\mathbf{L}^F}(v)|}|\{g \in \mathbf{G}^F \mid u^g \in v\mathbf{U}^F\}|.$$

By assumption

$$\left\{x \in \mathbf{G}^F \mid u^x \in v\mathbf{U}^F\right\} = \dot{\bigcup}_{i \in I_u} \left\{x \in \mathbf{G}^F \mid u^x \in u_i^{H_i}\right\}.$$

The size of the sets on the right hand side are given in the next lemma.

Lemma 4.17. *Let* G *be a finite group,* $H \leqslant G$ *a subgroup, and* $g, l \in G$ *such that* $g^{g_0} = l$ *for some* $g_0 \in G$*. Then*

$$\left\{x \in G \mid g^x \in l^H\right\} = g_0 C_G(l) H$$

with cardinality $|C_G(l)|\,[H : C_H(l)]$*.*

Proof. Let $x \in G$ be such that $g^x = l^h$ for some $h \in H$. Then $g^{x(g_0h)^{-1}} = g$, meaning that $x \in C_G(g)g_0H = g_0C_G(l)H$. The other inclusion is trivial.

The cardinality is now easily computed,

$$|g_0C_G(l)H| = |C_G(l)H| = \frac{|C_G(l)||H|}{|C_G(l) \cap H|} = |C_G(l)|\,[H : C_H(l)]\,.$$

□

It follows that

$$|\{x \in \mathbf{G}^F \mid u^x \in v\mathbf{U}^F\}| = \sum_{i \in I_u} |C_{\mathbf{G}^F}(u_i)|\,[H_i : C_{H_i}(u_i)]$$

and since for all $i \in I_u$ the elements u_i are conjugate to u we get the result. □

Thanks to [DLM92, (5.12) Lemma], we can slightly improve this result. That lemma states that if a unipotent element $u \in \mathbf{L}^F$ is $\mathbf{G}$-conjugate to $v \in u\mathbf{U}^F$, then u is $\mathbf{U}^F$-conjugate to v.

This gives the following Lemma.

Lemma 4.18. *If a unipotent element* $u \in \mathbf{L}^F$ *is* $\mathbf{G}$*-conjugate to* $v \in u\mathbf{U}^F$ *then*

$$u^{\mathbf{G}^F} \cap u\mathbf{U}^F = u^{\mathbf{U}^F}$$

Proof. The inclusion $u^{\mathbf{U}^F} \subseteq u^{\mathbf{G}^F} \cap u\mathbf{U}^F$ follows from the fact that $\mathbf{L}^F$ normalizes $\mathbf{U}^F$,

$$u^w = w^{-1}uw = u(w^{-1})^u\, w \in u\mathbf{U}^F$$

for all $w \in \mathbf{U}^F$.

The other inclusion comes from [DLM92, (5.12) Lemma]. $\square$

In general, there is not much to say for the classes contained in $u\mathbf{U}^F$ other than $u^{\mathbf{G}^F}$. However, for split Levi subgroups that have an abelian maximal unipotent subgroup, we can add the following results.

Lemma 4.19. *Assume that* $\mathbf{L}$ *is a split Levi subgroup with abelian maximal unipotent subgroup* $\mathbf{U}_L$*. We choose* $\mathbf{U}_L$ *such that the maximal unipotent subgroup of* $\mathbf{G}$ *is* $\mathbf{U}_0 = \mathbf{U}_L\mathbf{U}$*. Then,*

$$v^{\mathbf{U}_0^F} = v^{\mathbf{G}^F} \cap u\mathbf{U}^F$$

for $u \in \mathbf{U}_L^F$ *and* $v \in u\mathbf{U}^F$*.*

Proof. Since $u\mathbf{U}^F \subseteq \mathbf{U}_0^F$ it is clear that $v^{\mathbf{G}^F} \cap u\mathbf{U}^F \subseteq v^{\mathbf{U}_0^F}$.

For the other inclusion, we write $v = uw$ for $w \in \mathbf{U}^F$ and set $x = u_L u_U \in \mathbf{U}_0^F$ with $u_L \in \mathbf{U}_L^F$ and $u_U \in \mathbf{U}^F$. Then, because $\mathbf{U}_0^F$ normalizes $\mathbf{U}^F$ and $\mathbf{U}_L^F$ is abelian, it follows that

$$v^x = u^x w^x = u^{u_U} w^x = u(u_U^{-1})^u u_U w^x \in u\mathbf{U}^F.$$

$\square$

Then, we directly have:

Corollary 4.20. *For a split Levi subgroup* $\mathbf{L}$ *of* $\mathbf{G}$ *which has an abelian maximal unipotent subgroup, the 2-parameter Green function is given by*

$$\tilde{Q}_{\mathbf{L}}^{\mathbf{G}}(u,v) = \frac{|C_{\mathbf{G}^F}(u)|}{|\mathbf{U}^F||C_{\mathbf{L}^F}(v)|} \sum_{i\in I_u} \left[\mathbf{U}_0^F : C_{\mathbf{U}_0^F}(u_i)\right]$$

for $u_i \in v\mathbf{U}^F$ *such that* $v\mathbf{U}^F = \dot{\bigcup}_{i\in I_u} u_i^{\mathbf{U}_0^F}$ *and* u_i *are* $\mathbf{G}^F$*-conjugate to* u

It follows that for split Levi subgroups of type $A_1 \times \cdots \times A_1$ the 2-parameter Green functions can easily be computed once the fusion of the unipotent classes of $\mathbf{U}_0^F$ are known.

4.4 A note on the method of [Lue20]

The method discussed above does not assure to completely solve the problem of finding the unknowns introduced following the discussion after Remark 4.14. For example, for $\mathrm{Spin}_8^+(q)$ the system of equations has too many variables to have a unique solution. An ulterior equation is provided by the Gel'fand–Graev characters (see Section 5.2), when Conjecture 5.29 is valid. By Proposition 5.33 (c) we gain this information, in the twisted case, when "q is large enough".

Since the publication of the first version of [MaRo20] Lübeck, [Lue20] used a theoretically more involved method to also compute 2-parameter Green functions. In short, he uses the Springer correspondence to compute some generalised Green functions. Then he uses the fact that Lusztig induction of a generalised Green function is a generalised Green function (under some conditions). This provides him with the equations that are missing when considering only the ordinary Green functions, like here.

Unlike our elementary method, Lübeck's is sure to always give enough equations to solve the system. This is a trivial consequence of the fact that there are as many generalized Green functions as there are unipotent classes (the generalized Green functions are a basis of the space of class functions with unipotent support, by [Lu85, Corollary 9.11] and [Lu86b, Lemma 25.4]).

This way he finds the same result as in our Tables 52 and 53. However, as he states, with his method the "large q" requirement can be dropped for the Levi subgroups of type ${A_1}^3$ of $\mathrm{Spin}_8^+(q)$, so thanks to Lemma 4.6 all of our tables, computed in Part III, are valid for arbitrary (odd) q.

5 (Modified) Gel'fand–Graev characters

In this section, we denote by $\mathbf{G}$ a connected reductive group in characteristic p defined over $\mathbb{F}_q$ via the Frobenius map F (q is an integer power of p). We choose a maximally split torus of $\mathbf{G}$ denoted by $\mathbf{T}_0$, contained in an F-stable Borel subgroup $\mathbf{B}_0$. We denote by Φ the root system of $\mathbf{G}$ relative to $\mathbf{T}_0$ and we denote by $\mathbf{U}_0 := R_u(\mathbf{B}_0)$ the maximal unipotent subgroup of $\mathbf{G}$ with decomposition $\prod_{\alpha\in\Phi^+}\mathbf{U}_\alpha$. We denote by τ the permutation induced by F on the base Δ of Φ.

We present in this section an important family of characters of the finite group of Lie type $\mathbf{G}^F$ called the Gel'fand–Graev characters. Basically these characters are simply the induction of certain (regular) linear characters of the unipotent subgroup $\mathbf{U}_0^F$.

Although their definition doesn't require any deep theoretical construction, the Gel'fand–Graev characters have an interesting variety of properties that makes them an invaluable tool in the decomposition of the Deligne–Lusztig characters (see for example their use in [Bo11, Chapter 5.2]). Moreover, it is possible to parametrize them in a precise way revealing a tight relation to a certain type of unipotent elements (the regular ones), which are parametrized in the same way.

We describe in Section 5.1 the regular unipotent elements, their conjugacy classes and their parametrization. In Section 5.2 we define and parametrize the Gel'fand–Graev characters and list their properties. Finally, in Section 5.3 we slightly modify the definition of the latter in a way more useful for the computations of Parts II and III and deduce some helpful results.

The main references for this section are [DLM92] (and its follow-up [DLM97]), which gives a detailed description of the Gel'fand–Graev characters and their relation with the character table of finite groups of Lie type, and [DiMi20, Chapter 12], which gives a somehow more concise description of the same material but gives a good description of regular unipotent elements.

5.1 Regular unipotent elements

Definition 5.1. An element x of an algebraic group $\mathbf{G}$ is said to be *regular* if the dimension of its centraliser is minimal.

In the present case ($\mathbf{G}$ is reductive), this minimal dimension is $\mathrm{rk}(\mathbf{G})$ (see for example [MaTe11, Proposition 14.9]).

We are mainly interested in the case of regular unipotent elements. But, for future reference, we cite the following result on regular semisimple elements.

Proposition 5.2 ([MaTe11, Corollary 14.10])**.** *Let $\mathbf{G}$ be connected reductive with maximal torus $\mathbf{T}$ and root system Φ. For $s\in\mathbf{T}$ the following are equivalent:*

- *s is regular;*
- *$\alpha(s)\neq 1$ for all $\alpha\in\Phi$;*
- *$C_{\mathbf{G}}(s)^\circ = \mathbf{T}$.*

Now, we discuss the regular unipotent classes. Here is a list of their properties.

Proposition 5.3. *There exist regular unipotent elements (in connected reductive groups) and they form a single conjugacy class of $\mathbf{G}$,* [DiMi20, Corollary 12.2.4]. *Let $u\in\mathbf{U}_0$ be a regular unipotent element of $\mathbf{G}$. Then:*

- *The element decomposes as $u=\prod_{\alpha\in\Phi^+}u_\alpha(x_\alpha)$ with $x_\alpha\neq 0$ when α is a simple root.* [DiMi20, Proposition 12.2.2 (iv)]

- $C_{\mathbf{G}}(u) = Z(\mathbf{G})C_{\mathbf{U}_0}(u)$. [DiMi20, Lemma 12.2.3]
- *If the characteristic is good* [5] *for* $\mathbf{G}$ *then* $C_{\mathbf{U}_0}(u)$ *is connected.* [DiMi20, Proposition 12.2.7]

For example in the general/special linear groups (GL_n, SL_n) the matrices with ones on the diagonal and on the second diagonal (and zeros below the diagonal) are regular unipotent elements.

It follows directly from Proposition 1.53 and Proposition 5.3 that (in good characteristic) when $\mathbf{G}$ has connected centre there is a single regular unipotent class of $\mathbf{G}^F$, in general there are $|H^1(F, Z(\mathbf{G})/Z(\mathbf{G})^\circ)|$ many:

Proposition 5.4 ([DiMi20, Proposition 12.2.15])**.** *If the characteristic is good for* $\mathbf{G}$*, then the* $\mathbf{G}^F$*-conjugacy classes of regular unipotent elements are parametrised by the* F*-conjugacy classes* $H^1(F, Z(\mathbf{G})/Z(\mathbf{G})^\circ)$.

Remark 5.5. Notice that due to [DiMi20, Lemma 4.2.13 (ii)] there is a bijection between $H^1(F, Z(\mathbf{G})/Z(\mathbf{G})^\circ)$ and $H^1(F, Z(\mathbf{G}))$. From now on, we use the second set for readability reasons.

Notation 5.6. We denote by $\mathrm{Reg}_{\mathrm{uni}}(\mathbf{G}^F)$ the set of regular unipotent classes of $\mathbf{G}^F$. Also, we denote by U_z the set of regular unipotent elements of $\mathbf{G}^F_{\mathrm{uni}}$ parametrized by $z \in H^1(F, Z(\mathbf{G}))$.

The discussion that follows depends on the choice of this parametrization. We fix as representative of U_1 the element $u_1 := \prod_{\alpha \in \Delta} u_\alpha(1)$.

From now on we consider only the case of $\mathbf{G}$ in good characteristic. Therefore, by [DLM92, (3.2) Proposition (iii)] the sets U_z are precisely the regular unipotent classes of $\mathbf{G}^F$.

The next lemma is useful for actually computing the F-classes of the centre.

Lemma 5.7. *We have (recall that* $\mathscr{L}$ *is the Lang map)*

$$H^1(F, Z(\mathbf{G})) = Z(\mathbf{G})/\mathscr{L}(Z(\mathbf{G})).$$

Proof. By definition, two elements $z_1, z_2 \in Z(\mathbf{G})$ are in the same F-class of $Z(\mathbf{G})$ if there is a central element $x \in Z(\mathbf{G})$ such that

$$z_2 = F(x)z_1x^{-1} \Leftrightarrow z_2 = x^{-1}F(x)z_1 = \mathscr{L}(x)z_1.$$

The claim follows directly. □

The same constructions can be done in any F-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$. It is natural to search for a relation between the analogous objects of $\mathbf{G}$ and $\mathbf{L}$. It turns out that there is a canonical surjection between the F-classes of $Z(\mathbf{G})$ and $Z(\mathbf{L})$:

Lemma 5.8 ([DiMi20, Lemma 12.3.5])**.** *The inclusion* $Z(\mathbf{G}) \subseteq Z(\mathbf{L})$ *induces a surjective map*

$$\mathfrak{h}_{\mathbf{L}} : H^1(F, Z(\mathbf{G})) \to H^1(F, Z(\mathbf{L})).$$

---

[5] See [DiMi20, 12.2.6] for the definition of good characteristic. In this work the characteristic is always good.

5.2 Gel'fand–Graev characters

We start by defining the linear characters of $\mathbf{U}_0^F$ that we want to induce.

Lemma 5.9 ([DLM92, (2.2) Lemma]). *We have*

$$\mathbf{U}_0^F/[\mathbf{U}_0^F,\mathbf{U}_0^F] \cong \prod_{\omega\in\Delta/\tau} \mathbf{U}_\omega^F$$

where Δ/τ *is the set of orbits of* τ *on* Δ *and* $\mathbf{U}_\omega = \prod_{\alpha\in\omega}\mathbf{U}_\alpha$. *Moreover* $\mathbf{U}_\omega^F \cong \mathbb{F}_{q^{|\omega|}}^+$.

Definition 5.10. A linear character ϕ of $\mathbf{U}_0^F$ is called *regular* if it is non-trivial on each group $\mathbf{U}_\omega^F$ of Lemma 5.9.

Like the regular unipotent classes, the regular characters of $\mathbf{U}_0^F$ are parametrized by the F-classes of the centre.

Proposition 5.11 ([DiMi20, Proposition 12.3.2]). *The* $\mathbf{T}_0^F$*-orbits of regular characters of* $\mathbf{U}_0^F$ *are in one-to-one correspondence with* $H^1(F, Z(\mathbf{G}))$.

It is possible to build this correspondence by explicitly choosing a regular character of $\mathbf{U}_0^F$ (see discussion after [DiMi20, 12.3.2] and/or after [DLM92, (2.4) Theorem]).

Notation 5.12. For any $N\in\mathbb{N}$ we denote by $\chi_N:\mathbb{F}_{q^N}\to\mathbb{C}$ the linear character of $\mathbb{F}_{q^N}$ defined by

$$\chi_N(x) := \mathrm{e}^{\frac{2\pi i \mathrm{Tr}(x)}{p}}$$

for any $x\in\mathbb{F}_{q^N}$, where $\mathrm{Tr}:\mathbb{F}_{q^N}\to\mathbb{F}_p$ is the usual field trace.

In the particular case $N=1$ we denote $\phi := \chi_1$, and for any element $j\in\mathbb{F}_q$ we set $\phi_j(x) := \phi(jx)$. Clearly, ϕ_j runs through all the characters of $\mathbb{F}_q$ when j runs through the elements of $\mathbb{F}_q$. Analogously, we denote $\phi_{j_1}\times\phi_{j_2}\times\cdots\times\phi_{j_k}$ by $\phi_{j_1,j_2,\ldots,j_k}$.

We choose as regular character of $\mathbf{U}_0^F$ parametrized by $1\in H^1(F,Z(\mathbf{G}))$ the following product of characters (thanks to Lemma 5.9)

$$\psi_1 := \prod_{\omega\in\Delta/\tau}\chi_{|\omega|} : \prod_{\omega\in\Delta/\tau}\mathbb{F}_{q^{|\omega|}}^+ \cong \prod_{\omega\in\Delta/\tau}\mathbf{U}_\omega^F \to \mathbb{C}.$$

Then, by [DLM92, (2.4.10) and the following discussion] the group $\mathscr{L}_{\mathbf{T}_0}^{-1}(Z)$ acts transitively, by conjugation, on the set of regular characters of $\mathbf{U}_0^F$ and, more precisely, $\mathscr{L}_{\mathbf{T}_0}^{-1}(Z)/Z\mathbf{T}_0^F$ acts regularly on the same set. Therefore, for any $z\in H^1(F,Z(\mathbf{G}))$ we can choose a representative $t_z\in\mathscr{L}_{\mathbf{T}_0}^{-1}(z)$ and define the regular character

$$\psi_z := {}^{t_z}\psi_1$$

which is a representative of the $\mathbf{T}_0^F$-orbit Ψ_z of regular characters of $\mathbf{U}_0^F$ parametrized by z.

Definition 5.13. For $z\in H^1(F,Z(\mathbf{G}))$ we define the *Gel'fand-Graev character* of $\mathbf{G}^F$ by

$$\Gamma_z^{\mathbf{G}} := \mathrm{Ind}_{\mathbf{U}_0^F}^{\mathbf{G}^F}(\psi_z).$$

It is clear that the $\Gamma_z^{\mathbf{G}}$ are well defined for $z\in H^1(F,Z(\mathbf{G}))$ since they are defined up to $\mathbf{T}_0^F$-conjugacy of ψ_z.

Definition 5.14. We define $\sigma_z := \sum_{\psi\in\Psi_z}\psi(u_1)$, where Ψ_z is the $\mathbf{T}_0^F$-orbit of ψ_z and u_1 the chosen representative of the regular unipotent elements parametrized by $1\in H^1(F,Z(\mathbf{G}))$.

Remark 5.15. The main reason for introducing the Gel'fand–Graev characters is that, unlike the Deligne–Lusztig characters, they have different values on (almost all) unipotent splitting classes since regular characters of $\mathbf{U}_0^F$ already distinguish them. This means that they must have at least one irreducible constituent that also distinguishes between those splitting classes.

By definition, the Gel'fand–Graev characters always distinguish the regular unipotent classes of $\mathbf{G}^F$. The hope is, therefore, to use them to distinguish the values of the irreducible constituents of the Deligne–Lusztig characters on unipotent classes.

First of all, we have the following important result on Gel'fand–Graev characters.

Proposition 5.16 ([DLM92, (3.5) Theorem]). *Let* $\chi \in \operatorname{Irr}(\mathbf{G}^F)$ *and let* $z \in H^1(F, Z(\mathbf{G}))$. *Then (recall that* $\eta_{\mathbf{G}}$ *was given in Definition 1.56),*

$$\frac{1}{|U_z|}\sum_{u\in U_z}\chi(u) = \eta_{\mathbf{G}} \sum_{z'\in H^1(F,Z(\mathbf{G}))} \sigma_{zz'^{-1}}\langle D_{\mathbf{G}}\chi, \Gamma_{z'}\rangle.$$

Notice that in good characteristic U_z is a conjugacy class. Then, the left-hand side of the equation reduces to $\chi(u)$ for u a regular unipotent element parametrized by $z \in H^1(F, Z(\mathbf{G}))$.

We list now further properties that justify the use of these characters for the decomposition of Deligne–Lusztig characters.

Theorem 5.17 ([DiMi20, 12.3.4]). *The Gel'fand–Graev characters are multiplicity free.*

Proposition 5.18 ([DLM92, (3.6) Scholium (i)]). *The class functions* $\{\Gamma_z^{\mathbf{G}} \mid z \in H^1(F, Z(\mathbf{G}))\}$ *are linearly independent and distinct.*

Definition 5.19. We say that $\chi \in \operatorname{Irr}(\mathbf{G}^F)$ is *regular* if $\langle\chi, \Gamma_z^{\mathbf{G}}\rangle \neq 0$ for some $z \in H^1(F, Z(\mathbf{G}))$.

Definition 5.20. For any semisimple class (s) of $\mathbf{G}^{*F^*}$, let $\mathbf{T}^*$ be a maximally split torus of $C_{\mathbf{G}^*}(s)^\circ$ and define the following class function of $\mathbf{G}^F$

$$\chi_{(s)} = |W(s)^\circ|^{-1} \sum_{w\in W(s)^\circ} \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}_w^*} R_{\mathbf{T}_w^*}^{\mathbf{G}}(s)$$

where $\mathbf{T}_w^*$ is the F^*-stable maximal torus of $C_{\mathbf{G}^*}(s)^\circ$ obtained by twisting $\mathbf{T}^*$ with $w \in W(s)^\circ$ (this was introduced in Theorem 1.20).

Notice that by definition these class functions $\chi_{(s)}$ are orthogonal to each other for different conjugacy classes (s). Moreover we have the following crucial results.

Proposition 5.21 ([DLM92, (3.10) Proposition]). *The class functions* $\chi_{(s)}$ *have the following properties:*

(a) $\chi_{(s)}$ *is a proper character of* $\mathbf{G}^F$.

(b) *For* $z \in H^1(F, Z(\mathbf{G}))$, *we have* $\langle\chi_{(s)}, \Gamma_z^{\mathbf{G}}\rangle = 1$.

Notation 5.22. Let (s) be a semisimple class of $\mathbf{G}^{*F^*}$ and $z \in H^1(F, Z(\mathbf{G}))$, then we denote by $\chi_{(s),z}$ the unique common irreducible constituent of $\chi_{(s)}$ and $\Gamma_z^{\mathbf{G}}$.

Proposition 5.23 ([DLM92, (3.12) Proposition (i)]). *The set*

$$\{\chi_{(s),z} \mid (s) \text{ semisimple class of } \mathbf{G}^{*F^*}, z \in H^1(F, Z(\mathbf{G}))\}$$

contains all the regular characters of $\mathbf{G}^F$.

Thus, it is immediate from Proposition 5.21 and Proposition 5.23 that:

Corollary 5.24 ([DLM92, (3.14) Corollary])**.** *For each* $z \in H^1(F, Z(\mathbf{G}))$*, we have*

$$\Gamma_z^{\mathbf{G}} = \sum_{(s)} \chi_{(s),z}$$

where the sum is over the semisimple classes of $\mathbf{G}^{*F^*}$*.*

The explicit computation of the Gel'fand–Graev characters is tightly related with the computation of some (partial) Gauss sums. We give some theoretical information in this regard that will be used later. The explicit computations will be carried out in Lemma 5.58.

Lemma 5.25 ([DiMi20, Lemma 12.3.11])**.** *We have*

$$\sigma_z = \frac{\Gamma_z^{\mathbf{G}}(u_1)}{|Z(\mathbf{G})^F|}.$$

The next result can be used to verify the computations.

Lemma 5.26 ([DLM92, (3.7) Scholium])**.** *Let* σ *be the matrix whose* (z, z') *entry is* $\sigma_{zz'^{-1}}$ *for* $z, z' \in H^1(F, Z(\mathbf{G}))$*. Then,*

$$\langle \Gamma_z^{\mathbf{G}}, \Gamma_{z'}^{\mathbf{G}} \rangle = \frac{|\mathbf{G}^F|}{|\mathrm{U}_1|}((\sigma^{\mathrm{T}}\sigma)^{-1})_{z,z'}$$

where $^{\mathrm{T}}$ *denotes the matrix transpose.*

In good characteristic U_1 is a conjugacy class, then the formula becomes

$$\langle \Gamma_z^{\mathbf{G}}, \Gamma_{z'}^{\mathbf{G}} \rangle = |C_{\mathbf{G}^F}(u_1)|((\sigma^{\mathrm{T}}\sigma)^{-1})_{z,z'}.$$

Remark 5.27. Although the decomposition of Gel'fand–Graev characters is even more challenging than the decomposition of Deligne–Lusztig characters (without having the character table a priori), the precise knowledge of how their decomposition looks like gives the possibility of writing a system of equations for the values of regular characters at unipotent elements.

The question is if this gives enough equations to solve the system. It is hardly ever the case when $Z(\mathbf{G})$ is disconnected, since there are usually more (splitting) unipotent classes than there are Gel'fand–Graev characters. An example where the system is solvable is $\mathrm{SL}_2(q)$ when q is odd. There is one unipotent class of SL_2 (the regular one) that splits into two classes of $\mathrm{SL}_2(q)$ and two Gel'fand–Graev characters, for q odd.

More equations can be found by inspecting the relation between the regular characters of two groups related by a regular embedding.

Remark 5.28. It is possible to relate the regular/Gel'fand–Graev characters of $\mathbf{G}^F$ and $\tilde{\mathbf{G}}^F$:

(a) We fix a regular embedding $i : \mathbf{G} \to \tilde{\mathbf{G}}$ (and identify $\mathbf{G}$ with $i(\mathbf{G}) \subset \tilde{\mathbf{G}}$) and we denote by $(\tilde{\mathbf{G}}^*, \tilde{F}^*)$ the dual of $(\tilde{\mathbf{G}}, \tilde{F})$. Also, we denote by $i^* : \tilde{\mathbf{G}}^* \to \mathbf{G}^*$ the corresponding central isotypy (see [GeMa20, Section 1.7.11]) such that $i^* \circ \tilde{F}^* = F^* \circ i^*$.

Let (s) be a semisimple class of $\mathbf{G}^{*F^*}$ and we choose a class $(\tilde{s})$ of $\tilde{\mathbf{G}}^{*\tilde{F}^*}$ which lies over (s), i.e. $i^*(\tilde{s}) = s$. Then, by Proposition 3.13

$$\mathrm{Res}_{\mathbf{G}^F}^{\tilde{\mathbf{G}}^{\tilde{F}}} \chi_{(\tilde{s})} = \chi_{(s)}.$$

(b) The regular characters of $\mathbf{U}_0^F$ are all $\tilde{\mathbf{G}}^{\tilde{F}}$-conjugate. Then by transitivity of the ordinary induction we have for any $z \in H^1(F, Z(\mathbf{G}))$

$$\mathrm{Ind}_{\mathbf{G}^F}^{\tilde{\mathbf{G}}^{\tilde{F}}} \Gamma_z^{\mathbf{G}} = \Gamma^{\tilde{\mathbf{G}}}.$$

It follows from (a) and by Clifford theory that the regular characters of $\mathbf{G}^F$ in the same Lusztig series are $\tilde{\mathbf{G}}^F$-conjugates. This greatly reduces the number of unknown values of regular characters in the system discussed in Remark 5.27. Moreover, it follows that all the constituents of the restriction of regular characters of $\tilde{\mathbf{G}}^{\tilde{F}}$ are regular characters of $\mathbf{G}^F$.

For non-unipotent elements we use two "extensions" of the same principle. On one side we consider a modified version of the Gel'fand–Graev characters (discussed in Section 5.3) which are non-zero on elements of the form zu with u unipotent and z central. On the other side we use Lusztig induction/restriction from/to a Levi subgroup and the "modified" Gel'fand–Graev characters of those Levis to obtain similar systems of equations for elements of the form zu where this time z is central in the Levi.

This way we manage to write uniquely solvable systems of equations for almost all splitting classes of $\mathbf{G}^F$ (for $\mathrm{SL}_4(q)$ this method actually covers all the problematic cases, see Part II).

The main result that we need is about the relation between the regular characters of the group $\mathbf{G}^F$ and of a Levi subgroup $\mathbf{L}^F$ via Harish–Chandra/Lusztig restriction (Theorem 5.35 below). This follows by an analogous relation between the Gel'fand–Graev characters of the group $\mathbf{G}^F$ and of a Levi subgroup $\mathbf{L}^F$ via Harish–Chandra/Lusztig restriction. At the moment of the writing this is only a conjecture.

Conjecture 5.29 ([DLM92, (5.2)' Conjecture])**.** *Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$. For $z \in H^1(F, Z(\mathbf{G}))$, we have*

$$^*R_{\mathbf{L}}^{\mathbf{G}} \Gamma_z^{\mathbf{G}} = \varepsilon_{\mathbf{G}} \varepsilon_{\mathbf{L}} \Gamma_{z'}^{\mathbf{L}}$$

for some $z' \in H^1(F, Z(\mathbf{L}))$.

There is a tight relation between Gel'fand–Graev characters and characteristic functions of regular unipotent classes.

Notation 5.30. We denote by γ_g^G the class function of a finite group G that takes value $|C_G(g)|$ on the conjugacy class of $g \in G$ and zero on the other classes. More precisely, in good characteristic, for $z \in H^1(F, Z(\mathbf{G}))$ we set $\gamma_z^{\mathbf{G}^F} := \gamma_{u_z}^{\mathbf{G}^F}$ for a regular unipotent element $u_z \in \mathbf{G}^F$ parametrized by z.

Lemma 5.31 ([DLM92, (3.5)' Scholium])**.** *If the characteristic is good for $\mathbf{G}$, then for any $z \in H^1(F, Z(\mathbf{G}))$ we have*

$$D_{\mathbf{G}} \gamma_z^{\mathbf{G}^F} = \eta_{\mathbf{G}} \sum_{z' \in H^1(F, Z(\mathbf{G}))} \sigma_{zz'^{-1}} \Gamma_{z'}^{\mathbf{G}}.$$

The conjecture above is equivalent to the following one (the equivalence is proven later in Proposition 5.36).

Conjecture 5.32 ([DLM92, (5.2) Conjecture])**.** *Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$. For $z \in H^1(F, Z(\mathbf{G}))$, we have*

$$^*R_{\mathbf{L}}^{\mathbf{G}} \gamma_z^{\mathbf{G}^F} = \gamma_{z'}^{\mathbf{L}^F}$$

for some $z' \in H^1(F, Z(\mathbf{L}))$.

We list now the cases for which these conjectures have been proved.

Proposition 5.33. *Conjectures 5.29 and 5.32 are valid when either of the following conditions is satisfied.*

(a) *The centre of* $\mathbf{G}$ *is connected.*

(b) *The Levi* $\mathbf{L}$ *is split (contained in an* F*-stable parabolic subgroup).*

(c) *The characteristic* p *is good for* $\mathbf{G}$ *and* q *is "large enough"*[6].

Proof. (a) See [DiMi83, Théorème 4.4] and [DLM92, (5.4) Proposition].

(b) See [DLM92, (2.9) Theorem] and [DLM92, (5.3) Theorem].

(c) See [DLM97, 3.7 Theorem] and [Bo05, Theorem 15.2] for the determination of z', appearing in Conjecture 5.29. □

Remark 5.34. The precise determination of $z' \in H^1(F, Z(\mathbf{L}))$, given $z \in H^1(F, Z(\mathbf{G}))$, in the conjectures above is not an easy task. It depends on the arbitrary choice of the parametrization of the regular unipotent classes in both the group $\mathbf{G}^F$ and the Levi subgroup $\mathbf{L}^F$ considered. Clearly, the problem does not arise in the connected centre case (where there is only one regular unipotent class in $\mathbf{G}^F$ and $\mathbf{L}^F$). In general, the solution has been worked out by Bonnafé in [Bo05]. In summary, he introduces and explicitly constructs two main tools:

- a restriction map for regular unipotent classes ([Bo00, Section 15.A])
$$\mathrm{res}_{\mathbf{L}}^{\mathbf{G}} : \mathrm{Reg}_{\mathrm{uni}}(\mathbf{G}^F) \to \mathrm{Reg}_{\mathrm{uni}}(\mathbf{L}^F).$$
- a certain morphism ([Bo05, Section 12.C and Table 1]) which, in practice, associates an element of $Z(\mathbf{L})/Z(\mathbf{L})^\circ$ (and thus of $H^1(F, Z(\mathbf{L}))$) to each F-stable Levi subgroup $\mathbf{L}$, we will denote it by $z_{\mathbf{L}}$.

Recall that for any $z \in H^1(F, Z(\mathbf{G}))$ we choose a representative $t_z \in \mathscr{L}_{\mathbf{T}_0}^{-1}(z)$. Then $\mathrm{U}_z = {}^{t_z}\mathrm{U}_1$ and (because conjugation with t_z and $\mathrm{res}_{\mathbf{L}}^{\mathbf{G}}$ commute)
$$\mathrm{res}_{\mathbf{L}}^{\mathbf{G}}\mathrm{U}_z = (\mathrm{res}_{\mathbf{L}}^{\mathbf{G}}\mathrm{U}_1)_{\mathfrak{h}_{\mathbf{L}}(z)}.$$
Moreover, $\mathrm{res}_{\mathbf{L}}^{\mathbf{G}}$ is transitive:
$$\mathrm{res}_{\mathbf{L}}^{\mathbf{G}} = \mathrm{res}_{\mathbf{L}}^{\mathbf{L}'} \circ \mathrm{res}_{\mathbf{L}'}^{\mathbf{G}}$$
for an F-stable Levi subgroup $\mathbf{L}'$ of $\mathbf{G}$ containing $\mathbf{L}$, see [Bo04, Proposition 7.2 (c)].

Finally, if Conjecture 5.32 holds, by [Bo05, Theorem 15.2] (when the characteristic is good for $\mathbf{G}$) then
$$^*R_{\mathbf{L}}^{\mathbf{G}}\gamma_z^{\mathbf{G}^F} = \gamma_{\mathfrak{h}_{\mathbf{L}}(z)z_{\mathbf{L}}}^{\mathbf{L}^F}$$
when choosing $\mathrm{res}_{\mathbf{L}}^{\mathbf{G}}\mathrm{U}_1 \in \mathrm{Reg}_{\mathrm{uni}}(\mathbf{L}^F)$ as being the regular unipotent class parametrized by $1 \in H^1(F, Z(\mathbf{L}))$.

The elements $z_{\mathbf{L}}$ are the identity for split Levi subgroups while otherwise they were determined by Bonnafé, and are given in [Bo05, Table 1].

For split Levi subgroups it is easy to check explicitly that $\mathrm{res}_{\mathbf{L}}^{\mathbf{G}}(u_1)^{\mathbf{G}^F} = (u_1)^{\mathbf{L}^F}$ where u_1 is as in Notation 5.6. In the case of twisted Levi subgroups it is not clear how to explicitly compute the restriction of regular unipotent classes. In Parts II and III, for an F-stable Levi subgroup $\mathbf{L}$ of type (I, w) (see discussion before Definition 1.59), we find representatives of the regular unipotent classes of $\mathbf{L}^F$ in $\mathbf{L}_I^{Fw^{-1}}$. In this case, we need to explicitly find an element $g \in \mathbf{G}$ such that $g^{-1}F(g) = \dot{w}$ in order to compute $\mathrm{res}_{\mathbf{L}}^{\mathbf{G}}$. The generic determination of such g is (at the moment of the writing) still an open problem. This means that it will not be possible to fix unequivocally representatives of some splitting classes (that intersect twisted Levi subgroups) in the generic character tables computed in Parts II and III. We are able to explicitly find such g in all but one case of $\mathrm{Spin}_8^+(q)$.

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[6] The proof uses results of Lusztig valid only for q bigger than a constant which depends on the type of $\mathbf{G}$.

Unfortunately, we will need to use the conjectures above also for non-split Levi subgroups and for any q. It will be possible to use them anyway thanks to the explicit determination of the 2-parameter Green functions (see Section 4). It is easy to verify that Conjecture 5.32 is true in the cases we are interested in, allowing us to use Conjecture 5.29 and its consequences.

When these conjectures are verified we are allowed to use the next very useful result.

Theorem 5.35 ([DLM92, (6.2) Theorem]). *Suppose* $\mathbf{L}$ *is an* F*-stable Levi subgroup of* $\mathbf{G}$ *for which Conjecture 5.29 holds. Then we have*

$$ {}^*R_{\mathbf{L}}^{\mathbf{G}} \chi_{(s),z}^{\mathbf{G}} = \varepsilon_{\mathbf{G}} \varepsilon_{\mathbf{L}} \sum_{(t)} \chi_{(t),z'}^{\mathbf{L}} $$

where the sum is over the semisimple classes $(t) \in \mathbf{L}^{*F^*}$ *such that* $t \in (s)_{\mathbf{G}^{*F^*}}$ *and* z' *is as in Conjecture 5.29.*

The equivalence of the conjectures is mentioned without proof in [DLM92]. A proof was provided by Digne and Michel after requesting details on it. We present it here.

Proposition 5.36 (Digne–Michel). *If the characteristic is good for* $\mathbf{G}$ *(and the Mackey formula holds) then Conjectures 5.29 and 5.32 are equivalent.*

Proof. For readability reason, for this proof, we fix $H_{\mathbf{G}} = H^1(F, Z(\mathbf{G}))$, $H_{\mathbf{L}} = H^1(F, Z(\mathbf{L}))$, $\hat{H}_{\mathbf{G}} = \operatorname{Irr}(H_{\mathbf{G}})$ and $\hat{H}_{\mathbf{L}} = \operatorname{Irr}(H_{\mathbf{L}})$.

We start by introducing the Mellin transforms

$$ \gamma_\zeta^{\mathbf{G}^F} = \sum_{z \in H_{\mathbf{G}}} \zeta(z) \gamma_z^{\mathbf{G}^F}, \ \Gamma_\zeta^{\mathbf{G}} = \sum_{z \in H_{\mathbf{G}}} \zeta(z) \Gamma_z^{\mathbf{G}} \text{ and } \sigma_\zeta^{\mathbf{G}} = \sum_{z \in H_{\mathbf{G}}} \zeta(z) \sigma_z^{\mathbf{G}} $$

for $\zeta \in \hat{H}_{\mathbf{G}}$ (and also the analogous ones in $\mathbf{L}$). Then we apply Lemma 5.31 to these transforms

$$ D_{\mathbf{G}} \gamma_\zeta^{\mathbf{G}^F} = \sum_{z \in H_{\mathbf{G}}} \zeta(z) D_{\mathbf{G}} \gamma_z^{\mathbf{G}^F} = \eta_{\mathbf{G}} \sum_{z, z' \in H_{\mathbf{G}}} \zeta(z) \sigma_{zz'^{-1}} \Gamma_{z'}^{\mathbf{G}} = \eta_{\mathbf{G}} \sum_{z' \in H_{\mathbf{G}}} \zeta(z') \sum_{z \in H_{\mathbf{G}}} \zeta(zz'^{-1}) \sigma_{zz'^{-1}} \Gamma_{z'}^{\mathbf{G}} $$

and, finally, by evaluating the sums we get

$$ D_{\mathbf{G}} \gamma_\zeta^{\mathbf{G}^F} = \eta_{\mathbf{G}} \sigma_\zeta^{\mathbf{G}} \Gamma_\zeta^{\mathbf{G}}. \tag{$*$} $$

Analogously, we get the same formula in $\mathbf{L}$.

We apply now ${}^*R_{\mathbf{L}}^{\mathbf{G}}$ to the equation above, finding ${}^*R_{\mathbf{L}}^{\mathbf{G}} D_{\mathbf{G}} \gamma_\zeta^{\mathbf{G}^F} = \eta_{\mathbf{G}} \sigma_\zeta^{\mathbf{G}} \sum_{z \in H_{\mathbf{G}}} \zeta(z) {}^*R_{\mathbf{L}}^{\mathbf{G}} \Gamma_z^{\mathbf{G}}$. By Conjecture 5.29 and Remark 5.34, the right-hand side is $\eta_{\mathbf{G}} \varepsilon_{\mathbf{G}} \varepsilon_{\mathbf{L}} \sigma_\zeta^{\mathbf{G}} \sum_{z \in H_{\mathbf{G}}} \zeta(z) \Gamma_{\mathfrak{h}_{\mathbf{L}}(z) z_{\mathbf{L}}}^{\mathbf{L}}$. There are two possibilities. Either ζ factorizes through $\mathfrak{h}_{\mathbf{L}}$, i.e. there is $\zeta_{\mathbf{L}} \in \hat{H}_{\mathbf{L}}$ such that $\zeta = \zeta_{\mathbf{L}} \circ \mathfrak{h}_{\mathbf{L}}$, or it does not, in which case it can be checked that the sum is 0

$$ {}^*R_{\mathbf{L}}^{\mathbf{G}} D_{\mathbf{G}} \gamma_\zeta^{\mathbf{G}^F} = \begin{cases} \eta_{\mathbf{G}} \varepsilon_{\mathbf{G}} \varepsilon_{\mathbf{L}} \sigma_\zeta^{\mathbf{G}} |\ker(\mathfrak{h}_{\mathbf{L}})| \sum_{z \in H_{\mathbf{L}}} \zeta_{\mathbf{L}}(z_{\mathbf{L}})^{-1} \zeta_{\mathbf{L}}(z) \Gamma_z^{\mathbf{L}} & \zeta = \zeta_{\mathbf{L}} \circ \mathfrak{h}_{\mathbf{L}}, \\ 0 & \text{else.} \end{cases} $$

Thus, we assume that $\zeta = \zeta_{\mathbf{L}} \circ \mathfrak{h}_{\mathbf{L}}$, for $\zeta_{\mathbf{L}} \in \hat{H}_{\mathbf{L}}$. Therefore, we get

$$ {}^*R_{\mathbf{L}}^{\mathbf{G}} D_{\mathbf{G}} \gamma_\zeta^{\mathbf{G}^F} = \eta_{\mathbf{G}} \varepsilon_{\mathbf{G}} \varepsilon_{\mathbf{L}} \sigma_\zeta^{\mathbf{G}} \zeta_{\mathbf{L}}(z_{\mathbf{L}})^{-1} |\ker(\mathfrak{h}_{\mathbf{L}})| \Gamma_{\zeta_{\mathbf{L}}}^{\mathbf{L}}. $$

In this case, by applying twice [DLM97, Proposition 2.5] we get the relation $\eta_{\mathbf{G}} \sigma_\zeta^{\mathbf{G}} = \eta_{\mathbf{L}} \sigma_{\zeta_{\mathbf{L}}}^{\mathbf{L}}$. Hence, ${}^*R_{\mathbf{L}}^{\mathbf{G}} D_{\mathbf{G}} \gamma_\zeta^{\mathbf{G}^F} = \eta_{\mathbf{L}} \varepsilon_{\mathbf{G}} \varepsilon_{\mathbf{L}} \sigma_{\zeta_{\mathbf{L}}}^{\mathbf{L}} \zeta_{\mathbf{L}}(z_{\mathbf{L}})^{-1} |\ker(\mathfrak{h}_{\mathbf{L}})| \Gamma_{\zeta_{\mathbf{L}}}^{\mathbf{L}}$. Then, by commuting $D_{\mathbf{G}}$ and ${}^*R_{\mathbf{L}}^{\mathbf{G}}$ with Proposition 3.17, we get

$$ {}^*R_{\mathbf{L}}^{\mathbf{G}} \gamma_\zeta^{\mathbf{G}^F} = \eta_{\mathbf{L}} \sigma_{\zeta_{\mathbf{L}}}^{\mathbf{L}} \zeta_{\mathbf{L}}(z_{\mathbf{L}})^{-1} |\ker(\mathfrak{h}_{\mathbf{L}})| D_{\mathbf{L}} \Gamma_{\zeta_{\mathbf{L}}}^{\mathbf{L}}. $$

Now we use equation $(*)$ in $\mathbf{L}$ to get

$$^{*}R_{\mathbf{L}}^{\mathbf{G}}\gamma_{\zeta}^{\mathbf{G}^F} = \zeta_{\mathbf{L}}(z_{\mathbf{L}})^{-1}|\ker(\mathfrak{h}_{\mathbf{L}})|\gamma_{\zeta_{\mathbf{L}}}^{\mathbf{L}^F}$$

for ζ that factorizes through $\mathfrak{h}_{\mathbf{L}}$.

The result follows by taking the inverse Mellin transform $\gamma_z^{\mathbf{G}^F} = \frac{1}{|H_{\mathbf{G}}|}\sum_{\zeta\in\hat{H}_{\mathbf{G}}}\zeta(z^{-1})\gamma_{\zeta}^{\mathbf{G}^F}$,

$$^{*}R_{\mathbf{L}}^{\mathbf{G}}\gamma_z^{\mathbf{G}^F} = \frac{1}{|H_{\mathbf{G}}|}\sum_{\zeta\in\hat{H}_{\mathbf{G}}}\zeta(z^{-1})^{*}R_{\mathbf{L}}^{\mathbf{G}}\gamma_{\zeta}^{\mathbf{G}^F} = \frac{|\ker(\mathfrak{h}_{\mathbf{L}})|}{|H_{\mathbf{G}}|}\sum_{\zeta_{\mathbf{L}}\in\hat{H}_{\mathbf{L}}}\zeta_{\mathbf{L}}(\mathfrak{h}(z)^{-1}z_{\mathbf{L}}^{-1})\gamma_{\zeta_{\mathbf{L}}}^{\mathbf{L}^F} = \gamma_{\mathfrak{h}_{\mathbf{L}}(z)z_{\mathbf{L}}}^{\mathbf{L}^F}$$

since $|H_{\mathbf{G}}| = |H_{\mathbf{L}}||\ker(\mathfrak{h}_{\mathbf{L}})|$. □

Before ending this section we introduce another family of irreducible characters. They are the so-called semisimple characters, and they are as important (to our computations) as the regular characters.

Theorem 5.37 ([DLM92, (3.15) Theorem (i) and (ii)]). *We have the following:*

(a) *Let $\chi \in \mathrm{Irr}(\mathbf{G}^F)$. Then*

$$\frac{1}{|U_z|}\sum_{u\in U_z}\chi(u) = 0$$

for all $z \in H^1(F, Z(\mathbf{G}))$ unless $\chi = \pm D_{\mathbf{G}}\chi_{(s),z'}$ for some semisimple class $(s) \subset \mathbf{G}^{*F^*}$ and some $z' \in H^1(F, Z(\mathbf{G}))$.

(b) *For each pair $((s), z)$ as in* (a), *$\varrho_{(s),z} = \varepsilon_{\mathbf{G}}\varepsilon_{C_{\mathbf{G}^*}(s)}D_{\mathbf{G}}\chi_{(s),z}$ is an irreducible character of $\mathbf{G}^F$. Moreover $\varrho_{(s),z} = \varrho_{(s'),z'}$ if and only if $\chi_{(s),z} = \chi_{(s'),z'}$.*

Notice that the notation $\varrho_{(s),z}$ is justified by the fact that $D_{\mathbf{G}}$ preserves the Lusztig series (Corollary 3.19).

Definition 5.38. The characters

$$\{\varrho_{(s),z} \mid (s) \text{ semisimple class of } \mathbf{G}^{*F^*}, z \in H^1(F, Z(\mathbf{G}))\}$$

are called the *semisimple* characters of $\mathbf{G}^F$.

Remark 5.39. In view of Proposition 5.16, if the regular characters of a certain Lusztig series, say $\chi_{(s),z}$, are known then we know the value of the semisimple characters $\varrho_{(s),z}$ on regular unipotent elements. We have

$$\begin{aligned}\tfrac{1}{|U_{z'}|}\textstyle\sum_{u\in U_{z'}}\varrho_{(s),z}(u) &= \eta_{\mathbf{G}}\sum_{z''\in H^1(F,Z(\mathbf{G}))}\sigma_{z'z''^{-1}}\langle D_{\mathbf{G}}\varrho_{(s),z}, \Gamma_{z''}\rangle \\ &= \eta_{\mathbf{G}}\varepsilon_{\mathbf{G}}\varepsilon_{C_{\mathbf{G}^*}(s)}\sum_{z''\in H^1(F,Z(\mathbf{G}))}\sigma_{z'z''^{-1}}\langle \chi_{(s),z}, \Gamma_{z''}\rangle\end{aligned}$$

where the scalar products with the regular characters are known by hypothesis. Notice that, by definition, $\eta_{\mathbf{G}}\varepsilon_{\mathbf{G}} = \varepsilon_{R(\mathbf{G})}$ is equal 1 if $\mathbf{G}$ is semisimple.

Remark 5.40. By the discussion after the proof of [DLM92, (3.15) Theorem], a regular character $\chi_{(s),z}$ is also semisimple if and only if s is a regular semisimple element of $\mathbf{G}^*$. Hence the regular semisimple characters are

$$\{\chi_{(s),z} \mid (s) \text{ regular semisimple class of } \mathbf{G}^{*F^*}, z \in H^1(F, Z(\mathbf{G}))\}.$$

Thus, in good characteristic, $\chi_{(s),z}$ vanishes on regular unipotent classes unless it is regular semisimple.

Because of the tight relation between regular and semisimple characters, we can state an analogue of Theorem 5.35 for semisimple characters.

Theorem 5.41 ([DLM92, (6.4) Corollary])**.** *Suppose* $\mathbf{L}$ *is an* F*-stable Levi subgroup of* $\mathbf{G}$ *for which Conjecture 5.29 holds (and assume that the Mackey formula holds). Then we have*

$$^{*}R_{\mathbf{L}}^{\mathbf{G}}\,\varrho_{(s),z}^{\mathbf{G}} = \varepsilon_{C_{\mathbf{G}^*}(s)} \sum_{(t)} \varepsilon_{C_{\mathbf{L}^*}(t)} \varrho_{(t),z'}^{\mathbf{L}}$$

where the sum is over the semisimple classes $(t) \in \mathbf{L}^{*F^*}$ *such that* $t \in (s)_{\mathbf{G}^{*F^*}}$ *and* z' *is as in Conjecture 5.29.*

Remark 5.42. Notice that once the regular characters are identified (by taking scalar products with the Gel'fand–Graev characters) it is easy to identify the semisimple characters. In practice, we just need to inspect character degrees. This is direct from [GeMa20, Proposition 3.4.21], which states that the degree polynomials of $\rho \in \mathrm{Irr}(\mathbf{G}^F)$ and $D_{\mathbf{G}}(\rho)$ are related by

$$\mathbb{D}_{D_{\mathbf{G}}(\rho)}(\mathbf{q}) = \mathbf{q}^{|\Phi^+|}\mathbb{D}_{\rho}(\mathbf{q}^{-1}).$$

Remark 5.43. We already know one regular and one semisimple character. The Steinberg character is a regular character. This follows directly by taking the scalar product $\langle \Gamma_z^{\mathbf{G}}, \mathrm{St}_{\mathbf{G}} \rangle$ for any $z \in H^1(F, z(\mathbf{G}))$. By definition the only non-zero value on semisimple elements of the Gel'fand–Graev characters is $\Gamma_z^{\mathbf{G}}(1) = \frac{|\mathbf{G}^F|}{|\mathbf{U}_0^F|}$. While by Proposition 3.21 the only non-zero value on unipotent elements of the Steinberg character is $\mathrm{St}_{\mathbf{G}}(1) = |\mathbf{U}_0^F|$. Then the scalar product is equal to 1.

It follows by the definition of the Steinberg character that the trivial character is semisimple.

5.3 Modified Gel'fand–Graev characters

The idea behind the use of Gel'fand–Graev characters comes from the need for characters that distinguish between splitting unipotent classes. In this section, we make a step towards dropping the unipotent requirement, by modifying their definition. We consider characters that distinguish between splitting classes with elements whose semisimple part is central. Although the definition is as simple as it can be, they are a key ingredient for the decomposition of Deligne–Lusztig characters. For example, in $\mathrm{SL}_4(q)$ these modified Gel'fand–Graev characters encode all needed information to complete the character table.

Definition 5.44. Let ZU denote the subgroup of $\mathbf{G}^F$ which is the direct product of the centre $Z = Z(\mathbf{G}^F)$ and the unipotent subgroup $U = \mathbf{U}_0^F$. The *modified Gel'fand–Graev characters* of $\mathbf{G}^F$ are the characters of the form

$$\mathrm{Ind}_{ZU}^{\mathbf{G}^F}(\theta \times \psi)$$

where θ is a linear character of Z and ψ is a regular character of U.

If ψ is parametrized by $z \in H^1(F, Z)$, we write $\Gamma_{\theta,z}^{\mathbf{G}}$ for the corresponding modified Gel'fand–Graev character.

By the definition of induction, it is easy to see how the modified Gel'fand–Graev characters can be written in terms of the usual Gel'fand–Graev characters.

Lemma 5.45. *Let* $\theta \in \mathrm{Irr}(Z(\mathbf{G}^F))$ *and* $z \in H^1(F, Z(\mathbf{G}))$*. For* $g \in \mathbf{G}^F$ *with semisimple part* s *and unipotent part* u*, we have*

$$\Gamma_{\theta,z}^{\mathbf{G}}(g) = \begin{cases} \frac{1}{|Z(\mathbf{G}^F)|}\theta(s)\Gamma_z^{\mathbf{G}}(u) & s \in Z(\mathbf{G}^F) \\ 0 & \textit{else.} \end{cases}$$

Proof. Clearly, if $s \notin Z = Z(\mathbf{G}^F)$ then $\Gamma_{\theta,z}^{\mathbf{G}^F}(g) = 0$. So we assume that $s \in Z$ and let ψ be a regular character of $U = \mathbf{U}_0^F$ parametrized by z. Then, by definition of the induction of characters

$$\mathrm{Ind}_{ZU}^{\mathbf{G}^F}(\theta \times \psi)(g) = \frac{1}{|ZU|} \sum_{\substack{x \in \mathbf{G}^F, \\ {}^x g \in ZU}} \theta({}^x s)\psi({}^x u) = \frac{\theta(s)}{|Z|} \frac{1}{|U|} \sum_{\substack{x \in \mathbf{G}^F, \\ {}^x u \in U}} \psi({}^x u) = \frac{\theta(s)}{|Z|} \Gamma_z^{\mathbf{G}}(u).$$

□

It is easy to relate modified and non-modified Gel'fand–Graev characters more specifically.

Lemma 5.46. *Let $\theta, \theta' \in \mathrm{Irr}(Z(\mathbf{G}^F))$ and $z, z' \in H^1(F, Z(\mathbf{G}))$, then*

$$\langle \Gamma_{\theta,z}^{\mathbf{G}}, \Gamma_{\theta',z'}^{\mathbf{G}} \rangle = \delta_{\theta,\theta'} \frac{1}{|Z(\mathbf{G}^F)|} \langle \Gamma_z^{\mathbf{G}}, \Gamma_{z'}^{\mathbf{G}} \rangle.$$

Proof. By the discussion above, we can rewrite the scalar product as

$$\langle \Gamma_{\theta,z}^{\mathbf{G}}, \Gamma_{\theta',z'}^{\mathbf{G}} \rangle = \langle \Gamma_z^{\mathbf{G}}, \Gamma_{z'}^{\mathbf{G}} \rangle \sum_{x \in Z(\mathbf{G}^F)} \frac{\theta(x)}{|Z(\mathbf{G}^F)|} \frac{\theta'(x^{-1})}{|Z(\mathbf{G}^F)|}$$

and the result follows by the orthogonality relation for $\mathrm{Irr}(Z(\mathbf{G}^F))$. □

Corollary 5.47. *Let $z \in H^1(F, Z(\mathbf{G}))$. Then*

$$\sum_{\theta \in \mathrm{Irr}(Z(\mathbf{G}^F))} \Gamma_{\theta,z}^{\mathbf{G}} = \Gamma_z^{\mathbf{G}}.$$

Proof. By [Is76, (2.11) Lemma] the sum of all the linear characters of the centre is the regular character of the centre. So we have

$$\sum_{\theta \in \mathrm{Irr}(Z(\mathbf{G}^F))} \Gamma_{\theta,z}^{\mathbf{G}}(g) = \Gamma_z^{\mathbf{G}}(u) \sum_{\theta \in \mathrm{Irr}(Z(\mathbf{G}^F))} \frac{\theta(s)}{|Z(\mathbf{G}^F)|} = \Gamma_z^{\mathbf{G}}(u) \frac{\mathrm{reg}_{Z(\mathbf{G}^F)}(s)}{|Z(\mathbf{G}^F)|} = \begin{cases} \Gamma_z^{\mathbf{G}}(u) & s = 1 \\ 0 & \textit{else}, \end{cases}$$

for any $g \in \mathbf{G}^F$ with semisimple part s and unipotent part u. □

In other words, the modified Gel'fand–Graev characters partition the usual Gel'fand–Graev characters.

Remark 5.48. At the cost of having a support $|Z(\mathbf{G}^F)|$ times bigger, the modified Gel'fand–Graev characters contain more precise information than the non-modified ones. If in the previous section, we could get a system of at most $|H^1(F, Z(\mathbf{G}))|$ equations for any character of $\mathbf{G}^F$, now we get up to $|Z(\mathbf{G}^F)||H^1(F, Z(\mathbf{G}))|$ of them.

Although we introduced them as characters that distinguish certain conjugacy classes, it turns out that the modified Gel'fand–Graev characters can be explained/obtained with another construction. We end this section with a discussion on this equivalent construction. Theoretically, the information obtained is redundant. However, in practice, we gain an alternative, and in some cases more effective, way of performing some computations.

What follows is inspired by [Bo00, Chapter 1.7].

For every element of the centre of $\mathbf{G}^F$ we can define an action on $\mathrm{CF}(\mathbf{G}^F)$.

Definition 5.49. Let $z \in Z(\mathbf{G}^F)$ and $\chi \in \mathrm{CF}(\mathbf{G}^F)$. Then we define $\mathbf{t}_z^{\mathbf{G}}\chi \in \mathrm{CF}(\mathbf{G}^F)$ by $\mathbf{t}_z^{\mathbf{G}}\chi(g) := \chi(zg)$ for $g \in \mathbf{G}^F$.

It is easy to see that $\mathbf{t}_z^{\mathbf{G}} : \mathrm{CF}(\mathbf{G}^F) \to \mathrm{CF}(\mathbf{G}^F)$ is an isometry and $\mathbf{t}_z^{\mathbf{G}} \circ \mathbf{t}_{z'}^{\mathbf{G}} = \mathbf{t}_{zz'}^{\mathbf{G}}$ for all $z, z' \in Z(\mathbf{G}^F)$.

Notation 5.50. For any linear character $\varphi \in \mathrm{Irr}(Z(\mathbf{G}^F))$ we define the set

$$\mathrm{CF}(\mathbf{G}^F)^\varphi := \left\{\chi \in \mathrm{CF}(\mathbf{G}^F) \mid \mathbf{t}_z^{\mathbf{G}}\chi = \varphi(z)\chi, \text{ for all } z \in Z(\mathbf{G}^F)\right\}.$$

The interest of introducing this action of the centre comes from the next proposition.

Proposition 5.51. *We have*

$$\mathrm{CF}(\mathbf{G}^F) = \bigoplus_{\varphi \in \mathrm{Irr}(Z(\mathbf{G}^F))} \mathrm{CF}(\mathbf{G}^F)^\varphi$$

and the direct sum is orthogonal.

Proof. For any class function $\chi \in \mathrm{CF}(\mathbf{G}^F)$ and any $\varphi \in \mathrm{Irr}(Z(\mathbf{G}^F))$ the transform

$$\chi \mapsto \chi_\varphi := \frac{1}{|Z(\mathbf{G}^F)|} \sum_{z \in Z(\mathbf{G}^F)} \varphi(z^{-1})\mathbf{t}_z^{\mathbf{G}}\chi$$

projects onto $\mathrm{CF}(\mathbf{G}^F)^\varphi$ (easy to verify by applying $\mathbf{t}_z^{\mathbf{G}}$ to χ_φ).

By explicit computation we find,

$$\sum_{\varphi \in \mathrm{Irr}(Z(\mathbf{G}^F))} \chi_\varphi = \chi \quad \text{and} \quad (\chi_\varphi)_{\varphi'} = \delta_{\varphi,\varphi'}\chi_\varphi,$$

for $\varphi, \varphi' \in \mathrm{Irr}(Z(\mathbf{G}^F))$, which proves that $\mathrm{CF}(\mathbf{G}^F)$ is the direct sum from the statement.

Lastly, take $\chi_1 \in \mathrm{CF}(\mathbf{G}^F)^{\varphi_1}$ and $\chi_2 \in \mathrm{CF}(\mathbf{G}^F)^{\varphi_2}$ for $\varphi_1, \varphi_2 \in \mathrm{Irr}(Z(\mathbf{G}^F))$. Because $\mathbf{t}_z^{\mathbf{G}}$ is an isometry for all $z \in Z(\mathbf{G}^F)$, we get the orthogonality relation

$$\begin{aligned}\langle \chi_1, \chi_2 \rangle &= \frac{1}{|Z(\mathbf{G}^F)|} \sum_{z \in Z(\mathbf{G}^F)} \langle \mathbf{t}_z^{\mathbf{G}}\chi_1, \mathbf{t}_z^{\mathbf{G}}\chi_2 \rangle = \frac{1}{|Z(\mathbf{G}^F)|} \sum_{z \in Z(\mathbf{G}^F)} \langle \varphi_1(z)\chi_1, \varphi_2(z)\chi_2 \rangle \\ &= \sum_{z \in Z(\mathbf{G}^F)} \frac{\varphi_1(z^{-1})\varphi_2(z)}{|Z(\mathbf{G}^F)|} \langle \chi_1, \chi_2 \rangle = \langle \varphi_1, \varphi_2 \rangle \langle \chi_1, \chi_2 \rangle = \delta_{\varphi_1,\varphi_2}\langle \chi_1, \chi_2 \rangle.\end{aligned}$$

□

Notation 5.52. From now on, for all $\chi \in \mathrm{CF}(\mathbf{G}^F)$ and $\varphi \in \mathrm{Irr}(Z(\mathbf{G}^F))$ we denote by χ_φ the projection of χ to $\mathrm{CF}(\mathbf{G})^\varphi$, as in the proof above.

Remark 5.53. It is clear by the definition that for $z \in Z(\mathbf{G}^F)$ and $\varphi \in \mathrm{Irr}(Z(\mathbf{G}^F))$ the modified Gel'fand–Graev character $\Gamma_{\varphi,z}$ is the projection of the Gel'fand–Graev character Γ_z to $\mathrm{CF}(\mathbf{G}^F)^\varphi$, i.e. $\Gamma_{\varphi,z} = (\Gamma_z)_\varphi$.

The next result will be useful for applying the theory developed so far to the computations in Part III.

Proposition 5.54 ([Bo00, Lemma 1.7.5])**.** *If s is a semisimple element of $\mathbf{G}^{*F^*}$, then there exists a unique $\varphi \in \mathrm{Irr}(Z(\mathbf{G}^F))$, depending only on the $\mathbf{G}^{*F^*}$-conjugacy class of s, such that*

$$\mathbb{C}\mathscr{E}(\mathbf{G}^F, s) \subseteq \mathrm{CF}(\mathbf{G}^F)^\varphi.$$

Remark 5.55. In particular, every irreducible character $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ takes values related by $\chi(zu) = \varphi(z)\chi(u)$, for $\varphi \in \mathrm{Irr}(Z(\mathbf{G}^F))$ as in the previous statement, where $z \in Z(\mathbf{G}^F)$ and $u \in \mathbf{G}^F$ is unipotent. This character φ can be determined on any known class function in the span of the Lusztig series containing χ.

In general, for an element with Jordan decomposition su (s semisimple and u unipotent) we have analogous statements for the characters of $C_{\mathbf{G}}(s)^F$. When this centralizer is a Levi subgroup, $\mathbf{L}^F$, we can use these observations to gain further informations on the values of ${}^*R^{\mathbf{G}}_{\mathbf{L}}\chi$. Although, one must be careful that the constituents of ${}^*R^{\mathbf{G}}_{\mathbf{L}}\chi$ might not all lie in the same subspace $\mathrm{CF}(\mathbf{L}^F)^{\varphi}$ for one $\varphi \in \mathrm{Irr}(Z(\mathbf{L}^F))$.

Remark 5.56. By the last remark, if it is possible to determine $\varphi \in \mathrm{Irr}(Z(\mathbf{G}^F))$ corresponding to each semisimple conjugacy class of $\mathbf{G}^{*F^*}$, then it is unnecessary to know the modified Gel'fand–Graev characters of $\mathbf{G}^F$. This will be the case for $\mathrm{Spin}_8^+(q)$. However, as already discussed in the previous section, we will Lusztig restrict regular/semisimple characters to F-stable Levi subgroups. By Theorems 5.35 and 5.41 these restriction might not (and it is hardly ever the case) be irreducible. Then the determination of their decomposition in $\bigoplus_{\varphi \in \mathrm{Irr}(Z(\mathbf{L}^F))} \mathrm{CF}(\mathbf{G}^F)^{\varphi}$, for F-stable Levi subgroups $\mathbf{L}$, is computationally easier via scalar products with modified Gel'fand–Graev characters. This is true especially because we usually do not know $\mathrm{Irr}(\mathbf{L}^F)$ nor we know any class function in some Lusztig series of $\mathbf{L}^F$.

We end this section with an easy result that will be useful when we consider the Lusztig restriction of non-regular non-semisimple characters. In a few words, thanks to the orthogonality of Proposition 5.51 if a class function belongs to $\mathrm{CF}(\mathbf{G}^F)^{\varphi}$ for a certain $\varphi \in \mathrm{Irr}(Z(\mathbf{G}^F))$, then all its irreducible constituents belong to $\mathrm{CF}(\mathbf{G}^F)^{\varphi}$ too.

Lemma 5.57. *Let $\chi \in \mathrm{CF}(\mathbf{G}^F)^{\varphi}$ for $\varphi \in \mathrm{Irr}(Z(\mathbf{G}^F))$ such that its decomposition into irreducible constituents is*

$$\chi = \sum_{\theta \in \mathrm{Irr}(\mathbf{G}^F)} a_\theta \theta$$

for scalars $a_\theta \in \mathbb{C}$. Then, $\theta \in \mathrm{CF}(\mathbf{G}^F)^{\varphi}$ if $a_\theta \neq 0$.

Proof. By Proposition 5.54 for each $\theta \in \mathrm{Irr}(\mathbf{G}^F)$ there exists a $\varphi_\theta \in \mathrm{Irr}(Z(\mathbf{G}^F))$ such that $\theta \in \mathrm{CF}(\mathbf{G}^F)^{\varphi_\theta}$. By hypothesis, for all $z \in Z(\mathbf{G}^F)$ we have

$$\mathbf{t}_z\chi = \varphi(z)\chi.$$

We apply the decomposition on both sides of the equation:

$$\sum_{\theta \in \mathrm{Irr}(\mathbf{G}^F)} a_\theta \varphi_\theta(z)\theta = \sum_{\theta \in \mathrm{Irr}(\mathbf{G}^F)} a_\theta \varphi(z)\theta$$

which is equivalent to

$$\sum_{\theta \in \mathrm{Irr}(\mathbf{G}^F)} a_\theta \left(\varphi_\theta(z) - \varphi(z)\right) \theta = 0,$$

for all $z \in Z(\mathbf{G}^F)$. Because $\mathrm{Irr}(\mathbf{G}^F)$ is a basis of $\mathrm{CF}(\mathbf{G}^F)$, then we have

$$\varphi_\theta(z) - \varphi(z) = 0$$

for all θ such that $a_\theta \neq 0$ and for all $z \in Z(\mathbf{G}^F)$. This implies that $\varphi_\theta = \varphi$ for all θ such that $a_\theta \neq 0$. □

5.4 Some partial Gauss sums

The next lemma gives all the identities of ϕ (defined in Notation 5.12) needed to compute the Gel'fand–Graev characters in Parts II and III, and some other sums that appear in the computation of Gel'fand–Graev characters.

Denote the Legendre symbol of $x \in \mathbb{F}_q$ by

$$\eta(x) = \begin{cases} 0 & x = 0, \\ 1 & x \in (\mathbb{F}_q^\times)^2, \\ -1 & \text{else,} \end{cases}$$

and the corresponding Gauss sum

$$\mathcal{G}(\eta) = \sum_{x \in \mathbb{F}_q} \eta(x)\phi(x).$$

Lemma 5.58. *The character ϕ has the following properties (every symbol j_k, $k = 1, 2, \ldots$ denotes a fixed element of $\mathbb{F}_q^\times$):*

1. $\sum_{r_1 \in \mathbb{F}_q^\times} \phi(j_1 r_1) = -1$, $\phi(0) = 1$ *and* $\phi(x)\phi(y) = \phi(x+y)$ *for all* $x, y \in \mathbb{F}_q$.
2. $\sum_{x_1,\ldots,x_r \in \mathbb{F}_q^\times} \prod_{i=1,..,r} \phi(a_i x_i) = (-1)^r$ *for all* $a_i \in \mathbb{F}_q^\times$ *and* $r > 0$.
3. $\mathcal{G}(\eta) = \sum_{x \in \mathbb{F}_q} \phi(x^2)$.
4. $\mathcal{G}(\eta)^2 = \eta(-1)q$.
5. $\sum_{x \in \mathbb{F}_q} \phi(\alpha x^2) = \eta(\alpha) \sum_{x \in \mathbb{F}_q} \phi(x^2)$, $\forall \alpha \in \mathbb{F}_q^\times$.
6. $\sum_{\substack{r_1, r_2 \in \mathbb{F}_q^\times \\ r_1 r_2 \in \mu^k (\mathbb{F}_q^\times)^2}} \phi(j_1 r_1 + j_2 r_2) = \frac{q\eta\left(-j_1 j_2 \mu^k\right)+1}{2}$.
7. $\sum_{\substack{r_1, r_2 \in \mathbb{F}_q^\times \\ r_1 r_2 \in \mu^k (\mathbb{F}_q^\times)^2}} \phi(j_1 r_1) = -\frac{q-1}{2}$.
8. $\sum_{\substack{r_1, r_2, r_3, r_4 \in \mathbb{F}_q^\times \\ r_1 r_2 r_3 r_4 \in \mu^k (\mathbb{F}_q^\times)^2}} \phi(j_1 r_1 + j_2 r_2 + j_3 r_3) = -\frac{q-1}{2}$.
9. $\sum_{\substack{r_1, r_2, r_3, r_4 \in \mathbb{F}_q^\times \\ r_1 r_2 r_3 r_4 \in \mu^k (\mathbb{F}_q^\times)^2}} \phi(j_1 r_1) = -\frac{(q-1)^3}{2}$.
10. $\sum_{\substack{r_1, r_2, r_3, r_4, r_5 \in \mathbb{F}_q^\times \\ r_1 = a r_2 r_3 r_4 r_5^{-2}}} \phi(j_1 r_1) = -(q-1)^3$ *for* $a \in \mathbb{F}_q^\times$.
11. $\sum_{\substack{r_1, r_2, r_3 \in \mathbb{F}_q^\times \\ r_1 r_3 \in \mu^k (\mathbb{F}_q^\times)^2 \\ r_2 r_3 \in \mu^l (\mathbb{F}_q^\times)^2}} \phi(j_1 r_1 + j_2 r_2 + j_3 r_3) = -\frac{q\eta\left(-j_1 j_2 \mu^{k+l}\right)+q\eta\left(-j_1 j_3 \mu^k\right)+q\eta\left(-j_2 j_3 \mu^l\right)+1}{4}$.

12. $\sum\limits_{\substack{r_1,r_2,r_3\in\mathbb{F}_q^\times\\ r_1r_3\in\mu^k(\mathbb{F}_q^\times)^2\\ r_2r_3\in\mu^l(\mathbb{F}_q^\times)^2}} \phi(j_1r_1+j_2r_2) = \frac{q-1}{4}\left(q\eta\left(-j_1j_2\mu^{k+l}\right)+1\right).$

13. $\sum\limits_{\substack{r_1,r_2,r_3\in\mathbb{F}_q^\times\\ r_1r_3\in\mu^k(\mathbb{F}_q^\times)^2\\ r_2r_3\in\mu^l(\mathbb{F}_q^\times)^2}} \phi(j_1r_1) = -\frac{(q-1)^2}{4}.$

14. $\sum\limits_{\substack{r_1,r_2,r_3,r_4,r_5\in\mathbb{F}_q^\times\\ r_2r_4\in\mu^k(\mathbb{F}_q^\times)^2\\ r_3r_4\in\mu^l(\mathbb{F}_q^\times)^2\\ r_1=ar_2r_3r_4r_5^{-2}}} \phi(j_1r_1) = -\frac{(q-1)^3}{4}$ *for* $a\in\mathbb{F}_q^\times$.

15. $\sum\limits_{\substack{r_1,r_2,r_3,r_4,r_5\in\mathbb{F}_q^\times\\ t:=r_1r_5^2r_2^{-1}r_3^{-1}r_4^{-1}\neq-2,-4\\ t(t+4)\in\mu^k(\mathbb{F}_q^\times)^2}} \phi(j_1r_1) = -(q-1)^3\frac{q-3-\eta(\mu^k)-\eta(-\mu^k)}{2}.$

16. $\sum\limits_{\substack{r_1,r_2,r_3,r_4,r_5\in\mathbb{F}_q^\times\\ r_1r_2-r_1r_3r_4r_5^{-1}\in\mu^k(\mathbb{F}_q^\times)^2\\ -r_4r_5\in\mu^l(\mathbb{F}_q^\times)^2}} \phi(j_1r_1+j_2r_2+j_3r_3) = \frac{q-1}{4}\big(-q+2+q\eta(-j_1j_3\mu^{k+l})+ q\eta(-j_1j_2\mu^k)+q\eta(j_2j_3\mu^l)\big).$

17. $\sum\limits_{\substack{r_1,r_2,r_3\in\mathbb{F}_q^\times\\ r_1r_2^2r_3^3\in\mu^k(\mathbb{F}_q^\times)^4}} 4\,\phi(j_1r_1+j_2r_2+j_3r_3) = (q-1)+q\left(1+\eta(j_1j_3\mu^k)\right)\left(\pm\eta(j_2\alpha)\sqrt{q}-1\right)$
for $q\equiv 1 \pmod 4$ *and with* $\alpha=\sqrt{-j_1^{-1}j_3\mu^{-k}}$.

Other sums:

18. $\sum\limits_{\substack{r_1,r_2\in\mathbb{F}_q^\times\\ r_1r_2\in\mu^k(\mathbb{F}_q^\times)^2}} 1 = \frac{(q-1)^2}{2}.$

19. $\sum\limits_{\substack{r_1,r_2,r_3\in\mathbb{F}_q^\times\\ r_1r_3\in\mu^k(\mathbb{F}_q^\times)^2\\ r_2r_3\in\mu^l(\mathbb{F}_q^\times)^2}} 1 = \frac{(q-1)^3}{4}.$

20. $\sum\limits_{\substack{t\in\mathbb{F}_q^\times\setminus\{-2,-4\}\\ t(t+4)\in\mu^k(\mathbb{F}_q^\times)^2}} 1 = \frac{q-3-\eta(\mu^k)-\eta(-\mu^k)}{2}.$

Remark 5.59. It is essential for the proof of this lemma that the number of pairs $(x_1,x_2)\in\mathbb{F}_q\times\mathbb{F}_q$ which solve the equation

$$a_1x_1^2+a_2x_2^2=b,$$

for $a_1,a_2,b\in\mathbb{F}_q^\times$, is

$$q-\eta(-a_1a_2),$$

while for

$$a_1x_1^2+a_2x_2^2=0,$$

with $a_1,a_2\in\mathbb{F}_q^\times$, it is

$$q+(q-1)\eta(-a_1a_2).$$

This is Lemma 6.24 of [LiNi97].

Proof of 1. Clear. □

Proof of 2. This follows by induction and by noticing that

$$\sum_{x_1,\dots,x_r\in\mathbb{F}_q^\times}\prod_{i=1,..,r}\phi(a_ix_i)=\sum_{x_r\in\mathbb{F}_q^\times}\phi(a_rx_r)\sum_{x_1,\dots,x_{r-1}\in\mathbb{F}_q^\times}\prod_{i=1,..,r-1}\phi(a_ix_i).$$

□

Proof of 3.

$$\mathcal{G}(\eta)=\sum_{x\in(\mathbb{F}_q^\times)^2}\phi(x)-\sum_{x\in\mathbb{F}_q^\times\setminus(\mathbb{F}_q^\times)^2}\phi(x)=1+2\sum_{x\in(\mathbb{F}_q^\times)^2}\phi(x)=1+\sum_{x\in\mathbb{F}_q^\times}\phi(x^2)=\sum_{x\in\mathbb{F}_q}\phi(x^2).$$

□

Proof of 4. By point 3, we have

$$\mathcal{G}(\eta)^2=\sum_{x,y\in\mathbb{F}_q}\phi(x^2+y^2).$$

We can compute the sum by performing the "change of variable" $r=x^2+y^2$. Thanks to Remark 5.59 above, $r=0$ for $q+(q-1)\eta(-1)$ pairs (x,y) while r is equal to each non zero element of $\mathbb{F}_q$ for $q-\eta(-1)$ pairs (x,y). Then, we get

$$\mathcal{G}(\eta)^2=q+(q-1)\eta(-1)+(q-\eta(-1))\underbrace{\sum_{r\in\mathbb{F}_q^\times}\phi(r)}_{-1}=q\eta(-1)$$

□

Proof of 5. Notice that α makes the sum over all the squares change to the sum over all the non squares if $\alpha\in\mathbb{F}_q^\times\setminus(\mathbb{F}_q^\times)^2$. □

Proof of 6. To perform this sum over $r_1r_2\in\mu^k(\mathbb{F}_q^\times)^2$ introduce the "change of variable" $r_1=r_2\mu^kt^2$ with $t\in\mathbb{F}_q^\times$. The sum becomes:

$$\sum_{r_1r_2\in\mu^k(\mathbb{F}_q^\times)^2}\phi(j_1r_1+j_2r_2)=\frac{1}{2}\sum_{\substack{r_2\in\mathbb{F}_q^\times\\ t\in\mathbb{F}_q^\times}}\phi(r_2(j_1\mu^kt^2+j_2))=\frac{1}{2}\sum_{A_k}(q-1)-\frac{1}{2}\sum_{\overline{A_k}}1=\frac{q-1}{2}|A_k|-\frac{|\overline{A_k}|}{2}$$

where $A_k=\left\{t\in\mathbb{F}_q^\times\mid j_2+j_1\mu^kt^2=0\right\}$. The set A_k is non-empty, and of cardinality two, if $-j_1j_2^{-1}\in\mu^k(\mathbb{F}_q^\times)^2$:

$$|A_k|=1+\eta(-j_1j_2\mu^k),$$

and $|\overline{A_k}|=q-1-|A_k|$. The result follows. □

Proof of 7. Perform the change of variable $r_1=r_2\mu^kt^2$ with $t\in\mathbb{F}_q^\times$. The sum becomes:

$$\sum_{\substack{r_1,r_2\in\mathbb{F}_q^\times\\ r_1r_2\in\mu^k(\mathbb{F}_q^\times)^2}}\phi(j_1r_1)=\frac{1}{2}\sum_{t,r_2\in\mathbb{F}_q^\times}\phi(j_1r_2\mu^kt^2)=-\frac{1}{2}\sum_{t\in\mathbb{F}_q^\times}1=-\frac{q-1}{2}.$$

□

Proof of 8. Perform the change of variable $r_1 = r_2r_3r_4t^2\mu^k$ with $t \in \mathbb{F}_q^\times$. The sum becomes

$$\sum_{\substack{r_1,r_2,r_3,r_4\in\mathbb{F}_q^\times \\ r_1r_2r_3r_4\in\mu^k(\mathbb{F}_q^\times)^2}} \phi(j_1r_1 + j_2r_2 + j_3r_3) = \frac{1}{2}\sum_{t,r_2,r_3,r_4\in\mathbb{F}_q^\times} \phi(j_1r_2r_3r_4t^2\mu^k + j_2r_2 + j_3r_3) = -\frac{q-1}{2}$$

by applying three times property 1 (to the variables r_4, then r_3 and r_2). ☐

Proof of 9. Perform the change of variable $r_1 = r_2r_3r_4t^2\mu^k$ with $t \in \mathbb{F}_q^\times$. The sum becomes

$$\sum_{\substack{r_1,r_2,r_3,r_4\in\mathbb{F}_q^\times \\ r_1r_2r_3r_4\in\mu^k(\mathbb{F}_q^\times)^2}} \phi(j_1r_1) = \frac{1}{2}\sum_{t,r_2,r_3,r_4\in\mathbb{F}_q^\times} \phi(j_1r_2r_3r_4t^2\mu^k) = -\frac{(q-1)^3}{2}$$

by applying property 1. ☐

Proof of 10. Follows directly from property 1.

$$\sum_{\substack{r_1,r_2,r_3,r_4,r_5\in\mathbb{F}_q^\times \\ r_1=ar_2r_3r_4r_5^{-2}}} \phi(j_1r_1) = \sum_{r_2,r_3,r_4,r_5\in\mathbb{F}_q^\times} \phi(aj_1r_2r_3r_4r_5^{-2}) = -(q-1)^3$$

☐

Proof of 11. Perform the change of variables $r_1 = r_3\mu^kt_1^2$ and $r_2 = r_3\mu^lt_2^2$ with $t_1, t_2 \in \mathbb{F}_q^\times$. The sum becomes:

$$\sum_{\substack{r_1r_3\in\mu^k(\mathbb{F}_q^\times)^2 \\ r_2r_3\in\mu^l(\mathbb{F}_q^\times)^2}} \phi(j_1r_1+j_2r_2+j_3r_3) = \frac{1}{4}\sum_{t_1,t_2,r_3\in\mathbb{F}_q^\times} \phi(r_3(j_1\mu^kt_1^2+j_2\mu^lt_2^2+j_3)) = \frac{1}{4}(q-1)|B_{kl}| - \frac{1}{4}|\overline{B_{kl}}|$$

for $B_{kl} = \left\{(t_1,t_2) \in \mathbb{F}_q^\times \times \mathbb{F}_q^\times \mid j_1\mu^kt_1^2 + j_2\mu^lt_2^2 = -j_3\right\}$.

The cardinality is

$$\begin{aligned}|B_{kl}| &= \left|\left\{(t_1,t_2) \in \mathbb{F}_q \times \mathbb{F}_q \mid j_1\mu^kt_1^2 + j_2\mu^lt_2^2 = -j_3\right\}\right| - \left|\left\{t_1 \in \mathbb{F}_q^\times \mid j_1\mu^kt_1^2 = -j_3\right\}\right| - \\ &\quad \left|\left\{t_2 \in \mathbb{F}_q^\times \mid j_2\mu^lt_2^2 = -j_3\right\}\right| = q - \eta(-j_1j_2\mu^{k+l}) - |A_k| - |A_l| \\ &\quad = q - 2 - \eta(-j_1j_2\mu^{k+l}) - \eta(-j_1j_3\mu^k) - \eta(-j_2j_3\mu^l)\end{aligned}$$

where A_k is as in proof of 6, and $|\overline{B_{kl}}| = (q-1)^2 - |B_{kl}|$. The result follows. ☐

Proof of 12. Perform the change of variables $r_1 = r_3\mu^kt_1^2$ and $r_2 = r_3\mu^lt_2^2$ with $t_1, t_2 \in \mathbb{F}_q^\times$. The sum becomes:

$$\sum_{\substack{r_1,r_2,r_3\in\mathbb{F}_q^\times \\ r_1r_3\in\mu^k(\mathbb{F}_q^\times)^2 \\ r_2r_3\in\mu^l(\mathbb{F}_q^\times)^2}} \phi(j_1r_1 + j_2r_2) = \frac{1}{4}\sum_{t_1,t_2,r_3\in\mathbb{F}_q^\times} \phi(r_3(j_1\mu^kt_1^2 + j_2\mu^lt_2^2)) = \frac{1}{4}(q-1)|C_{kl}| - \frac{1}{4}|\overline{C_{kl}}|$$

for $C_{kl} = \left\{(t_1,t_2) \in \mathbb{F}_q^\times \times \mathbb{F}_q^\times \mid j_1\mu^kt_1^2 + j_2\mu^lt_2^2 = 0\right\}$.

The cardinality is

$$|C_{kl}| = \left|\left\{(t_1,t_2) \in \mathbb{F}_q \times \mathbb{F}_q \mid j_1\mu^kt_1^2 + j_2\mu^lt_2^2 = 0\right\}\right| - |\{(0,0)\}| = q - 1 + (q-1)\eta(-j_1j_2\mu^{k+l})$$

and $|\overline{C_{kl}}| = (q-1)^2 - |C_{kl}|$. The result follows. ☐

Proof of 13. Perform the change of variables $r_1 = r_3\mu^k t_1^2$ and $r_2 = r_3\mu^l t_2^2$ with $t_1, t_2 \in \mathbb{F}_q^\times$. The sum becomes:

$$\sum_{\substack{r_1,r_2,r_3\in\mathbb{F}_q^\times \\ r_1r_3\in\mu^k(\mathbb{F}_q^\times)^2 \\ r_2r_3\in\mu^l(\mathbb{F}_q^\times)^2}} \phi(j_1r_1) = \frac{1}{4}\sum_{t_1,t_2,r_3\in\mathbb{F}_q^\times} \phi(r_3j_1\mu^k t_1^2) = -\frac{1}{4}\sum_{t_1,t_2\in\mathbb{F}_q^\times} 1 = -\frac{(q-1)^2}{4}.$$

. □

Proof of 14. Perform the change of variables $r_2 = r_4\mu^k t_1^2$ and $r_3 = r_4\mu^l t_2^2$ with $t_1, t_2 \in \mathbb{F}_q^\times$. The sum becomes:

$$\sum_{\substack{r_1,r_2,r_3,r_4,r_5\in\mathbb{F}_q^\times \\ r_2r_4\in\mu^k(\mathbb{F}_q^\times)^2 \\ r_3r_4\in\mu^l(\mathbb{F}_q^\times)^2 \\ r_1=ar_2r_3r_4r_5^{-2}}} \phi(j_1r_1) = \sum_{\substack{r_2,r_3,r_4,r_5\in\mathbb{F}_q^\times \\ r_2r_4\in\mu^k(\mathbb{F}_q^\times)^2 \\ r_3r_4\in\mu^l(\mathbb{F}_q^\times)^2}} \phi(ar_2r_3r_4r_5^{-2}j_1) = \frac{1}{4}\sum_{t_1,t_2,r_4,r_5\in\mathbb{F}_q^\times} \phi(at_1^2t_2^2r_4^3r_5^{-2}j_1).$$

By redefining $t_3 = r_4t_1$ the sum is of the form of property 1. Then

$$\frac{1}{4}\sum_{t_1,t_2,r_4,r_5\in\mathbb{F}_q^\times} \phi(at_1^2t_2^2r_4^3r_5^{-2}j_1) = \frac{1}{4}\sum_{t_3,t_2,r_4,r_5\in\mathbb{F}_q^\times} \phi(at_3^2t_2^2r_4r_5^{-2}j_1) = -\frac{1}{4}\sum_{t_3,t_2,r_5\in\mathbb{F}_q^\times} 1 = -\frac{(q-1)^3}{4}.$$

□

Proof of 15. Perform the change of variable $t = r_1r_5^2r_2^{-1}r_3^{-1}r_4^{-1}$. Then

$$\sum_{\substack{r_1,r_2,r_3,r_4,r_5\in\mathbb{F}_q^\times \\ t:=r_1r_5^2r_2^{-1}r_3^{-1}r_4^{-1}\neq-2,-4 \\ t(t+4)\in\mu^k(\mathbb{F}_q^\times)^2}} \phi(j_1r_1) = \sum_{\substack{t,r_2,r_3,r_4,r_5\in\mathbb{F}_q^\times \\ t\neq-2,-4 \\ t(t+4)\in\mu^k(\mathbb{F}_q^\times)^2}} \phi(j_1tr_2r_3r_4r_5^{-2}) = -(q-1)^3 \sum_{\substack{t\in\mathbb{F}_q^\times \\ t\neq-2,-4 \\ t(t+4)\in\mu^k(\mathbb{F}_q^\times)^2}} 1.$$

The result follows from point 20. □

Proof of 16. As usual perform a change of variable, $r_2 = r_3r_4r_5^{-1} + r_1^{-1}t^2\mu^k$, however here one needs to be careful that r_2 cannot be 0, then

$$\sum_{\substack{r_1,r_2,r_3,r_4,r_5\in\mathbb{F}_q^\times \\ r_1r_2-r_1r_3r_4r_5^{-1}\in\mu^k(\mathbb{F}_q^\times)^2 \\ -r_4r_5\in\mu^l(\mathbb{F}_q^\times)^2}} \phi(j_1r_1 + j_2r_2 + j_3r_3) = \frac{1}{2}\Sigma_1 - \frac{1}{2}\Sigma_2$$

where

$$\Sigma_1 = \sum_{\substack{r_1,t,r_3,r_4,r_5\in\mathbb{F}_q^\times \\ -r_4r_5\in\mu^l(\mathbb{F}_q^\times)^2}} \phi\left(j_1r_1 + j_2(r_3r_4r_5^{-1} + r_1^{-1}t^2\mu^k) + j_3r_3\right)$$

and

$$\Sigma_2 = \sum_{\substack{r_1,t,r_3,r_4,r_5\in\mathbb{F}_q^\times \\ r_3=-r_1^{-1}r_4^{-1}r_5t^2\mu^k \\ -r_4r_5\in\mu^l(\mathbb{F}_q^\times)^2}} \phi(j_1r_1 + j_3r_3).$$

To compute Σ_1 perform the change of variable $r_4 = -r_5 s^2 \mu^l$,

$$\Sigma_1 = \frac{q-1}{2} \sum_{r_1,t,r_3,s \in \mathbb{F}_q^\times} \phi\left(j_1 r_1 - j_2 r_3 s^2 \mu^l + j_2 r_1^{-1} t^2 \mu^k + j_3 r_3\right) =$$

$$\frac{q-1}{2} \sum_{r_1,t \in \mathbb{F}_q^\times} \phi\left(r_1 (j_1 + j_2 t^2 \mu^k)\right) \sum_{r_3,s \in \mathbb{F}_q^\times} \phi\left(r_3 (j_3 - j_2 s^2 \mu^l)\right).$$

Now exactly as in the proof of 6 $\sum_{r,t \in \mathbb{F}_q^\times} \phi\left(r(a \pm bt^2)\right)) = q\eta(\mp ab) + 1$, then

$$\Sigma_1 = \frac{q-1}{2} \left(q\eta(-j_1 j_2 \mu^k) + 1\right) \left(q\eta(j_2 j_3 \mu^l) + 1\right).$$

For Σ_2 a similar computation must be done.

$$\Sigma_2 = \sum_{\substack{r_1,t,r_4,r_5 \in \mathbb{F}_q^\times \\ -r_4 r_5 \in \mu^l (\mathbb{F}_q^\times)^2}} \phi(j_1 r_1 - r_1^{-1} r_4^{-1} r_5 t^2 \mu^k j_3) = \frac{q-1}{2} \sum_{r_1,t,s \in \mathbb{F}_q^\times} \phi(j_1 r_1 + r_1^{-1} s^2 \mu^l t^2 \mu^k j_3) =$$

$$\frac{(q-1)^2}{2} \sum_{r_1,t \in \mathbb{F}_q^\times} \phi\left(r_1 (j_1 + \mu^l t^2 \mu^k j_3)\right) = \frac{(q-1)^2}{2} \left(q\eta(-j_1 j_3 \mu^{k+l}) + 1\right)$$

Gathering the pieces give the result. □

Proof of 17. Perform the change of variable $r_1 = r_2^2 r_3 \mu^k t^4$, then

$$\sum_{\substack{r_1,r_2,r_3 \in \mathbb{F}_q^\times \\ r_1 r_2^2 r_3^3 \in \mu^k (\mathbb{F}_q^\times)^4}} 4\,\phi(j_1 r_1 + j_2 r_2 + j_3 r_3) = \sum_{t,r_2,r_3 \in \mathbb{F}_q^\times} \phi(j_1 r_2^2 r_3 \mu^k t^4 + j_2 r_2 + j_3 r_3) =$$

$$\sum_{t,r_2 \in \mathbb{F}_q^\times} \phi(j_2 r_2) \sum_{r_3 \in \mathbb{F}_q^\times} \phi\left(r_3 (j_1 r_2^2 \mu^k t^4 + j_3)\right) = (q-1) \sum_{D_k} \phi(j_2 r_2) - \sum_{\overline{D_k}} \phi(j_2 r_2) = q - 1 + q \sum_{D_k} \phi(j_2 r_2)$$

where $D_k = \left\{(r_2, t) \in \mathbb{F}_q^\times \times \mathbb{F}_q^\times \mid j_1 r_2^2 \mu^k t^4 + j_3 = 0\right\}$.

If $j_1 j_3 \mu^k$ is not a square then $D_k = \varnothing$, and the expression becomes

$$q - 1 + q \frac{\left(1 + \eta(j_1 j_3 \mu^k)\right)}{2} \sum_{D_k} \phi(j_2 r_2).$$

On the other hand, if $j_1 j_3 \mu^k$ is a square then for each $t \in \mathbb{F}_q^\times$ there are exactly two $r_2 \in \mathbb{F}_q^\times$ that solve $j_1 r_2^2 \mu^k t^4 + j_3 = 0$, they are written formally as $\pm\sqrt{-j_1^{-1} j_3 \mu^{-k}} t^{-2}$. Denote $\alpha := \sqrt{-j_1^{-1} j_3 \mu^{-k}}$. The expression becomes

$$q - 1 + q \frac{\left(1 + \eta(j_1 j_3 \mu^k)\right)}{2} \sum_{t \in \mathbb{F}_q^\times} \left(\phi(j_2 \alpha t^2) + \phi(-j_2 \alpha t^2)\right) = q - 1 + q \left(1 + \eta(j_1 j_3 \mu^k)\right) \sum_{t \in \mathbb{F}_q^\times} \phi(j_2 \alpha t^2) =$$

$$q - 1 + q \left(1 + \eta(j_1 j_3 \mu^k)\right) (\pm\eta(j_2 \alpha)\sqrt{q} - 1)$$

by point 3, 4 and 5 of the lemma. □

Proof of 20. Notice that for $t=-2$, $t(t+4)=-4\in\mu^k(\mathbb{F}_q^\times)^2$ if $-\mu^k\in(\mathbb{F}_q^\times)^2$.

Then

$$\sum_{\substack{t\in\mathbb{F}_q^\times\setminus\{-2,-4\}\\ t(t+4)\in\mu^k(\mathbb{F}_q^\times)^2}} 1 = \sum_{\substack{t\in\mathbb{F}_q^\times\setminus\{-4\}\\ t(t+4)\in\mu^k(\mathbb{F}_q^\times)^2}} 1-\frac{1+\eta(-\mu^k)}{2} = \sum_{\substack{t\in\mathbb{F}_q^\times\\ t(t+4)\in\mu^k(\mathbb{F}_q^\times)^2}} 1-\frac{1+\eta(-\mu^k)}{2} \overset{t'=t+2}{=}$$
$$\frac{\left|\left\{(t',s)\in\mathbb{F}_q^\times\times\mathbb{F}_q^\times \mid t'^2-\mu^k s^2=4\right\}\right|}{2}-\frac{1+\eta(-\mu^k)}{2}=$$
$$\frac{\left|\left\{(t,s)\in\mathbb{F}_q\times\mathbb{F}_q \mid t^2-\mu^k s^2=4\right\}\right|-2}{2}-\frac{1+\eta(-\mu^k)}{2}=\frac{q-\eta(\mu^k)-2}{2}-\frac{1+\eta(-\mu^k)}{2}.$$

The result follows.

□

Remark 5.60. (a) To compute point 17 we used point 4, which is the reason for the ± sign appearing in the result. Fixing this sign requires to know the degree s of the field extension $\mathbb{F}_q$ over $\mathbb{F}_p$ $(q=p^s)$. By [LiNi97, Theorem 5.15] we have, for $q\equiv 1 \pmod 4$

$$\mathcal{G}(\eta)=(-1)^{s-1}\sqrt{q}.$$

Clearly, this sign could be a problem, since we would like to find "homogeneous" formulas in q in a generic character table. However, we can absorb it in α by eventually redefining $j_3\mapsto j_3\mu^{\pm 2}$, for example.

(b) Moreover, in the special cases where the sums are over fields of the form $\mathbb{F}_{q^2}$, the sign of the Gauss sum can be fixed more homogeneously. By [LiNi97, Theorem 5.16], we have

$$\mathcal{G}(\eta)=\sum_{x\in\mathbb{F}_{q^2}}\eta(x)\chi_2(x)=-(-1)^{\frac{q-1}{2}}q$$

where now η is the Legendre symbol of $\mathbb{F}_{q^2}$ and χ_2 is as in Notation 5.12.

6 Outline of the computation

In this section, we denote by $\mathbf{G}$ a simply connected semisimple linear algebraic group defined over $\mathbb{F}_q$ (q is a prime power), by F a Steinberg map of $\mathbf{G}$ with respect to its $\mathbb{F}_q$-rational structure. We are ultimately interested in the case where $\mathbf{G}$ is either SL_4 or Spin_8, q is odd and F is a Frobenius map. Then, we write $G = \mathbf{G}^F$ for either $\mathrm{SL}_4(q)$ or $\mathrm{Spin}_8^+(q)$. We write $\mathbf{G}^*$ for the dual group and F^* for the corresponding Frobenius map, then we also simply write G^* for $\mathbf{G}^{*F^*}$. Moreover, we fix a regular embedding $i : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ with corresponding Steinberg map $\tilde{F}$ of $\tilde{\mathbf{G}}$, where for SL_4 we have $\tilde{\mathbf{G}} = \mathrm{GL}_4$ and for Spin_8 we use the same $\tilde{\mathbf{G}}$ that is constructed in [GePf92].

Additionally, we denote by Φ the root system with base Δ and by W the Weyl group of $\mathbf{G}$ relative to a chosen maximally split torus $\mathbf{T}_0$ of $\mathbf{G}$. We also choose a reference F-stable Borel subgroup $\mathbf{B}_0$, containing $\mathbf{T}_0$, and we denote its unipotent radical by $\mathbf{U}_0$. Analogously, in the finite group, we write $T_0 = \mathbf{T}_0^F$, $B_0 = \mathbf{B}_0^F$ and $U_0 = \mathbf{U}_0^F$.

6.1 The starting point: conjugacy classes and almost characters of finite groups of Lie type

In this work, we start the computations with partial generic character tables computed by Frank Lübeck (with computer programs of his creation). These tables contain a finite number of rows labelled by the "character types" of G and a finite number of columns labelled by the "class types" of G. In both cases "finite number" is to be understood as "independent of q".

The way Lübeck computes these partial tables is similar to what he did for his thesis [Lue93] on the groups $\mathrm{CSp}_6(q)$ (q odd) and $\mathrm{Sp}_6(q)$ (q even). We give here a quick summary of his thesis and we refer to it for the details.

As seen in Section 2, to write a character table we need a list of representatives of conjugacy classes. By the Jordan decomposition of elements, we can write any element $g \in G$ as $g = su = us$, for s semisimple and $u \in C_{\mathbf{G}}(s)^\circ$ unipotent. Then, to write a list of conjugacy classes of G one can start by computing the semisimple classes, choose a representative $s \in G$ of each, then look for all the possible unipotent classes of $C_G(s)$ and choose a representative $u \in G$ for each. Notice that for a connected reductive group there are only finitely many unipotent classes (see [DiMi20, Theorem 12.1.11]).

It is clear that the number of conjugacy classes and of irreducible characters of a finite group of Lie type depends on q. To be able to write a generic character table that does not change size with q we unite the semisimple conjugacy classes in what are called types.

Definition 6.1. Two semisimple elements $s_1, s_2 \in G$ are said to belong to the same *semisimple class type* if their centralizers, in $\mathbf{G}$, $C_{\mathbf{G}}(s_1)$ and $C_{\mathbf{G}}(s_2)$ are G-conjugate.

Clearly the semisimple class types are unions of semisimple classes. But, although the number of semisimple classes does depend on q, the number of semisimple class types does not, by Remark 1.22.

Lübeck [Lue93] computes the semisimple classes/class types in Chapter 4.1, the unipotent classes in Chapter 4.2 and then discusses the mixed classes in Chapter 4.3.

In practice, to find representatives for the semisimple classes Lübeck takes the following steps. First, notice that every semisimple class of $\mathbf{G}$ has a representative in $\mathbf{T}_0$, as discussed in Section 1.1. Let us say that we look for a representative $s \in G$ contained in an F-stable torus $\mathbf{T}$ which is obtained from $\mathbf{T}_0$ by twisting with $w \in W$, where $\dot{w} = g^{-1}F(g)$ and ${}^g\mathbf{T}_0 = \mathbf{T}$. Then, analogously to the case of Levi subgroups (see discussion at the end of Section 1.5) the centralizer $C_G(s)$ is conjugate, by g, to $(C_{\mathbf{G}}(s^g))^{Fw^{-1}}$. Assume that $C_{\mathbf{G}}(s^g)$ has root system

Ψ, then we can find "representatives"[7] for the semisimple classes of G in $\mathbf{T}_0$ by imposing that $t \in \mathbf{T}_0$ satisfies ${}^{w}F(t) = t$, $\alpha(t) = 1$ for all $\alpha \in \Psi$ and $\alpha(t) \neq 1$ for all $\alpha \in \Phi \setminus \Psi$. Finally, one must be careful that several elements found this way might belong to the same G-class. Therefore, to get all the semisimple classes of G one needs to determine all possible closed root subsystem of Φ and all the possible twistings.

We discuss the unipotent conjugacy classes in detail in the next section, as they play a main role in the computations of Parts II and III.

Next, we want to find the irreducible characters of G. For this purpose, Lusztig introduced another basis of $\mathrm{CF}(G)$, for a group of Lie type G, consisting of so-called "almost characters". These almost characters are explicit linear combinations of irreducible characters. The change of basis between $\mathrm{Irr}(G)$ and the almost characters is given in terms of so-called "non-abelian Fourier matrices". The precise determination of the values of the almost characters is not an easy task. Conjecturally, the set of almost characters is equal, up to scalars, to yet another basis consisting of "characteristic functions of F-stable character sheaves", the definition of which greatly exceeds the scope of the present work (that consists in computing the complete character table with methods as elementary as possible). See [Ge18, Chapter 7] for a nice survey of these facts. Some of the almost characters coincide with some linear combinations of the Deligne–Lusztig characters, they are called *uniform almost characters*.

It turns out that for $\mathrm{GL}_n(q)$ the uniform almost characters are, up to signs, all the irreducible characters.

Theorem 6.2 ([DiMi20, Theorem 11.7.3]). *The irreducible characters of* $\mathbf{G}^F = \mathrm{GL}_n(q)$ *are (up to sign) the class functions*

$$R_\chi(s) := \frac{1}{|W_I|} \sum_{w \in W_I} \tilde{\chi}(ww_0) R^{\mathbf{G}}_{\mathbf{T}_{ww_0}}(s)$$

where (s) *runs over the* F*-stable semisimple conjugacy classes of* $\mathbf{G}$*; then* $C_{\mathbf{G}}(s)$ *is a Levi subgroup*[8] *parametrized by a pair* (I, w_0) *and with Weyl group* W_I*. The character* χ *runs over* w_0*-stable irreducible characters of* W_I *and* $\tilde{\chi}$ *stands for a real extension of* χ *to* $W_I \cdot \langle w_0 \rangle$.

Remark 6.3. For $\mathrm{SL}_n(q)$ the uniform almost characters can be considered as being the restriction from $\mathrm{GL}_n(q)$ of the functions $R_\chi(s)$ from Theorem 6.2, thanks to Proposition 3.13.

The values of the partial character table of $\mathrm{SL}_4(q)$, received from Lübeck, are computed this way. Clearly, the almost characters of $\mathrm{SL}_4(q)$ obtained this way are not irreducible, although they are true characters (up to sign). It follows, by Theorem 6.2 and Theorem 3.12 that every irreducible character of $\mathrm{SL}_4(q)$ is a constituent of one of these uniform almost characters.

For the computation of the uniform almost characters for other groups, in the connected centre case, we refer the reader to [GeMa20, Remark 2.4.17][9]. In general, the definition depends on the choice of the extension of the characters of a certain Weyl group. Lübeck uses the "preferred extension" as defined in [Lu86a, Chapter 17.2] for the uniform almost characters of $\mathrm{Spin}_8^+(q)$.

Remark 6.4. Thanks to [GeMa20, Example 2.4.18], also in type D_4, the uniform almost characters of $\tilde{\mathbf{G}}^{\tilde{F}}$ are irreducible, up to sign (apart from the unipotent ones, but these have already been taken care of in [GePf92]). Then, thanks to Proposition 3.13 we can see the uniform almost characters, of $\mathrm{Spin}_8^+(q)$, provided by Lübeck as the restriction of these irreducible characters of

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[7] The representatives so found are in $\mathbf{G}$ but might not be in G.

[8] In GL_n all centralizers are connected, moreover GL_n is its own dual group.

[9] Notice that there the uniform almost characters are written not with respect to pairs $(\mathbf{T}^*, s)$, as in Theorem 6.2, but to another formalism not introduced here.

$\tilde{\mathbf{G}}^{\tilde{F}}$. Like for $\mathrm{SL}_4(q)$, (previous Remark) these characters are, up to sign, true characters, and every non-unipotent irreducible character of $\mathrm{Spin}_8^+(q)$ is a constituent of one of them.

In any case, if we want the uniform almost characters we have to determine the Deligne–Lusztig characters.

To compute the values of Deligne–Lusztig characters at an element with Jordan decomposition su, one needs two things. One the one hand, we want all the pairs $(\mathbf{T}, \theta)$ up to geometric conjugacy, where $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$ and $\theta \in \mathrm{Irr}(\mathbf{T}^F)$. On the other hand, we need to know the (ordinary) Green functions of $C_{\mathbf{G}}(s)^\circ$ (by the character formula for Deligne–Lusztig characters, see for example [GeMa20, Theorem 2.2.16]).

Lübeck [Lue93] computes the Green functions in Chapter 5 and then uses them to compute the Deligne–Lusztig characters in Chapter 6.

By the discussion at the end of Section 1.5, we have that the F-stable maximal tori of $\mathbf{G}$ are parametrized (up to G-conjugacy) by the F-classes of W. Lübeck [Lue93] explains how he gets the list of F-stable maximal tori in Chapter 3. Then, he forms the pairs $(\mathbf{T}, s)$ in Chapter 4.1, in particular see "paragraph" (3) and Proposition 4.3(f). Analogously for $\mathbf{G}^*$, he computes the pairs $(\mathbf{T}^*, s)$ and associates them to the pairs $(\mathbf{T}, \theta)$ in Chapter 6.

Remark 6.5. We already mentioned before that the almost characters form a basis of the space of class functions of finite groups of Lie type. Therefore, in theory, the problem of finding the character table of a finite group of Lie type is solved by computing all the almost characters and then applying the change of basis to $\mathrm{Irr}(G)$. However, the computation of the non-uniform almost characters is (at the moment of the writing) still an open problem.

One possibility for completing the character table is to perform Harish–Chandra/Lusztig induction to create class functions that are orthogonal to the space of uniform functions. Unfortunately, the character formulas (Proposition 3.30) on an arbitrary element are not easy to use explicitly. In some cases it will be possible to do so for $\mathrm{SL}_4(q)$.

It seems that the best way to proceed, instead of character by character, is to work class by class. We will explain this in the next sections.

Notice that since $\mathrm{Irr}(G)$ is partitioned into Lusztig series, by Theorem 3.12, we can group the irreducible characters of G into types, exactly as for the semisimple classes.

Definition 6.6. Two Lusztig series (G, s_1) and (G, s_2) are said to belong to the same *type* when the elements s_1 and s_2 belong to the same semisimple class type of G^*.

In general we call *class types* and *character types*, respectively, the columns and rows of the generic character table. Notice that generic character tables need not be square.

6.2 Construction of the groups, the special case of simply connected groups

First of all, the next result shows how for simply connected groups an important property of semisimple groups is inherited by the finite group of Lie type.

Proposition 6.7 ([MaTe11, Theorem 24.15])**.** *Let* $\mathbf{G}$ *be a simply connected semisimple linear algebraic group with Steinberg endomorphism* $F : \mathbf{G} \to \mathbf{G}$*. Then* $\mathbf{G}^F = \langle \mathbf{G}^F_{\mathrm{uni}} \rangle$*, that is,* $\mathbf{G}^F$ *is generated by its unipotent elements.*

In particular, in our case the groups are F-split. This means that we get the Steinberg presentation (see Notation 1.47 and Remark 1.48) for G just by restricting the root maps of $\mathbf{G}$ to $\mathbb{F}_q$.

The first step in our computations will be to find a Steinberg presentation of G. By the above we just need to choose a faithful representation of G in order to write explicitly the root maps $u_\alpha : \mathbb{F}_q \to U_\alpha$, for all $\alpha \in \Phi$. For $\mathrm{SL}_4(q)$ we will use the natural matrix representation, while for $\mathrm{Spin}_8^+(q)$ we will build one following [Ge17]. Actually, it was already known by Steinberg how to write Steinberg presentations in general, up to choosing some signs. In [Ge17] these signs are fixed naturally.

We will need to work with Levi subgroups and centralizers of semisimple elements of G.

For a semisimple group of simply connected type the centralizers of semisimple elements are connected reductive groups:

Proposition 6.8 ([MaTe11, Theorem 14.16]). *Let* $\mathbf{G}$ *be connected reductive such that the derived group* $[\mathbf{G}, \mathbf{G}]$ *is simply connected, and* $s \in \mathbf{G}$ *a semisimple element. Then:*

(a) $C_{\mathbf{G}}(s)$ *is connected.*

(b) *If the order of* s *in* $\mathbf{G}/Z(\mathbf{G})$ *is finite, but not divisible by any torsion primes of* $\mathbf{G}$ *then* $[C_{\mathbf{G}}(s), C_{\mathbf{G}}(s)]$ *is again simply connected.*

In particular any Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ is the centralizer of a certain semisimple element in the connected component of its centre, to be precise $\mathbf{L} = C_{\mathbf{G}}(s)$ if $s \in Z(\mathbf{L})^\circ$ is semisimple of order prime to all torsion primes (for the definition of torsion primes see [MaTe11, Definition 14.14], there are none for type A_3 and 2 is the only torsion prime for D_4).

Proposition 6.9 ([MaTe11, Proposition 12.14]). *Let* $\mathbf{G}$ *be semisimple of simply connected type. Then, for any Levi subgroup* $\mathbf{L}$ *of* $\mathbf{G}$*, the derived subgroup* $[\mathbf{L}, \mathbf{L}]$ *is again of simply connected type.*

Together with Corollary 1.19, which states that $\mathbf{L} = [\mathbf{L}, \mathbf{L}]\, Z(\mathbf{L})^\circ$ this gives great informations about the structure and conjugacy classes of $\mathbf{L}$. In our case all proper Levi subgroups are of type $A_{n_1} \times \cdots \times A_{n_k}$ for some $n_1, ..., n_k$, then $[\mathbf{L}, \mathbf{L}] \cong \mathrm{SL}_{n_1} \times \cdots \times \mathrm{SL}_{n_k}$ and we just need to worry about the centre part, which is easily computed by Theorem 1.18 (b).

In the cases that we will consider, it is possible to identify an F-stable torus $\mathbf{T}$ of $\mathbf{L}$ such that

$$\mathbf{L} \cong (\mathrm{SL}_{n_1} \times \cdots \times \mathrm{SL}_{n_k}) \rtimes \mathbf{T}.$$

This also makes it easier to compute the fixed points $\mathbf{L}^F$. It is clear that if $\mathbf{L}$ is F-stable then its derived subgroup is too. It follows that

$$\mathbf{L}^F = [\mathbf{L}, \mathbf{L}]^F \rtimes \mathbf{T}^F.$$

For example when $\mathbf{L}$ is a split Levi subgroup then

$$\mathbf{L}^F \cong (\mathrm{SL}_{n_1}(q) \times \cdots \times \mathrm{SL}_{n_k}(q)) \rtimes \mathbf{T}^F$$

otherwise some finite special unitary groups can appear in the formula. The point is that we just need to compute the action of $\mathbf{T}^F$ on the well known special linear/unitary groups to get the structure and conjugacy classes of $\mathbf{L}^F$.

The above together with the discussion at the end of Section 1.5 (on the parametrization of F-stable Levi subgroups of $\mathbf{G}$) gives us the structure and the conjugacy classes of all the Levi subgroups of $\mathbf{G}^F$ that we need. Since the final goal is to find character values on splitting classes, we will use the Levi subgroups/centralizers that contain representatives of these splitting classes. These are the ones with disconnected centre.

6.3 Fusion of unipotent classes

Although we receive the list of conjugacy classes of G (with centralizers) already computed by Lübeck it is important for the scope of the present work to know the precise fusion of unipotent classes.

We explain here in detail how this is done. The method is actually quite simple and straightforward, and is easily applicable to any finite group of Lie type (for which the Chevalley relations are fixed).

First, recall that all the unipotent classes have a representative in U_0. Thus, the plan is to conjugate the unipotent elements in U_0 to one another until every element of U_0 has been put in some unipotent class of G. The number of unipotent classes of a finite group of Lie type is in theory known a priori (see for example [LiSe12]).

By the Bruhat decomposition (Theorem 1.36), we can write any element $g \in G$ uniquely as

$$g = utwu_w$$

for $u \in U_0$, $t \in T_0$, $w \in W$ and $u_w \in U_w$. Therefore, we can proceed in three steps. First, we compute the conjugacy classes of U_0. Then, we fuse them under conjugation with elements of T_0 (recall that U_0 is normalized by T_0). Finally, we use W to conjugate them further. Notice that thanks to the Chevalley relations we can do all this in Steinberg presentation, without having to explicitly write down matrices.

(1) **Conjugacy classes of U_0:** Notice that, by the commutation relations (Proposition 1.17), for a suitable numbering of the positive roots we have

$$U_k U_{k+1} \cdots U_N \triangleright U_{k+1} \cdots U_N$$

for all $k = 1, ..., N-1$, where $N = |\Phi^+|$ is the number of positive roots.

It follows that we can compute the conjugacy classes of U_0 with a “going down” procedure.

We compute the conjugacy classes of the subgroups $U_k \cdots U_N$ in a recursive way for k going from N to 1, with representatives $u_k(r_k) \cdots u_N(r_N)$ where $r_k \in \mathbb{F}_q^\times$ and $r_{k+1}, ..., r_N \in \mathbb{F}_q$. Then, we check that we have all the conjugacy classes in the subgroup by counting the elements taken into account and comparing it to $|U_k \cdots U_N| = q^{N-k+1}$.

In practice, this means the following. Denote by $u_0 = u_1(t_1) \cdots u_N(t_N)$ a generic element of U_0, where $t_1, ..., t_N$ are variables in $\mathbb{F}_q$. We start with the elements of U_N, i.e. $u_N(r_N)$ for $r_N \in \mathbb{F}_q$, and we conjugate them with u_0. Clearly, here there is nothing to do, U_N is in the centre of U_0. Thus, we have q distinct conjugacy classes of U_0. Now, assume that we have all the classes in $U_{k+1} \cdots U_N$ (accounting for q^{N-k} elements of U_0). Then, we consider $U_k \cdots U_N$, and we conjugate the elements $u_k(r_k)$ with u_0, where $r_k \in \mathbb{F}_q^\times$. The resulting element is of the form

$$u_k(r_k)u_{k+1}(f_{k+1}(r_k; t_1, ..., t_N)) \cdots u_N(f_N(r_k; t_1, ..., t_N)),$$

for certain relations f_{k+1},...,f_N between the variables. It is now easy to compute the centralizer order of $u_k(r_k)$ which is given by

$$|C_{U_0}(u_k(r_k))| = |\{t_1, ..., t_N \in \mathbb{F}_q \mid f_{k+1}(r_k; t_1, ..., t_N) = 0, ..., f_N(r_k; t_1, ..., t_N) = 0\}|.$$

We check if $\frac{q^N}{|C_{U_0}(u_k(r_k))|}(q-1) + q^{N-k}$ is equal to $|U_k \cdots U_N| = q^{N-k+1}$. In case of positive answer we can move to the next subgroup. Else, there are other classes to consider in this subgroup. By inspection of $u_k(r_k)^{u_0}$, we identify elements of the subgroup that are not in this orbit. Let us say that $u_k(r_k)u_j(r_j)$ are such elements, for j between k and N and $r_k, r_j \in \mathbb{F}_q^\times$. Then, we proceed again by conjugating with u_0, computing the centralizer and comparing the number of elements taken into account and the order of the subgroup

$$\frac{q^N}{|C_{U_0}(u_k(r_k))|}(q-1) + \frac{q^N}{|C_{U_0}(u_k(r_k)u_j(r_j))|}(q-1)^2 + q^{N-k} \overset{?}{=} q^{N-k+1}.$$

By continuing with this algorithm we obtain a list of representatives of the conjugacy classes of U_0, their centralizers in U_0 and their orbits in U_0. These informations are needed for computing induction of characters from U_0 to G and for point (3) below.

(2) **Unipotent classes in B_0:** Thanks to the Chevalley relations, we can conjugate the representatives found above by elements of T_0 and choose new representatives for the unipotent classes of B_0.

In practice, we write $h = \prod_{\alpha\in\Delta} h_\alpha(t_\alpha)$ for a generic element of T_0, with $t_\alpha \in \mathbb{F}_q^\times$ for all $\alpha \in \Delta$. Then, we conjugate each element of the list from point (1) with h, and choose an element from this orbit as representative. For example, like above, assume that $u_k(r_k)u_j(r_j)$ denote some representatives. Then we have

$$^{h}(u_k(r_k)u_j(r_j)) = u_k\left(r_k \prod_{\alpha\in\Delta} t_\alpha^{\langle\alpha_k,\alpha^\vee\rangle}\right) u_j\left(r_j \prod_{\alpha\in\Delta} t_\alpha^{\langle\alpha_j,\alpha^\vee\rangle}\right).$$

In our cases ($\mathrm{SL}_4(q)$ and $\mathrm{Spin}_8^+(q)$) we can always choose h in a way to put to 1 at least one of the arguments and to a chosen non-square of $\mathbb{F}_q$ for the other arguments.

Notice that conjugation with T_0 could also be done after conjugation by W (done in point (3) below). However, performing it at this point makes the list of unipotent elements and the number of variables in the system smaller and somehow easier to handle.

(3) **Unipotent classes of G:** Again, thanks to the Chevalley relations we can act on the list of unipotent classes with W. We write every element $w \in W$ as a product of the simple reflections. In `CHEVIE` ([MiChv]) this is automatically done with the commands `CoxeterGroup` and `CoxeterWords`. Then, we conjugate the representatives computed above in point (2) with the elements of W. This results in unordered products of elements $u_k(r_k)$. We reorder them with the commutation relations. Now, with help from the list of orbits computed in point (1) above, we identify which classes of U_0 belong to the same class of G.

For example, we write $w \in W$ in terms of the simple reflections as $w = s_{i_1}\cdots s_{i_l}$ and we conjugate the same example element from above (by abuse of notation)

$$^{w}(u_k(r_k)u_j(r_j)) = u_{w(k)}(\pm r_k)u_{w(j)}(\pm r_j).$$

If $w(k) > w(j)$ we reorder the expression with the commutation relations to obtain

$$u_{w(j)}(\pm r_j)u_{w(k)}(\pm r_k)\left[u_{w(k)}(\pm r_k), u_{w(j)}(\pm r_j)\right].$$

Finally we check to which elements this expression is U_0-conjugate.

The same procedure will be applied to find the unipotent classes of centralizers of semisimple elements. For the non split cases the computation is similar with the occasional replacing of q with q^2 (for example in subgroups of $D_4(q)$ of type $^2A_3(q)$).

6.4 2-parameter Green functions

We want to perform Harish–Chandra/Lusztig induction/restriction from/to a Levi subgroups $\mathbf{L}$ with disconnected centre. To this end, it is essential to compute the 2-parameter Green functions of $\mathbf{L}$. By knowing the fusion of the unipotent classes, we can directly determine $Q_{\mathbf{L}}^{\mathbf{G}}$ in the split Levi case, see Section 4.3. Then, we follow the discussion of Section 4.2 to get the rest of the 2-parameter Green functions for the twisted Levi subgroups.

6.5 Gel'fand–Graev, regular and semisimple characters

Once we know the conjugacy classes of G and of all the Levi subgroups with disconnected centre, especially the unipotent classes with their fusion, we can compute the (modified) Gel'fand–Graev characters. In practice, their determination is just a direct evaluation of the induction formula (Proposition 2.18). Explicitly, we have to evaluate some partial Gauss sums. All the needed cases are given in Lemma 5.58.

By definition, the modified Gel'fand–Graev characters distinguish the classes with representatives zu, where $z \in Z(\mathbf{L}^F)$ and u is unipotent in $\mathbf{L}^F$, for any F-stable Levi subgroup $\mathbf{L}$. Thus, they are a natural candidate to find the missing values in the partial character table of uniform almost characters. Moreover, they have nice properties that make them easy to work with. On one side, they have constituents in every Lusztig series (the regular characters), by Corollary 5.24. On the other side, they behave well with respect to Harish–Chandra/Lusztig restriction, see Proposition 5.33. Furthermore, their constituents, the regular characters, behave well with respect to Harish–Chandra/Lusztig restriction too, by Theorem 5.35.

Remark 6.10. These properties give us systems of equations for the values of characters. Consider a certain regular character $\chi_{z,(s)}^{\mathbf{G}}$ in the Lusztig series $\mathscr{E}(G, s)$ for a certain semisimple element $s \in G^*$ and $z \in H^1(F, Z(\mathbf{G}))$. By definition, $\langle \Gamma_z^{\mathbf{G}}, \chi_{(s),z}^{\mathbf{G}} \rangle = 1$. Then for every F-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ we get the equation

$$\langle \Gamma_{\mathfrak{h}_{\mathbf{L}}(z) z_{\mathbf{L}}}^{\mathbf{L}}, {}^*R_{\mathbf{L}}^{\mathbf{G}} \chi_{(s),z}^{\mathbf{G}} \rangle = |\{(t) \subset L^* \mid t \text{ semisimple such that } (t) \subset (s)\}|$$

by Theorem 5.35 and Remark 5.34 (if we choose $\mathrm{res}_{\mathbf{L}}^{\mathbf{G}} \mathrm{U}_1 \in \mathrm{Reg}_{\mathrm{uni}}(\mathbf{L}^F)$ as being the regular unipotent class parametrized by $1 \in H^1(F, Z(\mathbf{L}))$).

The regular characters $\chi_{(s),z}^{\mathbf{G}}$ can be identified directly thanks to the scalar product between the uniform almost characters and the Gel'fand–Graev characters. By Theorem 1.65 and Theorem 5.17, these are both multiplicity free.

By Remark 5.42, we can determine which characters are semisimple. Then, we also get similar equations for these characters. With the difference that the scalar product between semisimple and Gel'fand–Graev characters is zero. Notice, however, that for non-regular characters it is not always true that their Lusztig restriction does not contain regular constituents. We can check directly on the uniform almost characters if they do. In practice, we take scalar products between their Lusztig restriction and the Gel'fand–Graev characters of the Levi subgroups.

Once these equations are known, we can get similar ones for the modified Gel'fand–Graev characters, since they partition the ordinary ones. The interest being to include all the character values for elements of the form $zu \in Z(L)U_L$ in the system of equations and to get as many equations as there are elements of that form, where U_L denotes the maximal unipotent subgroup of $L = \mathbf{L}^F$.

We want to point out once again that we consider also the modified Gel'fand–Graev characters of the Levi subgroups with disconnected centre since they distinguish the splitting classes of $\mathbf{G}^F$ that they intersect. By taking scalar products as in the remark, we manage to get systems of equations involving the character values on every splitting class. The question is whether there are enough equations to solve uniquely the system. We will see in Section 18 that there is only one problematic case. That is, the values on some elements with decomposition su such that $C_{\mathbf{G}}(s)$ is not a Levi subgroup cannot be uniquely fixed.

As explained above, we want to perform Harish–Chandra/Lusztig restriction thanks to the character formula of Proposition 3.30 (b), to some Levi subgroups $\mathbf{L}^F$ of G. Since we only want to take scalar products with modified Gel'fand–Graev characters, we only need to consider the formula on elements of $\mathbf{L}^F$ with Jordan decomposition zv, with $z \in Z(\mathbf{L}^F)$ and v unipotent. In this case the formula becomes

$$({}^*R_{\mathbf{L}}^{\mathbf{G}}\psi)(zv) = \frac{|\mathbf{L}^F|}{|C_{\mathbf{G}}(z)^F|} \sum_{u \in C_{\mathbf{G}}(z)^F_{\text{uni}}} Q_{\mathbf{L}}^{C_{\mathbf{G}}(z)}(u, v^{-1})\, \psi(zu).$$

Thus, one just needs to find out what subgroup $C_{\mathbf{G}}(z)^F$ is, with Theorem 1.20 (a), and the computation can be done explicitly.

6.6 Decomposition of almost characters

As discussed in Section 6.1, to decompose the almost characters, one should compute the non-uniform almost characters via character sheaves, and perform the change of basis to $\mathrm{Irr}(G)$. However, one goal of this work is to try to use methods as elementary as possible. And the computation of character sheaves is, at the moment of the writing, still a theoretical challenge.

Another possibility is to create class functions which are orthogonal to the space of uniform functions, or even irreducible characters, by Harish–Chandra/Lusztig inducing some cleverly chosen class functions of some Levi subgroup. Although in theory this might give the means to complete the character table, the character formula in Proposition 3.30 (a) makes it difficult to actually perform these computations, in general. In some cases we will be able to use this procedure for $\mathrm{SL}_4(q)$.

There is a third possibility, which is the one that we are going to implement by default. Instead of trying to decompose one character at a time, we will work one class at a time, and compute the character values for each splitting class. The two important theoretical results used to find these values rely on Gel'fand–Graev theory and on regular embeddings.

On the one hand, thanks to the explicit computation of the 2-parameter Green functions we can apply Theorem 5.35 and Theorem 5.41 to all Levi subgroups that we will need to consider. Thus, we get relations

$${}^*R_{\mathbf{L}}^{\mathbf{G}}\chi_{(s),z}^{\mathbf{G}} = \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{L}} \sum_{(t)} \chi_{(t),\mathfrak{h}_{\mathbf{L}}(z)z_{\mathbf{L}}}^{\mathbf{L}}$$

for the regular characters, and

$${}^*R_{\mathbf{L}}^{\mathbf{G}}\varrho_{(s),z}^{\mathbf{G}} = \varepsilon_{C_{\mathbf{G}^*}(s)} \sum_{(t)} \varepsilon_{C_{\mathbf{L}^*}(t)}\varrho_{(t),\mathfrak{h}_{\mathbf{L}}(z)z_{\mathbf{L}}}^{\mathbf{L}}$$

for the semisimple characters, where the sum is over the semisimple classes (t) of L^* that fuse to (s) in G^*.

On the other hand, by Remark 6.3 and Remark 6.4 the existence of a regular embedding $\mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ gives the possibility to see the uniform almost characters of $\mathbf{G}^F$ as restrictions of the uniform almost characters of $\tilde{\mathbf{G}}^{\tilde{F}}$. In particular, Clifford theory can be applied, see Remark 3.15.

This is used to write down the general values of the irreducible constituents of every uniform almost characters. Explicitly, by Clifford theory, the irreducible constituents of the restriction $\chi|_{\mathbf{G}^F}$ form a $\tilde{\mathbf{G}}^{\tilde{F}}$-orbit for all $\chi \in \mathrm{Irr}(\tilde{\mathbf{G}}^{\tilde{F}})$. For example, assume that χ splits into two irreducible characters of $\mathbf{G}^F$. Write the restriction $\chi|_{\mathbf{G}^F} = \theta_1 + \theta_2$, with the constituents $\theta_1, \theta_2 \in \mathrm{Irr}(\mathbf{G}^F)$

being $\tilde{\mathbf{G}}^{\tilde{F}}$-conjugate. This implies that for classes $g^{\tilde{\mathbf{G}}^{\tilde{F}}}$, $g \in \mathbf{G}^F$, that do not split as classes of $\mathbf{G}^F$, $\theta_1(g) = \theta_2(g) = \chi(g)/2$. Otherwise, if the class $g^{\tilde{\mathbf{G}}^{\tilde{F}}}$ splits into the classes $g_1^{\mathbf{G}^F}$ and $g_2^{\mathbf{G}^F}$ then $\theta_1(g_1) = \theta_2(g_2)$ and $\theta_1(g_2) = \theta_2(g_1)$. With the information that $\chi(g_i) = \theta_1(g_i) + \theta_2(g_i)$ $(i = 1, 2)$, we can write this portion of the character table in the following way

	g_1	g_2
θ_1	f	$\chi(g) - f$
θ_2	$\chi(g) - f$	f

with f an unknown algebraic integer. This discussion can be generalized, for classes/characters that split into more than 2, easily. To explicitly compute the action of $\tilde{\mathbf{G}}^{\tilde{F}}$ on the splitting classes of $\mathbf{G}^F$ we use Remark 1.64. For each $z \in H^1(F, Z(\mathbf{G}))$ we choose an element $g_z \in \mathbf{G}$ such that $g_z^{-1}F(g_z) \in Z(\mathbf{G})$ represents z. Then, up to $\mathbf{G}^F$-conjugacy the action of $\tilde{\mathbf{G}}^{\tilde{F}}$ is given by conjugating with those elements g_z. We introduce these unknowns f in all the characters that decompose, at each splitting class.

Then, thanks to the system of equations discussed in Remark 6.10, most of the unknowns f can be determined.

In practice, we take the following steps:

(a) We write every irreducible character of $\mathbf{G}^F$ in terms of the uniform almost characters. To do this, we introduce unknowns for every splitting character at every splitting class.

(b) We identify which of these irreducible characters are regular:

For example, we consider a uniform almost character (known from Lübeck's table) which is the sum of two irreducible constituents $\chi_1, \chi_2 \in \mathscr{E}(\mathbf{G}^F, s)$ ($s \in \mathbf{G}^{*F^*}$ is also known from Lübeck's table). Then:

- By taking scalar products $\langle \Gamma_z, \chi_1 + \chi_2 \rangle$ (which must be 0 or 1) we determine if χ_1, χ_2 are regular characters, for $z \in H^1(F, Z(\mathbf{G}))$.
- If they are, we can choose $\langle \Gamma_1, \chi_1 \rangle = 1$. In other words, we have $\chi_1 = \chi_{(s),1}$.
- Then, we fix the unknowns of χ_1, on some unipotent elements, in such a way that the scalar product above is correct.
- This also fixes the corresponding unknowns of χ_2. Moreover, it becomes possible to determine $z \in H^1(F, Z(\mathbf{G}))$ for which $\chi_2 = \chi_{(s),z}$

(c) Next, we can determine which of these regular characters are also semisimple, by checking if $C_{\mathbf{G}^*}(s)^\circ$ is a torus (this is known from Lübeck's table).

(d) Due to Remark 5.42 it is also possible to identify the semisimple characters, by inspecting character degrees.

(e) At this point, we can already reduce the number of unknowns in the table by imposing ${}^{g_{z'}}\chi_{(s),z} = \chi_{(s),zz'}$ and ${}^{g_{z'}}\varrho_{(s),z} = \varrho_{(s),zz'}$ (since ${}^{g_{z'}}\Gamma_z = \Gamma_{zz'}$), where, by Remark 1.64, conjugation by $g_{z'} \in \mathbf{G}$ for all $z' \in H^1(F, Z(\mathbf{G}))$ is equivalent to conjugating with elements of $\tilde{\mathbf{G}}^{\tilde{F}}$.

(f) Regularity and semisimplicity both give us systems of equations. These involve some of the unknowns and the values of the Gel'fand–Graev characters of Levi subgroups (Theorems 5.35 and 5.41, see also Remark 6.10). We can use these equations to fix all the values of regular and semisimple characters on the unipotent classes of $\mathbf{G}^F$.

(g) For every semisimple class (s) of $\mathbf{G}^{*F^*}$, we determine $\varphi_s \in \operatorname{Irr}(Z(\mathbf{G}^F))$ such that

$$\mathbb{C}\mathscr{E}(\mathbf{G}^F, s) \subseteq \mathrm{CF}(\mathbf{G}^F)^{\varphi_s},$$

according to Proposition 5.54. To do this, we compare the values on central elements of any uniform almost character in the span of $\mathscr{E}(\mathbf{G}^F, s)$.

(h) Then, for $s \in \mathbf{G}^{*F^*}$ semisimple we directly obtain the character values for all $z \in Z(\mathbf{G}^F)$ and $u \in \mathbf{G}^F_{\text{uni}}$:

$$\chi_{(s),1}(zu) = \mathbf{t}_z \chi_{(s),1}(u) = \varphi_s(z)\chi_{(s),1}(u).$$

The same applies for the semisimple character $\varrho_{(s),1}$.

(i) Let $z \in \mathbf{G}^F$ be a semisimple element such that $\mathbf{L} = C_{\mathbf{G}}(z)$ is a Levi subgroup. Then, by Remark 4.3 for any $\chi \in \operatorname{Irr}(\mathbf{G}^F)$, we have $\big({}^*R^{\mathbf{G}}_{\mathbf{L}}\chi\big)(zu) = \chi(zu)$.

(j) When χ, from the previous point, is regular, we can determine the decomposition of ${}^*R^{\mathbf{G}}_{\mathbf{L}}\chi$ in $\bigoplus_{\varphi \in \operatorname{Irr}(Z(\mathbf{L}^F))} \mathrm{CF}(\mathbf{L})^{\varphi}$. In practice, we do it for the uniform almost character, say R, of which χ is a constituent and then apply Lemma 5.57. To do this, we use the fact that ${}^*R^{\mathbf{G}}_{\mathbf{L}}R$ has constituents in $\mathrm{CF}(\mathbf{L}^F)^{\varphi}$, for some $\varphi \in \operatorname{Irr}(Z(\mathbf{L}^F))$, if $\langle {}^*R^{\mathbf{G}}_{\mathbf{L}}R, \Gamma^{\mathbf{L}}_{\varphi,z}\rangle \neq 0$ for some $z \in H^1(F, Z(\mathbf{L}))$.

When χ is semisimple, we use the information for regular characters and the fact that $D_{\mathbf{G}}$ and ${}^*R^{\mathbf{G}}_{\mathbf{L}}$ commute to get the decomposition of ${}^*R^{\mathbf{G}}_{\mathbf{L}}\chi$ in $\bigoplus_{\varphi \in \operatorname{Irr}(Z(\mathbf{L}^F))} \mathrm{CF}(\mathbf{L})^{\varphi}$.

(k) Thanks to points (f), (i) and (j), we can relate (and fix all) the values $\chi(zu)$ to the values $({}^*R^{\mathbf{G}}_{\mathbf{L}}\chi)(u)$, for z such that $C_{\mathbf{G}}(z) = \mathbf{L}$ is a Levi subgroup and $u \in \mathbf{L}$ unipotent, when χ is regular or semisimple.

(l) At this point, we are left with unknown character values only on elements zu, with z semisimple such that $\mathbf{C} = C_{\mathbf{G}}(z)$ is not a Levi subgroup. By the character formula at the end of Section 6.5, these unknowns appear in ${}^*R^{\mathbf{G}}_{\mathbf{L}}\chi$ for every F-stable Levi subgroup contained in $\mathbf{C}$. Then, thanks to point (j) we can relate these unknowns to the values of $\big({}^*R^{\mathbf{G}}_{\mathbf{L}}\chi\big)(u)$ for all $u \in \mathbf{L}^F$ unipotent.

Remark 6.11. In $\mathrm{SL}_4(q)$ every irreducible character that we need to decompose is either regular or semisimple. Unfortunately, for $\mathrm{Spin}_8^+(q)$ this is not the case. Notice that the Lusztig restriction of such characters to any F-stable Levi subgroup might have regular/semisimple characters as constituents. So it is not clear what their scalar product with the Gel'fand–Graev characters (of the Levi subgroup) is. However, notice that by the discussion in [Bo00, beween Corollary 1.8.6 and (1.8.7)] all the characters in the same Lusztig series share with the regular/semisimple characters an important property. Namely, if conjugation with $g_z \in \mathbf{G}$, for $z \in H^1(F, Z(\mathbf{G}))$, fixes all the regular/semisimple characters of a Lusztig series, then it does so for all the elements of that series. In practice, the method to decompose these characters is similar to the one for regular/semisimple characters, and is better understood with an example. We treat one in Section 18.3.

For the computations in Part II on $\mathrm{SL}_4(q)$, we use a somehow different but equivalent approach. We want to highlight how the modified Gel'fand–Graev characters really do contain the missing information in the table of uniform almost characters. Then, we use the most basic remarks about the modified Gel'fand–Graev characters to complete the generic character table, and nothing else.

On the contrary, for $\mathrm{Spin}_8^+(q)$ we show how to work with the formalism of regular/semisimple characters to complete the character table.

Part II

The case of $\mathrm{SL}_4(q)$ for $q \equiv 1 \pmod 4$

This part of the work is devoted to completing the computation of the generic character table of $\mathrm{SL}_4(q)$ where q is a power of a prime such that $q \equiv 1 \pmod 4$. The starting point is the partial table consisting of the uniform almost characters of $\mathrm{SL}_4(q)$, computed automatically by computer programs of Frank Lübeck. These are integer linear combinations of the Deligne–Lusztig characters of $\mathrm{SL}_4(q)$, see Section 6.1.

This computation is important as an example of finite groups of Lie type built from a simple algebraic group with disconnected centre. It is small enough to be reasonable to try and compute the full table but big enough such that the theory of Deligne–Lusztig fails to yield the full table without auxiliary methods.

This case is also a preparation to the computation of more complicated tables, for example $D_4(q)$ (simply connected case, for q odd). Therefore, one of the ulterior goals is to try to use methods which are easily adaptable to other groups. For instance, no knowledge of generic character tables of subgroups of $\mathrm{SL}_4(q)$ is assumed.

The computations made to complete the character table include: the determination of modified Gel'fand–Graev characters of $\mathrm{SL}_4(q)$ and of two Levi subgroups L_1 and L_2, with disconnected centre, respectively of type $A_1(q)^2$ and $A_1(q^2)$, Harish–Chandra/Lusztig induction/restriction from/to L_1 and L_2, and Clifford theory.

The main information needed for most of these computations is the fusion of the unipotent classes. This means computing the conjugacy classes of a maximal unipotent subgroup of $\mathrm{SL}_4(q)$ and identify their $\mathrm{SL}_4(q)$-classes.

7 The simply connected group of type A_3 and the finite groups $\mathrm{SL}_4(q)$

For the rest of this part of the work we fix the following notation.

Notation 7.1. We denote by $\mathbf{G}$ the simply connected algebraic group of type A_3 defined over $\mathbb{F}_q$ (which is $\mathrm{SL}_4(\bar{\mathbb{F}}_q)$), where q is a prime power such that $q \equiv 1 \pmod 4$. Then, we denote by F the standard Frobenius morphism associated with the rational structure of $\mathbf{G}$, and by $G = \mathrm{SL}_4(q)$ the finite group of Lie type $\mathbf{G}^F$. This is the special linear group of 4×4 matrices with entries in $\mathbb{F}_q$ and determinant one.

We will denote by $\mathbf{B}_0$ the Borel subgroup of $\mathbf{G}$ consisting of upper-triangular matrices and by $\mathbf{U}_0 = R_u(\mathbf{B}_0)$ (the unipotent radical of $\mathbf{B}_0$) the maximal unipotent subgroup of $\mathbf{G}$ consisting of upper-triangular matrices with ones on the diagonal. Moreover, we write $\mathbf{T}_0$ for the maximally split torus of $\mathbf{G}$ consisting of diagonal matrices, such that $\mathbf{B}_0 = \mathbf{T}_0 \ltimes \mathbf{U}_0$, and its normalizer is denoted by $\mathbf{N}_0 = N_{\mathbf{G}}(\mathbf{T}_0)$.

The Weyl group of $\mathbf{G}$ is $W = \mathbf{N}_0/\mathbf{T}_0$. The root system associated to $\mathbf{G}$ and relative to $\mathbf{T}_0$ is of type A_3 and is denoted by Φ_{A_3}. A base of positive roots of Φ_{A_3} is denoted Δ_{A_3} and the set of positive roots relative to this base is denoted by $\Phi^+_{A_3}$. We denote by $\mathcal{I}$ the index set of Δ_{A_3}, and for all $i \in \mathcal{I}$ we denote the root subgroups by $\mathbf{U}_i := \mathbf{U}_{\alpha_i}$.

Analogously for the subgroups of the finite group G, we will write $B_0 = \mathbf{B}_0^F$, $U_0 = \mathbf{U}_0^F$, $T_0 = \mathbf{T}_0^F$, $N_0 = N_G(\mathbf{T}_0)$ and $U_i = U_{\alpha_i} = \mathbf{U}_i^F$.

When working with the finite fields, we will need to choose generators of $\mathbb{F}_q^\times$ and $\mathbb{F}_{q^2}^\times$. From now on $\mu \in \mathbb{F}_q^\times$ is fixed such that $\langle \mu \rangle = \mathbb{F}_q^\times$, and $\rho \in \mathbb{F}_{q^2}^\times$ such that $\langle \rho \rangle = \mathbb{F}_{q^2}^\times$ and $\rho^{q+1} = \mu$.

Lastly, when possible, the polynomial expressions in q will be given as product of the cyclotomic polynomials. In what follows, they will be denoted by $\Phi_1 = q - 1$, $\Phi_2 = q + 1$, $\Phi_3 = q^2 + q + 1$ and $\Phi_4 = q^2 + 1$. Moreover, we denote the k-th power of the n-th root of unity in $\mathbb{C}$ by $\zeta_n^k := \mathrm{e}^{2\pi i \frac{k}{n}}$.

It is easy to work with SL_4 thanks to its matrix representation. However, to avoid confusions and to make the results easier to read, we will write the elements of $\mathbf{G}$ and G using the Steinberg presentation.

7.1 Roots, Chevalley generators and Chevalley relations

The root system Φ_{A_3} of $\mathbf{G}$ with respect to the maximal torus $\mathbf{T}_0$ is of type A_3. It has a base of simple roots $\Delta_{A_3} = \{\alpha_1, \alpha_2, \alpha_3\}$ (with index set $\mathcal{I} = \{1, 2, 3\}$) with Cartan matrix

$$\langle \alpha_i, \alpha_j^\vee \rangle_{i,j \in \mathcal{I}} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

A_3

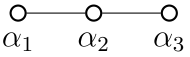

Figure 1: Dynkin diagram of type A_3.

The positive roots $\Phi_{A_3}^+$ of this root system are given in Table 1.

Table 1: Positive roots of Φ_{A_3} written as sum of simple roots and as images under simple reflections of simple roots.

α_1	α_1	α_1
α_2	α_2	α_2
α_3	α_3	α_3
α_4	$\alpha_1 + \alpha_2$	$s_{\alpha_1}(\alpha_2)$
α_5	$\alpha_2 + \alpha_3$	$s_{\alpha_3}(\alpha_2)$
α_6	$\alpha_1 + \alpha_2 + \alpha_3$	$s_{\alpha_3} s_{\alpha_1}(\alpha_2)$

We denote by Id_4 the identity matrix of SL_4 and by e_{ij} the 4×4 matrices with entry 1 in position (i, j) and 0 everywhere else. Then, for $i = 1, ..., 6$ the root maps $u_i := u_{\alpha_i} : \bar{\mathbb{F}}_q \to \mathbf{U}_i$ are given by

$$\begin{aligned} u_1(t) &:= \mathrm{Id}_4 + te_{12}, & u_4(t) &:= \mathrm{Id}_4 + te_{13}, \\ u_2(t) &:= \mathrm{Id}_4 + te_{23}, & u_5(t) &:= \mathrm{Id}_4 + te_{24}, \\ u_3(t) &:= \mathrm{Id}_4 + te_{34}, & u_6(t) &:= \mathrm{Id}_4 + te_{14}. \end{aligned}$$

Similarly, the opposite maximal unipotent subgroup $\mathbf{U}^-$ is generated by $v_i(t) := u_i(t)^{\mathrm{T}}$ for $i = 1, ..., 6$ and $t \in \bar{\mathbb{F}}_q$ ($^{\mathrm{T}}$ stands for the matrix transpose).

Then, the other Chevalley generators are given by (see Notation 1.47)

$$n_i(t) := u_i(t) v_i(-t^{-1}) u_i(t) \in \mathbf{N}_0, \text{ and}$$

$$h_i(t) := n_i(t) n_i(-1) \in \mathbf{T}_0,$$

for $t \in \bar{\mathbb{F}}_q^\times$ and $i = 1, ..., 6$. Then, $\mathbf{T}_0 = \langle h_i(t) \mid i \in \mathcal{I}, t \in \bar{\mathbb{F}}_q^\times \rangle$ and $\mathbf{N}_0 = \langle \mathbf{T}_0, n_i \mid i \in \mathcal{I} \rangle$, where $n_i = n_i(1)$ can be taken as representatives of the simple reflections $s_{\alpha_i} \in W$, for $i \in \mathcal{I}$.

Table 2: Action of the Weyl group W on $u_j(t)$ via conjugation with the representatives n_i of the simple reflections, i.e. $n_j u_i(t) n_j{}^{-1}$ for $t \in \bar{\mathbb{F}}_q$, $i = 1, 2, ..., 6$ and $j \in \mathcal{I}$.

	n_1	n_2	n_3
$u_1(t)$	$v_1(-t)$	$u_4(-t)$	$u_1(t)$
$u_2(t)$	$u_4(t)$	$v_2(-t)$	$u_5(-t)$
$u_3(t)$	$u_3(t)$	$u_5(t)$	$v_3(-t)$
$u_4(t)$	$u_2(-t)$	$u_1(t)$	$u_6(-t)$
$u_5(t)$	$u_6(t)$	$u_3(-t)$	$u_2(t)$
$u_6(t)$	$u_5(-t)$	$u_6(t)$	$u_4(t)$

Table 3: Commutation relations in $\mathbf{U}_0$ $(s, t \in \bar{\mathbb{F}}_q)$.

$$[u_1(t), u_2(s)] = u_4(ts)$$
$$[u_1(t), u_5(s)] = u_6(ts)$$
$$[u_2(t), u_3(s)] = u_5(ts)$$
$$[u_3(t), u_4(s)] = u_6(-ts)$$

Notation 7.2. In this part of the thesis, we will denote a general element of $\mathbf{T}_0$ by

$$h(t_1, t_2, t_3) := h_1(t_1) h_2(t_2) h_3(t_3)$$

for $t_1, t_2, t_3 \in \bar{\mathbb{F}}_q^\times$.

The centre of $\mathbf{G}$ is disconnected and is given by (it follows from Theorem 1.18 (b))

$$Z(\mathbf{G}) = \left\{ h(t, t^2, t^3) \mid t \in \bar{\mathbb{F}}_q^\times, t^4 = 1 \right\} \cong C_4.$$

Notice that for $q \equiv 1 \pmod 4$ the field $\mathbb{F}_q$ has four solutions to the equation $t^4 = 1$. This means that the centre of $\mathbf{G}$ is also the centre of the finite group $\mathrm{SL}_4(q)$ (since $Z(\mathbf{G}^F) = Z(\mathbf{G})^F$ by Proposition 1.60).

By straightforward computations, we can determine the Chevalley relations (see Section 1.4).

Proposition 7.3. *For $t \in \bar{\mathbb{F}}_q$, $s \in \bar{\mathbb{F}}_q^\times$, $i = 1, ..., 6$ and $j \in \mathcal{I}$ the following hold:*

- $h_j(s) u_i(t) h_j(s)^{-1} = u_i(s^{\langle \alpha_i, \alpha_j^\vee \rangle} t)$.
- *The action of W on $\mathbf{U}_0$ is given in Table 2.*
- *The commutation relations in $\mathbf{U}_0$ are given in Table 3.*
- *The action of W on $\mathbf{T}_0$ is given in Table 4.*

Table 4: Action of the Weyl group W on $\mathbf{T}_0$ via conjugation with the representatives n_i of the simple reflections, i.e. $n_i h_j(t) n_i^{-1}$ for $t \in \bar{\mathbb{F}}_q^\times$ and $i, j \in \mathcal{I}$.

	n_1	n_2	n_3
$h_1(t)$	$h_1(t^{-1})$	$h_1(t)h_2(t)$	$h_1(t)$
$h_2(t)$	$h_1(t)h_2(t)$	$h_2(t^{-1})$	$h_2(t)h_3(t)$
$h_3(t)$	$h_3(t)$	$h_2(t)h_3(t)$	$h_3(t^{-1})$

7.2 The finite special linear groups

Let F be the untwisted Frobenius endomorphism of $\mathbf{G}$ associated to the $\mathbb{F}_q$-structure of $\mathbf{G}$. Explicitly, F acts on the matrices in $\mathbf{G}$ by raising every entry to the q-th power. Then $G = \mathbf{G}^F$ is a finite group.

By the discussion after Proposition 1.57 the group G inherits the Chevalley generators and the Chevalley relations from $\mathbf{G}$ by restricting all formulas from Proposition 7.3 to $\mathbb{F}_q$.

Proposition 7.4. *The finite group of Lie type $G = \mathbf{G}^F$ has order ([MaTe11, Table 24.1])*

$$|G| = q^6 \Phi_1^3 \Phi_2^2 \Phi_3 \Phi_4.$$

It is generated by unipotent elements,

$$G = \langle u_i(t), v_i(t) \mid i \in \mathcal{I}, t \in \mathbb{F}_q \rangle .$$

It has a split BN-pair formed by the Borel subgroup (upper triangular matrices) $B_0 = T_0 \ltimes U_0$ with $T_0 = \mathbf{T}_0^F = \langle h_i(t) \mid i \in \mathcal{I},\, t \in \mathbb{F}_q^\times \rangle$, $U_0 = \mathbf{U}_0^F = \langle u_i(t) \mid i \in \mathcal{I},\, t \in \mathbb{F}_q \rangle$ and the normalizer of T_0, $N_0 = \mathbf{N}_0^F = \langle T_0, n_i \mid i \in \mathcal{I} \rangle$.

The action of the Weyl group $W = N_0/T_0$ on the unipotent elements is given in Table 2 when identifying the simple reflection s_{α_i} of W with the representative n_i for $i \in \mathcal{I}$.

The action of T_0 on the unipotent elements is given by

$$h_i(s) u_\alpha(t) h_i(s)^{-1} = u_\alpha(t s^{\langle \alpha, \alpha_i^\vee \rangle})$$

for $s \in \mathbb{F}_q^\times$, $t \in \mathbb{F}_q$ and $i \in \mathcal{I}$.

The commutation relations of the unipotent elements are given in Table 3.

Furthermore, W acts on T_0 as

$$n_i h_\alpha(t) n_i^{-1} = h_{s_{\alpha_i}(\alpha)}(t)$$

for $i \in \mathcal{I}$, $\alpha \in \Phi$ and $t \in \mathbb{F}_q^\times$ (where $h_{-\alpha}(t) = h_\alpha(t^{-1})$ and $h_{\alpha+\beta}(t) = h_\alpha(t) h_\beta(t)$), see Table 4.

The centre of G is $Z(G) = Z(\mathbf{G})^F = \{h(t, t^2, t^3) \mid t^4 = 1\} \cong C_4$, for $q \equiv 1 \pmod 4$.

Finally, the F-classes of the centre are

$$H^1(F, Z(\mathbf{G})) \cong Z(\mathbf{G}).$$

Notation 7.5. The elements of the centre, $Z = Z(G)$, will be denoted simply by

$$h_Z(k_a) := h\left(\mu^{\frac{q-1}{4} k_a}, \mu^{\frac{q-1}{2} k_a}, \mu^{3 \frac{q-1}{4} k_a}\right)$$

for $k_a = 0, 1, 2, 3$.

Remark 7.6. As discussed in Remark 1.64 we can associate to each element $z \in H^1(F, Z(\mathbf{G}))$ a representative $t_z \in \mathbf{T}_0$. We choose for $z = h_Z(k_a)$

$$t_{k_a} = h(\omega^{\frac{q^4-1}{4(q-1)} j}, \omega^{\frac{q^4-1}{2(q-1)} j}, \omega^{3 \frac{q^4-1}{4(q-1)} j}) \in \mathbf{T}_0$$

where $\omega \in \mathbb{F}_{q^4}^\times$ is a generator such that $\mu = \omega^{q^3+q^2+q+1}$, for $k_a = 0, 1, 2, 3$.

8 Fusion of unipotent classes

In this section, we compute the fusion of the unipotent classes. In other words, we explicitly write to which conjugacy class of G belong the elements of each conjugacy class of U_0.

As discussed in Section 6.3, we start by giving the conjugacy classes of U_0.

Remark 8.1. The numbering of the positive roots is such that $U_{k+1} \cdots U_6 \lhd U_k U_{k+1} \cdots U_6$ for each $k = 1, ..., 5$. This can be seen by the commutation relations in Table 3.

Moreover, every element $u \in U_0$ can be written uniquely as the ordered product

$$u = u_1(r_1)u_2(r_2) \cdots u_6(r_6)$$

with $r_i \in \mathbb{F}_q$.

Proposition 8.2. *The conjugacy classes of U_0 have representatives listed in the first column of Table 5. They are also written in Table 59, in the appendix, with their U_0-orbits.*

Remark 8.3. There are

$$2q^3 + q^2 - 2q$$

different conjugacy classes in U_0, in agreement with [GoRoe09, Table 1].

By acting with T_0 we obtain representatives of the unipotent classes of B_0.

Proposition 8.4. *Representatives of the unipotent classes of B_0 are given in the fourth column of Table 5.*

We give an example of the computation of the fusion from U_0 to B_0.

Example 8.5. Fusion of $u_1(r_1)u_2(r_2)u_3(r_3)$ to B_0, for $r_1, r_2, r_3 \in \mathbb{F}_q^\times$:

We have, by Proposition 7.4,

$$^{h_1(s_1)h_2(s_2)h_3(s_3)}(u_1(r_1)u_2(r_2)u_3(r_3)) = u_1(r_1 s_1^2 s_2^{-1})u_2(r_2 s_1^{-1} s_2^2 s_3^{-1})u_3(r_3 s_2^{-1} s_3^2)$$

where $s_1, s_2, s_3 \in \mathbb{F}_q^\times$.

It is clear that the argument of u_3 can be set to 1 by imposing $s_2 = r_3 s_3^2$. Then, the expression becomes

$$u_1(r_1 r_3^{-1} s_1^2 s_3^{-2})u_2(r_2 r_3^2 s_1^{-1} s_3^3)u_3(1).$$

Again, we can set the argument of u_2 to 1 by imposing $s_1 = r_2 r_3^2 s_3^3$. Then, we obtain

$$u_1(r_1 r_2^2 r_3^3 s_3^4)u_2(1)u_3(1).$$

Finally, the argument of u_1 can be set to 1 only if $r_1 r_2^2 r_3^3$ is a fourth power in $\mathbb{F}_q^\times$. Otherwise, it can be set to a representative of the coset $r_1 r_2^2 r_3^3 (\mathbb{F}_q^\times)^4$.

In other words, we proved that the unipotent elements $u_1(r_1)u_2(r_2)u_3(r_3)$ are conjugate to $u_1(\mu^k)u_2(1)u_4(1)$ if $r_1 r_2^2 r_3^3 \in \mu^k (\mathbb{F}_q^\times)^4$.

Notice that, this reflects the fact that the regular unipotent classes are parametrized by $H^1(F, Z)$ (Proposition 5.4) which in this case is isomorphic to Z.

We complete the fusion of the unipotent classes, by conjugating with elements of the Weyl group W. We conjugate only by elements of W that send a given representative to elements of U_0. We check at the end that the resulting list indeed contains representative of distinct classes (by explicit matrix computations). Also, the list coincides with the unipotent classes computed by Lübeck for the partial character table that he provided.

We give an example of the computation of the fusion from B_0 to G.

Table 5: Representatives u of the classes of U_0 and a representative in B_0. The representatives u are given with parameters $r_j \in \mathbb{F}_q^\times$ $(j = 1, ..., 6)$, and they are written in the order they are computed thanks to the algorithm discussed in Section 6.3. The second column contains the number of classes of each type. The third column contains the centralizer order in U_0 of u. The last two columns contain representatives of the B_0-class of u.

u	#classes	$\lvert C_{U_0}(u)\rvert$	Representative in B_0	Conditions
1	1	q^6	1	
$u_6(r_6)$	$q-1$	q^6	$u_6(1)$	
$u_5(r_5)$	$q-1$	q^5	$u_5(1)$	
$u_4(r_4)$	$q-1$	q^5	$u_4(1)$	
$u_4(r_4)u_5(r_5)$	$(q-1)^2$	q^5	$u_4(\mu^k)u_5(1)$	$r_4r_5 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_3(r_3)$	$q-1$	q^4	$u_3(1)$	
$u_3(r_3)u_4(r_4)$	$(q-1)^2$	q^4	$u_3(1)u_4(1)$	
$u_2(r_2)$	$q-1$	q^4	$u_2(1)$	
$u_2(r_2)u_3(r_3)$	$(q-1)^2$	q^3	$u_2(1)u_3(1)$	
$u_2(r_2)u_6(r_6)$	$(q-1)^2$	q^4	$u_2(\mu^k)u_6(1)$	$r_2r_6 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_1(r_1)$	$q-1$	q^4	$u_1(1)$	
$u_1(r_1)u_2(r_2)$	$(q-1)^2$	q^3	$u_1(1)u_2(1)$	
$u_1(r_1)u_3(r_3)$	$(q-1)^2$	q^4	$u_1(\mu^k)u_3(1)$	$r_1r_3 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_1(r_1)u_5(r_5)$	$(q-1)^2$	q^4	$u_1(1)u_5(1)$	
$u_1(r_1)u_2(r_2)u_3(r_3)$	$(q-1)^3$	q^3	$u_1(\mu^k)u_2(1)u_3(1)$	$r_1r_2^2r_3^3 \in \mu^k(\mathbb{F}_q^\times)^4$
$u_1(r_1)u_3(r_3)u_5(r_5)$	$(q-1)^3$	q^4	$u_1(\mu^k)u_3(1)u_5(1)$	$r_1r_3r_5^2 \in \mu^k(\mathbb{F}_q^\times)^4$

Example 8.6. Fusion of $u_1(\mu^k)u_3(1)$ to G, for $k = 0, 1$:

Conjugating with W yields the following list of conjugates in U_0,

$$u_1(\mu^k)u_3(1),\ u_2(\mu^k)u_6(1),\ u_3(\mu^k)u_1(1),\ u_4(-\mu^k)u_5(1),\ u_5(-\mu^k)u_4(1),\ u_6(\mu^k)u_2(1).$$

With the commutation relations, we reorder these elements and check to what element of B_0 they are conjugate (with Table 59 in the appendix).

In the ordered form, the list becomes (recall that $-1 \in (\mathbb{F}_q^\times)^2$)

$$u_1(\mu^k)u_3(1),\ u_2(\mu^k)u_6(1),\ u_4(\mu^k)u_5(1).$$

Therefore, these elements belong to the same conjugacy class of G.

Proposition 8.7. *There are 9 distinct unipotent conjugacy classes in G. They are fused from U_0 in G according to Table 7. A list of representatives with corresponding centralizer order in G is given in Table 6.*

Table 6: Representatives u of the unipotent classes of G and the order of their centralizer. They are ordered according to their Jordan normal form (partitions of 4), given in the first column.

	u	$\lvert C_G(u)\rvert$
1^4	1	$q^6\Phi_1^3\Phi_2^2\Phi_3\Phi_4$
21^2	$u_1(1)$	$q^6\Phi_1^2\Phi_2$
2^2	$u_1(\mu^k)u_3(1)$, $k=0,1$	$2q^5\Phi_1\Phi_2$
31	$u_1(1)u_2(1)$	$q^4\Phi_1$
4	$u_1(\mu^k)u_2(1)u_3(1)$, $k=0,1,2,3$	$4q^3$

Table 7: Representatives of the unipotent classes of G and their fusion from U_0. When not specified, the condition on the parameters is simply $r_i \in \mathbb{F}_q^\times$.

Representative in G	Representatives in U_0	Conditions
1	1	
$u_1(1)$	$u_1(r_1)$	
	$u_2(r_2)$	
	$u_3(r_3)$	
	$u_4(r_4)$	
	$u_5(r_5)$	
	$u_6(r_6)$	
$u_1(\mu^k)u_3(1)$	$u_1(r_1)u_3(r_3)$	$r_1r_3 \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_2(r_2)u_6(r_6)$	$r_2r_6 \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_4(r_4)u_5(r_5)$	$r_4r_5 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_1(1)u_2(1)$	$u_1(r_1)u_2(r_2)$	
	$u_1(r_1)u_5(r_5)$	
	$u_2(r_2)u_3(r_3)$	
	$u_3(r_3)u_4(r_4)$	
	$u_1(r_1)u_3(r_3)u_5(r_5)$	
$u_1(\mu^k)u_2(1)u_3(1)$	$u_1(r_1)u_2(r_2)u_3(r_3)$	$r_1r_2^2r_3^3 \in \mu^k(\mathbb{F}_q^\times)^4$

9 Levi subgroups with disconnected centre

The group $\mathrm{SL}_4(q)$ has two F-stable proper Levi subgroups with disconnected centre (up to conjugation).

For $I = \{\alpha_1, \alpha_3\}$ the associated Levi subgroup of $\mathbf{G}$ is $\mathbf{L}_I = \langle \mathbf{T}_0, \mathbf{U}_\alpha \mid \alpha \in \Phi_I \rangle$ with root system $\Phi_I = \pm I$.

Lemma 9.1. *The centre of* $\mathbf{L}_I$ *is disconnected and is given by*

$$Z(\mathbf{L}_I) = \left\{ h(\epsilon t, t^2, t) \mid \epsilon = \pm 1,\ t \in \bar{\mathbb{F}}_q^\times \right\}.$$

Proof. This is computed explicitly thanks to Theorem 1.18 (b). □

Lemma 9.2. *The Levi subgroup* $\mathbf{L}_I$ *has a semidirect product decomposition*

$$\mathbf{L}_I \cong \left(\mathrm{SL}_2(\bar{\mathbb{F}}_q) \times \mathrm{SL}_2(\bar{\mathbb{F}}_q) \right) \rtimes \left\langle h_2(t) \mid t \in \bar{\mathbb{F}}_q^\times \right\rangle.$$

Proof. By Corollary 1.19 we have $\mathbf{L}_I = [\mathbf{L}_I, \mathbf{L}_I] Z(\mathbf{L}_I)^\circ$. Because $\mathbf{G}$ is simply connected, $[\mathbf{L}_I, \mathbf{L}_I]$ is simply connected (by Proposition 6.9) of type $A_1 \times A_1$. This means that

$$[\mathbf{L}_I, \mathbf{L}_I] = \langle \mathbf{U}_\alpha \mid \alpha \in \Phi_I \rangle \cong \mathrm{SL}_2(\bar{\mathbb{F}}_q) \times \mathrm{SL}_2(\bar{\mathbb{F}}_q).$$

By definition of the h_α (see Notation 1.47), we have $h_{\alpha_1}(t), h_{\alpha_3}(t) \in [\mathbf{L}_I, \mathbf{L}_I]$ for any $t \in \bar{\mathbb{F}}_q^\times$. Then, we have

$$\mathbf{L}_I = [\mathbf{L}_I, \mathbf{L}_I] Z(\mathbf{L}_I)^\circ = [\mathbf{L}_I, \mathbf{L}_I] \langle h_2(t) \mid t \in \bar{\mathbb{F}}_q^\times \rangle$$

which is a semidirect product by the Chevalley relations (Proposition 7.3). □

Lemma 9.3. *The unipotent subgroup* $\mathbf{U}_I := \langle \mathbf{U}_{\alpha_1}, \mathbf{U}_{\alpha_3} \rangle$ *of* $\mathbf{L}_I$ *is abelian.*

Proof. It is straightforward from the commutation relations. □

Thanks to the command `TwistingElements` of CHEVIE ([MiChv]) we get that there are two $\mathbf{G}^F$-classes of F-stable Levi subgroups $\mathbf{G}$-conjugate to $\mathbf{L}_I$. These are of type $(I, 1_W)$ and $(I, s_{\alpha_2} s_{\alpha_1} s_{\alpha_3} s_{\alpha_2})$.

Notation 9.4. We denote by $\mathbf{L}_1$ the Levi subgroup of type $(I, 1)$ and by $\mathbf{L}_2$ the one of type $(I, s_{\alpha_2} s_{\alpha_1} s_{\alpha_3} s_{\alpha_2})$.

We denote the two finite Levi subgroups by $L_1 := \mathbf{L}_I^F = \mathbf{L}_1^F$ and $L_2 := \mathbf{L}_I^{F'} \cong \mathbf{L}_2^F$, where F' is the twisted Frobenius map obtained by composing F and conjugation with $s_{\alpha_2} s_{\alpha_1} s_{\alpha_3} s_{\alpha_2} \in W$.

We have that L_1 is a Levi of type $A_1(q)^2(q-1)$ and L_2 is of type $A_1(q^2)(q+1)$.

9.1 Split Levi subgroup

We describe here the properties of L_1.

It is easy to see that

$$L_1 = \langle T_1, U_\alpha \mid \alpha \in \Phi_I \rangle,$$

where $T_1 = \mathbf{T}_0^F = \{h(t_1, t_2, t_3) \mid t_1, t_2, t_3 \in \mathbb{F}_q^\times\}$ and $U_\alpha = \mathbf{U}_\alpha^F = \{u_\alpha(t) \mid t \in \mathbb{F}_q\}$ for all $\alpha \in \Phi_I$.

Proposition 9.5. *We have* $L_1 \cong (\mathrm{SL}_2(q) \times \mathrm{SL}_2(q)) \rtimes \{h_{\alpha_2}(t) \mid t \in \mathbb{F}_q^\times\}$*, and a maximal unipotent subgroup of* L_1 *is* $U_1 := \mathbf{U}_I^F = \langle U_{\alpha_1}, U_{\alpha_3} \rangle$.

Proof. It follows from Lemma 9.2 and Lemma 9.3, since every factor is F-stable. □

Table 8: Representatives of the unipotent classes of L_1, and of U_1 with their fusion to L_1

Repr. u_0 in L_1	$\lvert C_{L_1}(u_0)\rvert$	Repr. u in U_1	condition	$\lvert C_{U_1}(u)\rvert$
1	$\lvert L_1\rvert$	1		q^2
$u_1(1)$	$q^2\Phi_1^2\Phi_2$	$u_1(r_1)$		q^2
$u_3(1)$	$q^2\Phi_1^2\Phi_2$	$u_3(r_3)$		q^2
$u_1(\mu^k)u_3(1)$	$2q^2\Phi_1$	$u_1(r_1)u_3(r_3)$	$r_1r_3 \in \mu^k(\mathbb{F}_q^\times)^2$	q^2

Corollary 9.6. *The Levi subgroup L_1 has order $|L_1| = q^2\Phi_1^3\Phi_2^2$. Its maximal unipotent subgroup has order $|U_1| = q^2$.*

It follows from Proposition 9.5 that the unipotent conjugacy classes of $\mathrm{SL}_2(q) \times \mathrm{SL}_2(q)$ are the same in L_1 up to fusion.

Remark 9.7. The unipotent conjugacy classes of $\mathrm{SL}_2(q)$ have representatives $u_{\mathrm{SL}_2}(\mu^k)$ for u_{SL_2} an isomorphism from $\mathbb{F}_q^+$ to a maximal unipotent subgroup of $\mathrm{SL}_2(q)$ and $k = 0, 1$ (see for example [Bo11, Chapter 1.3]).

Proposition 9.8. *Representatives of the unipotent conjugacy classes of L_1 are listed in Table 8. Their centralizers (apart from the identity element) have order*

$$|C_{L_1}(u)| = (q-1)|C_{\mathrm{SL}_2(q)\times\mathrm{SL}_2(q)}(u)|/2$$

when seeing $u \in L_1$ as an element of $\mathrm{SL}_2(q) \times \mathrm{SL}_2(q)$.

Proof. Because of the preceding remark, all unipotent conjugacy classes of $\mathrm{SL}_2(q)\times\mathrm{SL}_2(q) \subset L_1$ have representatives being products of elements in a subset of $\{u_1(\mu^k),\ u_3(\mu^l) \mid k, l = 0, 1\}$.

By direct computation, it is easy to see that all the classes fuse in pairs (apart from the identity). In fact, conjugation with the elements $h_2(t)$ ($t \in \mathbb{F}_q^\times$) allows us to set to 1 the argument of exactly one of the factors $u_i(\mu^k)$.

Because of this fusion, the formula for the centralizer orders follows directly. □

Notation 9.9. We write Z_1 for the centre of L_1. We denote the elements of Z_1 by

$$h_{Z_1}(k_a, k_b) = h(\mu^{\frac{q-1}{2}k_a+k_b}, \mu^{2k_b}, \mu^{k_b})$$

for $k_a = 0, 1$, $k_b = 0, ..., q-2$.

Lemma 9.10. *The centre of* L_1 *is*

$$Z_1 = Z(\mathbf{L}_I)^F = \{h_{Z_1}(k_a, k_b) \mid k_a = 0, 1,\ k_b = 0, ..., q-2\} \cong C_2 \times C_{q-1}.$$

Proof. By Proposition 1.60 we have $Z_1 = Z(\mathbf{L}_I^F) = Z(\mathbf{L}_I)^F$. Let $t \in \bar{\mathbb{F}}_q^\times$. Then, it is clear that

$$F(h(\pm t, t^2, t)) = h(\pm t, t^2, t) \Leftrightarrow t^q = t \Leftrightarrow t \in \mathbb{F}_q^\times.$$

Every element of $\mathbb{F}_q^\times$ can be written as μ^k for $k = 0, ..., q-2$ and $-1 = \mu^{\frac{q-1}{2}}$. □

As we will work with Gel'fand–Graev characters, it is convenient to know the F-conjugacy classes of $Z(\mathbf{L}_I)$.

Lemma 9.11. *We have*

$$H^1(F, Z(\mathbf{L}_I)) \cong C_2$$

with $\ker(\mathfrak{h}_{L_1}) = \{h_Z(0), h_Z(2)\}$ *where* $\mathfrak{h}_{L_1}$ *is the canonical surjection from Lemma 5.8.*

Proof. By Lemma 5.7 we have $H^1(F, Z(\mathbf{L}_I)) = Z(\mathbf{L}_I)/\mathscr{L}(Z(\mathbf{L}_I))$. It is easy to see that $\mathscr{L}(Z(\mathbf{L}_I)) = \left\{h(t, t^2, t) \mid t \in \bar{\mathbb{F}}_q^\times\right\}$, since for elements of the maximally split torus $\mathbf{T}_0$ the map $\mathscr{L}$ acts as a $q-1$ power. In conclusion, $H^1(F, Z(\mathbf{L}_I)) \cong \{h(\pm 1, 1, 1)\}$.

Furthermore, we check that $h_Z(0), h_Z(2) \in \mathscr{L}(Z(\mathbf{L}_I))$. □

9.2 Twisted Levi subgroup

We describe here the properties of L_2.

In this case the twisted Frobenius F' has a non-trivial action on the root system Φ_I (α_1 and α_3 are interchanged). Explicitly F' acts on a generic element of $\mathbf{T}_0$ as

$$F'\left(h(t_1, t_2, t_3)\right) = h(t_3^q t_2^{-q}, t_2^{-q}, t_1^q t_2^{-q})$$

and on a generic element of $\mathbf{U}_I$ as

$$F'\left(u_1(t_1)u_3(t_3)\right) = u_1(t_3^q)u_3(t_1^q).$$

Then the root system Φ_2 of L_2 is of type A_1. It follows by explicit computations that

$$L_2 = \langle T_2, U_\alpha \mid \alpha \in \Phi_2 \rangle,$$

where the twisted torus is

$$T_2 = \mathbf{T}_0^{F'} = \{h\left(t_1, t_2, t_1^q t_2\right) \mid t_1, t_2 \in \mathbb{F}_{q^2}^\times \text{ s.t. } t_2^{q+1} = 1\}$$

and $U_\alpha = \mathbf{U}_I^{F'} = \{u_1(t)u_3(t^q) \mid t \in \mathbb{F}_{q^2}\}$ for α the positive root of Φ_2.

Proposition 9.12. *We have* $L_2 \cong \mathrm{SL}_2(q^2) \rtimes \{h(1, t, t) \mid t \in \mathbb{F}_{q^2}^\times \text{ s.t. } t^{q+1} = 1\}$.

Proof. This follows from the discussion above, where the part isomorphic to $\mathrm{SL}_2(q^2)$ has torus $\{h\left(t_1, 1, t_1^q\right) \mid t_1 \in \mathbb{F}_{q^2}^\times\}$ and unipotent subgroup $\{u_1(t)u_3(t^q) \mid t \in \mathbb{F}_{q^2}\}$. □

Corollary 9.13. *The Levi subgroup* L_2 *has order* $|L_2| = q^2\Phi_1\Phi_2^2\Phi_4$. *Its unipotent subgroup has order* $|U_2| = q^2$.

It follows from Proposition 9.12 that the unipotent conjugacy classes of $\mathrm{SL}_2(q^2)$ are the same in L_2 up to fusion. However, this time no fusion happens.

Proposition 9.14. *Representatives of the unipotent conjugacy classes of* L_2 *are listed in Table 9. The centralizers of the unipotent elements have order*

$$|C_{L_2}(u)| = (q+1)|C_{\mathrm{SL}_2(q^2)}(u)|$$

when seeing the unipotent element $u \in L_2$ *as an element of* $\mathrm{SL}_2(q^2)$.

Proof. We have that in $\mathrm{SL}_2(q^2)$ the unipotent elements $u_{\mathrm{SL}_2}(t)$ ($t \in \mathbb{F}_{q^2}^\times$) fall into two classes with representatives $u_{\mathrm{SL}_2}(1)$ and $u_{\mathrm{SL}_2}(\rho)$ depending whether t is a square of $\mathbb{F}_{q^2}^\times$ or not. These two representatives correspond in L_2, respectively, to $u_1(1)u_3(1)$ and $u_1(\rho)u_3(\rho^q)$. For $t \in \mathbb{F}_{q^2}^\times$ such that $t^{q+1} = 1$ we have

$$^{h(1,t,t)}\left(u_1(\rho)u_3(\rho^q)\right) = u_1\left(\rho t^{-1}\right) u_3\left(\rho^q t^{-q}\right).$$

Since t is a square in $\mathbb{F}_{q^2}^\times$ this expression is never equal to $u_1(1)u_3(1)$.

The centralizer order follows directly from Proposition 9.12. □

Table 9: Representatives of the unipotent classes of L_2, and of U_2 with their fusion to L_2

Repr. u_0 in L_2	$\lvert C_{L_2}(u_0)\rvert$	Repr. u in U_2	condition	$\lvert C_{U_2}(u)\rvert$
1	$q^2\Phi_1\Phi_2^2\Phi_4$	1		q^2
$u_1(\rho^k)u_3(\rho^{qk})$	$2q^2\Phi_2$	$u_1(\lambda)u_3(\lambda^q)$	$\lambda \in \rho^k(\mathbb{F}_{q^2}^\times)^2$ $(k = 0, 1)$	q^2

Notation 9.15. We write Z_2 for the centre of L_2. We denote the elements of Z_2 by

$$h_{Z_2}(k_a) = h(\rho^{-qk_a\frac{q-1}{2}}, \rho^{k_a(q-1)}, \rho^{k_a\frac{q-1}{2}})$$

for $k_a = 0, ..., 2q + 1$.

Lemma 9.16. *The centre of L_2 is*

$$Z_2 = Z(\mathbf{L}_I)^{F'} = \left\{h\left(\rho^{-qk_a\frac{q-1}{2}}, \rho^{k_a(q-1)}, \rho^{k_a\frac{q-1}{2}}\right) \mid k_a = 0, ..., 2q + 1\right\}.$$

Proof. By Proposition 1.60, we have $Z_2 = Z(\mathbf{L}_I^{F'}) = Z(\mathbf{L}_I)^{F'}$. Let $t \in \bar{\mathbb{F}}_q^\times$. Then, it is clear that

$$F'(h(\pm t, t^2, t)) = h(\pm t, t^2, t) \Leftrightarrow t^{2q+2} = 1 \Leftrightarrow t = \rho^{k\frac{q-1}{2}}$$

for $k = 0, ..., 2q + 1$ and $-1 = \rho^{\frac{q^2-1}{2}}$. □

We will work with Gel'fand–Graev characters, so it is convenient to know the F'-conjugacy classes of $Z(\mathbf{L}_I)$.

Lemma 9.17. *We have*

$$H^1(F', Z(\mathbf{L}_I)) \cong C_2$$

with $\ker(\mathfrak{h}_{L_1}) = \{h_Z(0), h_Z(2)\}$ *where* $\mathfrak{h}_{L_1}$ *is the canonical surjection from Lemma 5.8.*

The element $z_{\mathbf{L}_2}$ *(Remark 5.34) is* $h(-1, 1, 1)$*.*

Proof. By Lemma 5.7, we have $H^1(F', Z(\mathbf{L}_I)) = Z(\mathbf{L}_I)/\mathscr{L}'(Z(\mathbf{L}_I))$, where $\mathscr{L}'$ is the Lang map defined by F'. Explicitly, $\mathscr{L}'(Z(\mathbf{L}_I)) = \{h(t, t^2, t) \mid t \in \bar{\mathbb{F}}_q^\times\}$. Therefore, the centre $Z(\mathbf{L}_I)$ has two F'-conjugacy classes $H^1(F', Z(\mathbf{L}_I)) \cong \{h(\pm 1, 1, 1)\}$. Furthermore, we check that $h_Z(0), h_Z(2) \in \mathscr{L}'(Z(\mathbf{L}_I))$.

The element $z_{\mathbf{L}_2}$ is given explicitly by Bonnafé in [Bo05, Table 1]. □

By writing the general elements of $\mathbf{G}$ in Bruhat form (Theorem 1.36), it is possible to find a preimage to $g^{-1}F(g) = \dot{w}$ for $w = s_2s_1s_3s_2$.

Lemma 9.18. *The element*

$$g_w = u_4(-\rho)u_5(\rho)h_2(\rho^q - \rho)n_2n_1n_3n_2u_4(-1)u_5(1) \in \mathbf{G}$$

is such that $g_w^{-1}F(g_w) = n_2n_1n_3n_2$.

Proof. This is a straightforward verification by using the matrix representation of the Steinberg presentation. □

With this element, we can explicitly apply $\operatorname{res}^{\mathbf{G}}_{\mathbf{L}_2}$ to the regular unipotent elements of G.

Corollary 9.19. *We have*

$$\operatorname{res}^{\mathbf{G}}_{\mathbf{L}_2}\left((u_1(1)u_2(1)u_3(1))^G\right) = (u_1(\rho)u_3(\rho^q))^{L_2}.$$

Proof. First we conjugate the representatives of the regular unipotent classes of L_2 with g_w, from the previous lemma, to get representatives in $\mathbf{G}^F$. Then we can apply directly the definition of $\operatorname{res}^{\mathbf{G}}_{\mathbf{L}_2}$ from [Bo05, Chapter 15.A]. □

10 The 2-parameter Green functions

Following the discussion in Section 4, we can compute the 2-parameter Green functions for the two types of Levi subgroups with disconnected centre of $\mathrm{SL}_4(q)$ when q is odd.

Proposition 10.1. *Let q be an odd prime power, $\mathbf{G} = \mathrm{SL}_4$ when $q \equiv 3 \pmod 4$, or $\mathbf{G} = \mathrm{SL}_4/\langle \pm 1\rangle$ when $q \equiv 1 \pmod 4$, and F a split Frobenius with respect to an $\mathbb{F}_q$-structure on $\mathbf{G}$. Then*

$$\tilde{Q}^{\mathbf{G}}_{\mathbf{L}_1} = \begin{pmatrix} \Phi_3\Phi_4 & \Phi_2 & 1 & 1 & . & . & . \\ . & q^2 & . & . & 1 & . & . \\ . & q^2 & . & . & 1 & . & . \\ . & . & q\Phi_2 & . & \frac{1}{2}\Phi_1 & 1 & . \\ . & . & . & q\Phi_2 & \frac{1}{2}\Phi_1 & . & 1 \end{pmatrix}$$

for a split Levi subgroup $\mathbf{L}_1$ with $\mathbf{L}_1^F = A_1(q)^2(q-1)$, and

$$\tilde{Q}^{\mathbf{G}}_{\mathbf{L}_2} = \begin{pmatrix} \Phi_1^2\Phi_3 & -\Phi_1 & 1 & 1 & . & . & . \\ . & . & . & q\Phi_1 & -\frac{1}{2}\Phi_1 & 1 & . \\ . & . & q\Phi_1 & . & -\frac{1}{2}\Phi_1 & . & 1 \end{pmatrix}$$

for a non-split Levi subgroup $\mathbf{L}_2$ with $\mathbf{L}_2^F = A_1(q^2)(q+1)$.

In both cases the unipotent classes of $\mathbf{G}$ are ordered such that their Jordan forms are given by the partitions 1^4, 21^2, 2^2 (two classes), 31, 4 (two classes) of 4. The unipotent classes of the Levi subgroups have "same representatives" as the ones in Table 8 and Table 9 (see table in the proof).

Proof. In both cases, $G = \mathbf{G}^F$ has centre of order 2 and seven unipotent classes, since the classes of $\mathbf{G}$ with Jordan normal forms 2^2 and 4 both split into two classes in the finite group. As $\mathbf{L}_1$ is split of type A_1^2, the Levi subgroup $\mathbf{L}_1^F$ has five unipotent classes with representatives of Jordan types $(1^2, 1^2)$, $(1^2, 2)$, $(2, 1^2)$ and two of type $(2, 2)$. Finally, the class of regular unipotent elements of $\mathbf{L}_2$ splits into two $\mathbf{L}_2^F$-classes. The same computations of the previous sections, applied to the groups of the statements, give us the unipotent classes of $\mathbf{G}^F$, $\mathbf{L}_1^F$ and $\mathbf{L}_2^F$, and their fusion. We choose the following representatives of the unipotent classes (in a Steinberg presentation analogous to the one of Section 7.1):

$\mathbf{G}^F$	1, $u_1(1)$, $u_1(1)u_3(1)$, $u_1(\mu)u_3(1)$, $u_1(1)u_2(1)$, $u_1(1)u_2(1)u_3(1)$, $u_1(\mu)u_2(1)u_3(1)$
$\mathbf{L}_1^F$	1, $u_1(1)$, $u_3(1)$, $u_1(1)u_3(1)$, $u_1(\mu)u_3(1)$
$\mathbf{L}_2^F$	1, $u_1(1)u_3(1)$, $u_1(\rho)u_3(\rho^q)$

From the fusion of the unipotent classes and by Corollary 4.20, we compute the values $Q^{\mathbf{G}}_{\mathbf{L}_1}(u, v^{-1})$ explicitly. Thus, with respect to the ordering of the splitting classes given in the table, we find the stated result for $\tilde{Q}^{\mathbf{G}}_{\mathbf{L}_1}$.

We denote the representatives of regular unipotent classes of $\mathbf{L}_2$ by v_2 and v_3. Moreover, we set $u_3 := u_1(1)u_3(1)$ and $u_6 := u_1(1)u_2(1)u_3(1)$. By the known Green functions for $\tilde{\mathbf{L}}_2$ in $\tilde{\mathbf{G}}$ (see Example 4.8) and our considerations in Remark 4.14, there are two unknown values in $\tilde{Q}^{\mathbf{G}}_{\mathbf{L}_2}$, denoted by $a_1 := \tilde{Q}^{\mathbf{G}}_{\mathbf{L}_2}(u_3, v_2)$ and $a_2 := \tilde{Q}^{\mathbf{G}}_{\mathbf{L}_2}(u_6, v_2)$.

Consider the cuspidal class function ψ on $\mathbf{L}_2^F$ that takes values 1 and -1 on v_2, v_3 respectively, and zero everywhere else. By the Mackey formula this means, since $|N_{\mathbf{G}}(\mathbf{L}_2)^F : \mathbf{L}_2^F| = 2$, that $\langle R^{\mathbf{G}}_{\mathbf{L}_2}(\psi), R^{\mathbf{G}}_{\mathbf{L}_2}(\psi)\rangle = N\langle \psi, \psi\rangle$, with N being equal 1 or 2 (depending on whether v_2 and v_3 fuse in $\mathbf{G}^F$). On the other hand, the norm of $R^{\mathbf{G}}_{\mathbf{L}_2}(\psi)$ can be computed from the character formula using the (unknown) values of $\tilde{Q}^{\mathbf{G}}_{\mathbf{L}_2}(u, v)$ on the pairs of splitting classes. Then, the norm equation above gives

$$(2a_1 - q\Phi_1)^2 + q^2\Phi_1\Phi_2(2a_2 - 1)^2 = Nq^3\Phi_1.$$

Next, let ξ be the class function on $\mathbf{L}_1^F$ that takes values 1 resp. -1 on the two regular unipotent classes, respectively. Since ψ is cuspidal, the Mackey formula shows that $\langle R_{\mathbf{L}_1}^{\mathbf{G}}(\xi), R_{\mathbf{L}_2}^{\mathbf{G}}(\psi)\rangle = 0$ (see Remark 4.15). Using the known values of $\tilde{Q}_{\mathbf{L}_1}^{\mathbf{G}}$, this translates to $a_1 = q\Phi_1(1-a_2)$. This system of equations has rational solutions only for $N = 2$ (meaning that the classes of v_2 and v_3 don't fuse in $\mathbf{G}^F$). The solutions of this system are $(a_1, a_2) \in \{(q\Phi_1, 0), (0, 1)\}$. Both correspond to a matrix as in the statement, but in one case the second and third lines are interchanged. □

It is easy, to use the previous result to find $\tilde{Q}_{\mathbf{L}}^{\mathbf{G}}$ for $\mathrm{SL}_4(q)$ for $q \equiv 1 \pmod 4$ thanks to Proposition 4.13.

Proposition 10.2. *Let* $\mathbf{G} = \mathrm{SL}_4$ *and* $\mathbf{G}^F = \mathrm{SL}_4(q)$ *with* $q \equiv 1 \pmod 4$. *Then*

$$\tilde{Q}_{\mathbf{L}_1}^{\mathbf{G}} = \begin{pmatrix} \Phi_3\Phi_4 & \Phi_2 & 1 & 1 & . & . & . & . & . \\ . & q^2 & . & . & 1 & . & . & . & . \\ . & q^2 & . & . & 1 & . & . & . & . \\ . & . & q\Phi_2 & . & \frac{1}{2}\Phi_1 & 1 & . & 1 & . \\ . & . & . & q\Phi_2 & \frac{1}{2}\Phi_1 & . & 1 & . & 1 \end{pmatrix}$$

for a split Levi subgroup $\mathbf{L}_1$ *of type* A_1^2, *and*

$$\tilde{Q}_{\mathbf{L}_2}^{\mathbf{G}} = \begin{pmatrix} \Phi_1^2\Phi_3 & -\Phi_1 & 1 & 1 & . & . & . & . & . \\ . & . & . & q\Phi_1 & -\frac{1}{2}\Phi_1 & 1 & . & 1 & . \\ . & . & q\Phi_1 & . & -\frac{1}{2}\Phi_1 & . & 1 & . & 1 \end{pmatrix}$$

for a non-split Levi subgroup $\mathbf{L}_2$ *of type* A_1^2.

The unipotent classes are ordered according to Table 6, Table 8 and Table 9.

Proof. This directly follows from Proposition 10.1 and Lemma 4.13 applied to the surjection $i : \mathrm{SL}_4 \to \mathrm{SL}_4/\langle\pm 1\rangle$ with $\ker(i) = \pm 1$. Where we have checked the condition of the lemma, $\ker(i) \subset \{z^{-1}F'(z) \mid z \in Z(\mathbf{L}_I)\}$, explicitly (with the notation from the previous section). □

Remark 10.3. Conjecture 5.32 is verified for $\mathbf{L}_2$.

To fix the row order for a given choice of representative, in the last proposition, we apply Remark 5.34 with Lemma 9.17 and Corollary 9.19 to Proposition 10.2, i.e. the restriction of a Gel'fand–Graev character is a Gel'fand–Graev character.

Table 10: Non-zero values of the modified Gel'fand–Graev characters of G.

zu $k_a, k_c = 0,1,2,3,\ k_b = 0,1$	$\Gamma^{\mathbf{G}}_{j_a;j_b}(zu)$ $j_a, j_b = 0,1,2,3$
$h_Z(k_a)$	$\frac{i^{j_a k_a}}{4}\Phi_1^3\Phi_2^2\Phi_3\Phi_4$
$h_Z(k_a)u_1(1)$	$-\frac{i^{j_a k_a}}{4}\Phi_1^2\Phi_2\Phi_3$
$h_Z(k_a)u_1(\mu^{k_b})u_3(1)$	$\frac{i^{j_a k_a}}{4}\Phi_1\Phi_2\left(q^2(-1)^{j_b+k_b}+1\right)$
$h_Z(k_a)u_1(1)u_2(1)$	$\frac{i^{j_a k_a}}{4}\Phi_1\Phi_2$
$h_Z(k_a)u_1(\mu^{k_c})u_2(1)u_3(1)$	$\frac{i^{j_a k_a}}{4}\left[(q-1)+q\left(1+(-1)^{j_b+k_c}\right)\left((-1)^{\frac{q-1}{4}+\frac{j_b+k_c}{2}}\sqrt{q}-1\right)\right]$

11 Modified Gel'fand–Graev characters

With Lemma 5.58 it is straightforward to compute the induction of the regular linear characters of U (or ZU) to G.

Notation 11.1. In this section we denote by ϕ_{j_1,j_2,j_3} the linear character of U_0 defined by

$$\phi_{j_1,j_2,j_3}\left(u_1(r_1)u_2(r_2)u_3(r_3)\cdots u_6(r_6)\right) = \phi(j_1r_1 + j_2r_2 + j_3r_3)$$

where $j_1, j_2, j_3 \in \mathbb{F}_q^\times$ and $r_1, ..., r_6 \in \mathbb{F}_q$.

We keep the same notation also for linear characters of U_1 ($\phi_{j_1,0,j_3}$ with $j_1, j_3 \in \mathbb{F}_q^\times$).

Remark 11.2. Recall that in Remark 7.6, we have chosen a representative $t_j \in \mathscr{L}_{\mathbf{T}_0}^{-1}(z)$ for $z = h_Z(j) \in H^1(F, Z(\mathbf{G}))$, $j = 0,1,2,3$. We compute $\phi_j^{U_0} := {}^{t_j}\phi_{1,1,1} = \phi_{1,1,\mu^{-j}}$ and denote the Gel'fand-Graev character associated with $h_Z(j)$ by $\Gamma_j^{\mathbf{G}} = \mathrm{Ind}_{U_0}^{G}\phi_j^{U_0}$.

Then, the modified Gel'fand–Graev characters have two parameters j_a and j_b associated respectively to the Z-part and to $h_Z(j_b)$ (with same range $0,1,2,3$).

Proposition 11.3. *There are 16 different modified Gel'fand–Graev characters of* $\mathrm{SL}_4(q)$*. Their values are given in Table 10. The table contains only the non-zero values (which are precisely those on elements of the form zu for $z \in Z$ and u unipotent).*

Notice that following Remark 5.60 (a), we have hidden the ± sign from the values of $\Gamma_{j_b}^{\mathbf{G}}\left(u_1(\mu^k)u_2(1)u_3(1)\right)$ by redefining $j_b \mapsto j_b \pm 2$ when the sign is negative.

Remark 11.4. The norm of the Gel'fand–Graev characters (the non modified ones) is, for all $z \in H^1(F, Z(\mathbf{G}))$

$$\langle \Gamma_z^{\mathbf{G}}, \Gamma_z^{\mathbf{G}} \rangle = q^3 + q + 2,$$

which, by Corollary 5.24, is equal to the number of semisimple conjugacy classes of the dual group of G. This number is in agreement with the one given in [BrLue13, Theorem 4.1].

We compute now the (modified) Gel'fand–Graev characters of L_1.

Remark 11.5. By Lemma 9.11 the Gel'fand–Graev characters of L_1 are parametrized by $H^1(F, Z(\mathbf{L}_1)) \cong \{h(\pm 1, 1, 1)\}$. For $h_{Z_1}(j,1) = h((-1)^j, 1, 1)$ we choose a representative in $\mathscr{L}_{\mathbf{T}_0}^{-1}(h_{Z_1}(j,1))$

$$t_j = h(\rho^{-\frac{q+1}{2}j}, 1, 1) \in \mathbf{T}_0.$$

Then, a representative of the T_1-class of regular characters of U_1, parametrized by $h_{Z_1}(j,1)$, is $\phi_j^{U_1} = {}^{t_j}\phi_{1,0,1} = \phi_{\mu^j,0,1}$, for $j = 0,1$.

Table 11: Non-zero values of the modified Gel'fand–Graev characters of L_1.

$zu \in Z_1U_1$ $k_a, k_c = 0, 1,\ k_b = 0, ..., q-2$	$\Gamma^{\mathbf{L}_1}_{j_a,j_b;j_c}(zu)$ $j_a, j_c = 0, 1,\ j_b = 0, ..., q-2$
$h_{Z_1}(k_a, k_b)$	$\frac{(-1)^{j_ak_a}\zeta_{q-1}^{j_bk_b}}{2}\Phi_1^2\Phi_2^2$
$h_{Z_1}(k_a, k_b)u_1(1)$	$-\frac{(-1)^{j_ak_a}\zeta_{q-1}^{j_bk_b}}{2}\Phi_1\Phi_2$
$h_{Z_1}(k_a, k_b)u_3(1)$	$-\frac{(-1)^{j_ak_a}\zeta_{q-1}^{j_bk_b}}{2}\Phi_1\Phi_2$
$h_{Z_1}(k_a, k_b)u_1(\mu^{k_c})u_3(1)$	$\frac{(-1)^{j_ak_a}\zeta_{q-1}^{j_bk_b}}{2}(q(-1)^{j_c+k_c}+1)$

We denote the Gel'fand–Graev characters of L_1 by $\Gamma^{\mathbf{L}_1}_j = \mathrm{Ind}_{U_1}^{L_1}\phi_j^{U_1}$ for $j = 0, 1$.

By Lemma 9.10, there are $2(q-1)$ linear characters of Z_1 parametrized by $j_a = 0, 1$ and $j_b = 0, ..., q-2$.

Then, the modified Gel'fand–Graev characters of L_1 are denoted by $\Gamma^{\mathbf{L}_1}_{j_a,j_b;j_c}$ with two parameters $j_a = 0, 1$, $j_b = 0, ..., q-2$ coming from the Z_1-part and one parameter $j_c = 0, 1$ coming from $h_{Z_1}(j_c, 1) \in H^1(F, Z(\mathbf{L}_1))$.

Proposition 11.6. *There are* $4(q-1)$ *distinct modified Gel'fand–Graev characters of* L_1. *Their (non zero) values are given in Table 11.*

Remark 11.7. By explicit computation, the Gel'fand–Graev characters (the non modified ones) of L_1 have norm

$$\langle \Gamma^{\mathbf{L}_1}_z, \Gamma^{\mathbf{L}_1}_z \rangle = \Phi_1\Phi_4.$$

We verify that this agrees with Lemma 5.26.

One interesting aspect of these characters comes from the fact that the differences

$$\Gamma^{\mathbf{L}_1}_{j_a,j_b;0} - \Gamma^{\mathbf{L}_1}_{j_a,j_b;1}$$

have norm 2 (by explicit computation). Moreover, these differences are orthogonal to the space spanned by the Deligne–Lusztig characters of L_1, for all $j_a = 0, 1$ and $j_b = 0, ..., q-2$. The last assertion comes from the fact that every Deligne–Lusztig character takes the same values at splitting classes while the Gel'fand–Graev characters (and their difference) do not.

Remark 11.8. The Gel'fand–Graev characters $\Gamma^{\mathbf{L}_1}_{j_c}$ are related to $\Gamma^{\mathbf{G}}_{j_b}$ by Lemma 9.11. Explicitly,

$$\begin{aligned} {}^*R^{\mathbf{G}}_{\mathbf{L}_1}\Gamma^{\mathbf{G}}_0 &= \Gamma^{\mathbf{L}_1}_0, \\ {}^*R^{\mathbf{G}}_{\mathbf{L}_1}\Gamma^{\mathbf{G}}_1 &= \Gamma^{\mathbf{L}_1}_1, \\ {}^*R^{\mathbf{G}}_{\mathbf{L}_1}\Gamma^{\mathbf{G}}_2 &= \Gamma^{\mathbf{L}_1}_0, \\ {}^*R^{\mathbf{G}}_{\mathbf{L}_1}\Gamma^{\mathbf{G}}_3 &= \Gamma^{\mathbf{L}_1}_1. \end{aligned}$$

Finally, we compute the modified Gel'fand–Graev characters of L_2. The treatment is analogous to the case of L_1.

Remark 11.9. By Lemma 9.17 the Gel'fand–Graev characters of L_2 are parametrized by $H^1(F', Z(\mathbf{L}_2)) \cong \{h(\pm 1, 1, 1)\}$. We choose for $j = 0, 1$ a representative in $\mathscr{L}_{\mathbf{T}_0}^{-1}(h((-1)^j, 1, 1))$

$$t_j = h(\omega^{-\frac{q^2+1}{2}j}, 1, \omega^{-q\frac{q^2+1}{2}j}).$$

Table 12: Non-zero values of the modified Gel'fand–Graev characters of L_2

$hu \in Z_2U_2$ $k_a = 0, ..., 2q+1,\ k_b = 0, 1$	$\Gamma^{\mathbf{L}_2}_{j_a;j_b}$ $j_a = 0, ..., 2q+1,\ j_b = 0, 1$
$h_{Z_2}(k_a)$	$\frac{\zeta_{2q+2}^{j_a k_a}}{2}\Phi_1\Phi_2\Phi_4$
$h_{Z_2}(k_a)u_1(\rho^{k_b})u_3(\rho^{qk_b})$	$-\frac{\zeta_{2q+2}^{j_a k_a}}{2}\left(q(-1)^{j_b+k_b}+1\right)$

Then, the T_2-class of regular characters of U_2, parametrized by $h((-1)^j, 1, 1)$, has a representative ${}^{t_j}\chi_2$ (by abuse of notation, see Notation 5.12 and the discussion that follows it).

The Gel'fand–Graev characters of L_2 are given by $\Gamma^{\mathbf{L}_2}_j = \mathrm{Ind}_{U_2}^{L_2}({}^{t_j}\chi_2)$ for $j = 0, 1$.

By Lemma 9.16, there are $2q+2$ linear characters of Z_2, parametrized by $j_a = 0, ..., 2q+1$. Then, the modified Gel'fand–Graev characters are denoted by $\Gamma^{\mathbf{L}_2}_{j_a;j_b}$ with $j_a = 0, ..., 2q+1$ coming from the Z_2-part and $j_b = 0, 1$ coming from $h((-1)^{j_b}, 1, 1) \in H^1(F', Z(\mathbf{L}_2))$.

Proposition 11.10. *There are* $4(q+1)$ *modified Gel'fand–Graev characters of* L_2*, they are given in Table 12.*

Remark 11.11. By explicit computation, the Gel'fand–Graev characters (the non modified ones) of L_2 have norm

$$\langle \Gamma^{\mathbf{L}_2}_z, \Gamma^{\mathbf{L}_2}_z \rangle = \Phi_2\Phi_4.$$

We verify that this agrees with Lemma 5.26.

Like in the case of L_1, the differences

$$\Gamma^{\mathbf{L}_2}_{j_a;0} - \Gamma^{\mathbf{L}_2}_{j_a;1}$$

have norm 2 (by explicit computation) and are orthogonal to the space spanned by the Deligne–Lusztig characters of L_2, for all $j_a = 0, ..., 2q+1$. Again, the orthogonality comes from the fact that every Deligne–Lusztig character takes the same values at splitting classes while the Gel'fand–Graev characters (and their difference) do not.

Remark 11.12. The Gel'fand–Graev characters $\Gamma^{\mathbf{L}_2}_{j_b}$ are related to $\Gamma^{\mathbf{G}}_{j_b}$ by Lemma 9.17. Explicitly, we have

$$\begin{aligned} {}^*R^{\mathbf{G}}_{\mathbf{L}_2}\Gamma^{\mathbf{G}}_0 &= \Gamma^{\mathbf{L}_2}_0, \\ {}^*R^{\mathbf{G}}_{\mathbf{L}_2}\Gamma^{\mathbf{G}}_1 &= \Gamma^{\mathbf{L}_2}_1, \\ {}^*R^{\mathbf{G}}_{\mathbf{L}_2}\Gamma^{\mathbf{G}}_2 &= \Gamma^{\mathbf{L}_2}_0, \\ {}^*R^{\mathbf{G}}_{\mathbf{L}_2}\Gamma^{\mathbf{G}}_3 &= \Gamma^{\mathbf{L}_2}_1. \end{aligned}$$

Table 13: Splitting class types of G. With representatives (in Steinberg presentation) and indices range.

Name	Representative	indices range
$c_3(i_1)$	$h_Z(i_1)u_1(1)u_3(1)$	$i_1 = 0, 1, 2, 3$
$c_4(i_1)$	$h_Z(i_1)u_1(\mu)u_3(1)$	
$c_6(i_1)$	$h_Z(i_1)u_1(1)u_2(1)u_3(1)$	$i_1 = 0, 1, 2, 3$
$c_7(i_1)$	$h_Z(i_1)u_1(\mu)u_2(1)u_3(1)$	
$c_8(i_1)$	$h_Z(i_1)u_1(\mu^2)u_2(1)u_3(1)$	
$c_9(i_1)$	$h_Z(i_1)u_1(\mu^3)u_2(1)u_3(1)$	
$c_{16}(i_1, i_2)$	$h_{Z_1}(i_1, i_2)u_1(1)u_3(1)$	$i_1 = 0, 1,\ i_2 = 0, ..., q-2$
$c_{17}(i_1, i_2)$	$h_{Z_1}(i_1, i_2)u_1(\mu)u_3(1)$	$\frac{q-1}{4} \nmid \frac{q-1}{2} i_1 + i_2$
$c_{19}(i_1)$	$h_{Z_2}(i_1)u_1(1)u_3(1)$	$i_1 = 0, ..., 2q+1$
$c_{20}(i_1)$	$h_{Z_2}(i_1)u_1(\rho^q)u_3(\rho)$	$\frac{q+1}{2} \nmid \frac{q+3}{4} i_1$

12 Decomposition of almost characters

Notation 12.1. To distinguish the notation for regular characters $\chi_{(s),z}$ of G, L_1 and L_2, we add a superscript

$$\chi^{\mathbf{G}}_{(s),z},\ \chi^{\mathbf{L}_1}_{(s),z},\ \chi^{\mathbf{L}_2}_{(s),z}.$$

Following the discussion of Section 6.6, we introduce unknowns f in every character that decomposes at every splitting class of the partial character table.

In this section, we describe how to use the modified Gel'fand–Graev characters to find these unknowns f. As already announced at the end of Section 6.6, we want to put the stress on the importance of the Gel'fand–Graev characters. Therefore, we will present the most basilar procedure to decompose the uniform almost characters. To this end, we use only the properties of regular characters.

We start by fixing the notation used to identify the characters and classes in the table provided by Lübeck.

Notation 12.2. The conjugacy classes of G are divided into class types (see Section 6.1). The class types are denoted by c_i with $i = 1, ..., 29$. To identify a particular class of a class type suitable indices are specified $c_i(j, k, l, ...)$.

Notation 12.3. The uniform almost characters of G are denoted by R_i with $i = 1, ..., 54$. If a family of uniform almost characters is associated to a semisimple class type of $G^* = \mathbf{G}^{*F^*}$ whose representatives are parametrized by indices $j, k, l, ...$, then the particular almost character associated with a particular semisimple class is denoted by $R_i(j, k, l, ...)$.

Tables 13 and 14 show respectively the conjugacy class types of $\mathrm{SL}_4(q)$ that split and the uniform almost characters that are not irreducible. The notation $h^*(t_1, t_2, t_3)$ in Table 14 stands for $h_{\alpha_1^*}(t_1)h_{\alpha_2^*}(t_2)h_{\alpha_3^*}(t_3)$ where $\alpha_1^*, \alpha_2^*, \alpha_3^*$ are, respectively, the dual roots of $\alpha_1, \alpha_2, \alpha_3$.

12.1 Decomposition of R_{15} and R_{18}

The almost characters R_{15} and R_{18} belong to the Lusztig series $\mathscr{E}(G, s_{15})$ (known from Lübeck's table). They even belong the same Harish–Chandra series by Lemma 12.6, below.

Table 14: Non irreducible almost characters of G. With range of indices.

Name	Norm	Representative for associated ss. class	indices range
R_{15}	2	$s_{15} = h^*(1,-1,1)$	
R_{18}	2	$s_{15} = h^*(1,-1,1)$	
R_{20}	2	$s_{20} = h^*(1,-1,1)$ (twisted)	
R_{21}	2	$s_{20} = h^*(1,-1,1)$ (twisted)	
$R_{33}(k_1)$	2	$s_{33}(k_1) = h^*(-1, \mu^{k_1}, -1)$	$k_1 = 0, ..., q-2$, $\frac{q-1}{4} \nmid k_1$
$R_{35}(k_1)$	2	$s_{35}(k_1) = h^*(-1, \rho^{k_1 \frac{q+1}{2}}, -1)$	$k_1 = 0, ..., 2q+1$, $2 \nmid k_1$
$R_{37}(k_1)$	2	$s_{37}(k_1) = h^*(-1, \rho^{k_1 \frac{q-1}{2}}, -1)$	$k_1 = 0, ..., 2q-3$, $2 \nmid k_1$, $\frac{q+1}{2} \nmid k_1$
$R_{39}(k_1)$	2	$s_{39}(k_1) = h^*(-1, \rho^{k_1(q-1)}, -1)$	$k_1 = 0, ..., q$, $\frac{q+1}{2} \nmid k_1$
$R_{41}(k_1)$	2	$s_{41}(k_1) = h^*(-1, \mu^{k_1}, -1)$	$k_1 = 0, ..., q-2$, $\frac{q-1}{4} \nmid k_1$
R_{43}	4	$s_{43} = h^*(\mu^{\frac{q-1}{4}}, \mu^{\frac{q-1}{4}}, \mu^{\frac{q-1}{4}})$	
$R_{47}(k_1)$	4	$s_{47}(k_1) = h^*((-1)^{k_1}, (-1)^{k_1}, (-1)^{k_1})$	$k_1 = 0, 1$
R_{51}	4	$s_{51} = h^*(\mu^{3\frac{q-1}{4}}, \mu^{3\frac{q-1}{4}}, \mu^{3\frac{q-1}{4}})$	

Lemma 12.4. *The irreducible constituents of R_{15} are regular characters and those of R_{18} are not.*

Proof. The scalar product with the (non-modified) Gel'fand–Graev characters gives

$$\left\langle R_{15}, \Gamma_z^{\mathbf{G}} \right\rangle_G = 1, \qquad \left\langle R_{18}, \Gamma_z^{\mathbf{G}} \right\rangle_G = 0,$$

for all $z \in Z = H^1(F, Z(\mathbf{G}))$.

We know by Remark 5.28 and Clifford theory that the regular characters of any Lusztig series form a unique $\tilde{\mathbf{G}}^F$-orbit. Then, the scalar product implies that the constituents of R_{15} are regular while those of R_{18} are not. ☐

To continue, we need to know the precise parametrization of the regular characters of R_{15}. We denote by $\chi^{\mathbf{G}}_{(s_{15}),0}$ the irreducible constituent common to R_{15} and $\Gamma^{\mathbf{G}}_0$.

By imposing $\left\langle \chi^{\mathbf{G}}_{(s_{15}),0}, \Gamma^{\mathbf{G}}_0 \right\rangle_G = 1$ one finds a relation between the unknowns introduced for the classes $c_3(0)$ and $c_6(0)$.

With this relation, we can compute

$$\left\langle \chi^{\mathbf{G}}_{(s_{15}),0}, \Gamma^{\mathbf{G}}_0 \right\rangle_G = 1, \quad \left\langle \chi^{\mathbf{G}}_{(s_{15}),0}, \Gamma^{\mathbf{G}}_1 \right\rangle_G = 0, \quad \left\langle \chi^{\mathbf{G}}_{(s_{15}),0}, \Gamma^{\mathbf{G}}_2 \right\rangle_G = 1, \quad \left\langle \chi^{\mathbf{G}}_{(s_{15}),0}, \Gamma^{\mathbf{G}}_3 \right\rangle_G = 0.$$

It follows that the other constituent of R_{15} is contained in $\Gamma^{\mathbf{G}}_1$ and $\Gamma^{\mathbf{G}}_3$. It will be denoted by $\chi^{\mathbf{G}}_{(s_{15}),1}$.

Lemma 12.5. *The class (s_{15}) of G^* splits into two classes of L_1^* and*

$${}^*R^{\mathbf{G}}_{\mathbf{L}_1} \left(\chi^{\mathbf{G}}_{(s_{15}),0} - \chi^{\mathbf{G}}_{(s_{15}),1} \right) = \Gamma^{\mathbf{L}_1}_{0,0;0} - \Gamma^{\mathbf{L}_1}_{0,0;1}.$$

Proof. By explicit computation, we get $\left\langle {}^*R^{\mathbf{G}}_{\mathbf{L}_1} R_{15}, \Gamma^{\mathbf{L}_1}_{j_a,j_b;j_c} \right\rangle_{L_1} = 3\delta_{j_a,0}\delta_{j_b,0}$.

Since the restrictions ${}^*R^{\mathbf{G}}_{\mathbf{L}_1}\chi^{\mathbf{G}}_{(s_{15}),0}$ and ${}^*R^{\mathbf{G}}_{\mathbf{L}_1}\chi^{\mathbf{G}}_{(s_{15}),1}$ are parametrized by the same semisimple classes of L_1^*, they have the same number of irreducible constituents. Also, the modified Gel'fand–Graev characters and the Harish–Chandra restriction of regular characters are multiplicity free. This, together with the fact that only one irreducible constituent is common to both $\Gamma^{\mathbf{L}_1}_{j_a,j_b;0}$ and $\Gamma^{\mathbf{L}_1}_{j_a,j_b;1}$ for all $j_a = 0,1$ and $j_b = 0,...,q-2$ (see Remark 11.7), forces ${}^*R^{\mathbf{G}}_{\mathbf{L}_1}\chi^{\mathbf{G}}_{(s_{15}),i}$ to have two constituents $\chi^{\mathbf{L}_1}_{(t_{15}),i}$ and $\chi^{\mathbf{L}_1}_{(t'_{15}),i}$, $i = 0,1$. These must be such that $\chi^{\mathbf{L}_1}_{(t_{15}),0} = \chi^{\mathbf{L}_1}_{(t_{15}),1}$ and $\chi^{\mathbf{L}_1}_{(t'_{15}),0} \neq \chi^{\mathbf{L}_1}_{(t'_{15}),1}$.

Then, the two conclusions follow:

$$ {}^*R^{\mathbf{G}}_{\mathbf{L}_1}(\chi^{\mathbf{G}}_{(s_{15}),0} - \chi^{\mathbf{G}}_{(s_{15}),1}) = \chi^{\mathbf{L}_1}_{(t'_{15}),0} - \chi^{\mathbf{L}_1}_{(t'_{15}),1} = \Gamma^{\mathbf{L}_1}_{0,0;0} - \Gamma^{\mathbf{L}_1}_{0,0;1}, $$

and (s_{15}) splits into two classes of L_1^*, denoted here (t_{15}) and (t'_{15}). □

Lemma 12.5 and the formula for Harish–Chandra restriction directly give the unknowns f for all the classes of type c_{16}. Moreover, we get relations between the unknowns of $c_3(j)$ and $c_6(j)$, for each $j = 0,1,2,3$.

It remains to find four unknowns $f_{6,j}$, $j = 0,1,2,3$, one for each class of type c_6. Relations among them are found thanks to the scalar products of $\chi_{(s_{15}),0}$ and the modified Gel'fand–Graev characters. These are known by Lemma 12.4 and by explicitly computing scalar products between R_{15} and the modified Gel'fand–Graev characters.

We get the linear system of equations

$$ \sum_{j=0,1,2,3} f_{6,j}\zeta_4^{kj} = 0, \quad k = 0,1,2,3. $$

The unique solution is given by $f_{6,j} = 0$ for $j = 0,1,2,3$.

Lemma 12.6. *The irreducible constituents of R_{15} and R_{18} are related by*

$$ R^{\mathbf{G}}_{\mathbf{L}_1}\left(\Gamma^{\mathbf{L}_1}_{0,0;0} - \Gamma^{\mathbf{L}_1}_{0,0;1}\right) = \chi^{\mathbf{G}}_{(s_{15}),0} - \chi^{\mathbf{G}}_{(s_{15}),1} + \psi_{18} - \psi'_{18} $$

where ψ_{18} and ψ'_{18} denote the constituents of R_{18}.

Proof. By explicit computation, we get $\left\langle R^{\mathbf{G}}_{\mathbf{L}_1}\left(\Gamma^{\mathbf{L}_1}_{0,0;0} - \Gamma^{\mathbf{L}_1}_{0,0;1}\right), R^{\mathbf{G}}_{\mathbf{L}_1}\left(\Gamma^{\mathbf{L}_1}_{0,0;0} - \Gamma^{\mathbf{L}_1}_{0,0;1}\right)\right\rangle_G = 4$. Then, since the degree of $R^{\mathbf{G}}_{\mathbf{L}_1}\left(\Gamma^{\mathbf{L}_1}_{0,0;0} - \Gamma^{\mathbf{L}_1}_{0,0;1}\right)$ is zero, it must be that

$$ R^{\mathbf{G}}_{\mathbf{L}_1}\left(\Gamma^{\mathbf{L}_1}_{0,0;0} - \Gamma^{\mathbf{L}_1}_{0,0;1}\right) = \theta - \theta' + \theta'' - \theta''' $$

for irreducible character θ, θ', θ'', θ''' of G belonging to the same Harish–Chandra series.

Taking the scalar products with $\chi^{\mathbf{G}}_{(s_{15}),0}$ and $\chi^{\mathbf{G}}_{(s_{15}),1}$ proves the first part of the formula.

The only other possible irreducible characters that might belong to the same Harish–Chandra series are R_{17} and the constituents of R_{18} as they belong to the same Lusztig series (they are parametrized by the same semisimple class in the dual group, by Table 12). It follows that the other constituents must come from R_{18}, since R_{17} is irreducible. □

Evaluating the expression of the previous lemma one finds the unknowns f for the constituents of R_{18} on all splitting classes of type c_3, c_4, c_6, c_7, c_8, c_9, c_{16} and c_{17}.

For the classes c_{19} and c_{20} one needs to consider the Levi L_2 and Lusztig restriction.

Lemma 12.7. *The class (s_{15}) of G^* does not split as class of L_2^* and we have*

$$ {}^*R^{\mathbf{G}}_{\mathbf{L}_2}\left(\chi^{\mathbf{G}}_{(s_{15}),0} - \chi^{\mathbf{G}}_{(s_{15}),1}\right) = \Gamma^{\mathbf{L}_2}_{0;0} - \Gamma^{\mathbf{L}_2}_{0;1}. $$

Proof. This proof is analogous to the one of Lemma 12.5.

By direct computation, we find $\left\langle {}^*R^{\mathbf{G}}_{\mathbf{L}_2}R_{15}, \Gamma^{\mathbf{L}_2}_{j_a;j_b}\right\rangle_{L_2} = \delta_{j_a,0}$. Recall that the functions $\Gamma^{\mathbf{L}_2}_{j_a;j_b}$ are multiplicity free. This is true also for ${}^*R^{\mathbf{G}}_{\mathbf{L}_2}\chi^{\mathbf{G}}_{(s_{15}),k}$ $(k = 0,1)$. Since only one irreducible constituent is not common to both $\Gamma^{\mathbf{L}_2}_{j_a;0}$ and $\Gamma^{\mathbf{L}_2}_{j_a;1}$ for all $j_a = 0,...,2q+1$, by Remark 11.11.

This proves both the formula and that (s_{15}) does not split in L_2^*. □

Gathering all the information obtained so far gives the following result.

Proposition 12.8. *The values of the class functions $\chi^{\mathbf{G}}_{(s_{15}),0} - \chi^{\mathbf{G}}_{(s_{15}),1}$ and $\psi_{18} - \psi'_{18}$ (orthogonal to R_{15} and R_{18}) are as given in Table 15.*

Table 15: Certain class functions orthogonal to R_{15} and R_{18}.

	$\chi^{\mathbf{G}}_{(s_{15}),0} - \chi^{\mathbf{G}}_{(s_{15}),1}$	$\psi_{18} - \psi'_{18}$
c_3	q^3	q^2
c_4	$-q^3$	$-q^2$
c_6	0	q
c_7	0	$-q$
c_8	0	q
c_9	0	$-q$
c_{16}	q	q
c_{17}	$-q$	$-q$
c_{19}	q	$-q$
c_{20}	$-q$	q

12.2 Decomposition of R_{20} and R_{21}

The discussion for R_{20} and R_{21} is completely analogous to that for R_{15} and R_{18}. We point out only the differences.

Lemma 12.9. *The irreducible constituents of R_{20} are regular characters and those of R_{21} are not.*

Like after Lemma 12.4 we can write the decomposition $R_{20} = \chi^{\mathbf{G}}_{(s_{20}),0} + \chi^{\mathbf{G}}_{(s_{20}),1}$.

Lemma 12.10. *The class (s_{20}) of G^* does not split as class of L_1^* and we have*

$$ {}^*R^{\mathbf{G}}_{\mathbf{L}_1}\left(\chi^{\mathbf{G}}_{(s_{20}),0} - \chi^{\mathbf{G}}_{(s_{20}),1}\right) = \Gamma^{\mathbf{L}_1}_{1,0;0} - \Gamma^{\mathbf{L}_1}_{1,0;1}.$$

This follows from $\left\langle {}^*R^{\mathbf{G}}_{\mathbf{L}_1} R_{20}, \Gamma^{\mathbf{L}_1}_{j_a,j_b;j_c}\right\rangle_{\mathbf{L}_1} = \delta_{j_a,1}\delta_{j_b,0}$.

Lemma 12.11. *The class (s_{20}) of G^* splits into two classes of L_2^* and we have*

$$ {}^*R^{\mathbf{G}}_{\mathbf{L}_2}\left(\chi^{\mathbf{G}}_{(s_{20}),0} - \chi^{\mathbf{G}}_{(s_{20}),1}\right) = \Gamma^{\mathbf{L}_2}_{q+1;0} - \Gamma^{\mathbf{L}_2}_{q+1;1}.$$

This follows from $\left\langle {}^*R^{\mathbf{G}}_{\mathbf{L}_2} R_{20}, \Gamma^{\mathbf{L}_2}_{j_a;j_b}\right\rangle_{L_2} = 3\delta_{j_a,q+1}$.

Lemma 12.12. *The irreducible constituents of R_{20} and R_{21} are related by*

$$R^{\mathbf{G}}_{\mathbf{L}_1}\left(\Gamma^{\mathbf{L}_1}_{1,0;0} - \Gamma^{\mathbf{L}_1}_{1,0;1}\right) = \chi^{\mathbf{G}}_{(s_{20}),0} - \chi^{\mathbf{G}}_{(s_{20}),1} + \psi_{20} - \psi'_{20}$$

where ψ_{20} and ψ'_{20} denote the constituents of R_{20}.

Proposition 12.13. *The values of the class functions $\chi^{\mathbf{G}}_{(s_{20}),0} - \chi^{\mathbf{G}}_{(s_{20}),1}$ and $\psi_{21} - \psi'_{21}$ (orthogonal to R_{20} and R_{21}) are given in Table 16.*

12.3 Decomposition of R_{33}, R_{35} and R_{41}

The same computations are applied for these cases, however it turns out that no Lusztig restriction is needed to achieve the decomposition.

Lemma 12.14. *The irreducible constituents of $R_{33}(j)$, $R_{35}(k)$ and $R_{41}(l)$ are regular characters for all j, k, l in their respective range (see Table 14).*

Table 16: Certain class functions orthogonal to R_{20} and R_{21}.

	$\chi^{\mathbf{G}}_{(s_{20}),0} - \chi^{\mathbf{G}}_{(s_{20}),1}$	$\psi_{21} - \psi'_{21}$
$c_3(j)$	$q^3(-1)^j$	$q^2(-1)^j$
$c_4(j)$	$-q^3(-1)^j$	$-q^2(-1)^j$
$c_6(j)$	0	$q(-1)^j$
$c_7(j)$	0	$-q(-1)^j$
$c_8(j)$	0	$q(-1)^j$
$c_9(j)$	0	$-q(-1)^j$
$c_{16}(j,l)$	$q(-1)^j$	$q(-1)^j$
$c_{17}(j,l)$	$-q(-1)^j$	$-q(-1)^j$
$c_{19}(j)$	$q(-1)^j$	$-q(-1)^j$
$c_{20}(j)$	$-q(-1)^j$	$q(-1)^j$

This is proved like Lemma 12.4, and like in the discussion that follows it, we can write the decompositions $R_{33}(j) = \chi^{\mathbf{G}}_{(s_{33}(j)),0} + \chi^{\mathbf{G}}_{(s_{33}(j)),1}$, $R_{33}(j) = \chi^{\mathbf{G}}_{(s_{35}(k)),0} + \chi^{\mathbf{G}}_{(s_{35}(k)),1}$ and $R_{37}(l) = \chi^{\mathbf{G}}_{(s_{37}(l)),0} + \chi^{\mathbf{G}}_{(s_{37}(l)),1}$.

Lemma 12.15. *The classes* $(s_{33}(j))$ *of* G^* *all split into four classes of* L_1^* *and*

$$^*R^{\mathbf{G}}_{\mathbf{L}_1}\left(\chi^{\mathbf{G}}_{(s_{33}(j)),0} - \chi^{\mathbf{G}}_{(s_{33}(j)),1}\right) = \Gamma^{\mathbf{L}_1}_{0,2j;0} - \Gamma^{\mathbf{L}_1}_{0,2j;1} + \Gamma^{\mathbf{L}_1}_{0,q-1+2j;0} - \Gamma^{\mathbf{L}_1}_{0,q-1+2j;1}.$$

Proof. The scalar product with the modified Gel'fand–Graev characters gives

$$\left\langle {}^*R^{\mathbf{G}}_{\mathbf{L}_1} R_{33}(j), \Gamma^{\mathbf{L}_1}_{j_a,j_b;j_c}\right\rangle_{L_1} = \delta_{j_a,0}\delta_{j_b,2j} + \delta_{j_a,0}\delta_{j_b,q-1+2j} + 4\delta_{j_a,j \bmod 2}\delta_{j_b,0}.$$

Again, the result follows by the multiplicity freeness of ${}^*R^{\mathbf{G}}_{\mathbf{L}_1} R_{33}(j)$ and $\Gamma^{\mathbf{L}_1}_{j_a,j_b;j_c}$ and the fact that $\Gamma^{\mathbf{L}_1}_{j_a,j_b;0}$ and $\Gamma^{\mathbf{L}_1}_{j_a,j_b;1}$ differ only by one constituent. □

Lemma 12.16. *The classes* $(s_{35}(j))$ *of* G^* *all split into two classes of* L_1^* *and*

$$^*R^{\mathbf{G}}_{\mathbf{L}_1}\left(\chi^{\mathbf{G}}_{(s_{35}(j)),0} - \chi^{\mathbf{G}}_{(s_{35}(j)),1}\right) = \Gamma^{\mathbf{L}_1}_{0,j;0} - \Gamma^{\mathbf{L}_1}_{0,j;1} + \Gamma^{\mathbf{L}_1}_{1,q-1+j;0} - \Gamma^{\mathbf{L}_1}_{1,q-1+j;1}.$$

This is proven by

$$\left\langle {}^*R^{\mathbf{G}}_{\mathbf{L}_1} R_{35,j}, \Gamma^{\mathbf{L}_1}_{j_a,j_b;j_c}\right\rangle_{L_1} = \delta_{j_a,0}\delta_{j_b,j} + \delta_{j_a,1}\delta_{j_b,q-1+j}.$$

Lemma 12.17. *The classes* $(s_{41}(j))$ *of* G^* *all split into two classes of* L_1^* *and*

$$^*R^{\mathbf{G}}_{\mathbf{L}_1}\left(\chi^{\mathbf{G}}_{(s_{41}(j)),0} - \chi^{\mathbf{G}}_{(s_{41}(j)),1}\right) = \Gamma^{\mathbf{L}_1}_{1,2j;0} - \Gamma^{\mathbf{L}_1}_{1,2j;1} + \Gamma^{\mathbf{L}_1}_{1,q-1+2j;0} - \Gamma^{\mathbf{L}_1}_{1,q-1+2j;1}.$$

This is proven by

$$\left\langle {}^*R^{\mathbf{G}}_{\mathbf{L}_1} R_{41,j}, \Gamma^{\mathbf{L}_1}_{j_a,j_b;j_c}\right\rangle_{L_1} = \delta_{j_a,1}\delta_{j_b,2j} + \delta_{j_a,1}\delta_{j_b,q-1+2j}.$$

The main difference with the cases of R_{15}, R_{18}, R_{20} and R_{21} is that we have the following norms:

$$\left\langle R^{\mathbf{G}}_{\mathbf{L}_1}\left(\Gamma^{\mathbf{L}_1}_{0,2j;0} - \Gamma^{\mathbf{L}_1}_{0,2j;1}\right), R^{\mathbf{G}}_{\mathbf{L}_1}\left(\Gamma^{\mathbf{L}_1}_{0,2j;0} - \Gamma^{\mathbf{L}_1}_{0,2j;1}\right)\right\rangle_G = 2,$$

$$\left\langle R^{\mathbf{G}}_{\mathbf{L}_1}\left(\Gamma^{\mathbf{L}_1}_{0,k;0} - \Gamma^{\mathbf{L}_1}_{0,k;1}\right), R^{\mathbf{G}}_{\mathbf{L}_1}\left(\Gamma^{\mathbf{L}_1}_{0,k;0} - \Gamma^{\mathbf{L}_1}_{0,k;1}\right)\right\rangle_G = 2,$$

$$\left\langle R^{\mathbf{G}}_{\mathbf{L}_1}\left(\Gamma^{\mathbf{L}_1}_{1,2l;0} - \Gamma^{\mathbf{L}_1}_{1,2l;1}\right), R^{\mathbf{G}}_{\mathbf{L}_1}\left(\Gamma^{\mathbf{L}_1}_{1,2l;0} - \Gamma^{\mathbf{L}_1}_{1,2l;1}\right)\right\rangle_G = 2,$$

for j, k, l in the range given in Table 14 for (respectively) $R_{33}(j)$, $R_{35}(k)$ and $R_{41}(l)$. Then, $R^{\mathbf{G}}_{\mathbf{L}_1}\left(\Gamma^{\mathbf{L}_1}_{0,2j;0} - \Gamma^{\mathbf{L}_1}_{0,2j;1}\right)$, $R^{\mathbf{G}}_{\mathbf{L}_1}\left(\Gamma^{\mathbf{L}_1}_{0,k;0} - \Gamma^{\mathbf{L}_1}_{0,k;1}\right)$ and $R^{\mathbf{G}}_{\mathbf{L}_1}\left(\Gamma^{\mathbf{L}_1}_{1,2l;0} - \Gamma^{\mathbf{L}_1}_{1,2l;1}\right)$ are exactly the orthogonal class functions needed to decompose $R_{33}(j)$, $R_{35}(k)$ and $R_{41}(l)$.

Table 17: Certain class functions orthogonal to $R_{33}(k)$, $R_{35}(k)$ and $R_{41}(k)$ for k in the ranges given in Table 14.

	$\chi^{\mathbf{G}}_{(s_{33}(k)),0} - \chi^{\mathbf{G}}_{(s_{33}(k)),1}$	$\chi^{\mathbf{G}}_{(s_{35}(k)),0} - \chi^{\mathbf{G}}_{(s_{35}(k)),1}$	$\chi^{\mathbf{G}}_{(s_{41}(k)),0} - \chi^{\mathbf{G}}_{(s_{41}(k)),1}$
$c_3(j)$	$q^2(q+1)(-1)^{kj}$	$q^2(q+1)i^{3kl}$	$q^2(q+1)(-1)^{(k+1)j}$
$c_4(j)$	$-q^2(q+1)(-1)^{kj}$	$-q^2(q+1)i^{3kl}$	$-q^2(q+1)(-1)^{(k+1)j}$
$c_6(j)$	$q(-1)^{kj}$	qi^{3kl}	$q(-1)^{(k+1)j}$
$c_7(j)$	$-q(-1)^{kj}$	$-qi^{3kl}$	$-q(-1)^{(k+1)j}$
$c_8(j)$	$q(-1)^{kj}$	qi^{3kl}	$q(-1)^{(k+1)j}$
$c_9(j)$	$-q(-1)^{kj}$	$-qi^{3kl}$	$-q(-1)^{(k+1)j}$
$c_{16}(j,l)$	$q\zeta_{q-1}^{2kl} + q\zeta_{q-1}^{-2kl}$	$q\zeta_{q-1}^{kl} + q\zeta_{q-1}^{-kl}(-1)^{kj}$	$q\zeta_{q-1}^{2kl}(-1)^{kj} + q\zeta_{q-1}^{-2kl}(-1)^{kj}$
$c_{17}(j,l)$	$-q\zeta_{q-1}^{2kl} - q\zeta_{q-1}^{-2kl}$	$-q\zeta_{q-1}^{kl} - q\zeta_{q-1}^{-kl}(-1)^{kj}$	$-q\zeta_{q-1}^{2kl}(-1)^{kj} - q\zeta_{q-1}^{-2kl}(-1)^{kj}$
$c_{19}(j)$	0	0	0
$c_{20}(j)$	0	0	0

Proposition 12.18. *The values of the class functions* $\chi^{\mathbf{G}}_{(s_{33}(j)),0} - \chi^{\mathbf{G}}_{(s_{33}(j)),1}$, $\chi^{\mathbf{G}}_{(s_{35}(j)),0} - \chi^{\mathbf{G}}_{(s_{35}(j)),1}$ *and* $\chi^{\mathbf{G}}_{(s_{41}(j)),0} - \chi^{\mathbf{G}}_{(s_{41}(j)),1}$ *(orthogonal to* $R_{33}(j)$, $R_{35}(j)$ *and* $R_{41}(j)$*) are given in Table 17.*

12.4 Decomposition of R_{39}

Lemma 12.19. *The irreducible constituents of* $R_{39}(j)$ *are regular characters for all* j *in the range given in Table 14.*

This is proved as for Lemma 12.4. We get the decomposition $R_{39}(j) = \chi^{\mathbf{G}}_{(s_{39}(j)),0} + \chi^{\mathbf{G}}_{(s_{39}(j)),1}$.

Lemma 12.20. *The classes* $(s_{39}(j))$ *of* G^* *do not split as classes of* L_1^* *and*

$$^*R^{\mathbf{G}}_{\mathbf{L}_1}\left(\chi^{\mathbf{G}}_{(s_{39}(j)),0} - \chi^{\mathbf{G}}_{(s_{39}(j)),1}\right) = 0.$$

This comes from $\left\langle {}^*R^{\mathbf{G}}_{\mathbf{L}_1} R_{39,j}, \Gamma^{\mathbf{L}_1}_{j_a,j_b;j_c} \right\rangle_{L_1} = 2\delta_{j_a,j+1 \bmod 2}\delta_{j_b,0}$.

Lemma 12.21. *The classes* $(s_{39}(j))$ *of* G^* *each split into three classes of* L_2^* *and*

$$^*R^{G}_{\mathbf{L}_2}\left(\chi^{\mathbf{G}}_{(s_{39}(j)),0} - \chi^{\mathbf{G}}_{(s_{39}(j)),1}\right) = \Gamma^{\mathbf{L}_2}_{q+1+2j;0} - \Gamma^{\mathbf{L}_2}_{q+1+2j;1} + \Gamma^{\mathbf{L}_2}_{q+1-2j;0} - \Gamma^{\mathbf{L}_2}_{q+1-2j;1}.$$

This comes from $\left\langle {}^*R^{\mathbf{G}}_{\mathbf{L}_2} R_{39}(j), \Gamma^{\mathbf{L}_2}_{j_a;j_b} \right\rangle_{L_2} = \delta_{j_a,q+1+2j} + \delta_{j_a,q+1-2j} + 2\delta_{j_a,(j+1)(q+1) \bmod 2(q+1)}$.

Proposition 12.22. *The values of the class functions* $\chi^{\mathbf{G}}_{(s_{39}(j)),0} - \chi^{\mathbf{G}}_{(s_{39}(j)),1}$ *(orthogonal to* $R_{39}(j)$*) are given in Table 18.*

12.5 Decomposition of R_{37}

The character R_{37} has degree $\Phi_1^3\Phi_2\Phi_3$. Therefore, by Proposition 3.28 its constituents are cuspidal (they have the same degree polynomial since they are $\mathrm{GL}_4(q)$-conjugate by Clifford theory). So their Harish–Chandra restriction to L_1 is identically zero.

Lemma 12.23. *The classes* $(s_{37}(j))$ *of* G^* *do not intersect* L_1^*, *however they split into two classes of* L_2^* *and*

$$^*R^{\mathbf{G}}_{\mathbf{L}_2}\left(\chi^{\mathbf{G}}_{(s_{37}(j)),0} - \chi^{\mathbf{G}}_{(s_{37}(j)),1}\right) = \Gamma^{\mathbf{L}_2}_{j;0} - \Gamma^{\mathbf{L}_2}_{j;1} + \Gamma^{\mathbf{L}_2}_{qj;0} - \Gamma^{\mathbf{L}_2}_{qj;1}.$$

Table 18: Certain class functions orthogonal to $R_{39}(k)$ for k in the range given in Table 14.

	$\chi^{\mathbf{G}}_{(s_{39}(k)),0} - \chi^{\mathbf{G}}_{(s_{39}(k)),1}$
$c_3(j)$	$-q^2(q-1)(-1)^{j(k+1)}$
$c_4(j)$	$q^2(q-1)(-1)^{j(k+1)}$
$c_6(j)$	$q(-1)^{j(k+1)}$
$c_7(j)$	$-q(-1)^{j(k+1)}$
$c_8(j)$	$q(-1)^{j(k+1)}$
$c_9(j)$	$-q(-1)^{j(k+1)}$
$c_{16}(j,l)$	0
$c_{17}(j,l)$	0
$c_{19}(j)$	$q\zeta_{2q+2}^{j(q+1+2k)} + q\zeta_{2q+2}^{j(q+1-2k)}$
$c_{20}(j)$	$-q\zeta_{2q+2}^{j(q+1+2k)} - q\zeta_{2q+2}^{j(q+1-2k)}$

Table 19: Certain class functions orthogonal to $R_{37}(k)$ for k in the range given in Table 14.

	$\chi^{\mathbf{G}}_{(s_{37}(k)),0} - \chi^{\mathbf{G}}_{(s_{37}(k)),1}$
$c_3(j)$	$-q^2(q-1)i^{3jk}$
$c_4(j)$	$q^2(q-1)i^{3jk}$
$c_6(j)$	qi^{3jk}
$c_7(j)$	$-qi^{3jk}$
$c_8(j)$	qi^{3jk}
$c_9(j)$	$-qi^{3jk}$
$c_{16}(j,l)$	0
$c_{17}(j,l)$	0
$c_{19}(j)$	$q\zeta_{2q+2}^{j(q+1+2k)} + q\zeta_{2q+2}^{j(q+1-2k)}$
$c_{20}(j)$	$-q\zeta_{2q+2}^{j(q+1+2k)} - q\zeta_{2q+2}^{j(q+1-2k)}$

This comes from $\left\langle {}^*R^{\mathbf{G}}_{\mathbf{L}_2} R_{37}(j), \Gamma^{\mathbf{L}_2}_{j_a;j_b} \right\rangle_{L_2} = \delta_{j_a,j} + \delta_{j_a,qj}$.

Proposition 12.24. *The values of the class functions $\chi^{\mathbf{G}}_{(s_{37}(j)),0} - \chi^{\mathbf{G}}_{(s_{37}(j)),1}$ (orthogonal to $R_{37}(j)$) are given in Table 19.*

12.6 Decomposition of R_{43}, R_{47} and R_{51}

These are the last almost characters left to be decomposed. With the knowledge of the rest of the character table and the Gel'fand–Graev characters of G it is possible to decompose them.

Lemma 12.25. *The irreducible constituents of R_{43}, R_{47} and R_{51} are regular characters.*

This is proved similarly to Lemma 12.4, with the difference that here there are four distinct constituents instead of two.

Once this is known, the decomposition follows since there are 16 irreducible constituents to be found and 16 Gel'fand–Graev characters.

In practice, we can remove from the modified Gel'fand–Graev characters all the constituents but one. This is one of the characters that we are looking for.

12.7 Closing remarks

We complete the discussion on the decomposition of uniform almost characters of $\mathrm{SL}_4(q)$ by pointing out all the information that we could have used directly from the theory of Gel'fand–Graev characters.

First of all, we can say that the constituents of R_{18} and R_{21} are semisimple. This is a consequence of the fact that any Lusztig series contains semisimple characters that are the dual of the regular characters of the same series (by definition). And, there are no other possibilities in those Lusztig series. Equivalently, we can see it by their character degrees (see Remark 5.42). Moreover, the constituents of R_{33}, R_{35}, R_{37}, R_{39}, R_{41}, R_{43}, R_{47} and R_{51} are regular semisimple characters. This is easy to see, because there are no other characters in their Lusztig series. Equivalently, from Lübeck's table we know that they belong to Lusztig series parametrized by regular semisimple elements of G^*. More precisely, the centralizers in $\mathbf{G}^*$ of those semisimple elements are indicated as being tori of $\mathbf{G}^*$.

Once the regular /semisimple characters are identified, we can use Theorem 5.37 and Remark 5.39 to directly fix their character values on regular unipotent elements. At the same time, we fix the values of the Harish-Chandra restriction to L_1 of the regular/semisimple characters on the regular unipotent elements of L_1. Since there are only two unipotent classes of SL_4 that split in $\mathrm{SL}_4(q)$, at this point, all character values at unipotent elements are fixed.

Secondly, thanks to scalar products with the modified Gel'fand–Graev characters we can identify the decomposition of every character χ and $R_{\mathbf{L}}^{\mathbf{G}}\chi$ in $\bigoplus_{\varphi\in\mathrm{Irr}(Z(G))}\mathrm{CF}(G)^{\varphi}$, respectively in $\bigoplus_{\varphi\in\mathrm{Irr}(Z(L))}\mathrm{CF}(L)^{\varphi}$ for $L = L_1$ or $L = L_2$. With this information, we can fix every character value on non-unipotent elements (see Remark 5.55).

We will use this approach in the case of $\mathrm{Spin}_8^+(q)$. However, notice that in $\mathrm{SL}_4(q)$ all the non-irreducible uniform almost character are either regular or semisimple. This is no longer the case for $\mathrm{Spin}_8^+(q)$.

Part III

The case of $\mathrm{Spin}_8^+(q)$ for q odd

In this part of the work, we want to construct the generic character table of $\mathrm{Spin}_8^+(q)$, with q odd. To this goal, we apply the theory of Gel'fand–Graev characters to decompose the uniform almost characters of $\mathrm{Spin}_8^+(q)$. However, there is a new obstacle arising in the case of $\mathrm{Spin}_8^+(q)$ compared to $\mathrm{SL}_4(q)$. One crucial fact of the decomposition of the uniform almost characters of $\mathrm{SL}_4(q)$ is that their constituents are all either regular or semisimple. This is no longer true for $\mathrm{Spin}_8^+(q)$. It is, nevertheless, possible to write systems of equations involving the values of any character and the values of modified Gel'fand–Graev characters of the Levi subgroups of $\mathrm{Spin}_8^+(q)$.

In theory, the system of equations arising this way fixes the values for those elements that have semisimple part whose centralizer is a Levi subgroup. This comes from the fact that, by their definition there are as many modified Gel'fand–Graev characters in any Levi subgroup $\mathbf{L}^F$ as there are classes with that property, i.e. such that the centralizer of the semisimple part of a representative of the class is $\mathbf{L}$.

The problem arises for elements whose semisimple part has a centralizer which is not a Levi subgroup of Spin_8. However, the number of these cases is small enough that it seems that some ad-hoc methods might fix them.

13 The simply connected group of type D_4 and the finite groups $\mathrm{Spin}_8^+(q)$

Like for SL_4, we first need to determine Chevalley generators and relations for the semisimple simply connected group Spin_8. We use for this the construction developed by Geck in [Ge17], that results in a 16-dimensional faithful representation.

From now on we fix the following notation.

Notation 13.1. We denote by $\mathbf{G}$ a simply connected simple algebraic group of type D_4 defined over $\mathbb{F}_q$ (which is $\mathrm{Spin}_8(\bar{\mathbb{F}}_q)$), where q is an odd prime power. Then, we denote by F the Frobenius morphism associated with the rational structure of $\mathbf{G}$, and by $G = \mathrm{Spin}_8^+(q)$ the finite group of Lie type $\mathbf{G}^F$.

We denote by $\mathbf{B}_0$ the Borel subgroup of $\mathbf{G}$ consisting of upper-triangular matrices and by $\mathbf{U}_0 = R_u(\mathbf{B}_0)$ (the unipotent radical of $\mathbf{B}_0$) the maximal unipotent subgroup of $\mathbf{G}$ consisting of upper-triangular matrices with ones on the diagonal (with respect to the representation built from [Ge17]). Moreover, we write $\mathbf{T}_0$ for the maximally split torus of $\mathbf{G}$ consisting of diagonal matrices, such that $\mathbf{B}_0 = \mathbf{T}_0 \ltimes \mathbf{U}_0$, and we denote its normalizer by $\mathbf{N}_0 = N_{\mathbf{G}}(\mathbf{T}_0)$.

The Weyl group of $\mathbf{G}$ is $W = \mathbf{N}_0/\mathbf{T}_0$. The root system associated to $\mathbf{G}$ and relative to $\mathbf{T}_0$ is of type D_4 and is denoted by Φ_{D_4}. A base of positive roots of Φ_{D_4} is denoted by Δ_{D_4} and the set of positive roots relative to this base is denoted by $\Phi_{D_4}^+$. We denote by $\mathcal{I}$ the index set of Δ_{D_4}, and for all $i \in \mathcal{I}$ we denote the roots subgroups by $\mathbf{U}_i := \mathbf{U}_{\alpha_i}$.

Analogously for the subgroups of the finite group G, we will write $B_0 = \mathbf{B}_0^F$, $U_0 = \mathbf{U}_0^F$, $T_0 = \mathbf{T}_0^F$, $N_0 = N_G(\mathbf{T}_0)$ and $U_i = U_{\alpha_i} = \mathbf{U}_i^F$.

When working with the finite fields, we will need to choose generators of $\mathbb{F}_q^\times$ and $\mathbb{F}_{q^2}^\times$. As in the previous part of the work, $\mu \in \mathbb{F}_q^\times$ is fixed such that $\langle \mu \rangle = \mathbb{F}_q^\times$, and $\rho \in \mathbb{F}_{q^2}^\times$ such that $\langle \rho \rangle = \mathbb{F}_{q^2}^\times$ and $\rho^{q+1} = \mu$.

Again, when possible, any polynomial expressions in q will be given as product of the cyclotomic polynomials. They will be denoted by $\Phi_1 = q - 1$, $\Phi_2 = q + 1$, $\Phi_3 = q^2 + q + 1$,

$\Phi_4 = q^2 + 1$ and $\Phi_6 = q^2 - q + 1$.

Notice, before we start, that D_4 is the only indecomposable root system that has a diagram permutation of order 3. We call *triality automorphism* the graph automorphism of $\mathbf{G}$ arising from the permutation of order 3 of the external nodes of the Dynkin diagram of type D_4, see [Ca72, Chapter 12.2] for details on automorphisms of Chevalley groups (p. 200 for graph automorphisms). We denote, both, the triality permutation and the induced triality automorphism by τ.

13.1 Explicit construction of the algebraic group

Geck gives in [Ge17] an explicit identification of the BN pair of $\mathbf{G}$. He does it by representing the elements of $\mathbf{U}_0$, $\mathbf{N}_0 = N_{\mathbf{G}}(\mathbf{T}_0)$ and $\mathbf{T}_0$ as maps over a vector space whose basis elements are indexed by W-obits of "minuscule" weights. This results in a 16-dimensional faithful representation of $\mathbf{G}$. We follow here this construction step-by-step to get a Steinberg presentation of $\mathbf{G}$.

13.1.1 Roots and weights

The root system Φ_{D_4} of $\mathbf{G}$ of type D_4 with respect to a maximal torus $\mathbf{T}_0$ has a base of simple roots $\Delta_{D_4} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ (with index set $\mathcal{I} = \{1, 2, 3, 4\}$) with Cartan matrix

$$\langle \alpha_i, \alpha_j^\vee \rangle_{i,j\in\mathcal{I}} = \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

D_4

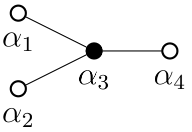

Figure 2: Dynking diagram of type D_4. The white dots represent the roots for which the associated fundamental dominant weights are minuscule.

The positive roots $\Phi^+_{D_4}$ of this root system are given in Table 20.

The construction of the algebraic group starts with the identification of the minuscule weights. See [Ge17, Section 2 (after Remark 2.1) and Table 1] for the definition and the list of all minuscule dominant weights. For D_4 there are three minuscule weights λ_1, λ_2 and λ_4, where λ_i is the fundamental dominant weight associated to α_i, i.e. $\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$ for $i, j \in \mathcal{I}$.

Then, we need the union of the W-orbits of the minuscule dominant weights, this set will be denoted Ψ. From [Ge17, Theorem 4.11] we have that $\mathbb{Z}\Psi$ is isomorphic to the character group of $\mathbf{T}_0$. Consequently, the fundamental group of $\mathbf{G}$ is trivial (hence $\mathbf{G}$ is simply connected) if $\mathbb{Z}\Psi$ is equal to the weight lattice Λ. It is easy to see that, if one considers the union of the W-orbits of all three λ_1, λ_2 and λ_4 then $\mathbb{Z}\Psi = \Lambda$. However, one can choose more economically only λ_1 and λ_2 and still have $\mathbb{Z}\Psi = \Lambda$.

From now on, Ψ denotes the union of the W-orbits of the minuscule dominant weights λ_1 and λ_2. The elements of Ψ are listed in Table 21 partially ordered according to the partial order $\succeq$ defined by $\mu \succeq \mu'$ if $\mu = \mu'$ or $\mu - \mu'$ is a sum of positive roots. The enumeration of the elements of $\Psi = \{\mu_1, ..., \mu_{16}\}$ is chosen such that $i < j$ whenever $\mu_j \succeq \mu_i$.

Table 20: Positive roots written as sums of simple roots and as images under simple reflections of simple roots.

α_1	α_1	α_1
α_2	α_2	α_2
α_3	α_3	α_3
α_4	α_4	α_4
α_5	$\alpha_1 + \alpha_3$	$s_{\alpha_1}(\alpha_3)$
α_6	$\alpha_2 + \alpha_3$	$s_{\alpha_2}(\alpha_3)$
α_7	$\alpha_3 + \alpha_4$	$s_{\alpha_4}(\alpha_3)$
α_8	$\alpha_1 + \alpha_2 + \alpha_3$	$s_{\alpha_2}s_{\alpha_1}(\alpha_3)$
α_9	$\alpha_1 + \alpha_3 + \alpha_4$	$s_{\alpha_4}s_{\alpha_1}(\alpha_3)$
α_{10}	$\alpha_2 + \alpha_3 + \alpha_4$	$s_{\alpha_4}s_{\alpha_2}(\alpha_3)$
α_{11}	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$	$s_{\alpha_4}s_{\alpha_2}s_{\alpha_1}(\alpha_3)$
α_{12}	$\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4$	$s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_1}(\alpha_3)$

13.1.2 Chevalley generators

Following [Ge17], we can construct a faithful representation of the algebraic group Spin_8 and determine its Chevalley generators. To lighten the notation all the bars over the maps in [Ge17] were dropped here.

Let $k = \bar{\mathbb{F}}_q$, for q odd, and M a free k-module with basis $\{z_\mu \mid \mu \in \Psi\}$ (to be precise, let $M_{\mathbb{Z}} = \langle z_\mu \mid \mu \in \Psi\rangle_{\mathbb{Z}}$ then $M = k \otimes M_{\mathbb{Z}}$). Then, for $i \in \mathcal{I}$ and $t \in k$ we define the maps $u_i(t), v_i(t) \in \mathrm{GL}(M)$ by

$$u_i(t) : z_\mu \mapsto \begin{cases} z_\mu + t z_{\mu+\alpha_i} & \mu + \alpha_i \in \Psi, \\ z_\mu & else, \end{cases}$$

$$v_i(t) : z_\mu \mapsto \begin{cases} z_\mu + t z_{\mu-\alpha_i} & \mu - \alpha_i \in \Psi, \\ z_\mu & else. \end{cases}$$

Then, it is proved in [Ge17, Theorem 4.11 and Example 5.6] that

$$\mathbf{G} = G_k(\Psi) := \langle u_i(t), v_i(t) \mid i \in \mathcal{I}, t \in k\rangle$$

is the simply connected Chevalley group of type D_4. We give now the details of the construction of these maps and, in particular, of the Steinberg presentation of $\mathbf{G}$.

The construction of the Chevalley groups in [Ge17] gives a way of writing down elements explicitly fixing, among others, the signs occurring in the Chevalley relations.

For $i \in \mathcal{I}$, we define the nilpotent linear maps of M

$$e_i : z_\mu \mapsto \begin{cases} z_{\mu+\alpha_i} & \mu + \alpha_i \in \Psi, \\ 0 & else, \end{cases}$$

$$f_i : z_\mu \mapsto \begin{cases} z_{\mu-\alpha_i} & \mu - \alpha_i \in \Psi, \\ 0 & else. \end{cases}$$

Then, we set $u_i(t) = \mathrm{id}_M + te_i$ and $v_i(t) = \mathrm{id}_M + tf_i$, for $t \in k$.

Then, [Ge17, Lemma 4.8] states that there is a unique group isomorphism $W \to \mathbf{N}_0/\mathbf{T}_0$. It is such that $s_i \mapsto n_i\mathbf{T}_0$ for all $i \in \mathcal{I}$ ($s_i = s_{\alpha_i}$ are the simple reflections of W), with $n_i = n_i(1)$ where $n_i(t) = u_i(t)v_i(-t^{-1})u_i(t)$ are maps acting on M as

$$n_i(t) : z_\mu \mapsto \begin{cases} tz_{\mu+\alpha_i} & \mu + \alpha_i \in \Psi, \\ -t^{-1}z_{\mu-\alpha_i} & \mu - \alpha_i \in \Psi, \\ z_\mu & else, \end{cases}$$

Table 21: W-orbits of minuscule weights λ_1 and λ_2 (ordering given by the partial order $\succeq$ from "big" to "small"), and their scalar products with the simple coroots.

μ	$\langle \mu, \alpha_1^\vee \rangle$	$\langle \mu, \alpha_2^\vee \rangle$	$\langle \mu, \alpha_3^\vee \rangle$	$\langle \mu, \alpha_4^\vee \rangle$
λ_1	1	0	0	0
$\lambda_1 - \alpha_1$	-1	0	1	0
$\lambda_1 - \alpha_1 - \alpha_3$	0	1	-1	1
$\lambda_1 - \alpha_1 - \alpha_2 - \alpha_3$	0	-1	0	1
$\lambda_1 - \alpha_1 - \alpha_3 - \alpha_4$	0	1	0	-1
$\lambda_1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$	0	-1	1	-1
$\lambda_1 - \alpha_1 - \alpha_2 - 2\alpha_3 - \alpha_4$	1	0	-1	0
$\lambda_1 - 2\alpha_1 - \alpha_2 - 2\alpha_3 - \alpha_4$	-1	0	0	0
λ_2	0	1	0	0
$\lambda_2 - \alpha_2$	0	-1	1	0
$\lambda_2 - \alpha_2 - \alpha_3$	1	0	-1	1
$\lambda_2 - \alpha_1 - \alpha_2 - \alpha_3$	-1	0	0	1
$\lambda_2 - \alpha_2 - \alpha_3 - \alpha_4$	1	0	0	-1
$\lambda_2 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$	-1	0	1	-1
$\lambda_2 - \alpha_1 - \alpha_2 - 2\alpha_3 - \alpha_4$	0	1	-1	0
$\lambda_2 - \alpha_1 - 2\alpha_2 - 2\alpha_3 - \alpha_4$	0	-1	0	0

for $t \in k^\times$.

To have an effective description of the elements of $\mathbf{G}$, we need root maps from $\bar{\mathbb{F}}_q$ to the root subgroup $\mathbf{U}_\alpha$ for each $\alpha \in \Phi_{D_4}^+$.

Set $u_i(t) := u_{\alpha_i}(t) := \mathrm{id}_M + te_{\alpha_i}$ for $i = 1, ..., 12$, and $\alpha_i \in \Phi_{D_4}^+$. To define $e_j := e_{\alpha_j}$, for $j = 1, ..., 12$, we choose a sequence $i, i_1, i_2, ..., i_l \in \mathcal{I}$ such that $\alpha_j = s_{\alpha_{i_1}} s_{\alpha_{i_2}} \cdots s_{\alpha_{i_l}}(\alpha_i)$ as in Table 20. Then, we define

$$e_j := e_{\alpha_j} := n_{i_1} n_{i_2} \cdots n_{i_l} e_i n_{i_l}^{-1} \cdots n_{i_2}^{-1} n_{i_1}^{-1}$$

for $j = 1, ..., 12$.

Notice that e_j are defined up to a sign and that they perform the following transformation on M

$$e_j(z_\mu) = \begin{cases} \pm z_{\mu+\alpha_i} & \mu + \alpha_j \in \Psi, \\ 0 & \mathit{else}. \end{cases}$$

It is now possible to explicitly compute ${}^{n_i}e_j$ for all $i \in \mathcal{I}$ and $j = 1, ..., 12$, given in Table 22 (on the left). This in turn is used to get the explicit action of the Weyl group W on $\mathbf{U}_0$, this is shown in Table 22 (on the right) for all $\alpha \neq \alpha_i$ (cases with equality are not needed in what follows).

Table 22: Action of the Weyl group W on the maps e_i, $u_i(t)$ via conjugation with the representatives n_j of the simple reflections, for $j \in \mathcal{I}$, $i = 1, ..., 12$, $e_i = e_{\alpha_i}$ and $u_i(t) = u_{\alpha_i}(t)$.

	n_1	n_2	n_3	n_4
e_1	$-$	$+e_1$	$-e_5$	$+e_1$
e_2	$+e_2$	$-$	$-e_6$	$+e_2$
e_3	$+e_5$	$+e_6$	$-$	$+e_7$
e_4	$+e_4$	$+e_4$	$-e_7$	$-$
e_5	$-e_3$	$+e_8$	$+e_1$	$+e_9$
e_6	$+e_8$	$-e_3$	$+e_2$	$+e_{10}$
e_7	$+e_9$	$+e_{10}$	$+e_4$	$-e_3$
e_8	$-e_6$	$-e_5$	$+e_8$	$+e_{11}$
e_9	$-e_7$	$+e_{11}$	$+e_9$	$-e_5$
e_{10}	$+e_{11}$	$-e_7$	$+e_{10}$	$-e_6$
e_{11}	$-e_{10}$	$-e_9$	$+e_{12}$	$-e_8$
e_{12}	$+e_{12}$	$+e_{12}$	$-e_{11}$	$+e_{12}$

	n_1	n_2	n_3	n_4
$u_1(t)$	$-$	$u_1(t)$	$u_5(-t)$	$u_1(t)$
$u_2(t)$	$u_2(t)$	$-$	$u_6(-t)$	$u_2(t)$
$u_3(t)$	$u_5(t)$	$u_6(t)$	$-$	$u_7(t)$
$u_4(t)$	$u_4(t)$	$u_4(t)$	$u_7(-t)$	$-$
$u_5(t)$	$u_3(-t)$	$u_8(t)$	$u_1(t)$	$u_9(t)$
$u_6(t)$	$u_8(t)$	$u_3(-t)$	$u_2(t)$	$u_{10}(t)$
$u_7(t)$	$u_9(t)$	$u_{10}(t)$	$u_4(t)$	$u_3(-t)$
$u_8(t)$	$u_6(-t)$	$u_5(-t)$	$u_8(t)$	$u_{11}(t)$
$u_9(t)$	$u_7(-t)$	$u_{11}(t)$	$u_9(t)$	$u_5(-t)$
$u_{10}(t)$	$u_{11}(t)$	$u_7(-t)$	$u_{10}(t)$	$u_6(-t)$
$u_{11}(t)$	$u_{10}(-t)$	$u_9(-t)$	$u_{12}(t)$	$u_8(-t)$
$u_{12}(t)$	$u_{12}(t)$	$u_{12}(t)$	$u_{11}(-t)$	$u_{12}(t)$

To compute the commutation relations of $\mathbf{U}_0$, $[u_\alpha(t), u_\beta(s)] = u_\alpha(-t)u_\beta(-s)u_\alpha(t)u_\beta(s)$, we use [Ge17, Proposition 4.2] stating that, when they are non-trivial, they are of the form $[u_\alpha(t), u_\beta(s)] = u_{\alpha+\beta}(cts)$ for $c \in \{\pm 1\}$ given by $[e_\alpha, e_\beta] = ce_{\alpha+\beta}$ (as a Lie subalgebra of $\mathrm{End}_k(M)$). The non-trivial Lie brackets $[e_\alpha, e_\beta]$ are

$$\begin{aligned}
[e_1, e_3] &= e_5 \\
[e_1, e_6] &= e_8 \\
[e_1, e_7] &= e_9 \\
[e_1, e_{10}] &= e_{11} \\
[e_2, e_3] &= e_6 \\
[e_2, e_5] &= e_8 \\
[e_2, e_7] &= e_{10} \\
[e_2, e_9] &= e_{11}
\end{aligned}$$

$$\begin{aligned}
[e_3, e_4] &= -e_7 \\
[e_3, e_{11}] &= e_{12} \\
[e_4, e_5] &= e_9 \\
[e_4, e_6] &= e_{10} \\
[e_4, e_8] &= e_{11} \\
[e_5, e_{10}] &= -e_{12} \\
[e_6, e_9] &= -e_{12} \\
[e_7, e_8] &= -e_{12}.
\end{aligned}$$

We describe now the semisimple elements contained in the maximally split torus $\mathbf{T}_0$. The latter is of the form $\mathbf{T}_0 = \langle h_i(t) \mid i \in \mathcal{I},\, t \in k^\times \rangle$, where by [Ge17, Lemma 4.4] the semisimple elements $h_i(t)$ are represented by maps of M given by

$$h_i(t) = n_i(t)n_i(-1) : z_\mu \mapsto t^{\langle \mu, \alpha_i^\vee \rangle} z_\mu.$$

These elements normalize $\mathbf{U}_0$. For $i \in \mathcal{I}$, $\alpha \in \Phi^+$, $t \in k$ and $s \in k^\times$, we get ([Ge17, Lemma 4.6])

$$h_i(s)u_\alpha(t)h_i(s)^{-1} = u_\alpha(ts^{\langle \alpha, \alpha_i^\vee \rangle}).$$

Notation 13.2. From now on we write

$$h(t_1, t_2, t_3, t_4) := h_1(t_1)h_2(t_2)h_3(t_3)h_4(t_4)$$

for a generic element of $\mathbf{T}_0$, for $t_1, t_2, t_3, t_4 \in k^\times$.

Finally, the centre of $\mathbf{G}$ is isomorphic to $C_2 \times C_2$ by [Ge17, Corollary 4.9 and Example 5.7], (but also by explicit computation with Theorem 1.18 (b))

$$Z(\mathbf{G}) = \left\{ h(t_1, t_2, 1, t_1t_2) \mid t_1^2 = t_2^2 = 1 \right\}.$$

Notice that (unlike in the case of SL_4) for every odd q the equation $t_1^2 = t_2^2 = 1$ in $\bar{\mathbb{F}}_q$ has all its solutions in $\mathbb{F}_q$ and therefore $Z(\mathbf{G})$ is also the centre of the finite group $\mathbf{G}^F$.

Table 23: Action of the Weyl group W on $\mathbf{T}_0$ via conjugation with the representatives n_i of the simple reflections, i.e. $n_i h_j(t) n_i^{-1}$ for $t \in \bar{\mathbb{F}}_q^\times$ and $i, j \in \mathcal{I}$.

	n_1	n_2	n_3	n_4
$h_1(t)$	$h_1(t^{-1})$	$h_1(t)$	$h_1(t)h_3(t)$	$h_1(t)$
$h_2(t)$	$h_2(t)$	$h_2(t^{-1})$	$h_2(t)h_3(t)$	$h_2(t)$
$h_3(t)$	$h_1(t)h_3(t)$	$h_2(t)h_3(t)$	$h_3(t^{-1})$	$h_4(t)h_3(t)$
$h_4(t)$	$h_4(t)$	$h_4(t)$	$h_3(t)h_4(t)$	$h_4(t^{-1})$

13.2 The finite spin groups $\mathrm{Spin}_8^+(q)$

Let F be the untwisted Frobenius endomorphism of $\mathbf{G}$ associated to the $\mathbb{F}_q$-structure of $\mathbf{G}$. Then $G = \mathbf{G}^F$ is a finite group.

By the discussion after Proposition 1.57 the group G inherits the Chevalley generators and the Chevalley relations from $\mathbf{G}$ by restricting all formulas from Section 13.1.2 to parameters from $\mathbb{F}_q$.

Proposition 13.3. *The finite group of Lie type $G = \mathbf{G}^F$ has order ([MaTe11, Table 24.1])*

$$|G| = q^{12}\Phi_1^4\Phi_2^4\Phi_3\Phi_4^2\Phi_6.$$

It is generated by unipotent elements,

$$G = \langle u_i(t), v_i(t) \mid i \in \mathcal{I}, t \in \mathbb{F}_q \rangle .$$

It has a split BN-pair formed by the Borel subgroup $B_0 = T_0 \ltimes U_0$ with $T_0 = \mathbf{T}_0^F = \langle h_i(t) \mid i \in \mathcal{I},\, t \in \mathbb{F}_q^\times \rangle$, $U_0 = \mathbf{U}_0^F = \langle u_i(t) \mid i \in \mathcal{I},\, t \in \mathbb{F}_q \rangle$ and the normalizer of T_0, $N_0 = \mathbf{N}_0^F = \langle T_0, n_i \mid i \in \mathcal{I} \rangle$.

The action of the Weyl group $W = N_0/T_0$ on the unipotent elements is given in Table 22 when identifying the simple reflections s_{α_i} of W with their representatives n_i for $i \in \mathcal{I}$.

The action of T_0 on the unipotent elements is given by

$$h_i(s)u_\alpha(t)h_i(s)^{-1} = u_\alpha(ts^{\langle \alpha, \alpha_i^\vee \rangle})$$

for $s \in \mathbb{F}_q^\times$, $t \in \mathbb{F}_q$ and $i \in \mathcal{I}$.

The non trivial commutation relations between unipotent elements of U_0 are (they directly follow from the discussion in Section 13.1.2)

$$\begin{aligned}
[u_1(t), u_3(s)] &= u_5(ts), & [u_3(t), u_4(s)] &= u_7(-ts),\\
[u_1(t), u_6(s)] &= u_8(ts), & [u_3(t), u_{11}(s)] &= u_{12}(ts),\\
[u_1(t), u_7(s)] &= u_9(ts), & [u_4(t), u_5(s)] &= u_9(ts),\\
[u_1(t), u_{10}(s)] &= u_{11}(ts), & [u_4(t), u_6(s)] &= u_{10}(ts),\\
[u_2(t), u_3(s)] &= u_6(ts), & [u_4(t), u_8(s)] &= u_{11}(ts),\\
[u_2(t), u_5(s)] &= u_8(ts), & [u_5(t), u_{10}(s)] &= u_{12}(-ts),\\
[u_2(t), u_7(s)] &= u_{10}(ts), & [u_6(t), u_9(s)] &= u_{12}(-ts),\\
[u_2(t), u_9(s)] &= u_{11}(ts), & [u_7(t), u_8(s)] &= u_{12}(-ts).
\end{aligned}$$

The centre of G is $Z(G) = Z(\mathbf{G}) = \{h(t_1, t_2, 1, t_1t_2) \mid t_1^2 = t_2^2 = 1\}$.

The F-classes of $Z(\mathbf{G})$ are $H^1(F, Z(\mathbf{G})) \cong Z(\mathbf{G})$.

Notation 13.4. The elements of the centre, $Z = Z(G)$, will be denoted simply by

$$h_Z(k_a, k_b) := h\left(\mu^{\frac{q-1}{2}k_a}, \mu^{\frac{q-1}{2}k_b}, 1, \mu^{\frac{q-1}{2}(k_a+k_b)}\right)$$

for $k_a, k_b = 0, 1$.

Remark 13.5. As discussed in Remark 1.64 we can associate to each element $z \in H^1(F, Z(\mathbf{G}))$ a representative $g_z \in \mathbf{T}_0$. We choose for $z = h_Z(k_a, k_b)$

$$g_{(k_a,k_b)} = h\left(\rho^{-\frac{q+1}{2}k_a}, \rho^{-\frac{q+1}{2}k_b}, 1, \rho^{-\frac{q+1}{2}(k_a+k_b)}\right) \in \mathbf{T}_0,$$

for $k_a, k_b = 0, 1$.

14 Fusion of unipotent classes

We compute, in this section, the fusion of the unipotent classes. In other words we explicitly write to which conjugacy class of G belong the elements of each conjugacy class of U_0.

As discussed in Section 6.3, we start by giving the conjugacy classes of U_0.

Remark 14.1. The numbering of the positive roots is such that $U_{k+1} \cdots U_{12} \lhd U_k U_{k+1} \cdots U_{12}$ for each $k = 1, ..., 11$. This can be seen by the commutation relations in Proposition 13.3.

Moreover, every element $u \in U_0$ can be written uniquely as the ordered product

$$u = u_1(r_1) u_2(r_2) \cdots u_{12}(r_{12})$$

with $r_i \in \mathbb{F}_q$.

Proposition 14.2. *The conjugacy classes of U are listed in Table 24, in the first column. They are also listed in the appendix with their U_0-orbits in Table 60.*

Remark 14.3. There are

$$2q^5 + 5q^4 - 4q^3 - 4q^2 + 2q$$

different conjugacy classes in U_0, in agreement with [GoRoe09, Table 1].

By acting with T_0 we obtain the representatives of the unipotent classes of B_0.

Proposition 14.4. *Representatives of the unipotent classes of B_0 are given in the fourth column of Table 24.*

We give an example of the computation of the fusion from U_0 to B_0.

Example 14.5. Fusion of $u_1(r_1)u_2(r_2)u_4(r_4)$ to B_0, for $r_1, r_2, r_4 \in \mathbb{F}_q^\times$:

We have, by Proposition 13.3,

$$^{h_1(s_1)h_2(s_2)h_3(s_3)h_4(s_4)}(u_1(r_1)u_2(r_2)u_4(r_4)) = u_1(r_1 s_1^2 s_3^{-1}) u_2(r_2 s_2^2 s_3^{-1}) u_4(r_4 s_3^{-1} s_4^2)$$

where $s_1, s_2, s_3, s_4 \in \mathbb{F}_q^\times$.

It is clear that the argument of u_4 can be set to 1 by imposing $s_3 = r_4 s_4^2$. Then, the expression becomes

$$u_1(r_1 r_4^{-1} s_1^2 s_4^{-2}) u_2(r_2 r_4^{-1} s_2^2 s_4^{-2}) u_4(1)$$

The first and/or second arguments can be set to 1 only if, respectively, $r_1 r_4$ and/or $r_2 r_4$ are squares of $\mathbb{F}_q^\times$. Otherwise, they can be set to a chosen non-square of $\mathbb{F}_q^\times$, for example μ.

In other words, $u_1(r_1)u_2(r_2)u_4(r_4)$ is conjugate to $u_1(\mu^k)u_2(\mu^l)u_4(1)$ if $r_1 r_4 \in \mu^k(\mathbb{F}_q^\times)^2$ and $r_2 r_4 \in \mu^l(\mathbb{F}_q^\times)^2$.

The next step is the fusion from B_0 to G. This is performed thanks to the action of the Weyl group (Table 22) on the representatives of B_0. In practice, we write a list of elements of W in reduced form and lift them to a product of the n_i in N_0. Then, we conjugate the representatives of B_0 with these elements of N_0 (only with those that do not concern the blank cases of Table 22).

This results in a list of elements of U_0 in an unordered product of the u_i. We reorder the products with the commutation relations in Proposition 13.3, and use Table 60 to find which element of U_0 are conjugate.

We give an example of the computation of the fusion from B_0 to G.

Table 24: Representatives of the conjugacy classes of U_0 and their representative in B_0, with $r_1, ..., r_{12} \in \mathbb{F}_q^\times$. They are listed in the order in which they are computed thanks to the algorithm discussed in Section 6.3. The second column contains the number of classes of each type. The third column contains the centralizer order in U_0 of u. The last two columns contain representatives of the B_0-class of u. (Continues on the next page)

u	#classes	$\lvert C_{U_0}(u)\rvert$	Representative in B_0	Conditions
1	1	q^{12}	1	
$u_{12}(r_{12})$	$q-1$	q^{12}	$u_{12}(1)$	
$u_{11}(r_{11})$	$q-1$	q^{11}	$u_{11}(1)$	
$u_{10}(r_{10})$	$q-1$	q^{10}	$u_{10}(1)$	
$u_9(r_9)$	$q-1$	q^{10}	$u_9(1)$	
$u_9(r_9)u_{10}(r_{10})$	$(q-1)^2$	q^{10}	$u_9(\mu^k)u_{10}(1)$	$r_9r_{10} \in \mu^k(\mathbb{F}_q^\times)^2$
$u_8(r_8)$	$q-1$	q^{10}	$u_8(1)$	
$u_8(r_8)u_9(r_9)$	$(q-1)^2$	q^{10}	$u_8(\mu^k)u_9(1)$	$r_8r_9 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_8(r_8)u_{10}(r_{10})$	$(q-1)^2$	q^{10}	$u_8(\mu^k)u_{10}(1)$	$r_8r_{10} \in \mu^k(\mathbb{F}_q^\times)^2$
$u_8(r_8)u_9(r_9)u_{10}(r_{10})$	$(q-1)^3$	q^{10}	$u_8(\mu^k)u_9(\mu^l)u_{10}(1)$	$r_8r_{10} \in \mu^k(\mathbb{F}_q^\times)^2$, $r_9r_{10} \in \mu^l(\mathbb{F}_q^\times)^2$
$u_7(r_7)$	$q-1$	q^9	$u_7(1)$	
$u_7(r_7)u_8(r_8)$	$(q-1)^2$	q^8	$u_7(1)u_8(1)$	
$u_7(r_7)u_{11}(r_{11})$	$(q-1)^2$	q^9	$u_7(\mu^k)u_{11}(1)$	$r_7r_{11} \in \mu^k(\mathbb{F}_q^\times)^2$
$u_6(r_6)$	$q-1$	q^9	$u_6(1)$	
$u_6(r_6)u_7(r_7)$	$(q-1)^2$	q^9	$u_6(\mu^k)u_7(1)$	$r_6r_7 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_6(r_6)u_9(r_9)$	$(q-1)^2$	q^8	$u_6(1)u_9(1)$	
$u_6(r_6)u_{11}(r_{11})$	$(q-1)^2$	q^9	$u_6(\mu^k)u_{11}(1)$	$r_6r_{11} \in \mu^k(\mathbb{F}_q^\times)^2$
$u_6(r_6)u_7(r_7)u_9(r_9)$	$(q-1)^3$	q^8	$u_6(\mu^k)u_7(1)u_9(1)$	$r_6r_7 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_6(r_6)u_7(r_7)u_{11}(r_{11})$	$(q-1)^3$	q^9	$u_6(\mu^k)u_7(\mu^l)u_{11}(1)$	$r_6r_{11} \in \mu^k(\mathbb{F}_q^\times)^2$, $r_7r_{11} \in \mu^l(\mathbb{F}_q^\times)^2$
$u_5(r_5)$	$q-1$	q^9	$u_5(1)$	
$u_5(r_5)u_6(r_6)$	$(q-1)^2$	q^9	$u_5(\mu^k)u_6(1)$	$r_5r_6 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_5(r_5)u_7(r_7)$	$(q-1)^2$	q^9	$u_5(\mu^k)u_7(1)$	$r_5r_7 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_5(r_5)u_{10}(r_{10})$	$(q-1)^2$	q^8	$u_5(1)u_{10}(1)$	
$u_5(r_5)u_{11}(r_{11})$	$(q-1)^2$	q^9	$u_5(\mu^k)u_{11}(1)$	$r_5r_{11} \in \mu^k(\mathbb{F}_q^\times)^2$
$u_5(r_5)u_6(r_6)u_7(r_7)$	$(q-1)^3$	q^8	$u_5(\mu^k)u_6(\mu^l)u_7(1)$	$r_5r_7 \in \mu^k(\mathbb{F}_q^\times)^2$, $r_6r_7 \in \mu^l(\mathbb{F}_q^\times)^2$
$u_5(r_5)u_6(r_6)u_{10}(r_{10})$	$(q-1)^3$	q^8	$u_5(\mu^k)u_6(1)u_{10}(1)$	$r_5r_6 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_5(r_5)u_6(r_6)u_{11}(r_{11})$	$(q-1)^3$	q^9	$u_5(\mu^k)u_6(\mu^l)u_{11}(1)$	$r_5r_{11} \in \mu^k(\mathbb{F}_q^\times)^2$, $r_6r_{11} \in \mu^l(\mathbb{F}_q^\times)^2$
$u_5(r_5)u_7(r_7)u_{10}(r_{10})$	$(q-1)^3$	q^8	$u_5(\mu^k)u_7(1)u_{10}(1)$	$r_5r_7 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_5(r_5)u_7(r_7)u_{11}(r_{11})$	$(q-1)^3$	q^9	$u_5(\mu^k)u_7(\mu^l)u_{11}(1)$	$r_5r_{11} \in \mu^k(\mathbb{F}_q^\times)^2$, $r_7r_{11} \in \mu^l(\mathbb{F}_q^\times)^2$
$u_5(r_5)u_6(r_6)u_7(r_7)u_{11}(r_{11})$	$(q-1)^4$	q^8	$u_5(\mu^k)u_6(\mu^l)u_7(\mu^m)u_{11}(1)$	$r_5r_{11} \in \mu^k(\mathbb{F}_q^\times)^2$, $r_6r_{11} \in \mu^l(\mathbb{F}_q^\times)^2$, $r_7r_{11} \in \mu^m(\mathbb{F}_q^\times)^2$
$u_4(r_4)$	$q-1$	q^8	$u_4(1)$	
$u_4(r_4)u_5(r_5)$	$(q-1)^2$	q^6	$u_4(1)u_5(1)$	
$u_4(r_4)u_6(r_6)$	$(q-1)^2$	q^6	$u_4(1)u_6(1)$	
$u_4(r_4)u_8(r_8)$	$(q-1)^2$	q^7	$u_4(1)u_8(1)$	
$u_4(r_4)u_{12}(r_{12})$	$(q-1)^2$	q^8	$u_4(\mu^k)u_{12}(1)$	$r_4r_{12} \in \mu^k(\mathbb{F}_q^\times)^2$
$u_4(r_4)u_5(r_5)u_6(r_6)$	$(q-1)^3$	q^6	$u_4(1)u_5(\mu^k)u_6(1)$	$r_5r_6 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_3(r_3)$	$q-1$	q^8	$u_3(1)$	
$u_3(r_3)u_4(r_4)$	$(q-1)^2$	q^5	$u_3(1)u_4(1)$	
$u_3(r_3)u_8(r_8)$	$(q-1)^2$	q^8	$u_3(\mu^k)u_8(1)$	$r_3r_8 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_3(r_3)u_9(r_9)$	$(q-1)^2$	q^8	$u_3(\mu^k)u_9(1)$	$r_3r_9 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_3(r_3)u_{10}(r_{10})$	$(q-1)^2$	q^8	$u_3(\mu^k)u_{10}(1)$	$r_3r_{10} \in \mu^k(\mathbb{F}_q^\times)^2$
$u_3(r_3)u_{11}(r_{11})$	$(q-1)^2$	q^8	$u_3(1)u_{11}(1)$	
$u_3(r_3)u_4(r_4)u_8(r_8)$	$(q-1)^3$	q^5	$u_3(\mu^k)u_4(1)u_8(1)$	$r_3r_8 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_3(r_3)u_8(r_8)u_9(r_9)$	$(q-1)^3$	q^8	$u_3(\mu^k)u_8(\mu^l)u_9(1)$	$r_3r_9 \in \mu^k(\mathbb{F}_q^\times)^2$, $r_8r_9 \in \mu^l(\mathbb{F}_q^\times)^2$
$u_3(r_3)u_8(r_8)u_{10}(r_{10})$	$(q-1)^3$	q^8	$u_3(\mu^k)u_8(\mu^l)u_{10}(1)$	$r_3r_{10} \in \mu^k(\mathbb{F}_q^\times)^2$, $r_8r_{10} \in \mu^l(\mathbb{F}_q^\times)^2$
$u_3(r_3)u_8(r_8)u_{11}(r_{11})$	$(q-1)^3$	q^8	$u_3(\mu^k)u_8(1)u_{11}(1)$	$r_3r_8 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_3(r_3)u_9(r_9)u_{10}(r_{10})$	$(q-1)^3$	q^8	$u_3(\mu^k)u_9(\mu^l)u_{10}(1)$	$r_3r_{10} \in \mu^k(\mathbb{F}_q^\times)^2$, $r_9r_{10} \in \mu^l(\mathbb{F}_q^\times)^2$
$u_3(r_3)u_9(r_9)u_{11}(r_{11})$	$(q-1)^3$	q^8	$u_3(\mu^k)u_9(1)u_{11}(1)$	$r_3r_9 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_3(r_3)u_{10}(r_{10})u_{11}(r_{11})$	$(q-1)^3$	q^8	$u_3(\mu^k)u_{10}(1)u_{11}(1)$	$r_3r_{10} \in \mu^k(\mathbb{F}_q^\times)^2$
$u_3(r_3)u_8(r_8)u_9(r_9)u_{10}(r_{10})$	$(q-1)^4$	q^8	$u_3(\mu^k)u_8(\mu^l)u_9(\mu^m)u_{10}(1)$	$r_3r_{10} \in \mu^k(\mathbb{F}_q^\times)^2$, $r_8r_{10} \in \mu^l(\mathbb{F}_q^\times)^2$, $r_9r_{10} \in \mu^m(\mathbb{F}_q^\times)^2$
$u_3(r_3)u_8(r_8)u_9(r_9)u_{11}(r_{11})$	$(q-1)^4$	q^8	$u_3(\mu^k)u_8(\mu^l)u_9(1)u_{11}(1)$	$r_3r_9 \in \mu^k(\mathbb{F}_q^\times)^2$, $r_8r_9 \in \mu^l(\mathbb{F}_q^\times)^2$
$u_3(r_3)u_8(r_8)u_{10}(r_{10})u_{11}(r_{11})$	$(q-1)^4$	q^8	$u_3(\mu^k)u_8(\mu^l)u_{10}(1)u_{11}(1)$	$r_3r_{10} \in \mu^k(\mathbb{F}_q^\times)^2$, $r_8r_{10} \in \mu^l(\mathbb{F}_q^\times)^2$
$u_3(r_3)u_9(r_9)u_{10}(r_{10})u_{11}(r_{11})$	$(q-1)^4$	q^8	$u_3(\mu^k)u_9(\mu^l)u_{10}(1)u_{11}(1)$	$r_3r_{10} \in \mu^k(\mathbb{F}_q^\times)^2$, $r_9r_{10} \in \mu^l(\mathbb{F}_q^\times)^2$
$u_3(r_3)u_8(r_8)u_9(r_9)u_{10}(r_{10})u_{11}(r_{11})$	$(q-1)^5$	q^8	$u_3(r_3r_{11}^2r_8^{-1}r_9^{-1}r_{10}^{-1}\mu^k\mu^l)\times$ $u_8(\mu^k)u_9(\mu^l)u_{10}(1)u_{11}(1)$	$r_8r_{10} \in \mu^k(\mathbb{F}_q^\times)^2$, $r_9r_{10} \in \mu^l(\mathbb{F}_q^\times)^2$

u	#classes	$\lvert C_{U_0}(u)\rvert$	Representative in B_0	Conditions
$u_2(r_2)$	$q-1$	q^8	$u_2(1)$	
$u_2(r_2)u_3(r_3)$	$(q-1)^2$	q^5	$u_2(1)u_3(1)$	
$u_2(r_2)u_4(r_4)$	$(q-1)^2$	q^8	$u_2(\mu^k)u_4(1)$	$r_2r_4 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_2(r_2)u_5(r_5)$	$(q-1)^2$	q^6	$u_2(1)u_5(1)$	
$u_2(r_2)u_7(r_7)$	$(q-1)^2$	q^6	$u_2(1)u_7(1)$	
$u_2(r_2)u_9(r_9)$	$(q-1)^2$	q^7	$u_2(1)u_9(1)$	
$u_2(r_2)u_{12}(r_{12})$	$(q-1)^2$	q^8	$u_2(\mu^k)u_{12}(1)$	$r_2r_{12} \in \mu^k(\mathbb{F}_q^\times)^2$
$u_2(r_2)u_3(r_3)u_4(r_4)$	$(q-1)^3$	q^5	$u_2(\mu^k)u_3(1)u_4(1)$	$r_2r_4 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_2(r_2)u_3(r_3)u_9(r_9)$	$(q-1)^3$	q^5	$u_2(1)u_3(\mu^k)u_9(1)$	$r_3r_9 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_2(r_2)u_4(r_4)u_5(r_5)$	$(q-1)^3$	q^6	$u_2(\mu^k)u_4(1)u_5(1)$	$r_2r_4 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_2(r_2)u_4(r_4)u_7(r_7)$	$(q-1)^3$	q^6	$u_2(\mu^k)u_4(1)u_7(1)$	$r_2r_4 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_2(r_2)u_4(r_4)u_9(r_9)$	$(q-1)^3$	q^7	$u_2(\mu^k)u_4(1)u_9(1)$	$r_2r_4 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_2(r_2)u_4(r_4)u_{12}(r_{12})$	$(q-1)^3$	q^8	$u_2(\mu^k)u_4(\mu^l)u_{12}(1)$	$r_2r_{12} \in \mu^k(\mathbb{F}_q^\times)^2$, $r_4r_{12} \in \mu^l(\mathbb{F}_q^\times)^2$
$u_2(r_2)u_5(r_5)u_7(r_7)$	$(q-1)^3$	q^6	$u_2(1)u_5(\mu^k)u_7(1)$	$r_5r_7 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_2(r_2)u_3(r_3)u_4(r_4)u_9(r_9)$	$(q-1)^4$	q^5	$u_2(\mu^k)u_3(\mu^l)u_4(1)u_9(1)$	$r_2r_4 \in \mu^k(\mathbb{F}_q^\times)^2$, $r_3r_9 \in \mu^l(\mathbb{F}_q^\times)^2$
$u_2(r_2)u_4(r_4)u_5(r_5)u_7(r_7)$	$(q-1)^4$	q^6	$u_2(\mu^k)u_4(1)u_5(\mu^l)u_7(1)$	$r_2r_4 \in \mu^k(\mathbb{F}_q^\times)^2$, $r_5r_7 \in \mu^l(\mathbb{F}_q^\times)^2$
$u_1(r_1)$	$q-1$	q^8	$u_1(1)$	
$u_1(r_1)u_2(r_2)$	$(q-1)^2$	q^8	$u_1(\mu^k)u_2(1)$	$r_1r_2 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_1(r_1)u_3(r_3)$	$(q-1)^2$	q^5	$u_1(1)u_3(1)$	
$u_1(r_1)u_4(r_4)$	$(q-1)^2$	q^8	$u_1(\mu^k)u_4(1)$	$r_1r_4 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_1(r_1)u_6(r_6)$	$(q-1)^2$	q^6	$u_1(1)u_6(1)$	
$u_1(r_1)u_7(r_7)$	$(q-1)^2$	q^6	$u_1(1)u_7(1)$	
$u_1(r_1)u_{10}(r_{10})$	$(q-1)^2$	q^7	$u_1(1)u_{10}(1)$	
$u_1(r_1)u_{12}(r_{12})$	$(q-1)^2$	q^8	$u_1(\mu^k)u_{12}(1)$	$r_1r_{12} \in \mu^k(\mathbb{F}_q^\times)^2$
$u_1(r_1)u_2(r_2)u_3(r_3)$	$(q-1)^3$	q^5	$u_1(\mu^k)u_2(1)u_3(1)$	$r_1r_2 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_1(r_1)u_2(r_2)u_4(r_4)$	$(q-1)^3$	q^7	$u_1(\mu^k)u_2(\mu^l)u_4(1)$	$r_1r_4 \in \mu^k(\mathbb{F}_q^\times)^2$, $r_2r_4 \in \mu^l(\mathbb{F}_q^\times)^2$
$u_1(r_1)u_2(r_2)u_6(r_6)$	$(q-1)^3$	q^6	$u_1(\mu^k)u_2(1)u_6(1)$	$r_1r_2 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_1(r_1)u_2(r_2)u_7(r_7)$	$(q-1)^3$	q^6	$u_1(\mu^k)u_2(1)u_7(1)$	$r_1r_2 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_1(r_1)u_2(r_2)u_{10}(r_{10})$	$(q-1)^3$	q^7	$u_1(\mu^k)u_2(1)u_{10}(1)$	$r_1r_2 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_1(r_1)u_2(r_2)u_{12}(r_{12})$	$(q-1)^3$	q^8	$u_1(\mu^k)u_2(\mu^l)u_{12}(1)$	$r_1r_{12} \in \mu^k(\mathbb{F}_q^\times)^2$, $r_2r_{12} \in \mu^l(\mathbb{F}_q^\times)^2$
$u_1(r_1)u_3(r_3)u_4(r_4)$	$(q-1)^3$	q^5	$u_1(\mu^k)u_3(1)u_4(1)$	$r_1r_4 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_1(r_1)u_3(r_3)u_{10}(r_{10})$	$(q-1)^3$	q^5	$u_1(1)u_3(\mu^k)u_{10}(1)$	$r_3r_{10} \in \mu^k(\mathbb{F}_q^\times)^2$
$u_1(r_1)u_4(r_4)u_6(r_6)$	$(q-1)^3$	q^6	$u_1(\mu^k)u_4(1)u_6(1)$	$r_1r_4 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_1(r_1)u_4(r_4)u_7(r_7)$	$(q-1)^3$	q^6	$u_1(\mu^k)u_4(1)u_7(1)$	$r_1r_4 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_1(r_1)u_4(r_4)u_{10}(r_{10})$	$(q-1)^3$	q^7	$u_1(\mu^k)u_4(1)u_{10}(1)$	$r_1r_4 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_1(r_1)u_4(r_4)u_{12}(r_{12})$	$(q-1)^3$	q^8	$u_1(\mu^k)u_4(\mu^l)u_{12}(1)$	$r_1r_{12} \in \mu^k(\mathbb{F}_q^\times)^2$, $r_4r_{12} \in \mu^l(\mathbb{F}_q^\times)^2$
$u_1(r_1)u_6(r_6)u_7(r_7)$	$(q-1)^3$	q^6	$u_1(1)u_6(\mu^k)u_7(1)$	$r_6r_7 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_1(r_1)u_2(r_2)u_3(r_3)u_4(r_4)$	$(q-1)^4$	q^4	$u_1(\mu^k)u_2(\mu^l)u_3(1)u_4(1)$	$r_1r_4 \in \mu^k(\mathbb{F}_q^\times)^2$, $r_2r_4 \in \mu^l(\mathbb{F}_q^\times)^2$
$u_1(r_1)u_2(r_2)u_3(r_3)u_{10}(r_{10})$	$(q-1)^4$	q^5	$u_1(\mu^k)u_2(1)u_3(\mu^l)u_{10}(1)$	$r_1r_2 \in \mu^k(\mathbb{F}_q^\times)^2$, $r_3r_{10} \in \mu^l(\mathbb{F}_q^\times)^2$
$u_1(r_1)u_2(r_2)u_4(r_4)u_6(r_6)$	$(q-1)^4$	q^6	$u_1(\mu^k)u_2(\mu^l)u_4(1)u_6(1)$	$r_1r_4 \in \mu^k(\mathbb{F}_q^\times)^2$, $r_2r_4 \in \mu^l(\mathbb{F}_q^\times)^2$
$u_1(r_1)u_2(r_2)u_4(r_4)u_7(r_7)$	$(q-1)^4$	q^6	$u_1(\mu^k)u_2(\mu^l)u_4(1)u_7(1)$	$r_1r_4 \in \mu^k(\mathbb{F}_q^\times)^2$, $r_2r_4 \in \mu^l(\mathbb{F}_q^\times)^2$
$u_1(r_1)u_2(r_2)u_4(r_4)u_{12}(r_{12})$	$(q-1)^4$	q^7	$u_1(\mu^k)u_2(\mu^l)u_4(\mu^m)u_{12}(1)$	$r_1r_{12} \in \mu^k(\mathbb{F}_q^\times)^2$, $r_2r_{12} \in \mu^l(\mathbb{F}_q^\times)^2$, $r_4r_{12} \in \mu^m(\mathbb{F}_q^\times)^2$
$u_1(r_1)u_2(r_2)u_6(r_6)u_7(r_7)$	$(q-1)^4$	q^6	$u_1(\mu^k)u_2(1)u_6(\mu^l)u_7(1)$	$r_1r_2 \in \mu^k(\mathbb{F}_q^\times)^2$, $r_6r_7 \in \mu^l(\mathbb{F}_q^\times)^2$
$u_1(r_1)u_3(r_3)u_4(r_4)u_{10}(r_{10})$	$(q-1)^4$	q^5	$u_1(\mu^k)u_3(\mu^l)u_4(1)u_{10}(1)$	$r_1r_4 \in \mu^k(\mathbb{F}_q^\times)^2$, $r_3r_{10} \in \mu^l(\mathbb{F}_q^\times)^2$
$u_1(r_1)u_4(r_4)u_6(r_6)u_7(r_7)$	$(q-1)^4$	q^6	$u_1(\mu^k)u_4(1)u_6(\mu^l)u_7(1)$	$r_1r_4 \in \mu^k(\mathbb{F}_q^\times)^2$, $r_6r_7 \in \mu^l(\mathbb{F}_q^\times)^2$
$u_1(r_1)u_2(r_2)u_4(r_4)u_6(r_6)u_7(r_7)$	$(q-1)^5$	q^6	$u_1(1)u_2(\mu^k)\times$ $u_4(r_4r_6r_2^{-1}r_7^{-1}\mu^{k-l})\times$ $u_6(\mu^l)u_7(1)$	$r_1r_2 \in \mu^k(\mathbb{F}_q^\times)^2$, $r_6r_7 \in \mu^l(\mathbb{F}_q^\times)^2$

Example 14.6. Fusion of $u_1(\mu^k)u_2(1)u_3(1)$ to G, for $k = 0, 1$:

Conjugation with W yields the following list of conjugates in U_0,

$$u_1(\mu^k)u_2(1)u_3(1),\ u_1(\mu^k)u_2(1)u_7(1),\ u_2(\mu^k)u_1(1)u_3(1),\ u_2(\mu^k)u_1(1)u_7(-1),$$
$$u_3(\mu^k)u_8(-1)u_4(1),\ u_5(-\mu^k)u_6(-1)u_4(1),\ u_6(\mu^k)u_5(1)u_4(1),\ u_8(-\mu^k)u_3(1)u_4(1).$$

With the commutation relations, we reorder these elements and check to what element of B_0 they are conjugate (with Table 60).

In the ordered form the list becomes

$$u_1(\mu^k)u_2(1)u_3(1),\ u_1(\mu^k)u_2(1)u_7(1),\ u_1(1)u_2(\mu^k)u_3(1),\ u_1(1)u_2(\mu^k)u_7(-1),$$
$$u_3(\mu^k)u_4(1)u_8(-1)u_{11}(1),\ u_4(1)u_5(-\mu^k)u_6(-1)u_9(\mu^k)u_{10}(1)u_{12}(-\mu^k),$$
$$u_4(1)u_5(1)u_6(\mu^k)u_9(-1)u_{10}(-\mu^k)u_{12}(-\mu^k),\ u_3(1)u_4(1)u_8(-\mu^k)u_{11}(\mu^k).$$

From Table 25, it is easy to see that the first and third elements are already in the same class (same for the second and fourth elements).

For the other elements, we need Table 60 to identify to which class they belong.

Finally,

$$u_1(\mu^k)u_2(1)u_3(1),\ u_1(\mu^k)u_2(1)u_7(1),\ u_3(-\mu^k)u_4(1)u_8(1),\ u_4(1)u_5(\mu^k)u_6(1)$$

belong to the same conjugacy class of G.

Lübeck computed a parametrization of the unipotent classes of G and their centralizers. He gives lists the unipotent classes in terms of Jordan normal forms of representatives in $\mathrm{SO}_8^+(q)$.[10] There are 28 distinct unipotent classes in G. Therefore, once a list of 28 candidate representatives is found (and every element of U_0 is conjugate to one of these) it is not checked that they are indeed not conjugate to each other.[11]

The representatives computed here are associated to those computed by Lübeck thanks to their Jordan normal form (after projecting to SO_8). Only for two classes this is not enough to identify them. The two classes with Jordan blocks $3^2 1^2$ have different centralizer, with orders, $2q^8(q-1)^2$ and $2q^8(q+1)^2$. By explicit computation, one of the classes is centralized only by two elements of the maximally split torus T_0. Then, it must be the one with centralizer $2q^8(q+1)^2$.

Proposition 14.7. *There are 28 distinct unipotent conjugacy classes in G. They fuse in G according to Table 26. A list of representatives with corresponding centralizer in G is given in Table 25.*

Remark 14.8. Although the "same" representatives of the unipotent classes can be chosen for all q odd, it is obvious from Table 26 that the fusion of classes is not the same. This is a consequence of -1 being a square of $\mathbb{F}_q^\times$ for the congruence $q \equiv 1 \pmod 4$ but not for $q \equiv 3 \pmod 4$.

The triality automorphism τ of G leaves invariant the unipotent subgroup U_0. It is possible to compute the permutation of the unipotent classes performed by τ, once the fusion of unipotent classes is known (Table 26 and Table 48).

We end this section with an example of the action of the triality on the unipotent classes. This action is summarized in Table 48.

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[10] The spin group is related to the orthogonal group by the short exact sequence $1 \to C_2 \to \mathrm{Spin}_8 \to \mathrm{SO}_8 \to 1$. In other words there is a projection from Spin_8 to SO_8.

[11] The number of unipotent classes can be computed knowing representatives of the unipotent conjugacy classes of SO_8. See for example [LiSe12, Proposition 3.19].

Table 25: Representatives of the unipotent classes of G and the order of their centralizer.

Jordan blocks	Representative u	$\|C_G(u)\|$
1^8	1	$q^{12}\Phi_1^4\Phi_2^4\Phi_3\Phi_4^2\Phi_6$
$2^2 1^4$	$u_1(1)$	$q^{12}\Phi_1^3\Phi_2^3$
2^4+	$u_1(\mu^k)u_2(1)$, $k=0,1$	$2q^{10}\Phi_1^2\Phi_2^2\Phi_4$
2^4-	$u_1(\mu^k)u_4(1)$, $k=0,1$	$2q^{10}\Phi_1^2\Phi_2^2\Phi_4$
31^5	$u_2(\mu^k)u_4(1)$, $k=0,1$	$2q^{10}\Phi_1^2\Phi_2^2\Phi_4$
$32^2 1$	$u_1(\mu^k)u_2(\mu^l)u_4(1)$, $k,l=0,1$	$4q^{10}\Phi_1\Phi_2$
$3^2 1^2$	$u_1(1)u_3(1)$	$2q^8\Phi_1^2$
$3^2 1^2$	$u_1(\mu)u_2(1)u_4(1)u_{12}(1)$	$2q^8\Phi_2^2$
4^2+	$u_1(\mu^k)u_2(1)u_3(1)$, $k=0,1$	$2q^6\Phi_1\Phi_2$
4^2-	$u_1(\mu^k)u_3(1)u_4(1)$, $k=0,1$	$2q^6\Phi_1\Phi_2$
51^3	$u_2(\mu^k)u_3(1)u_4(1)$, $k=0,1$	$2q^6\Phi_1\Phi_2$
53	$u_1(\mu^k)u_2(1)u_3(\mu^l)u_{10}(1)$, $k,l=0,1$	$4q^6$
71	$u_1(\mu^k)u_2(\mu^l)u_3(1)u_4(1)$, $k,l=0,1$	$4q^4$

Example 14.9. Triality sends the element $u_1(1)u_2(1)u_3(\mu)u_{10}(1)$ to $u_2(1)u_4(1)u_3(\mu)u_{10}(1)$ which can be reordered with the commutation relations to give $u_2(1)u_3(\mu)u_4(1)u_7(\mu)u_{10}(1)$.

Thanks to Table 60, the latter is conjugate to $u_2(1)u_3(\mu)u_4(1)u_9(1)$.

Finally, by Table 26, we have that this element is conjugate to $u_1(\mu)u_2(1)u_3(1)u_{10}(1)$ if $q \equiv 1 \pmod 4$ or to $u_1(\mu)u_2(1)u_3(\mu)u_{10}(1)$ if $q \equiv 3 \pmod 4$.

Table 26: Representatives of the unipotent classes of G and their fusion from U_0. When not specified, the condition on the parameters is simply $r_i \in \mathbb{F}_q^\times$. (continues on the next page)

Representative in G	Representatives in U	Conditions
1	1	
$u_1(1)$	$u_1(r_1)$	
	$u_2(r_2)$	
	$u_3(r_3)$	
	$u_4(r_4)$	
	$u_5(r_5)$	
	$u_6(r_6)$	
	$u_7(r_7)$	
	$u_8(r_8)$	
	$u_9(r_9)$	
	$u_{10}(r_{10})$	
	$u_{11}(r_{11})$	
	$u_{12}(r_{12})$	
$u_1(\mu^k)u_2(1)$	$u_1(r_1)u_2(r_2)$	$r_1r_2 \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_3(r_3)u_8(r_8)$	$-r_3r_8 \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_4(r_4)u_{12}(r_{12})$	$-r_4r_{12} \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_5(r_5)u_6(r_6)$	$r_5r_6 \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_7(r_7)u_{11}(r_{11})$	$-r_7r_{11} \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_9(r_9)u_{10}(r_{10})$	$r_9r_{10} \in \mu^k(\mathbb{F}_q^\times)^2$
$u_1(\mu^k)u_4(1)$	$u_1(r_1)u_4(r_4)$	$r_1r_4 \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_2(r_2)u_{12}(r_{12})$	$-r_2r_{12} \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_3(r_3)u_9(r_9)$	$-r_3r_9 \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_5(r_5)u_7(r_7)$	$r_5r_7 \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_6(r_6)u_{11}(r_{11})$	$-r_6r_{11} \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_8(r_8)u_{10}(r_{10})$	$r_8r_{10} \in \mu^k(\mathbb{F}_q^\times)^2$
$u_2(\mu^k)u_4(1)$	$u_2(r_2)u_4(r_4)$	$r_2r_4 \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_1(r_1)u_{12}(r_{12})$	$-r_1r_{12} \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_3(r_3)u_{10}(r_{10})$	$-r_3r_{10} \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_5(r_5)u_{11}(r_{11})$	$-r_5r_{11} \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_6(r_6)u_7(r_7)$	$r_6r_7 \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_8(r_8)u_9(r_9)$	$r_8r_9 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_1(\mu^k)u_2(\mu^l)u_4(1)$	$u_1(r_1)u_2(r_2)u_4(r_4)$	$r_1r_4 \in \mu^k(\mathbb{F}_q^\times)^2$, $r_2r_4 \in \mu^l(\mathbb{F}_q^\times)^2$
	$u_1(r_1)u_2(r_2)u_{12}(r_{12})$	$-r_1r_{12} \in \mu^l(\mathbb{F}_q^\times)^2$, $-r_2r_{12} \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_1(r_1)u_4(r_4)u_{12}(r_{12})$	$-r_1r_{12} \in \mu^l(\mathbb{F}_q^\times)^2$, $-r_4r_{12} \in \mu^{k+l}(\mathbb{F}_q^\times)^2$
	$u_2(r_2)u_4(r_4)u_{12}(r_{12})$	$-r_2r_{12} \in \mu^k(\mathbb{F}_q^\times)^2$, $-r_4r_{12} \in \mu^{k+l}(\mathbb{F}_q^\times)^2$
	$u_3(r_3)u_8(r_8)u_9(r_9)$	$-r_3r_9 \in \mu^k(\mathbb{F}_q^\times)^2$, $r_8r_9 \in \mu^l(\mathbb{F}_q^\times)^2$
	$u_3(r_3)u_8(r_8)u_{10}(r_{10})$	$-r_3r_{10} \in \mu^l(\mathbb{F}_q^\times)^2$, $r_8r_{10} \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_3(r_3)u_9(r_9)u_{10}(r_{10})$	$-r_3r_{10} \in \mu^l(\mathbb{F}_q^\times)^2$, $r_9r_{10} \in \mu^{k+l}(\mathbb{F}_q^\times)^2$
	$u_5(r_5)u_6(r_6)u_7(r_7)$	$r_5r_7 \in \mu^k(\mathbb{F}_q^\times)^2$, $r_6r_7 \in \mu^l(\mathbb{F}_q^\times)^2$
	$u_5(r_5)u_6(r_6)u_{11}(r_{11})$	$-r_5r_{11} \in \mu^l(\mathbb{F}_q^\times)^2$, $-r_6r_{11} \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_5(r_5)u_7(r_7)u_{11}(r_{11})$	$-r_5r_{11} \in \mu^l(\mathbb{F}_q^\times)^2$, $-r_7r_{11} \in \mu^{k+l}(\mathbb{F}_q^\times)^2$
	$u_6(r_6)u_7(r_7)u_{11}(r_{11})$	$-r_6r_{11} \in \mu^k(\mathbb{F}_q^\times)^2$, $-r_7r_{11} \in \mu^{k+l}(\mathbb{F}_q^\times)^2$
	$u_8(r_8)u_9(r_9)u_{10}(r_{10})$	$r_8r_{10} \in \mu^k(\mathbb{F}_q^\times)^2$, $r_9r_{10} \in \mu^{k+l}(\mathbb{F}_q^\times)^2$
	$u_3(r_3)u_8(r_8)u_9(r_9)u_{10}(r_{10})u_{11}(r_{11})$	$-r_3r_9 \in \mu^k(\mathbb{F}_q^\times)^2$, $r_8r_9 \in \mu^l(\mathbb{F}_q^\times)^2$, $r_3r_{11}^2r_8^{-1}r_9^{-1}r_{10}^{-1} = -4$

Representative in G	Representatives u in U	Conditions
$u_1(1)u_3(1)$	$u_1(r_1)u_3(r_3)$	
	$u_1(r_1)u_6(r_6)$	
	$u_1(r_1)u_7(r_7)$	
	$u_1(r_1)u_{10}(r_{10})$	
	$u_2(r_2)u_3(r_3)$	
	$u_2(r_2)u_5(r_5)$	
	$u_2(r_2)u_7(r_7)$	
	$u_2(r_2)u_9(r_9)$	
	$u_3(r_3)u_4(r_4)$	
	$u_3(r_3)u_{11}(r_{11})$	
	$u_4(r_4)u_5(r_5)$	
	$u_4(r_4)u_6(r_6)$	
	$u_4(r_4)u_8(r_8)$	
	$u_5(r_5)u_{10}(r_{10})$	
	$u_6(r_6)u_9(r_9)$	
	$u_7(r_7)u_8(r_8)$	
	$u_1(r_1)u_2(r_2)u_6(r_6)$	
	$u_1(r_1)u_2(r_2)u_{10}(r_{10})$	
	$u_3(r_3)u_8(r_8)u_{11}(r_{11})$	
	$u_5(r_5)u_6(r_6)u_{10}(r_{10})$	
	$u_1(r_1)u_4(r_4)u_7(r_7)$	
	$u_1(r_1)u_4(r_4)u_{10}(r_{10})$	
	$u_3(r_3)u_9(r_9)u_{11}(r_{11})$	
	$u_5(r_5)u_7(r_7)u_{10}(r_{10})$	
	$u_2(r_2)u_4(r_4)u_7(r_7)$	
	$u_2(r_2)u_4(r_4)u_9(r_9)$	
	$u_3(r_3)u_{10}(r_{10})u_{11}(r_{11})$	
	$u_6(r_6)u_7(r_7)u_9(r_9)$	
	$u_3(r_3)u_8(r_8)u_9(r_9)u_{11}(r_{11})$	
	$u_3(r_3)u_8(r_8)u_{10}(r_{10})u_{11}(r_{11})$	
	$u_3(r_3)u_9(r_9)u_{10}(r_{10})u_{11}(r_{11})$	
	$u_1(r_1)u_2(r_2)u_4(r_4)u_{12}(r_{12})$	$r_1r_2r_4r_{12} \in (\mathbb{F}_q^\times)^2$
	$u_3(r_3)u_8(r_8)u_9(r_9)u_{10}(r_{10})$	$r_3r_8r_9r_{10} \in (\mathbb{F}_q^\times)^2$
	$u_5(r_5)u_6(r_6)u_7(r_7)u_{11}(r_{11})$	$r_5r_6r_7r_{11} \in (\mathbb{F}_q^\times)^2$
	$u_3(r_3)u_8(r_8)u_9(r_9)u_{10}(r_{10})u_{11}(r_{11})$	$\delta := r_3r_{11}^2r_8^{-1}r_9^{-1}r_{10}^{-1} = -2$ if $q = 1 \pmod 4$ or $\delta \neq -2, -4$ and $\delta(\delta+4) \in (\mathbb{F}_q^\times)^2$
$u_1(\mu)u_2(1)u_4(1)u_{12}(1)$	$u_1(r_1)u_2(r_2)u_4(r_4)u_{12}(r_{12})$	$r_1r_2r_4r_{12} \in \mu(\mathbb{F}_q^\times)^2$
	$u_3(r_3)u_8(r_8)u_9(r_9)u_{10}(r_{10})$	$r_3r_8r_9r_{10} \in \mu(\mathbb{F}_q^\times)^2$
	$u_5(r_5)u_6(r_6)u_7(r_7)u_{11}(r_{11})$	$r_5r_6r_7r_{11} \in \mu(\mathbb{F}_q^\times)^2$
	$u_3(r_3)u_8(r_8)u_9(r_9)u_{10}(r_{10})u_{11}(r_{11})$	$\delta := r_3r_{11}^2r_8^{-1}r_9^{-1}r_{10}^{-1} = -2$ if $q = 3 \pmod 4$ or $\delta \neq -2, -4$ and $\delta(\delta+4) \in \mu(\mathbb{F}_q^\times)^2$
$u_1(\mu^k)u_2(1)u_3(1)$	$u_1(r_1)u_2(r_2)u_3(r_3)$	$r_1r_2 \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_1(r_1)u_2(r_2)u_7(r_7)$	$r_1r_2 \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_3(r_3)u_4(r_4)u_8(r_8)$	$-r_3r_8 \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_4(r_4)u_5(r_5)u_6(r_6)$	$r_5r_6 \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_1(r_1)u_2(r_2)u_4(r_4)u_7(r_7)$	$r_1r_2 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_1(\mu^k)u_3(1)u_4(1)$	$u_1(r_1)u_3(r_3)u_4(r_4)$	$r_1r_4 \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_1(r_1)u_4(r_4)u_6(r_6)$	$r_1r_4 \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_2(r_2)u_3(r_3)u_9(r_9)$	$-r_3r_9 \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_2(r_2)u_5(r_5)u_7(r_7)$	$r_5r_7 \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_1(r_1)u_2(r_2)u_4(r_4)u_6(r_6)$	$r_1r_4 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_2(\mu^k)u_3(1)u_4(1)$	$u_1(r_1)u_3(r_3)u_{10}(r_{10})$	$-r_3r_{10} \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_1(r_1)u_6(r_6)u_7(r_7)$	$r_6r_7 \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_2(r_2)u_3(r_3)u_4(r_4)$	$r_2r_4 \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_2(r_2)u_4(r_4)u_5(r_5)$	$r_2r_4 \in \mu^k(\mathbb{F}_q^\times)^2$
	$u_1(r_1)u_2(r_2)u_4(r_4)u_6(r_6)u_7(r_7)$	$r_6r_7 \in \mu^k(\mathbb{F}_q^\times)^2$, $r_4r_6r_2^{-1}r_7^{-1} = 1$
$u_1(\mu^k)u_2(1)u_3(\mu^l)u_{10}(1)$	$u_1(r_1)u_2(r_2)u_3(r_3)u_{10}(r_{10})$	$r_1r_2 \in \mu^k(\mathbb{F}_q^\times)^2$, $r_3r_{10} \in \mu^l(\mathbb{F}_q^\times)^2$
	$u_1(r_1)u_2(r_2)u_6(r_6)u_7(r_7)$	$r_1r_2 \in \mu^k(\mathbb{F}_q^\times)^2$, $-r_6r_7 \in \mu^l(\mathbb{F}_q^\times)^2$
	$u_2(r_2)u_3(r_3)u_4(r_4)u_9(r_9)$	$-r_2r_4 \in \mu^l(\mathbb{F}_q^\times)^2$, $-r_3r_9 \in \mu^{k+l}(\mathbb{F}_q^\times)^2$
	$u_2(r_2)u_4(r_4)u_5(r_5)u_7(r_7)$	$-r_2r_4 \in \mu^l(\mathbb{F}_q^\times)^2$, $r_5r_7 \in \mu^{k+l}(\mathbb{F}_q^\times)^2$
	$u_1(r_1)u_3(r_3)u_4(r_4)u_{10}(r_{10})$	$r_1r_4 \in \mu^{k+l}(\mathbb{F}_q^\times)^2$, $r_3r_{10} \in \mu^l(\mathbb{F}_q^\times)^2$
	$u_1(r_1)u_4(r_4)u_6(r_6)u_7(r_7)$	$r_1r_4 \in \mu^{k+l}(\mathbb{F}_q^\times)^2$, $-r_6r_7 \in \mu^l(\mathbb{F}_q^\times)^2$
	$u_1(r_1)u_2(r_2)u_4(r_4)u_6(r_6)u_7(r_7)$	$r_1r_2 - r_1r_4r_6r_7^{-1} \in \mu^k(\mathbb{F}_q^\times)^2$, $-r_6r_7 \in \mu^l(\mathbb{F}_q^\times)^2$
$u_1(\mu^k)u_2(\mu^l)u_3(1)u_4(1)$	$u_1(r_1)u_2(r_2)u_3(r_3)u_4(r_4)$	$r_1r_4 \in \mu^k(\mathbb{F}_q^\times)^2$, $r_2r_4 \in \mu^l(\mathbb{F}_q^\times)^2$

Table 27: F-stable Levi subgroups with disconnected centre and their fixed points.

$\mathbf{L}$	Type	$\mathbf{L}^F$
$\mathbf{M}_1$	$(\{\alpha_2, \alpha_3, \alpha_4\}, 1)$	$M_1 = A_3(q)(q-1)$
$\mathbf{M}_2$	$(\{\alpha_2, \alpha_3, \alpha_4\}, s_1 s_3 s_2 s_4 s_3 s_1)$	$M_2 = {}^2A_3(q)(q+1)$
$\mathbf{M}_3$	$(\{\alpha_1, \alpha_2, \alpha_4\}, 1)$	$M_3 = A_1(q)^3(q-1)$
$\mathbf{M}_4$	$(\{\alpha_1, \alpha_2, \alpha_4\}, s_3 s_2 s_4 s_3 s_1 s_3 s_2 s_4 s_3)$	$M_4 = A_1(q)^3(q+1)$
$\mathbf{L}_1$	$(\{\alpha_2, \alpha_4\}, 1)$	$L_1 = A_1(q)^2(q-1)^2$
$\mathbf{L}_2$	$(\{\alpha_2, \alpha_4\}, s_3 s_2 s_4 s_3)$	$L_2 = A_1(q^2)(q^2-1)$
$\mathbf{L}_3$	$(\{\alpha_2, \alpha_4\}, s_1)$	$L_3 = A_1(q)^2(q^2-1)$
$\mathbf{L}_4$	$(\{\alpha_2, \alpha_4\}, s_1 s_3 s_2 s_4 s_3 s_1 s_3 s_2 s_4 s_3)$	$L_4 = A_1(q)^2(q+1)^2$
$\mathbf{L}_5$	$(\{\alpha_2, \alpha_4\}, s_3 s_2 s_4 s_3 s_1)$	$L_5 = A_1(q^2)(q^2+1)$

15 Levi subgroups/centralizers with disconnected centre

In $\mathbf{G}$ there are nine F-stable proper Levi subgroups with disconnected centre, up to conjugacy and up to triality. Four of them are maximal (denoted $\mathbf{M}_i$, $i = 1, 2, 3, 4$), four are minimal (denoted $\mathbf{L}_i$, $i = 1, 2, 3, 4$) and one is both minimal and maximal (denoted $\mathbf{L}_5$) w.r.t. inclusion of Levi subgroups with disconnected centre. They are listed in Table 27.

Of these, $\mathbf{M}_3$ and $\mathbf{M}_4$ are the only Levi subgroups that are stable under the action of the triality automorphism. We give its action on their unipotent classes in Table 49 (see Example 14.9 for the computation). The other Levi subgroups are permuted by the triality automorphism τ. So we can do all the computations for just one Levi in each τ-orbit, and the results are easily adaptable for the others.

Notation 15.1. We denote by $I_1 = \{\alpha_2, \alpha_3, \alpha_4\}$, $I_2 = \{\alpha_1, \alpha_2, \alpha_4\}$ and $I_3 = \{\alpha_2, \alpha_4\}$ the bases of the root systems of the Levi subgroups considered below.

We can easily compute, thanks to Theorem 1.18 (b) the following centres

$$Z(\mathbf{L}_{I_1}) = \{h(\epsilon t^2, \epsilon t, t^2, t) \mid \epsilon = \pm 1,\ t \in \bar{\mathbb{F}}_q^\times\},$$
$$Z(\mathbf{L}_{I_2}) = \{h(\epsilon_1 t, \epsilon_2 t, t^2, t) \mid \epsilon_1, \epsilon_2 = \pm 1,\ t \in \bar{\mathbb{F}}_q^\times\},$$
$$Z(\mathbf{L}_{I_3}) = \{h(t_1, \epsilon t_2, t_2^2, t_2) \mid \epsilon = \pm 1,\ t_1, t_2 \in \bar{\mathbb{F}}_q^\times\}.$$

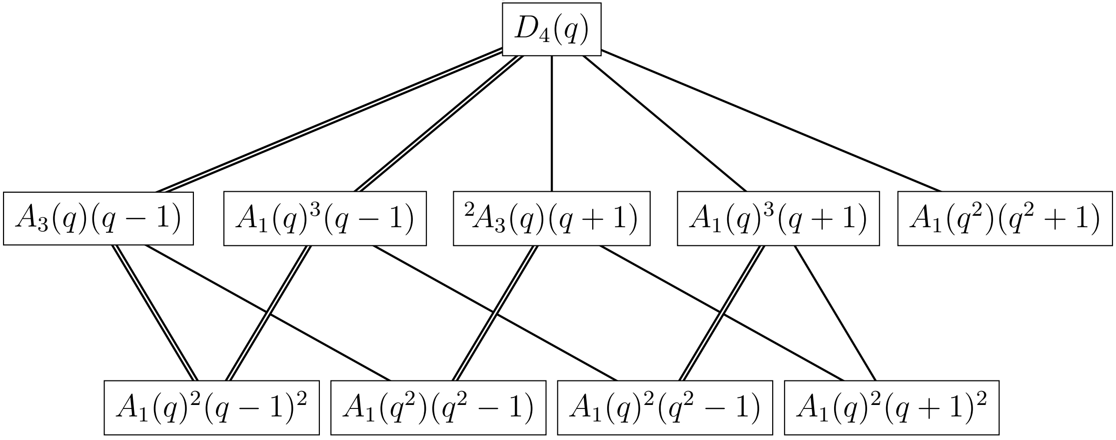
For each case below, if we discuss the Levi of type (I_i, w_i) we denote, without explicit mention, by $F' = F \circ w_i^{-1}$ and by ϕ the automorphism induced by F' on $X(\mathbf{T}_0) \otimes_{\mathbb{Z}} \mathbb{R}$ (see discussion before Definition 1.54). Also, we denote by $\mathscr{L}'$ the Lang map associated with F'. In this section, we denote by $\omega \in \mathbb{F}_{q^4}^\times$ a generator of $\mathbb{F}_{q^4}^\times$.

The treatment is analogous to that of Section 9, for $\mathrm{SL}_4(q)$. Therefore, we omit the details that require the same computations and give only the results.

For the decomposition of the uniform almost characters, we also need another subgroup which is not a Levi subgroup. By the algorithm of Borel–de Siebenthal [MaTe11, Chapter 13] there is a semisimple element $s \in \mathbf{G}$ ($s \in \mathbf{T}_0$) with centralizer of type A_1^4 (to be precise $s = h(-1, -1, 1, -1)$). This centralizer, that we denote by $\mathbf{C} = C_{\mathbf{G}}(s)$, is not a Levi subgroup of $\mathbf{G}$ since it is not conjugate to any $\mathbf{L}_I$ for $I \subseteq \Delta_{D_4}$. Notice that $\mathbf{C}$ is a semisimple group ($Z(\mathbf{C})^\circ = 1$) but it is not simply connected (see page 139 for details). However, the "order polynomial" of finite groups of Lie type does not distinguish between isogeny types (see discussion after [MaTe11, Corollary 24.6]), thus $|C| = |\,\mathrm{SL}_2(q)|^4$ where $C := \mathbf{C}^F$.

We discuss now the determination of $\mathrm{res}_{\mathbf{L}}^{\mathbf{G}}$ for the Levi subgroups $\mathbf{L}$ introduced above.

Figure 3: Subgroup lattice of Levi subgroups with disconnected centre of $\mathrm{Spin}_8^+(q)$, up to triality. The lines represent inclusions: a single line indicates a twisted Levi subgroup and a double line corresponds to a split Levi subgroup.



Remark 15.2. To compute $\mathrm{res}^{\mathbf{G}}_{\mathbf{L}}$, we use a "zig-zag" procedure that can be followed graphically in Figure 3. We make extensive use of the transitivity of $\mathrm{res}^{\mathbf{G}}_{\mathbf{L}}$. It is easy to check from the definition given in [Bo05, Section 15.A] that for split Levi subgroups the class of $\prod_{\alpha\in\Delta_{\mathbf{G}^F}} u_\alpha(1)$ is sent to the class of $\prod_{\alpha\in\Delta_{\mathbf{L}^F}} u_\alpha(1)$ where $\Delta_{\mathbf{G}^F}$ and $\Delta_{\mathbf{L}^F}$ are respectively bases of the root system of $\mathbf{G}^F$ and $\mathbf{L}^F$ (with $\Delta_{\mathbf{L}^F} \subseteq \Delta_{\mathbf{G}^F}$).

For clarity reasons, in the twisted cases, we are going to compute representatives of the unipotent classes, in $\mathbf{G}^{F'}$ (instead of $\mathbf{G}^F$) for F' some twisted Frobenius associated to the twisted Levi subgroups. To be able to compare these representatives of the regular unipotent classes with $\mathrm{res}^{\mathbf{G}}_{\mathbf{L}}$, we need to compute a preimage under the Lang map of the twisting element associated with $\mathbf{L}$.

We start with $L_2 = A_1(q^2)(q^2-1)$ since as Levi subgroup of $M_1 = A_3(q)(q-1)$ its treatment is analogous to the one of $A_1(q^2)(q+1)$ as Levi subgroup of $\mathrm{SL}_4(q)$. Lemma 9.18 gives a hint for finding an element $g_{w_2} \in \mathbf{G}$ such that $g_{w_2}^{-1}F(g_{w_2})$ represents the twisting element $s_3s_2s_4s_3$ in $N_{\mathbf{G}}(\mathbf{T}_0)^F$. The idea is to change the indices in Lemma 9.18 according to

$$1,2,3,4,5,6 \mapsto 2,3,4,6,7,10.$$

We obtain the element

$$g = u_6(-\rho)u_7(\rho)h_3(\rho^q-\rho)n_3n_2n_4n_3u_6(-1)u_7(1),$$

where ρ is our chosen generator of $\mathbb{F}_{q^2}^\times$, such that $g^{-1}F(g) = n_3n_2n_4n_3h_3(-1)h_4(-1)$. With this element, we can compute that

$$\mathrm{res}^{\mathbf{G}}_{\mathbf{L}_2}\left((u_1(1)u_2(1)u_3(1)u_4(1))^G\right) = (u_2(\rho)u_4(\rho^q))^{L_2}.$$

Because $\mathbf{L}_2$ is split in $\mathbf{M}_2$ and $\mathrm{res}^{\mathbf{G}}_{\mathbf{L}_2}$ is transitive, we also get the restriction to M_2. Next, we treat $\mathbf{L}_3$ and $\mathbf{L}_4$. They are both twisted Levi subgroups of, respectively, $\mathbf{M}_3$ and $\mathbf{M}_4$, with twisting element s_1. As such the only possibilities for a preimage of $g^{-1}F(g) = n_1$ are (in Bruhat form) $g = u_1(r_1)h(t_1,t_2,t_3,t_4)$ or $g = u_1(r_1)h(t_1,t_2,t_3,t_4)n_1u_1(r_1')$ for some $r_1, r_1' \in \bar{\mathbb{F}}_q$ and $t_1,t_2,t_3,t_4 \in \bar{\mathbb{F}}_q^\times$. A quick check gives a candidate, which in turn allows us to compute $\mathrm{res}^{\mathbf{M}_3}_{\mathbf{L}_3}$ and $\mathrm{res}^{\mathbf{M}_4}_{\mathbf{L}_4}$. Since $\mathbf{M}_3$ is a split Levi of $\mathbf{G}$, we get $\mathrm{res}^{\mathbf{G}}_{\mathbf{L}_3} = \mathrm{res}^{\mathbf{M}_3}_{\mathbf{L}_3} \circ \mathrm{res}^{\mathbf{G}}_{\mathbf{M}_3}$. Again, thanks to $\mathbf{L}_3$ being split in $\mathbf{M}_4$ and the transitivity of $\mathrm{res}^{\mathbf{G}}_{\mathbf{L}_3}$, we also get $\mathrm{res}^{\mathbf{G}}_{\mathbf{M}_4}$.

At the moment of the writing, no solution has been found for computing $\mathrm{res}^{\mathbf{G}}_{\mathbf{L}_5}$. The reason being the size of the system of equation to solve, and no educated guess (on the form of g_{w_5}) being apparent.

M_1 (split)	
Root system $\Phi_{M_1} \cong \Phi_{A_3}$	$\pm\{\alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_7, \alpha_{10}\}$
Structure	$M_1 \cong \mathrm{SL}_4(q) \rtimes \langle h_1(t) \mid t \in \mathbb{F}_q^\times \rangle$
Order $\|M_1\|$	$q^6\Phi_1^4\Phi_2^2\Phi_3\Phi_4$
Centre $Z_{M_1} = Z(\mathbf{L}_{I_1}^F)$	$\{h_{Z_{M_1}}(k_a, k_b) \mid k_a = 0, 1,\ k_b = 0, ..., q-2\}$ $h_{Z_{M_1}}(k_a, k_b) := h(\mu^{2k_b+\frac{q-1}{2}k_a}, \mu^{k_b+\frac{q-1}{2}k_a}, \mu^{2k_b}, \mu^{k_b})$
Unipotent subgroup	$U_{M_1} = \langle U_2, U_3, U_4, U_6, U_7, U_{10} \rangle$
Commutation relations	$[u_2(t_1), u_3(t_2)] = u_6(t_1t_2),\ t_1, t_2 \in \mathbb{F}_q,$ $[u_2(t_1), u_7(t_2)] = u_{10}(t_1t_2),\ t_1, t_2 \in \mathbb{F}_q,$ $[u_3(t_1), u_4(t_2)] = u_7(-t_1t_2),\ t_1, t_2 \in \mathbb{F}_q,$ $[u_4(t_1), u_6(t_2)] = u_{10}(t_1t_2),\ t_1, t_2 \in \mathbb{F}_q.$
Unipotent classes	Table 28
$H^1(F, Z(\mathbf{L}_{I_1}))$	$\{h((-1)^{k_a}, (-1)^{k_a}, 1, 1) \mid k_a = 0, 1\}$
$\ker(\mathfrak{h}_{M_1})$	$\{h_Z(0,0), h_Z(0,1)\}$
Repr. $t_z = \mathscr{L}_{\mathbf{T}_0}^{-1}(z)$ for $z = h((-1)^{k_a}, (-1)^{k_a}, 1, 1)$	$h(\rho^{\frac{q+1}{2}k_a}, \rho^{-\frac{q+1}{2}k_a}, 1, 1)$
Regular character of U_{M_1} parametrized by $h((-1)^{k_a}, (-1)^{k_a}, 1, 1)$	$\phi^{M_1}_{\mu^{k_a},1,1} : u_2(t_2)u_3(t_3)u_4(t_4) \mapsto \phi\left(\mu^{k_a}t_2 + t_3 + t_4\right)$

Table 28: Representatives of the unipotent classes of M_1, and of U_{M_1} with their fusion to M_1.

Repr. u_0 in M_1	$\|C_{M_1}(u_0)\|$	Repr. u in U_{M_1}	condition	$\|C_{U_{M_1}}(u)\|$
1	$q^6\Phi_1^4\Phi_2^2\Phi_3\Phi_4$	1		q^6
$u_2(1)$	$q^6\Phi_1^3\Phi_2$	$u_2(r_2)$		q^4
		$u_3(r_3)$		q^4
		$u_4(r_4)$		q^4
		$u_6(r_6)$		q^5
		$u_7(r_7)$		q^5
		$u_{10}(r_{10})$		q^6
$u_2(\mu^k)u_4(1)$	$2q^5\Phi_1^2\Phi_2$	$u_2(r_2)u_4(r_4)$	$r_2r_4 \in \mu^k(\mathbb{F}_q^\times)^2$	q^4
		$u_3(r_3)u_{10}(r_{10})$	$-r_3r_{10} \in \mu^k(\mathbb{F}_q^\times)^2$	q^4
		$u_6(r_6)u_7(r_7)$	$r_6r_7 \in \mu^k(\mathbb{F}_q^\times)^2$	q^5
$u_2(1)u_3(1)$	$q^4\Phi_1^2$	$u_2(r_2)u_3(r_3)$		q^3
		$u_2(r_2)u_7(r_7)$		q^4
		$u_3(r_3)u_4(r_4)$		q^3
		$u_4(r_4)u_6(r_6)$		q^4
		$u_2(r_2)u_4(r_4)u_6(r_6)$		q^4
$u_2(\mu^k)u_3(1)u_4(1)$	$2q^3\Phi_1$	$u_2(r_2)u_3(r_3)u_4(r_4)$	$r_2r_4 \in \mu^k(\mathbb{F}_q^\times)^2$	q^3

M_2 (twisted by $s_1s_3s_2s_4s_3s_1$)	
Action of ϕ on Φ_{I_1}	$\alpha_2 \leftrightarrow \alpha_4$, $\alpha_3 \leftrightarrow \alpha_3$, $\alpha_6 \leftrightarrow \alpha_7$, $\alpha_{10} \leftrightarrow \alpha_{10}$
Root system Φ_{M_2} (of type B_2)	$\pm\{\beta_1 := \frac{1}{2}(\alpha_2+\alpha_4), \beta_2 := \alpha_3, \frac{1}{2}(\alpha_6+\alpha_7), \alpha_{10}\} =$ $= \pm\{\beta_1, \beta_2, \beta_1+\beta_2, 2\beta_1+\beta_2\}$
Action of F' on $\mathbf{T}_0$	$F'\left(h(t_1,t_2,t_3,t_4)\right) = h(t_1^{-q}, t_4^q t_1^{-q}, t_3^q t_1^{-2q}, t_2^q t_1^{-q})$
Action of F' on $\mathbf{U}_{\mathbf{M}_2}$	$F'\left(u_2(t_2)u_3(t_3)u_4(t_4)u_6(t_6)u_7(t_7)u_{10}(t_{10})\right) =$ $u_4(t_2^q)u_3(t_3^q)u_2(t_4^q)u_7(t_6^q)u_6(t_7^q)u_{10}(t_{10}^q)$
$\mathbf{U}_{\beta_1}^F$ $\mathbf{U}_{\beta_2}^F$ $\mathbf{U}_{\beta_3}^F$ $\mathbf{U}_{\beta_4}^F$	$\langle u_{\beta_1}(t) \mid t \in \mathbb{F}_{q^2}\rangle$ with $u_{\beta_1}(t) := u_{\alpha_2}(t)u_{\alpha_4}(t^q)$ $\langle u_{\beta_2}(t) \mid t \in \mathbb{F}_{q}\rangle$ with $u_{\beta_2}(t) := u_{\alpha_3}(t)$ $\langle u_{\beta_3}(t) \mid t \in \mathbb{F}_{q^2}\rangle$ with $u_{\beta_3}(t) := u_{\alpha_6}(t)u_{\alpha_7}(t^q)$ $\langle u_{\beta_4}(t) \mid t \in \mathbb{F}_{q}\rangle$ with $u_{\beta_4}(t) := u_{\alpha_{10}}(t)$
Structure	$M_2 \cong \mathrm{SU}_4(q) \rtimes \langle h(\rho^{(q-1)k_a}, 1, \rho^{2qk_a}, \rho^{(q-1)k_a}) \mid k_a = 0, ..., q\rangle$
Order $\lvert M_2\rvert$	$q^6\Phi_1^2\Phi_2^4\Phi_4\Phi_6$
Centre $Z_{M_2} = Z(\mathbf{L}_{I_1}^{F'})$	$\{h_{Z_{M_2}}(k_a,k_b) \mid k_a = 0,1,\ k_b = 0,...,q\}$ $h_{Z_{M_2}}(k_a,k_b) := h\left(\rho^{\frac{q^2-1}{2}k_a+2(q-1)k_b}, \rho^{\frac{q^2-1}{2}k_a+(q-1)k_b}, \rho^{2(q-1)k_b}, \rho^{(q-1)k_b}\right)$
Unipotent subgroup	$U_{M_2} = \langle \mathbf{U}_\beta^F \mid \beta \in \Phi_{M_2}^+\rangle$
Commutation relations	$[u_{\beta_1}(t_1), u_{\beta_2}(t_2)] = u_{\beta_3}(t_1t_2)u_{\beta_4}(t_1^{q+1}t_2)$, $t_1 \in \mathbb{F}_{q^2}$, $t_2 \in \mathbb{F}_q$, $[u_{\beta_1}(t_1), u_{\beta_3}(t_2)] = u_{\beta_4}(\mathrm{Tr}(t_1)\mathrm{Tr}(t_2) - \mathrm{Tr}(t_1t_2))$, $t_1, t_2 \in \mathbb{F}_{q^2}$.
Unipotent classes	Table 29
$H^1(F', Z(\mathbf{L}_{I_1}))$	$\{h((-1)^{k_a}, (-1)^{k_a}, 1, 1) \mid k_a = 0, 1\}$
$\ker(\mathfrak{h}_{M_2})$	$\{h_Z(0,0), h_Z(0,1)\}$
Repr. $t_z = \mathscr{L}_{\mathbf{T}_0}'^{-1}(z)$ for $z = h((-1)^{k_a}, (-1)^{k_a}, 1, 1)$	$h(\rho^{-\frac{q-1}{2}k_a}, \rho^{-\frac{q+1}{2}k_a}, \rho^{-qk_a}, \rho^{-qk_a})$
Regular character of U_{M_2} parametrized by $h((-1)^{k_a}, (-1)^{k_a}, 1, 1)$	$\phi_{\rho^{k_a},1}^{M_2} : u_{\beta_1}(t_1)u_{\beta_2}(t_2) \mapsto \chi_2(\rho^{k_a}t_1)\phi(t_2)$
$z_{\mathbf{M}_2}$ (Remark 5.34)	$h(-1,-1,1,1)$
$\mathrm{res}_{\mathbf{M}_2}^{\mathbf{G}}$	$\mathrm{res}_{\mathbf{M}_2}^{\mathbf{G}}\left((u_1(1)u_2(1)u_3(1)u_4(1))^G\right) = (u_{\beta_1}(\rho)u_{\beta_2}(1))^{M_2}$

Table 29: Representatives of the unipotent classes of M_2, and of U_{M_2} with their fusion to M_2.

Repr. u_0 in M_2	$\lvert C_{M_2}(u_0)\rvert$	Repr. u in U_{M_2}	condition	$\lvert C_{U_{M_2}}(u)\rvert$
1	$q^6\Phi_1^2\Phi_2^4\Phi_4\Phi_6$	1		q^6
$u_{\beta_2}(1)$	$q^6\Phi_1\Phi_2^3$	$u_{\beta_2}(t_1)$	$t_1 \in \mathbb{F}_q^\times$	q^4
		$u_{\beta_4}(t_1)$	$t_1 \in \mathbb{F}_q^\times$	q^6
$u_{\beta_1}(\rho^k)$	$2q^5\Phi_1\Phi_2^2$	$u_{\beta_1}(t_1)$	$t_1 \in \rho^k(\mathbb{F}_{q^2}^\times)^2$	q^4
		$u_{\beta_3}(t_1)$	$t_1 \in \rho^k(\mathbb{F}_{q^2}^\times)^2$	q^5
		$u_{\beta_2}(t_1)u_{\beta_4}(t_2)$	$t_1t_2 \in -\mu^k(\mathbb{F}_q^\times)^2$	q^4
$u_{\beta_1}(\rho)u_{\beta_3}(1)$	$q^4\Phi_2^2$	$u_{\beta_1}(t_1)u_{\beta_3}(t_2)$	$t_2t_1^{-1} \notin \mathbb{F}_q^\times$	q^5
$u_{\beta_1}(\rho^k)u_{\beta_2}(1)$	$2q^3\Phi_2$	$u_{\beta_1}(t_1)u_{\beta_2}(t_2)$	$t_1 \in \rho^k(\mathbb{F}_{q^2}^\times)^2$, $t_2 \in \mathbb{F}_q^\times$	q^3

M_3 (split)	
Root system Φ_{M_3}	$\pm\{\alpha_1, \alpha_2, \alpha_4\}$
Structure	$M_3 \cong \mathrm{SL}_2(q)^3 \rtimes \langle h_1(t) \mid t \in \mathbb{F}_q^\times \rangle$
Order $\lvert M_3 \rvert$	$q^3\Phi_1^4\Phi_2^3$
Centre $Z_{M_3} = Z(\mathbf{L}_{I_2}^F)$	$\{h_{Z_{M_3}}(k_a, k_b, k_c) \mid k_a = 0, ..., q-2,\ k_b, k_c = 0, 1\}$ $h_{Z_{M_3}}(k_a, k_b, k_c) := h(\mu^{k_a+\frac{q-1}{2}k_b}, \mu^{k_a+\frac{q-1}{2}k_c}, \mu^{2k_a}, \mu^{k_a})$
Unipotent subgroup	$U_{M_3} = \langle U_1, U_2, U_4 \rangle$, it is abelian
Unipotent classes	Table 30
Triality on U_{M_3}	Table 49
$H^1(F, Z(\mathbf{L}_{I_2}))$	$\{h((-1)^{k_a}, (-1)^{k_b}, 1, 1) \mid k_a, k_b = 0, 1\}$
$\ker(\mathfrak{h}_{M_3})$	$\{h_Z(0, 0)\}$
Repr. $t_z = \mathscr{L}_{\mathbf{T}_0}^{-1}(z)$ for $z = h((-1)^{k_a}, (-1)^{k_b}, 1, 1)$	$h(\rho^{-\frac{q+1}{2}k_a}, \rho^{-\frac{q+1}{2}k_b}, 1, 1)$
Regular character of U_{M_3} parametrized by $h((-1)^{k_a}, (-1)^{k_b}, 1, 1)$	$\phi^{M_3}_{\mu^{k_a}, \mu^{k_b}, 1} : u_1(t_1)u_2(t_2)u_4(t_4) \mapsto \phi\left(\mu^{k_a}t_1 + \mu^{k_b}t_2 + t_4\right)$

Table 30: Representatives of the unipotent classes of M_3, and of U_{M_3} with their fusion to M_3. Because U_{M_3} is abelian all the centralizers of elements in U_{M_3} have order q^3.

Repr. u_0 in M_3	$\lvert C_{M_3}(u_0) \rvert$	Repr. u in U_{M_3}	condition
1	$q^3\Phi_1^4\Phi_2^3$	1	
$u_1(1)$	$q^3\Phi_1^3\Phi_2^2$	$u_1(r_1)$	
$u_2(1)$	$q^3\Phi_1^3\Phi_2^2$	$u_2(r_2)$	
$u_4(1)$	$q^3\Phi_1^3\Phi_2^2$	$u_4(r_4)$	
$u_1(\mu^k)u_2(1)$	$2q^3\Phi_1^2\Phi_2$	$u_1(r_1)u_2(r_2)$	$r_1r_2 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_1(\mu^k)u_4(1)$	$2q^3\Phi_1^2\Phi_2$	$u_1(r_1)u_4(r_4)$	$r_1r_4 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_2(\mu^k)u_4(1)$	$2q^3\Phi_1^2\Phi_2$	$u_2(r_2)u_4(r_4)$	$r_2r_4 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_1(\mu^k)u_2(\mu^l)u_4(1)$	$4q^3\Phi_1$	$u_1(r_1)u_2(r_2)u_4(r_4)$	$r_1r_4 \in \mu^k(\mathbb{F}_q^\times)^2$, $r_2r_4 \in \mu^l(\mathbb{F}_q^\times)^2$

M_4 (twisted by $s_3s_2s_4s_3s_1s_3s_2s_4s_3$)	
Action of ϕ on Φ_{I_2}	id
Root system $\Phi_{M_4} = \Phi_{I_2}$	$\pm\{\alpha_1, \alpha_2, \alpha_4\}$
Action of F' on $\mathbf{T}_0$	$F'\left(h(t_1, t_2, t_3, t_4)\right) = h(t_1^q t_3^{-q}, t_2^q t_3^{-q}, t_3^{-q}, t_4^q t_3^{-q})$
Action of F' on $\mathbf{U_{M_4}}$	$F'\left(u_1(t_1)u_2(t_2)u_4(t_4))\right) = u_1(t_1^q)u_2(t_2^q)u_4(t_4^q)$
Structure	$M_4 \cong \mathrm{SL}_2(q)^3 \rtimes \langle h(\rho^{qk_a}, \rho^{qk_a}, \rho^{(q-1)k_a}, \rho^{qk_a}) \mid k_a = 0, ..., q\rangle$
Order $\lvert M_4\rvert$	$q^3\Phi_1^3\Phi_2^4$
Centre $Z_{M_4} = Z(\mathbf{L}_{I_2}^{F'})$	$\{h_{Z_{M_4}}(k_a, k_b, k_c) \mid k_a = 0, ..., q,\ k_b, k_c = 0, 1\}$ $h_{Z_{M_4}}(k_a, k_b, k_c) := h(\rho^{\frac{q^2-1}{2}k_b+(q-1)k_a}, \rho^{\frac{q^2-1}{2}k_c+(q-1)k_a}, \rho^{2(q-1)k_a}, \rho^{(q-1)k_a})$
Unipotent subgroup	$U_{M_4} = \langle U_1, U_2, U_4\rangle$, it is abelian
Unipotent classes	Table 31
Triality on U_{M_4}	Table 49
$H^1(F, Z(\mathbf{L}_{I_2}))$	$\{h((-1)^{k_a}, (-1)^{k_b}, 1, 1) \mid k_a, k_b = 0, 1\}$
$\ker(\mathfrak{h}_{M_4})$	$\{h_Z(0, 0)\}$
Repr. $t_z = \mathscr{L}_{\mathbf{T}_0}'^{-1}(z)$ for $z = h((-1)^{k_a}, (-1)^{k_b}, 1, 1)$	$h(\rho^{-\frac{q+1}{2}k_a}, \rho^{-\frac{q+1}{2}k_b}, 1, 1)$
Regular character of U_{M_4} parametrized by $h((-1)^{k_a}, (-1)^{k_b}, 1, 1)$	$\phi^{M_4}_{\mu^{k_a}, \mu^{k_b}, 1} : u_1(t_1)u_2(t_2)u_4(t_4) \mapsto \phi\left(\mu^{k_a}t_1 + \mu^{k_b}t_2 + t_4\right)$
$z_{\mathbf{M_4}}$ (Remark 5.34)	1
$\mathrm{res}^{\mathbf{G}}_{\mathbf{M_4}}$	$\mathrm{res}^{\mathbf{G}}_{\mathbf{M_4}}\left((u_1(1)u_2(1)u_3(1)u_4(1))^G\right) = (u_1(1)u_2(1)u_4(1))^{M_4}$

Table 31: Representatives of the unipotent classes of M_4, and of U_{M_4} with their fusion to M_4. Because U_{M_4} is abelian all the centralizers of elements in U_{M_4} have order q^3.

Repr. u_0 in M_4	$\lvert C_{M_4}(u_0)\rvert$	Repr. u in U_{M_4}	condition
1	$q^3\Phi_1^3\Phi_2^4$	1	
$u_1(1)$	$q^3\Phi_1^2\Phi_2^3$	$u_1(r_1)$	
$u_2(1)$	$q^3\Phi_1^2\Phi_2^3$	$u_2(r_2)$	
$u_4(1)$	$q^3\Phi_1^2\Phi_2^3$	$u_4(r_4)$	
$u_1(\mu^k)u_2(1)$	$2q^3\Phi_1\Phi_2^2$	$u_1(r_1)u_2(r_2)$	$r_1r_2 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_1(\mu^k)u_4(1)$	$2q^3\Phi_1\Phi_2^2$	$u_1(r_1)u_4(r_4)$	$r_1r_4 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_2(\mu^k)u_4(1)$	$2q^3\Phi_1\Phi_2^2$	$u_2(r_2)u_4(r_4)$	$r_2r_4 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_1(\mu^k)u_2(\mu^l)u_4(1)$	$4q^3\Phi_2$	$u_1(r_1)u_2(r_2)u_4(r_4)$	$r_1r_4 \in \mu^k(\mathbb{F}_q^\times)^2$, $r_2r_4 \in \mu^l(\mathbb{F}_q^\times)^2$

L_1 (split)	
Root system Φ_{L_1}	$\pm\{\alpha_2, \alpha_4\}$
Structure	$L_1 \cong \mathrm{SL}_2(q)^2 \rtimes \langle h_1(t), h_3(t) \mid t \in \mathbb{F}_q^\times \rangle$
Order $\lvert L_1 \rvert$	$q^2\Phi_1^4\Phi_2^2$
Centre $Z_{L_1} = Z(\mathbf{L}_{I_3}^F)$	$\{h_{Z_{L_1}}(k_a, k_b, k_c) \mid k_a, k_b = 0, ..., q-2,\ k_c = 0, 1\}$ $h_{Z_{L_1}}(k_a, k_b, k_c) := h(\mu^{k_b}, \mu^{k_a+\frac{q-1}{2}k_c}, \mu^{2k_a}, \mu^{k_a})$
Unipotent subgroup	$U_{L_1} = \langle U_2, U_4 \rangle$, it is abelian
Unipotent classes	Table 32
$H^1(F, Z(\mathbf{L}_{I_3}))$	$\{h(1, (-1)^{k_a}, 1, 1) \mid k_a = 0, 1\}$
$\ker(\mathfrak{h}_{L_1})$	$\{h_Z(0,0), h_Z(0,1)\}$
Repr. $t_z = \mathscr{L}_{\mathbf{T}_0}^{-1}(z)$ for $z = h(1, (-1)^{k_a}, 1, 1)$	$h(1, \rho^{-\frac{q+1}{2}k_a}, 1, 1)$
Regular character of U_{L_1} parametrized by $h(1, (-1)^{k_a}, 1, 1)$	$\phi^{L_1}_{\mu^{k_a},1} : u_2(t_2)u_4(t_4) \mapsto \phi\left(\mu^{k_a}t_2 + t_4\right)$

Table 32: Representatives of the unipotent classes of L_1, and of U_{L_1} with their fusion to L_1. Because U_{L_1} is abelian all the centralizers of elements in U_{L_1} have order q^2.

Repr. u_0 in L_1	$\lvert C_{L_1}(u_0) \rvert$	Repr. u in U_{L_1}	condition
1	$q^2\Phi_1^4\Phi_2^2$	1	
$u_2(1)$	$q^2\Phi_1^3\Phi_2$	$u_2(r_2)$	
$u_4(1)$	$q^2\Phi_1^3\Phi_2$	$u_4(r_4)$	
$u_2(\mu^k)u_4(1)$	$2q^2\Phi_1^2$	$u_2(r_2)u_4(r_4)$	$r_2r_4 \in \mu^k(\mathbb{F}_q^\times)^2$

L_2 (twisted by $w_2 = s_3s_2s_4s_3$)	
Action of ϕ on Φ_{I_3}	$\alpha_2 \leftrightarrow \alpha_4$
Root system Φ_{L_2} (of type A_1)	$\pm\{\beta := \frac{1}{2}(\alpha_2 + \alpha_4)\}$
Action of F' on $\mathbf{T}_0$	$F'\left(h(t_1, t_2, t_3, t_4)\right) = h(t_1^q, t_1^q t_3^{-q} t_4^q, t_1^{2q} t_3^{-q}, t_1^q t_2^q t_3^{-q})$
Action of F' on $\mathbf{U}_{\mathbf{L}_2}$	$F'\left(u_2(t_2)u_4(t_4)\right) = u_4(t_2^q)u_2(t_4^q)$
$\mathbf{U}_\beta^F$	$\langle u_\beta(t) \mid t \in \mathbb{F}_{q^2}\rangle$ with $u_\beta(t) := u_{\alpha_2}(t)u_{\alpha_4}(t^q)$
Structure	$L_2 \cong \mathrm{SL}_2(q^2) \rtimes \langle h(\rho^{(q-1)k_a}, 1, \rho^{2qk_a}\mu^{k_b}, \rho^{(q-1)k_a}) \mid k_a = 0, ..., q,\ k_b = 0, ..., q-2\rangle$
Order $\lvert L_2\rvert$	$q^2\Phi_1^2\Phi_2^2\Phi_4$
Centre $Z_{L_2} = Z(\mathbf{L}_{I_3}^{F'})$	$\{h_{Z_{L_2}}(k_a, k_b) \mid k_a = 0, ..., q^2,\ k_b = 0, 1\}$ $h_{Z_{L_2}}(k_a, k_b) := h(\mu^{k_a}, \rho^{k_a - \frac{q^2-q}{2}k_b}, \rho^{2k_a+(q-1)k_b}, \rho^{k_a + \frac{(q-1)}{2}k_b})$
Unipotent subgroup	$U_{L_2} = \mathbf{U}_\beta^F$, it is abelian
Unipotent classes	Table 33
$H^1(F', Z(\mathbf{L}_{I_3}))$	$\{h(1, (-1)^{k_a}, 1, 1) \mid k_a = 0, 1\}$
$\ker(\mathfrak{h}_{L_2})$	$\{h_Z(0,0), h_Z(0,1)\}$
Repr. $t_z = \mathscr{L}_{\mathbf{T}_0}'^{-1}(z)$ for $z = h(1, (-1)^{k_a}, 1, 1)$	$h(1, \omega^{-\frac{q^2+1}{2}k_a}, 1, \omega^{-q\frac{q^2+1}{2}k_a})$
Regular character of U_{L_2} parametrized by $h(1, (-1)^{k_a}, 1, 1)$	$\phi_{\rho^{k_a}}^{L_2} : u_\beta(t_1) \mapsto \chi_2\left(\rho^{k_a}t_1\right)$
$z_{\mathbf{L}_2}$ (Remark 5.34)	$h(1, -1, 1, 1)$
$g_{w_2} \in \mathbf{G}$ with $g_{w_2}^{-1}F(g_{w_2}) = \dot{w}_2$	$h_3\left(\rho^{\frac{q+1}{2}}\right)h_4\left(\rho^{\frac{q+1}{2}}\right)u_6(-\rho)u_7(\rho)h_3(\rho^q - \rho)n_3n_2n_4n_3u_6(-1)u_7(1)$ $\left(g_{w_2}^{-1}F(g_{w_2}) = n_3n_2n_4n_3\right)$
$\mathrm{res}_{\mathbf{L}_2}^{\mathbf{G}}$	$\mathrm{res}_{\mathbf{L}_2}^{\mathbf{G}}\left((u_1(1)u_2(1)u_3(1)u_4(1))^G\right) = (u_\beta(\rho))^{L_2}$

Table 33: Representatives of the unipotent classes of L_2, and of U_{L_2} with their fusion to L_2. Because U_{L_2} is abelian all the centralizers of elements in U_{L_2} have order q^2.

Repr. u_0 in L_2	$\lvert C_{L_2}(u_0)\rvert$	Repr. u in U_{L_2}	condition
1	$q^2\Phi_1^2\Phi_2^2\Phi_4$	1	
$u_\beta(\rho^k)$	$2q^2\Phi_1\Phi_2$	$u_\beta(\lambda)$	$\lambda \in \rho^k(\mathbb{F}_{q^2}^\times)^2$ $(k = 0, 1)$

L_3 (twisted by s_1)	
Action of ϕ on Φ_{I_3}	id
Root system Φ_{L_3} (of type $A_1 \times A_1$)	$\pm\{\alpha_2, \alpha_4\}$
Action of F' on $\mathbf{T}_0$	$F'\left(h(t_1, t_2, t_3, t_4)\right) = h(t_1^{-q} t_3^q, t_2^q, t_3^q, t_4^q)$
Action of F' on $\mathbf{U}_{\mathbf{L}_3}$	$F'\left(u_2(t_2)u_4(t_4)\right) = u_2(t_2^q)u_4(t_4^q)$
$\mathbf{U}_\alpha^{F'}$ for $\alpha \in \Phi_{L_3}$	$\{u_\alpha(t) \mid t \in \mathbb{F}_q\}$
Structure	$L_3 \cong \mathrm{SL}_2(q)^2 \rtimes \langle h(\rho^{(q-1)k_a+k_b}, 1, \mu^{k_b}, 1) \mid k_a = 0, ..., q,\ k_b = 0, ..., q-2\rangle$
Order $\lvert L_3\rvert$	$q^2\Phi_1^3\Phi_2^3$
Centre $Z_{L_3} = Z(\mathbf{L}_{I_3}^{F'})$	$\{h_{Z_{L_3}}(k_a, k_b, k_c) \mid k_a = 0, ..., q-2,\ k_b = 0, ..., q,\ k_c = 0, 1\}$ $h_{Z_{L_3}}(k_a, k_b, k_c) := h(\rho^{2k_a+(q-1)k_b}, \mu^{k_a+\frac{q-1}{2}k_c}, \mu^{2k_a}, \mu^{k_a})$
Unipotent subgroup	$U_{L_3} = \langle U_2, U_4\rangle$, it is abelian
Unipotent classes	Table 34
$H^1(F', Z(\mathbf{L}_{I_3}))$	$\{h(1, (-1)^{k_a}, 1, 1) \mid k_a = 0, 1\}$
$\ker(\mathfrak{h}_{L_3})$	$\{h_Z(0, 0), h_Z(0, 1)\}$
Repr. $t_z = \mathscr{L}_{\mathbf{T}_0}'^{-1}(z)$ for $z = h(1, (-1)^{k_a}, 1, 1)$	$h(1, \rho^{-\frac{q+1}{2}k_a}, 1, 1)$
Regular character of U_{L_3} parametrized by $h(1, (-1)^{k_a}, 1, 1)$	$\phi_{\mu^{k_a},1}^{L_3} : u_2(t_2)u_4(t_4) \mapsto \phi\left(\mu^{k_a}t_2 + t_4\right)$
$z_{\mathbf{L}_3}$ (Remark 5.34)	1
$g_{w_3} \in \mathbf{G}$ with $g_{w_3}^{-1}F(g_{w_3}) = \dot{w}_3$	$u_1(r)h_3(\omega^{q^{\frac{q^2+1}{2}}})n_1u_1(\rho^{\frac{q-1}{2}})$ with $r^q - r = \omega^{q(q+1)\frac{q^2+1}{2}}$ $\left(g_{w_3}^{-1}F(g_{w_3}) = n_1\right)$
$\mathrm{res}_{\mathbf{L}_3}^{\mathbf{G}}$	$\mathrm{res}_{\mathbf{L}_3}^{\mathbf{G}}\left((u_1(1)u_2(1)u_3(1)u_4(1))^G\right) = (u_2(1)u_4(1))^{L_3}$

Table 34: Representatives of the unipotent classes of L_3, and of U_{L_3} with their fusion to L_3. Because U_{L_3} is abelian all the centralizers of elements in U_{L_3} have order q^2.

Repr. u_0 in L_3	$\lvert C_{L_3}(u_0)\rvert$	Repr. u in U_{L_3}	condition
1	$q^2\Phi_1^3\Phi_2^3$	1	
$u_2(1)$	$q^2\Phi_1^2\Phi_2^2$	$u_2(r_2)$	
$u_4(1)$	$q^2\Phi_1^2\Phi_2^2$	$u_4(r_4)$	
$u_2(\mu^k)u_4(1)$	$2q^2\Phi_1\Phi_2$	$u_2(r_2)u_4(r_4)$	$r_2r_4 \in \mu^k(\mathbb{F}_q^\times)^2$

L_4 (twisted by $s_1s_3s_2s_4s_3s_1s_3s_2s_4s_3$)	
Action of ϕ on Φ_{I_3}	id
Root system Φ_{L_4} (of type $A_1 \times A_1$)	$\pm\{\alpha_2, \alpha_4\}$
Action of F' on $\mathbf{T}_0$	$F'\left(h(t_1, t_2, t_3, t_4)\right) = h(t_1^{-q}, t_2^q t_3^{-q}, t_3^{-q}, t_4^q t_3^{-q})$
Action of F' on $\mathbf{U}_{\mathbf{L}_3}$	$F'\left(u_2(t_2)u_4(t_4)\right) = u_2(t_2^q)u_4(t_4^q)$
Structure	$L_4 \cong \mathrm{SL}_2(q)^2 \rtimes \langle h(\rho^{(q-1)k_a}, \rho^{-k_b}, \rho^{(q-1)k_b}, \rho^{-k_b}) \mid k_a, k_b = 0, ..., q\rangle$
Order $\lvert L_4\rvert$	$q^2\Phi_1^2\Phi_2^4$
Centre $Z_{L_4} = Z(\mathbf{L}_{I_3}^{F'})$	$\{h_{Z_{L_4}}(k_a, k_b, k_c) \mid k_a, k_b = 0, ..., q,\ k_c = 0, 1\}$ $h_{Z_{L_4}}(k_a, k_b, k_c) := h(\rho^{(q-1)k_b}, \rho^{(q-1)k_a + \frac{q^2-1}{2}k_c}, \rho^{2(q-1)k_a}, \rho^{(q-1)k_a})$
Unipotent subgroup	$U_{L_4} = \langle U_2, U_4\rangle$, it is abelian
Unipotent classes	Table 35
$H^1(F', Z(\mathbf{L}_{I_3}))$	$\{h(1, (-1)^{k_a}, 1, 1) \mid k_a = 0, 1\}$
$\ker(\mathfrak{h}_{L_4})$	$\{h_Z(0, 0), h_Z(0, 1)\}$
Repr. $t_z = \mathscr{L}_{\mathbf{T}_0}^{\prime -1}(z)$ for $z = h(1, (-1)^{k_a}, 1, 1)$	$h(1, \rho^{-\frac{q+1}{2}k_a}, 1, 1)$
Regular character of U_{L_4} parametrized by $h(1, (-1)^{k_a}, 1, 1)$	$\phi^{L_4}_{\mu^{k_a},1} : u_2(t_2)u_4(t_4) \mapsto \phi\left(\mu^{k_a}t_2 + t_4\right)$
$z_{\mathbf{L}_4}$ (Remark 5.34)	1
$\mathrm{res}^{\mathbf{G}}_{\mathbf{L}_4}$	$\mathrm{res}^{\mathbf{G}}_{\mathbf{L}_4}\left((u_1(1)u_2(1)u_3(1)u_4(1))^G\right) = (u_2(1)u_4(1))^{L_4}$

Table 35: Representatives of the unipotent classes of L_4, and of U_{L_4} with their fusion to L_4. Because U_{L_4} is abelian all the centralizers of elements in U_{L_4} have order q^2.

Repr. u_0 in L_4	$\lvert C_{L_4}(u_0)\rvert$	Repr. u in U_{L_4}	condition
1	$q^2\Phi_1^2\Phi_2^4$	1	
$u_2(1)$	$q^2\Phi_1\Phi_2^3$	$u_2(r_2)$	
$u_4(1)$	$q^2\Phi_1\Phi_2^3$	$u_4(r_4)$	
$u_2(\mu^k)u_4(1)$	$2q^2\Phi_2^2$	$u_2(r_2)u_4(r_4)$	$r_2r_4 \in \mu^k(\mathbb{F}_q^\times)^2$

L_5 (twisted by $s_3s_2s_4s_3s_1$)	
Action of ϕ on Φ_{I_3}	$\alpha_2 \leftrightarrow \alpha_4$
Root system Φ_{L_5} (of type A_1)	$\pm\{\beta := \frac{1}{2}(\alpha_2 + \alpha_4)\}$
Action of F' on $\mathbf{T}_0$	$F'\left(h(t_1, t_2, t_3, t_4)\right) = h(t_1^{-q}t_3^q, t_1^{-q}t_4^q, t_1^{-2q}t_3^q, t_1^{-q}t_2^q)$
Action of F' on $\mathbf{U}_{\mathbf{L}_2}$	$F'\left(u_2(t_2)u_4(t_4)\right) = u_4(t_2^q)u_2(t_4^q)$
$\mathbf{U}_\beta^F$	$\langle u_\beta(t) \mid t \in \mathbb{F}_{q^2}\rangle$ with $u_\beta(t) := u_{\alpha_2}(t)u_{\alpha_4}(t^q)$
Structure	$L_5 \cong \mathrm{SL}_2(q^2) \rtimes \langle h(\omega^{(q^2-1)k_a}, \omega^{(q-1)k_a}, \omega^{-(q-1)(q^2-1)k_a}, \omega^{-q^2(q-1)k_a}) \mid k_a = 0, ..., q^2\rangle$
Order $\lvert L_5\rvert$	$q^2\Phi_1\Phi_2\Phi_4^2$
Centre $Z_{L_5} = Z(\mathbf{L}_{I_3}^{F'})$	$\{h_{Z_{L_5}}(k_a, k_b) \mid k_a = 0, 1,\ k_b = 0, ..., q^2\}$ $h_{Z_{L_5}}(k_a, k_b) := h(\omega^{\frac{q^4-1}{2}k_a+(q+1)(q^2-1)k_b}, \omega^{\frac{q^4-1}{2}k_a+(q^2-1)k_b}, \omega^{2(q^2-1)k_b}, \omega^{(q^2-1)k_b})$
Unipotent subgroup	$U_{L_5} = \mathbf{U}_\beta^F$, it is abelian
Unipotent classes	Table 36
$H^1(F', Z(\mathbf{L}_{I_5}))$	$\{h(1, (-1)^{k_a}, 1, 1) \mid k_a = 0, 1\}$
$\ker(\mathfrak{h}_{L_5})$	$\{h_Z(0, 0), h_Z(0, 1)\}$
Repr. $t_z = \mathscr{L}_{\mathbf{T}_0}^{\prime -1}(z)$ for $z = h(1, (-1)^{k_a}, 1, 1)$	$h(1, \omega^{-\frac{q^2+1}{2}k_a}, 1, \omega^{-q\frac{q^2+1}{2}k_a})$
Regular character of U_{L_5} parametrized by $h(1, (-1)^{k_a}, 1, 1)$	$\phi_{\rho^{k_a}}^{L_5} : u_\beta(t_1) \mapsto \chi_2\left(\rho^{k_a}t_1\right)$
$z_{\mathbf{L}_5}$ (Remark 5.34)	$h(1, -1, 1, 1)$
$g_{w_5} \in \mathbf{G}$ with $g_{w_5}^{-1}F(g_{w_5}) = \dot{w}_5$	[??]
$\mathrm{res}_{\mathbf{L}_5}^{\mathbf{G}}$	[??]

Table 36: Representatives of the unipotent classes of L_5, and of U_{L_5} with their fusion to L_5. Because U_{L_5} is abelian all the centralizers of elements in U_{L_5} have order q^2.

Repr. u_0 in L_5	$\lvert C_{L_5}(u_0)\rvert$	Repr. u in U_{L_5}	condition
1	$q^2\Phi_1\Phi_2\Phi_4^2$	1	
$u_2(\rho^k)u_4(\rho^{qk})$	$2q^2\Phi_4$	$u_2(\lambda)u_4(\lambda^q)$	$\lambda \in \rho^k(\mathbb{F}_{q^2}^\times)^2$ $(k = 1, 2)$

$C_{\mathbf{G}}(s)^F$ of type $A_1(q)^4$	
Root system $\Phi_{\mathbf{C}}$ (of type A_1^4)	$\pm\{\alpha_1, \alpha_2, \alpha_4, \alpha_{12}\}$
Structure	$\mathbf{C} = \langle \mathbf{U}_\alpha \mid \alpha \in \Phi_{\mathbf{C}} \rangle$ (it is semisimple) $C \cong \mathrm{SL}_2(q)^3 \rtimes \langle h_3(t_1), u_{\pm\alpha_{12}}(t_2) \mid t_1 \in \mathbb{F}_q^\times,\ t_2 \in \mathbb{F}_q \rangle$
Order $\lvert C \rvert$	$q^4\Phi_1^4\Phi_2^4$
Centre $Z(C) = Z(\mathbf{C})$	$\{h((-1)^{k_a}, (-1)^{k_b}, 1, (-1)^{k_c}) \mid k_a, k_b, k_c = 0, 1\}$
Unipotent subgroup	$U_C = \langle U_1, U_2, U_4, U_{12} \rangle$, it is abelian
Unipotent classes	Table 37
Repr. $t_z = \mathscr{L}_{\mathbf{T}_0}^{-1}(z)$ for $z = h((-1)^{k_a}, (-1)^{k_b}, 1, (-1)^{k_c})$	$h(\rho^{-\frac{q+1}{2}k_a}, \rho^{-\frac{q+1}{2}k_b}, 1, \rho^{-\frac{q+1}{2}k_c})$
Regular character of U_C parametrized by $h((-1)^{k_a}, (-1)^{k_b}, 1, (-1)^{k_c})$	$\phi^C_{\mu^{k_a},\mu^{k_b},\mu^{k_c},1} : u_1(t_1)u_2(t_2)u_4(t_4)u_{12}(t_{12}) \mapsto \phi\left(\mu^{k_a}t_1 + \mu^{k_b}t_2 + \mu^{k_c}t_4 + t_{12}\right)$

Table 37: Representatives of the unipotent classes of C, and of U_C with their fusion to C. Because U_C is abelian all the centralizers of elements in U_C have order q^4.

Repr. u_0 in C	$\lvert C_C(u_0) \rvert$	Repr. u in U_C	condition
1	$q^4\Phi_1^4\Phi_2^4$	1	
$u_1(1)$	$q^4\Phi_1^3\Phi_2^3$	$u_1(r_1)$	
$u_2(1)$	$q^4\Phi_1^3\Phi_2^3$	$u_2(r_2)$	
$u_4(1)$	$q^4\Phi_1^3\Phi_2^3$	$u_4(r_4)$	
$u_{12}(1)$	$q^4\Phi_1^3\Phi_2^3$	$u_{12}(r_{12})$	
$u_1(\mu^k)u_2(1)$	$2q^4\Phi_1^2\Phi_2^2$	$u_1(r_1)u_2(r_2)$	$r_1r_2 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_1(\mu^k)u_4(1)$	$2q^4\Phi_1^2\Phi_2^2$	$u_1(r_1)u_4(r_4)$	$r_1r_4 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_1(\mu^k)u_{12}(1)$	$2q^4\Phi_1^2\Phi_2^2$	$u_1(r_1)u_{12}(r_{12})$	$r_1r_{12} \in \mu^k(\mathbb{F}_q^\times)^2$
$u_2(\mu^k)u_4(1)$	$2q^4\Phi_1^2\Phi_2^2$	$u_2(r_2)u_4(r_4)$	$r_2r_4 \in \mu^k(\mathbb{F}_q^\times)^2$
$u_2(\mu^k)u_{12}(1)$	$2q^4\Phi_1^2\Phi_2^2$	$u_2(r_2)u_{12}(r_{12})$	$r_2r_{12} \in \mu^k(\mathbb{F}_q^\times)^2$
$u_4(\mu^k)u_{12}(1)$	$2q^4\Phi_1^2\Phi_2^2$	$u_4(r_4)u_{12}(r_{12})$	$r_4r_{12} \in \mu^k(\mathbb{F}_q^\times)^2$
$u_1(\mu^k)u_2(\mu^l)u_4(1)$	$4q^4\Phi_1\Phi_2$	$u_1(r_1)u_2(r_2)u_4(r_4)$	$r_1r_4 \in \mu^k(\mathbb{F}_q^\times)^2$, $r_2r_4 \in \mu^l(\mathbb{F}_q^\times)^2$
$u_1(\mu^k)u_2(\mu^l)u_{12}(1)$	$4q^4\Phi_1\Phi_2$	$u_1(r_1)u_2(r_2)u_{12}(r_{12})$	$r_1r_{12} \in \mu^k(\mathbb{F}_q^\times)^2$, $r_2r_{12} \in \mu^l(\mathbb{F}_q^\times)^2$
$u_1(\mu^k)u_4(\mu^l)u_{12}(1)$	$4q^4\Phi_1\Phi_2$	$u_1(r_1)u_4(r_4)u_{12}(r_{12})$	$r_1r_{12} \in \mu^k(\mathbb{F}_q^\times)^2$, $r_4r_{12} \in \mu^l(\mathbb{F}_q^\times)^2$
$u_2(\mu^k)u_4(\mu^l)u_{12}(1)$	$4q^4\Phi_1\Phi_2$	$u_2(r_2)u_4(r_4)u_{12}(r_{12})$	$r_2r_{12} \in \mu^k(\mathbb{F}_q^\times)^2$, $r_4r_{12} \in \mu^l(\mathbb{F}_q^\times)^2$
$u_1(\mu^k)u_2(\mu^l)u_4(\mu^m)u_{12}(1)$	$8q^4$	$u_1(r_1)u_2(r_2)u_4(r_4)u_{12}(r_{12})$	$r_1r_{12} \in \mu^k(\mathbb{F}_q^\times)^2$, $r_2r_{12} \in \mu^l(\mathbb{F}_q^\times)^2$, $r_4r_{12} \in \mu^m(\mathbb{F}_q^\times)^2$

Remark 15.3. There is a surjective homomorphism of algebraic groups $i_{\mathbf{C}} : (\mathrm{SL}_2(q))^4 \to \mathbf{C}$ given by sending the generators of $(\mathrm{SL}_2)^4$ to those of $\mathbf{C}$.

We construct $i_{\mathbf{C}}$ explicitly now. We fix root maps of SL_2

$$u^{+}_{\mathrm{SL}_2} : \bar{\mathbb{F}}_q \to \mathrm{SL}_2,\ t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad u^{-}_{\mathrm{SL}_2} : \bar{\mathbb{F}}_q \to \mathrm{SL}_2,\ t \mapsto \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

such that $\mathrm{SL}_2 = \langle u^{+}_{\mathrm{SL}_2}(t), u^{-}_{\mathrm{SL}_2}(t) \mid t \in \bar{\mathbb{F}}_q \rangle$, then for $t \in \bar{\mathbb{F}}_q$ the elements $u^{\pm}_1(t) = (u^{\pm}_{\mathrm{SL}_2}(t), 1, 1, 1)$, $u^{\pm}_2(t) = (1, u^{\pm}_{\mathrm{SL}_2}(t), 1, 1)$, $u^{\pm}_3(t) = (1, 1, u^{\pm}_{\mathrm{SL}_2}(t), 1)$ and $u^{\pm}_4(t) = (1, 1, 1, u^{\pm}_{\mathrm{SL}_2}(t))$ are generators of $(\mathrm{SL}_2)^4$. For $t \in \bar{\mathbb{F}}_q^{\times}$, we denote by $h'_i(t) = u^{+}_i(t)u^{-}_i(-t^{-1})u^{+}_i(t)$ the generators of the maximally split torus that normalizes the "unitriangular matrices" of $(\mathrm{SL}_4)^4$. Then for $t_1, t_2, t_3, t_4 \in \bar{\mathbb{F}}_q^{\times}$, we simply write $h'(t_1, t_2, t_3, t_4) = h'_1(t_1)h'_2(t_2)h'_3(t_3)h'_4(t_4)$.

Then, $i_{\mathbf{C}}$ is defined by $i_{\mathbf{C}}(u^{\pm}_1(t)) = u_{\pm\alpha_1}(t)$, $i_{\mathbf{C}}(u^{\pm}_2(t)) = u_{\pm\alpha_2}(t)$, $i_{\mathbf{C}}(u^{\pm}_3(t)) = u_{\pm\alpha_{12}}(t)$ and $i_{\mathbf{C}}(u^{\pm}_4(t)) = u_{\pm\alpha_4}(t)$ for all $t \in \bar{\mathbb{F}}_q$.

It follows, by an easy computation, that for $t_1, t_2, t_3, t_4 \in \bar{\mathbb{F}}_q^{\times}$ we have

$$i_{\mathbf{C}}\left(h'(t_1, t_2, t_3, t_4)\right) = h(t_1t_3, t_2t_3, t_3^2, t_4t_3)$$

which implies that the kernel of the morphism has order 2 and is given by

$$\ker(i_{\mathbf{C}}) = \{h'(\varepsilon, \varepsilon, \varepsilon, \varepsilon) \mid \varepsilon = \pm 1\} \leq Z\left((\mathrm{SL}_2)^4\right).$$

Finally, notice that $i_{\mathbf{C}}$ satisfies the conditions to apply Proposition 4.12, in order to compute the 2-parameter Green functions of $\mathbf{C}^F$ (in the next section).

16 The 2-parameter Green functions

In this section, we determine the two parameter Green functions for the subgroups introduced in the previous section (the nine Levi subgroups $M_1, ..., M_4, L_1, ..., L_5$ and the centralizer C).

Here is the analogue of Proposition 10.1 for $\mathrm{SU}_4(q)$, which we need for $\mathrm{Spin}_8^+(q)$ since the group Spin_8 has Levi subgroups of type ${}^2A_3(q)$.

Proposition 16.1. *Let q be an odd prime power and $\mathbf{G} = \mathrm{SL}_4$ when $q \equiv 1 \pmod 4$, or $\mathbf{G} = \mathrm{SL}_4 / \langle \pm 1 \rangle$ when $q \equiv 3 \pmod 4$, and F a twisted Frobenius with respect to an $\mathbb{F}_q$-structure on $\mathbf{G}$. Then*

$$\tilde{Q}_{\mathbf{L}_2}^{\mathbf{G}} = \begin{pmatrix} \Phi_2^2\Phi_6 & \Phi_2 & 1 & 1 & . & . & . \\ . & . & q\Phi_2 & . & \frac{1}{2}\Phi_2 & 1 & . \\ . & . & . & q\Phi_2 & \frac{1}{2}\Phi_2 & . & 1 \end{pmatrix}$$

for a split Levi subgroup $\mathbf{L}_2$ with $\mathbf{L}_2^F = A_1(q^2)(q-1)$, and

$$\tilde{Q}_{\mathbf{L}_4}^{\mathbf{G}} = \begin{pmatrix} \Phi_4\Phi_6 & -\Phi_1 & 1 & 1 & . & . & . \\ . & q^2 & . & . & 1 & . & . \\ . & q^2 & . & . & 1 & . & . \\ . & . & . & q\Phi_1 & -\frac{1}{2}\Phi_2 & 1 & . \\ . & . & q\Phi_1 & . & -\frac{1}{2}\Phi_2 & . & 1 \end{pmatrix}$$

for a non-split Levi subgroup $\mathbf{L}_4$ with $\mathbf{L}_4^F = A_1(q)^2(q+1)$.

The unipotent classes are ordered as in Table 29, Table 33 and Table 35 (where we see $\mathbf{G}$, $\mathbf{L}_2$ and $\mathbf{L}_4$ as Levi subgroups of Spin_8*).*

Proof. Instead of doing the same computation twice for the stated groups and congruences, we did them once for the Levi subgroups $A_1(q^2)(q^2-1)$ and $A_1(q)^2(q+1)^2$ inside the Levi ${}^2A_3(q)(q+1)$ of $\mathrm{Spin}_8^+(q)$. Then, Lemma 4.11 assures that the solution is what we claim. The argument is analogous to the proof of Proposition 10.1. By explicit calculations one finds the result for the split case, by Corollary 4.20. Then, thanks to scalar products of Lusztig-induced cuspidal class functions and the norm equation one can fix the unknowns for the non-split case. □

As for Proposition 10.1 one can use Remark 5.34 to fix the order of the rows.

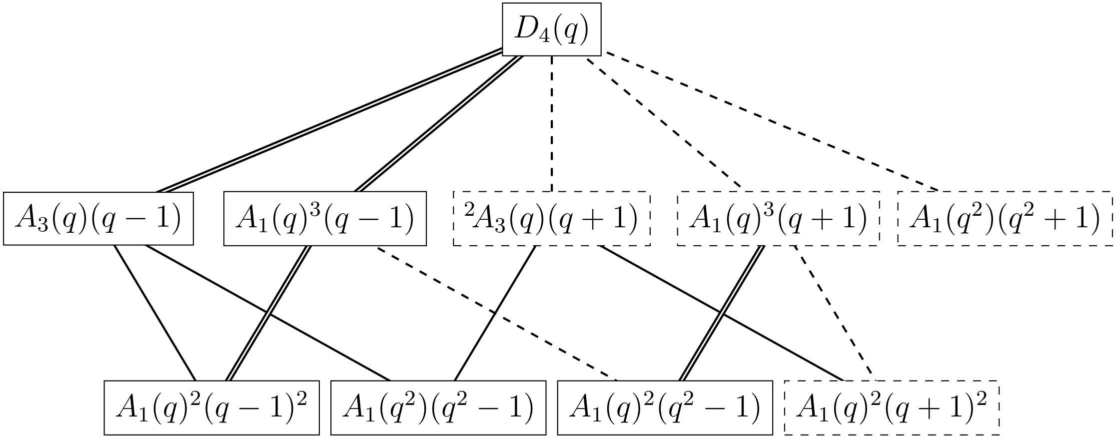
We consider now $\mathbf{G} = \mathrm{Spin}_8$ with a split Frobenius F, such that $\mathbf{G}^F = \mathrm{Spin}_8^+(q)$ with q odd. Moreover, let $\mathbf{G} \to \tilde{\mathbf{G}}$ be a regular embedding. Of the 13 unipotent classes of $\mathbf{G}$ six split into two and three split into four classes in $\mathbf{G}^F$ (see Table 25). By the general discussion in Section 4.2 this means that, every splitting unipotent class of a Levi subgroup $\mathbf{L}$ introduces at least 9 unknowns in the matrix $\tilde{Q}_{\mathbf{L}}^{\mathbf{G}}$. It turns out that this number can be reduced to 5 thanks to the diagonal action of $\tilde{\mathbf{L}}^F$. Therefore, we are missing 5 equations per Levi subgroup and splitting class of that Levi subgroup.

The types of maximal Levi subgroups with disconnected centre of $\mathrm{Spin}_8^+(q)$ are $M_1 = A_3(q)(q-1)$, $M_2 = {}^2A_3(q)(q+1)$, $M_3 = A_1(q)^3(q-1)$, $M_4 = A_1(q)^3(q+1)$ and $L_5 = A_1(q^2)(q^2+1)$. It turns out that to compute the Green functions for these Levi subgroups we need also to consider minimal Levi subgroups with disconnected centre $L_1 = A_1(q)^2(q-1)^2$, $L_2 = A_1(q^2)(q^2-1)$, $L_3 = A_1(q)^2(q^2-1)$ and $L_4 = A_1(q)^2(q+1)^2$. Figure 4 summarizes the situation by showing the diagram of inclusions for these Levi subgroups. Each of these has one splitting unipotent class, and we denote by f_i the (cuspidal) class function of L_i that has values 1, -1 on the splitting classes and 0 elsewhere.

The reason why we need L_1 to L_4 is because we want to emulate the proof of Proposition 10.1. While

$$\langle R_{\mathbf{L}_i}^{\mathbf{G}} f_i,\, R_{\mathbf{L}_j}^{\mathbf{G}} f_j \rangle = 0 \qquad \text{for } i, j = 1, \ldots, 5,\ i \neq j$$

Figure 4: Subgroup lattice of Levi subgroups with disconnected centre of $\mathrm{Spin}_8^+(q)$, up to triality. The lines represent inclusions: a single line corresponds to Levis already treated in Propositions 10.1 and 16.1, a double line indicates a split Levi subgroup and, therefore, that the Green function can be computed explicitly using Proposition 4.2, dashed lines indicate non-split Levi subgroups for which the Green functions must be computed with methods similar to those used in the proof of Proposition 10.1.



by the Mackey formula, an analogous formula for M_i, $i = 1, \ldots, 4$, need not hold since a class function f of M_i taking 1, -1 on splitting unipotent classes is not cuspidal. However,

$$\left\langle R^{\mathbf{G}}_{\mathbf{M}_i} f,\, R^{\mathbf{G}}_{\mathbf{L}_j} f_j \right\rangle = 0 \quad \text{for all } i, j \text{ such that } L_j \nsubseteq M_i$$

again, by the Mackey formula.

In conclusion, the plan to get the 2-parameter Green functions of $\mathrm{Spin}_8^+(q)$ is the following. First, we determine $\tilde{Q}^{\mathbf{G}}_{\mathbf{L}}$ for the split Levi subgroups $\mathbf{L} = \mathbf{M}_1, \mathbf{M}_3, \mathbf{L}_1$ by explicit computations (see Section 4.3). Second, we compute $\tilde{Q}^{\mathbf{G}}_{\mathbf{L}_i}$ for $i = 2, \ldots, 5$ thanks to Propositions 10.1 and 16.1 and Lemma 4.6. Then using this knowledge and Lemma 4.6 we get $\tilde{Q}^{\mathbf{G}}_{\mathbf{L}}$ for the maximal Levi subgroups $\mathbf{L} = \mathbf{M}_2, \mathbf{M}_4$.

Unfortunately, for the non-split Levi subgroups only 4 of the 5 equations (one of which is the norm equation) needed to find the unknowns can be obtained by the procedure described above. However, Digne, Lehrer and Michel prove in [DLM97, Thmeorem 3.7] and [DLM92, Conjectures 5.2 and 5.2'] that, when q is "large enough" ($q > q_0$ for q_0 a constant depending only on the Dynkin diagram), for all regular unipotent elements u of $\tilde{\mathbf{G}}^F$, $\tilde{Q}^{\mathbf{G}}_{\mathbf{L}}(u, v) = 1$ for exactly one regular unipotent class $(v)^{\mathbf{L}^F}$ and $\tilde{Q}^{\mathbf{G}}_{\mathbf{L}}(u, v') = 0$ for all other regular unipotent elements v' of $\mathbf{L}^F$. This gives the last equation.

Theorem 16.2. *The 2-parameter Green functions for* $\mathrm{Spin}_8^+(q)$, *q odd, are given in the tables in the appendix:*

$\tilde{Q}^{\mathbf{G}}_{\mathbf{M}_1}$	*Table 50*
$\tilde{Q}^{\mathbf{G}}_{\mathbf{M}_2}$	*Table 51*
$\tilde{Q}^{\mathbf{G}}_{\mathbf{M}_3}$	*Table 52*
$\tilde{Q}^{\mathbf{G}}_{\mathbf{M}_4}$	*Table 53*
$\tilde{Q}^{\mathbf{G}}_{\mathbf{L}_1}$	*Table 54*
$\tilde{Q}^{\mathbf{G}}_{\mathbf{L}_2}$	*Table 55*
$\tilde{Q}^{\mathbf{G}}_{\mathbf{L}_3}$	*Table 56*
$\tilde{Q}^{\mathbf{G}}_{\mathbf{L}_4}$	*Table 57*
$\tilde{Q}^{\mathbf{G}}_{\mathbf{L}_5}$	*Table 58*

The unipotent classes of $\mathrm{Spin}_8^+(q)$ *are ordered according to Table 25, while those of the Levi subgroups are ordered according to all the tables in the previous section.*

Remark 16.3. By the discussion in Section 4.4, every matrix computed here is valid for any odd q.

In practice, it is possible to use the result of Lübeck ([Lue20]) and Lemma 4.6 to find the same 2-parameter Green functions that we found here, but without the "large q" condition.

Proof. By Proposition 10.1 and Proposition 16.1 we know $\tilde{Q}_{\mathbf{L}_1}^{\mathbf{M}_1}$, $\tilde{Q}_{\mathbf{L}_2}^{\mathbf{M}_1}$, $\tilde{Q}_{\mathbf{L}_2}^{\mathbf{M}_2}$ and $\tilde{Q}_{\mathbf{L}_4}^{\mathbf{M}_2}$. By an analogous computation of those propositions we get $\tilde{Q}_{\mathbf{L}_1}^{\mathbf{M}_3}$, $\tilde{Q}_{\mathbf{L}_3}^{\mathbf{M}_3}$, $\tilde{Q}_{\mathbf{L}_3}^{\mathbf{M}_4}$ and $\tilde{Q}_{\mathbf{L}_4}^{\mathbf{M}_4}$ (we use Corollary 4.20 for the split Levi subgroups).

Next, $\tilde{Q}_{\mathbf{M}_1}^{\mathbf{G}}$, $\tilde{Q}_{\mathbf{M}_3}^{\mathbf{G}}$ and $\tilde{Q}_{\mathbf{L}_1}^{\mathbf{G}}$ are computed directly thanks to the fusion of unipotent classes (Section 14) of $\mathbf{G}^F$ and the discussion of Section 4.3. Thanks to Lemma 4.6, we directly obtain $\tilde{Q}_{\mathbf{L}_2}^{\mathbf{G}} = \tilde{Q}_{\mathbf{L}_2}^{\mathbf{M}_1} \cdot \tilde{Q}_{\mathbf{M}_1}^{\mathbf{G}}$ and $\tilde{Q}_{\mathbf{L}_3}^{\mathbf{G}} = \tilde{Q}_{\mathbf{L}_3}^{\mathbf{M}_3} \cdot \tilde{Q}_{\mathbf{M}_3}^{\mathbf{G}}$.

The situation is now graphically summarized in Figure 4. The Levi subgroups in solid boxes are those with known Green functions (which are valid for all odd q), while those in dashed boxes have to be found and, unfortunately, will be shown to be valid only for large enough q. The reason is the following. Both $\tilde{Q}_{\mathbf{L}_4}^{\mathbf{G}}$ and $\tilde{Q}_{\mathbf{L}_5}^{\mathbf{G}}$ have 5 unknown entries and in total we get 7 scalar products of the form

$$\langle R_{\mathbf{L}_i}^{\mathbf{G}} f_i,\, R_{\mathbf{L}_j}^{\mathbf{G}} f_j \rangle = 0 \quad \text{for } i \neq j,$$

plus two norm equations. Since (as can be seen from Figure 4) equations of the form

$$\langle R_{\mathbf{M}_i}^{\mathbf{G}} f,\, R_{\mathbf{L}_j}^{\mathbf{G}} f_j \rangle = 0 \qquad \text{for } L_j \nsubseteq M_i$$

do not provide new information, a solution is provided by the above mentioned result of Digne, Lehrer and Michel (valid for q large enough), see Proposition 5.33.

Setting $\tilde{Q}_{\mathbf{L}}^{\mathbf{G}}(u, v)$ to 1 or 0 for pairs of regular unipotent elements allows us to solve the system of equations for $\mathbf{L}_4$ and $\mathbf{L}_5$. Then, the Lusztig restriction of Gel'fand–Graev characters allows us to select the only correct solution. We mention, however, that the norm equation now yields two rational solutions. Only one of them satisfies $|C_{\mathbf{L}^F}(v)|\tilde{Q}_{\mathbf{L}}^{\mathbf{G}}(u, v) \in \mathbb{Z}$, which has to hold by the definition of the modified Green functions.

Thanks to Lemma 4.6 we obtain the equations $\tilde{Q}_{\mathbf{L}_2}^{\mathbf{G}} = \tilde{Q}_{\mathbf{L}_2}^{\mathbf{M}_2} \cdot \tilde{Q}_{\mathbf{M}_2}^{\mathbf{G}}$ and $\tilde{Q}_{\mathbf{L}_4}^{\mathbf{G}} = \tilde{Q}_{\mathbf{L}_4}^{\mathbf{M}_2} \cdot \tilde{Q}_{\mathbf{M}_2}^{\mathbf{G}}$ which are enough to determine the $\tilde{Q}_{\mathbf{M}_2}^{\mathbf{G}}$.

Similarly for $\mathbf{M}_4$ we have the equations $\tilde{Q}_{\mathbf{L}_3}^{\mathbf{G}} = \tilde{Q}_{\mathbf{L}_3}^{\mathbf{M}_4} \cdot \tilde{Q}_{\mathbf{M}_4}^{\mathbf{G}}$ and $\tilde{Q}_{\mathbf{L}_4}^{\mathbf{G}} = \tilde{Q}_{\mathbf{L}_4}^{\mathbf{M}_4} \cdot \tilde{Q}_{\mathbf{M}_4}^{\mathbf{G}}$. However this time, we also use the fact that $\tilde{Q}_{\mathbf{M}_4}^{\mathbf{G}}$ is invariant under triality to greatly reduce the number of unknowns in the system. Finally, we can solve the system for $\tilde{Q}_{\mathbf{M}_4}^{\mathbf{G}}$, completing the proof. □

To use the character formula for Lusztig restriction (Proposition 3.30 (b)) we need also the 2-parameter Green functions $\tilde{Q}_{\mathbf{L}}^{\mathbf{C}}$ for the centralizer $\mathbf{C}$ that is not a Levi subgroup, described in the previous section.

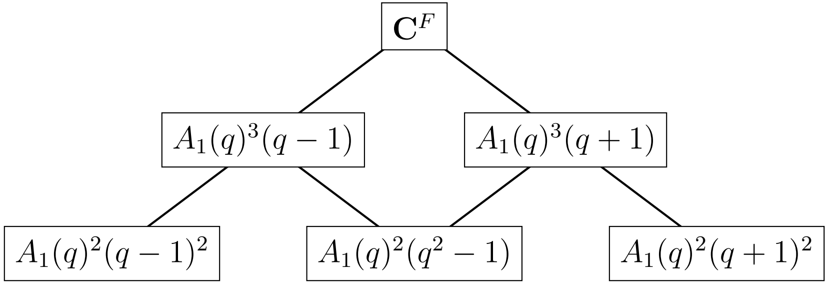
The Levi subgroups common to both $\mathbf{G}$ and $\mathbf{C}$ that we consider are $\mathbf{M}_3$, $\mathbf{M}_4$, $\mathbf{L}_1$, $\mathbf{L}_3$ and $\mathbf{L}_4$. The situation is graphically summarized in Figure 5.

The subgroup $\mathbf{C}^F$ has 41 unipotent elements. This makes it impractical to print the tables for $\tilde{Q}_{\mathbf{L}}^{\mathbf{C}}$ here. However, their determination is made easy thanks to the surjective morphism $(\mathrm{SL}_2(q))^4 \to \mathbf{C}$, discussed in Remark 15.3, and Proposition 4.12.

Let $\mathbf{G} = \prod_i \mathbf{G}_i$ be a semisimple algebraic group which is the product of some simple algebraic groups $\mathbf{G}_i$. Let F_i be a Steinberg endomorphism of $\mathbf{G}_i$. Then, $F = \prod_i F_i$ is a Steinberg endomorphism of $\mathbf{G}$ such that

$$F : \prod_i \mathbf{G}_i \to \prod_i \mathbf{G}_i,\ (g_1, g_2, ...) \mapsto (F_1(g_1), F_2(g_2), ...).$$

Figure 5: Levi subgroups common to both $\mathbf{G}$ and $\mathbf{C}$ that we consider.



Denote by $\mathscr{L}_i$ the Lang map of $\mathbf{G}_i$ determined by F_i ($\mathscr{L}_i(g_i) = g_i^{-1}F_i(g_i)$ for $g_i \in \mathbf{G}_i$). Then, $\mathscr{L} = \prod_i \mathscr{L}_i$ is the Lang map of $\mathbf{G}$ associated to F. Denote by $\mathbf{U}_i$ maximal F-stable unipotent subgroups of $\mathbf{G}_i$, and $\mathbf{U} = \prod_i \mathbf{U}_i$ a maximal F-stable unipotent subgroup of $\mathbf{G}$.

In this situation we can apply the following proposition.

Proposition 16.4 ([DiMi20, Proposition 8.1.9 (ii)]). *Let $\mathbf{X}, \mathbf{X}'$ be two varieties. Then, for $g \in \mathrm{Aut}(\mathbf{X})$ and $g' \in \mathrm{Aut}(\mathbf{X}')$ automorphisms of finite order, we have*

$$\mathrm{Trace}\,(g \times g', H_c^*(\mathbf{X} \times \mathbf{X}')) = \mathrm{Trace}\,(g, H_c^*(\mathbf{X}))\,\mathrm{Trace}\,(g', H_c^*(\mathbf{X}'))$$

It follows directly that, for an F-stable Levi subgroup $\mathbf{L} \leq \mathbf{G}$, with decomposition $\mathbf{L} = \prod_i \mathbf{L}_i$ we have for all $u \in \mathbf{G}^F_{\mathrm{uni}}$ and $v \in \mathbf{L}^F_{\mathrm{uni}}$

$$Q^{\mathbf{G}}_{\mathbf{L}}(u, v) = \prod_i Q^{\mathbf{G}_i}_{\mathbf{L}_i}(u_i, v_i)$$

where $u_i \in (\mathbf{G}_i)_{\mathrm{uni}}$ and $v_i \in (\mathbf{L}_i)_{\mathrm{uni}}$ such that $u = (u_1, u_2, ...)$ and $v = (v_1, v_2, ...)$.

Analogously, for $\tilde{Q}^{\mathbf{G}}_{\mathbf{L}}(u, v) := |v^{\mathbf{L}^F}| Q^{\mathbf{G}}_{\mathbf{L}}(u, v^{-1})$ the same relation holds

$$\tilde{Q}^{\mathbf{G}}_{\mathbf{L}}(u, v) = \prod_i \tilde{Q}^{\mathbf{G}_i}_{\mathbf{L}_i}(u_i, v_i)$$

since $|v^{\mathbf{L}^F}| = \prod_i |v_i^{\mathbf{L}_i^F}|$.

In the present case, the above discussion means that all the 2-parameter Green functions of $(\mathrm{SL}_2(q))^4$ are determined by those of $\mathrm{SL}_2(q)$. But in $\mathrm{SL}_2(q)$ the only proper Levi subgroups are maximal tori. Thus, we need just the ordinary Green functions of $\mathrm{SL}_2(q)$. These are known, since those of $\mathrm{GL}_2(q)$ are known (they can be found in `CHEVIE`) and by Remark 4.14.

There are (up to conjugacy) two F-stable maximal tori of SL_2 (the Weyl group of SL_2 is the symmetric group S_2), denote them by $\mathbf{T}_1$ for the split one and by $\mathbf{T}_2$ for the twisted one. With the notation as in Remark 15.3, $\mathrm{SL}_2(q)$ has three unipotent classes with representatives 1, $u^+_{\mathrm{SL}_2}(1)$ and $u^+_{\mathrm{SL}_2}(\mu)$ and the values of its ordinary Green functions are:

u	$Q^{\mathrm{SL}_2}_{\mathbf{T}_1}(u)$	$Q^{\mathrm{SL}_2}_{\mathbf{T}_2}(u)$
1	Φ_2	$-\Phi_1$
$u^+_{\mathrm{SL}_2}(1)$	1	1
$u^+_{\mathrm{SL}_2}(\mu)$	1	1

This together with Proposition 4.12 allows us to write all the 2-parameter Green functions of $\mathbf{C}^F$ that we need. Therefore, although we do not display these Green functions here, because the dimension is excessive, we have proved:

Theorem 16.5. *The matrices $\tilde{Q}^{\mathbf{C}}_{\mathbf{M}_3}$, $\tilde{Q}^{\mathbf{C}}_{\mathbf{M}_4}$, $\tilde{Q}^{\mathbf{C}}_{\mathbf{L}_1}$ $\tilde{Q}^{\mathbf{C}}_{\mathbf{L}_3}$ and $\tilde{Q}^{\mathbf{C}}_{\mathbf{L}_4}$ are determined by the ordinary Green functions of $\mathrm{SL}_2(q)$.*

17 Modified Gel'fand–Graev characters

In this section, we give all the modified Gel'fand–Graev characters for G and for the Levi subgroups $L_1, \ldots, L_5, M_1, \ldots, M_4$ of $\mathrm{Spin}_8^+(q)$. These can be computed explicitly by using Lemma 5.58. In the results, we denote the k-th power of the n-th root of unity in $\mathbb{C}$ by $\zeta_n^k := \mathrm{e}^{2\pi i \frac{k}{n}}$.

We use the notation from Section 15 for the next proposition. Also, for modified Gel'fand–Graev characters we separate the indices coming from the "centre part" from those coming from the "unipotent part" with a semicolon ";".

Proposition 17.1. *The Gel'fand–Graev characters of G are given by*

$$\Gamma_{j_c,j_d}^{\mathbf{G}} = \mathrm{Ind}_{U_0}^{G} \phi_{\mu^{j_c},\mu^{j_d},1,\mu^{j_c+j_d}}$$

for $j_c, j_d = 1, 2$.

The modified Gel'fand–Graev characters of G are given in Table 38.

The Gel'fand–Graev characters of M_1 are given by

$$\Gamma_{j_c}^{\mathbf{M}_1} = \mathrm{Ind}_{U_{M_1}}^{M_1} \phi_{\mu^{j_c},1,1}^{M_1}$$

for $j_c = 1, 2$.

The Gel'fand–Graev characters of M_2 are given by

$$\Gamma_{j_c}^{\mathbf{M}_2} = \mathrm{Ind}_{U_{M_2}}^{M_2} \phi_{\rho^{j_c},1}^{M_2}$$

for $j_c = 1, 2$.

The Gel'fand–Graev characters of M_i, $i = 3, 4$, are given by

$$\Gamma_{j_d,j_e}^{\mathbf{M}_i} = \mathrm{Ind}_{U_{M_i}}^{M_i} \phi_{\mu^{j_d},\mu^{j_e},1}^{M_i}$$

for $j_d, j_e = 1, 2$.

The Gel'fand–Graev characters of L_i, $i = 1, 3, 4$, are given by

$$\Gamma_{j_d}^{\mathbf{L}_i} = \mathrm{Ind}_{U_{L_i}}^{L_i} \phi_{\mu^{j_d},1}^{L_i}$$

for $j_d = 1, 2$.

The Gel'fand–Graev characters of L_i, $i = 2, 5$, are given by

$$\Gamma_{j_c}^{\mathbf{L}_i} = \mathrm{Ind}_{U_{L_i}}^{L_i} \phi_{\rho^{j_c}}^{L_i}$$

for $j_c = 1, 2$.

The modified Gel'fand–Graev characters of the Levi subgroups are given in Tables 39 to 47.

Remark 17.2. By explicit computations we notice the following:

(a) The norm of the Gel'fand–Graev characters $\Gamma_{j_c,j_d}^{\mathbf{G}}$ are

$$q^4 + 3q^2$$

in agreement with [BrLue13, Theorem 4.1].

(b) The differences $\Gamma_{\theta;0}^{\mathbf{L}_i} - \Gamma_{\theta;1}^{\mathbf{L}_i}$ are precisely the difference of two regular characters of L_i, for all $\theta \in \mathrm{Irr}(Z(L_i))$ and for all $i = 1, ..., 5$.

Table 38: Non-zero values of the modified Gel'fand–Graev characters of G.

zu $k_a, k_b, k_c, k_d = 0, 1$	$\Gamma^{\mathbf{G}}_{j_a,j_b;j_c,j_d}(zu)$ $j_a, j_b, j_c, j_d = 0, 1$
$h_Z(k_a, k_b)$	$\frac{(-1)^{j_ak_a+j_bk_b}}{4}\Phi_1^4\Phi_2^4\Phi_3\Phi_4^2\Phi_6$
$h_Z(k_a, k_b)u_1(1)$	$-\frac{(-1)^{j_ak_a+j_bk_b}}{4}\Phi_1^3\Phi_2^3\Phi_4^2$
$h_Z(k_a, k_b)u_1(\mu^{k_c})u_2(1)$	$\frac{(-1)^{j_ak_a+j_bk_b}}{4}\Phi_1^2\Phi_2^2\Phi_4\left(q^3(-1)^{j_c+j_d+k_c+\frac{q-1}{2}}+1\right)$
$h_Z(k_a, k_b)u_1(\mu^{k_c})u_4(1)$	$\frac{(-1)^{j_ak_a+j_bk_b}}{4}\Phi_1^2\Phi_2^2\Phi_4\left(q^3(-1)^{j_d+k_c+\frac{q-1}{2}}+1\right)$
$h_Z(k_a, k_b)u_2(\mu^{k_c})u_4(1)$	$\frac{(-1)^{j_ak_a+j_bk_b}}{4}\Phi_1^2\Phi_2^2\Phi_4\left(q^3(-1)^{j_c+k_c+\frac{q-1}{2}}+1\right)$
$h_Z(k_a, k_b)u_1(\mu^{k_c})u_2(\mu^{k_d})u_4(1)$	$-\frac{(-1)^{j_ak_a+j_bk_b}}{4}\Phi_1\Phi_2\left(1+q^3(-1)^{j_c+j_d+k_c+k_d+\frac{q-1}{2}}+\right.$ $\left.+q^3(-1)^{j_d+k_c+\frac{q-1}{2}}+q^3(-1)^{j_c+k_d+\frac{q-1}{2}}\right)$
$h_Z(k_a, k_b)u_1(1)u_3(1)$	$\frac{(-1)^{j_ak_a+j_bk_b}}{4}\Phi_1^2\Phi_2\Phi_3$
$h_Z(k_a, k_b)u_1(\mu)u_2(1)u_4(1)u_{12}(1)$	$-\frac{(-1)^{j_ak_a+j_bk_b}}{4}\Phi_1\Phi_2^2\Phi_6$
$h_Z(k_a, k_b)u_1(\mu^{k_c})u_2(1)u_3(1)$	$-\frac{(-1)^{j_ak_a+j_bk_b}}{4}\Phi_1\Phi_2\left(q^2(-1)^{j_c+j_d+k_c+\frac{q-1}{2}}+1\right)$
$h_Z(k_a, k_b)u_1(\mu^{k_c})u_3(1)u_4(1)$	$-\frac{(-1)^{j_ak_a+j_bk_b}}{4}\Phi_1\Phi_2\left(q^2(-1)^{j_d+k_c+\frac{q-1}{2}}+1\right)$
$h_Z(k_a, k_b)u_2(\mu^{k_c})u_3(1)u_4(1)$	$-\frac{(-1)^{j_ak_a+j_bk_b}}{4}\Phi_1\Phi_2\left(q^2(-1)^{j_c+k_c+\frac{q-1}{2}}+1\right)$
$h_Z(k_a, k_b)u_1(\mu^{k_c})u_2(1)u_3(\mu^{k_d})u_{10}(1)$	$-\frac{(-1)^{j_ak_a+j_bk_b}}{4}\Phi_1\Phi_2$
$h_Z(k_a, k_b)u_1(\mu^{k_c})u_2(\mu^{k_d})u_3(1)u_4(1)$	$\frac{(-1)^{j_ak_a+j_bk_b}}{4}\left(1+q(-1)^{j_c+j_d+k_c+k_d+\frac{q-1}{2}}+\right.$ $\left.+q(-1)^{j_d+k_c+\frac{q-1}{2}}+q(-1)^{j_c+k_d+\frac{q-1}{2}}\right)$

Table 39: Non-zero values of the modified Gel'fand–Graev characters of M_1.

zu $k_a, k_c = 0, 1,$ $k_b = 0, ..., q-2$	$\Gamma^{\mathbf{M_1}}_{j_a,j_b;j_c}(zu)$ $j_a, j_c = 0, 1,$ $j_b = 0, ..., q-2$
$h_{Z_{M_1}}(k_a, k_b)$	$\frac{(-1)^{j_ak_a}\zeta_{q-1}^{j_bk_b}}{2}\Phi_1^3\Phi_2^2\Phi_3\Phi_4$
$h_{Z_{M_1}}(k_a, k_b)u_2(1)$	$-\frac{(-1)^{j_ak_a}\zeta_{q-1}^{j_bk_b}}{2}\Phi_1^2\Phi_2\Phi_3$
$h_{Z_{M_1}}(k_a, k_b)u_2(\mu^{k_c})u_4(1)$	$\frac{(-1)^{j_ak_a}\zeta_{q-1}^{j_bk_b}}{2}\Phi_1\Phi_2\left(q^2(-1)^{j_c+k_c+\frac{q-1}{2}}+1\right)$
$h_{Z_{M_1}}(k_a, k_b)u_2(1)u_3(1)$	$\frac{(-1)^{j_ak_a}\zeta_{q-1}^{j_bk_b}}{2}\Phi_1\Phi_2$
$h_{Z_{M_1}}(k_a, k_b)u_2(\mu^{k_c})u_3(1)u_4(1)$	$-\frac{(-1)^{j_ak_a}\zeta_{q-1}^{j_bk_b}}{2}\left(q(-1)^{j_c+k_c+\frac{q-1}{2}}+1\right)$

Table 40: Non-zero values of the modified Gel'fand–Graev characters of M_2.

zu $k_a, k_c = 0, 1$, $k_b = 0, ..., q$	$\Gamma^{\mathbf{M_2}}_{j_a,j_b;j_c}(zu)$ $j_a, j_c = 0, 1$, $j_b = 0, ..., q$
$h_{Z_{M_2}}(k_a, k_b)$	$\frac{(-1)^{j_a k_a} \zeta_{q+1}^{j_b k_b}}{2} \Phi_1^2 \Phi_2^3 \Phi_4 \Phi_6$
$h_{Z_{M_2}}(k_a, k_b) u_{\beta_2}(1)$	$-\frac{(-1)^{j_a k_a} \zeta_{q+1}^{j_b k_b}}{2} \Phi_1 \Phi_2^2 \Phi_6$
$h_{Z_{M_2}}(k_a, k_b) u_{\beta_1}(\rho^{k_c})$	$-\frac{(-1)^{j_a k_a} \zeta_{q+1}^{j_b k_b}}{2} \Phi_1 \Phi_2 \left(q^2(-1)^{j_c+k_c+\frac{q-1}{2}} + 1\right)$
$h_{Z_{M_2}}(k_a, k_b) u_{\beta_1}(\rho) u_{\beta_3}(1)$	$-\frac{(-1)^{j_a k_a} \zeta_{q+1}^{j_b k_b}}{2} \Phi_1 \Phi_2$
$h_{Z_{M_2}}(k_a, k_b) u_{\beta_1}(\rho^{k_c}) u_{\beta_2}(1)$	$\frac{(-1)^{j_a k_a} \zeta_{q+1}^{j_b k_b}}{2} \left(q(-1)^{j_c+k_c+\frac{q-1}{2}} + 1\right)$

Table 41: Non-zero values of the modified Gel'fand–Graev characters of M_3.

zu $k_a = 0, ..., q-2$, $k_b, k_c, k_d, k_e = 0, 1$	$\Gamma^{\mathbf{M_3}}_{j_a,j_b,j_c;j_d,j_e}(zu)$ $j_a = 0, ..., q-2$, $j_b, j_c, j_d, j_e = 0, 1$
$h_{Z_{M_3}}(k_a, k_b, k_c)$	$\frac{(-1)^{j_b k_b + j_c k_c} \zeta_{q-1}^{j_a k_a}}{4} \Phi_1^3 \Phi_2^3$
$h_{Z_{M_3}}(k_a, k_b, k_c) u_1(1)$	$-\frac{(-1)^{j_b k_b + j_c k_c} \zeta_{q-1}^{j_a k_a}}{4} \Phi_1^2 \Phi_2^2$
$h_{Z_{M_3}}(k_a, k_b, k_c) u_2(1)$	$-\frac{(-1)^{j_b k_b + j_c k_c} \zeta_{q-1}^{j_a k_a}}{4} \Phi_1^2 \Phi_2^2$
$h_{Z_{M_3}}(k_a, k_b, k_c) u_4(1)$	$-\frac{(-1)^{j_b k_b + j_c k_c} \zeta_{q-1}^{j_a k_a}}{4} \Phi_1^2 \Phi_2^2$
$h_{Z_{M_3}}(k_a, k_b, k_c) u_1(\mu^{k_d}) u_2(1)$	$\frac{(-1)^{j_b k_b + j_c k_c} \zeta_{q-1}^{j_a k_a}}{4} \Phi_1 \Phi_2 \left(q(-1)^{j_d+j_e+k_d+\frac{q-1}{2}} + 1\right)$
$h_{Z_{M_3}}(k_a, k_b, k_c) u_1(\mu^{k_d}) u_4(1)$	$\frac{(-1)^{j_b k_b + j_c k_c} \zeta_{q-1}^{j_a k_a}}{4} \Phi_1 \Phi_2 \left(q(-1)^{j_d+k_d+\frac{q-1}{2}} + 1\right)$
$h_{Z_{M_3}}(k_a, k_b, k_c) u_2(\mu^{k_d}) u_4(1)$	$\frac{(-1)^{j_b k_b + j_c k_c} \zeta_{q-1}^{j_a k_a}}{4} \Phi_1 \Phi_2 \left(q(-1)^{j_e+k_d+\frac{q-1}{2}} + 1\right)$
$h_{Z_{M_3}}(k_a, k_b, k_c) u_1(\mu^{k_d}) u_2(\mu^{k_e}) u_4(1)$	$-\frac{(-1)^{j_b k_b + j_c k_c} \zeta_{q-1}^{j_a k_a}}{4} \left(1 + q(-1)^{j_d+j_e+k_d+k_e+\frac{q-1}{2}} + \right.$ $\left. + q(-1)^{j_d+k_d+\frac{q-1}{2}} + q(-1)^{j_e+k_e+\frac{q-1}{2}}\right)$

Table 42: Non-zero values of the modified Gel'fand–Graev characters of M_4.

zu $k_a = 0, ..., q,\ k_b, k_c, k_d, k_e = 0, 1$	$\Gamma^{\mathbf{M}_4}_{j_a,j_b,j_c;j_d,j_e}(zu)$ $j_a = 0, ..., q,\ j_b, j_c, j_d, j_e = 0, 1$
$h_{Z_{M_4}}(k_a, k_b, k_c)$	$\frac{(-1)^{j_bk_b+j_ck_c}\zeta_{q+1}^{j_ak_a}}{4}\Phi_1^3\Phi_2^3$
$h_{Z_{M_4}}(k_a, k_b, k_c)u_1(1)$	$-\frac{(-1)^{j_bk_b+j_ck_c}\zeta_{q+1}^{j_ak_a}}{4}\Phi_1^2\Phi_2^2$
$h_{Z_{M_4}}(k_a, k_b, k_c)u_2(1)$	$-\frac{(-1)^{j_bk_b+j_ck_c}\zeta_{q+1}^{j_ak_a}}{4}\Phi_1^2\Phi_2^2$
$h_{Z_{M_4}}(k_a, k_b, k_c)u_4(1)$	$-\frac{(-1)^{j_bk_b+j_ck_c}\zeta_{q+1}^{j_ak_a}}{4}\Phi_1^2\Phi_2^2$
$h_{Z_{M_4}}(k_a, k_b, k_c)u_1(\mu^{k_d})u_2(1)$	$\frac{(-1)^{j_bk_b+j_ck_c}\zeta_{q+1}^{j_ak_a}}{4}\Phi_1\Phi_2\left(q(-1)^{j_d+j_e+k_d+\frac{q-1}{2}}+1\right)$
$h_{Z_{M_4}}(k_a, k_b, k_c)u_1(\mu^{k_d})u_4(1)$	$\frac{(-1)^{j_bk_b+j_ck_c}\zeta_{q+1}^{j_ak_a}}{4}\Phi_1\Phi_2\left(q(-1)^{j_d+k_d+\frac{q-1}{2}}+1\right)$
$h_{Z_{M_4}}(k_a, k_b, k_c)u_2(\mu^{k_d})u_4(1)$	$\frac{(-1)^{j_bk_b+j_ck_c}\zeta_{q+1}^{j_ak_a}}{4}\Phi_1\Phi_2\left(q(-1)^{j_e+k_d+\frac{q-1}{2}}+1\right)$
$h_{Z_{M_4}}(k_a, k_b, k_c)u_1(\mu^{k_d})u_2(\mu^{k_e})u_4(1)$	$-\frac{(-1)^{j_bk_b+j_ck_c}\zeta_{q+1}^{j_ak_a}}{4}\Big(1+q(-1)^{j_d+j_e+k_d+k_e+\frac{q-1}{2}}+$ $+q(-1)^{j_d+k_d+\frac{q-1}{2}}+q(-1)^{j_e+k_e+\frac{q-1}{2}}\Big)$

Table 43: Non-zero values of the modified Gel'fand–Graev characters of L_1.

zu $k_a, k_b = 0, ..., q-2,\ k_c, k_d = 0, 1$	$\Gamma^{\mathbf{L}_1}_{j_a,j_b,j_c;j_d}(zu)$ $j_a, j_b = 0, ..., q-2,\ j_c, j_d = 0, 1$
$h_{Z_{L_1}}(k_a, k_b, k_c)$	$\frac{(-1)^{j_ck_c}\zeta_{q-1}^{j_ak_a+j_bk_b}}{2}\Phi_1^2\Phi_2^2$
$h_{Z_{L_1}}(k_a, k_b, k_c)u_2(1)$	$-\frac{(-1)^{j_ck_c}\zeta_{q-1}^{j_ak_a+j_bk_b}}{2}\Phi_1\Phi_2$
$h_{Z_{L_1}}(k_a, k_b, k_c)u_4(1)$	$-\frac{(-1)^{j_ck_c}\zeta_{q-1}^{j_ak_a+j_bk_b}}{2}\Phi_1\Phi_2$
$h_{Z_{L_1}}(k_a, k_b, k_c)u_2(\mu^{k_d})u_4(1)$	$\frac{(-1)^{j_ck_c}\zeta_{q-1}^{j_ak_a+j_bk_b}}{2}\left(q(-1)^{j_d+k_d+\frac{q-1}{2}}+1\right)$

Table 44: Non-zero values of the modified Gel'fand–Graev characters of L_2.

zu $k_a = 0, ..., q^2,\ k_b, k_c = 0, 1$	$\Gamma^{\mathbf{L}_2}_{j_a,j_b;j_c}(zu)$ $j_a = 0, ..., q^2, j_b, j_c = 0, 1$
$h_{Z_{L_2}}(k_a, k_b)$	$\frac{(-1)^{j_bk_b}\zeta_{q^2-1}^{j_ak_a}}{2}\Phi_1\Phi_2\Phi_4$
$h_{Z_{L_2}}(k_a, k_b)u_\beta(\rho^{k_c})$	$-\frac{(-1)^{j_bk_b}\zeta_{q^2-1}^{j_ak_a}}{2}\left(q(-1)^{j_c+k_c+\frac{q-1}{2}}+1\right)$

Table 45: Non-zero values of the modified Gel'fand–Graev characters of L_3.

zu $k_a = 0, ..., q-2,\ k_b = 0, ..., q,\ k_c, k_d = 0, 1$	$\Gamma^{\mathbf{L}_3}_{j_a,j_b,j_c;j_d}(zu)$ $j_a = 0, ..., q-2,\ j_b = 0, ..., q,\ j_c, j_d = 0, 1$
$h_{Z_{L_3}}(k_a, k_b, k_c)$	$\frac{(-1)^{j_ck_c}\zeta_{q-1}^{j_ak_a}\zeta_{q+1}^{j_bk_b}}{2}\Phi_1^2\Phi_2^2$
$h_{Z_{L_3}}(k_a, k_b, k_c)u_2(1)$	$-\frac{(-1)^{j_ck_c}\zeta_{q-1}^{j_ak_a}\zeta_{q+1}^{j_bk_b}}{2}\Phi_1\Phi_2$
$h_{Z_{L_3}}(k_a, k_b, k_c)u_4(1)$	$-\frac{(-1)^{j_ck_c}\zeta_{q-1}^{j_ak_a}\zeta_{q+1}^{j_bk_b}}{2}\Phi_1\Phi_2$
$h_{Z_{L_3}}(k_a, k_b, k_c)u_2(\mu^{k_d})u_4(1)$	$\frac{(-1)^{j_ck_c}\zeta_{q-1}^{j_ak_a}\zeta_{q+1}^{j_bk_b}}{2}\left(q(-1)^{j_d+k_d+\frac{q-1}{2}}+1\right)$

Table 46: Non-zero values of the modified Gel'fand–Graev characters of L_4.

zu $k_a, k_b = 0, ..., q,\ k_c, k_d = 0, 1$	$\Gamma^{\mathbf{L}_4}_{j_a,j_b,j_c;j_d}(zu)$ $j_a, j_b = 0, ..., q,\ j_c, j_d = 0, 1$
$h_{Z_{L_4}}(k_a, k_b, k_c)$	$\frac{(-1)^{j_ck_c}\zeta_{q+1}^{j_ak_a+j_bk_b}}{2}\Phi_1^2\Phi_2^2$
$h_{Z_{L_4}}(k_a, k_b, k_c)u_2(1)$	$-\frac{(-1)^{j_ck_c}\zeta_{q+1}^{j_ak_a+j_bk_b}}{2}\Phi_1\Phi_2$
$h_{Z_{L_4}}(k_a, k_b, k_c)u_4(1)$	$-\frac{(-1)^{j_ck_c}\zeta_{q+1}^{j_ak_a+j_bk_b}}{2}\Phi_1\Phi_2$
$h_{Z_{L_4}}(k_a, k_b, k_c)u_2(\mu^{k_d})u_4(1)$	$\frac{(-1)^{j_ck_c}\zeta_{q+1}^{j_ak_a+j_bk_b}}{2}\left(q(-1)^{j_d+k_d+\frac{q-1}{2}}+1\right)$

Table 47: Non-zero values of the modified Gel'fand–Graev characters of L_5.

zu $k_a, k_c = 0, 1,\ k_b = 0, ..., q^2$	$\Gamma^{\mathbf{L}_5}_{j_a,j_b;j_c}(zu)$ $j_a, j_c = 0, 1,\ j_b = 0, ..., q^2$
$h_{Z_{L_5}}(k_a, k_b)$	$\frac{(-1)^{j_ak_a}\zeta_{q^2+1}^{j_bk_b}}{2}\Phi_1\Phi_2\Phi_4$
$h_{Z_{L_5}}(k_a, k_b)u_\beta(\rho^{k_c})$	$-\frac{(-1)^{j_ak_a}\zeta_{q^2+1}^{j_bk_b}}{2}\left(q(-1)^{j_c+k_c+\frac{q-1}{2}}+1\right)$

18 Decomposition of almost characters

In this section, we discuss the case $q \equiv 1 \pmod 4$. For the other odd congruence, the discussion is completely analogous.

In the generic character table of $G = \mathrm{Spin}_8^+(q)$ there are 579 irreducible character types, of which 14 are unipotent, and 237 class types. Of these class types 134 come from splitting classes of $\mathbf{G} = \mathrm{Spin}_8$, we will denote them generically by c_i. Each one of these classes intersects at least one of the Levi subgroups introduced in Section 15 (up to triality).

In the table of uniform almost characters, provided by Lübeck, there are 182 (non-unipotent) types of characters which are not irreducible. We want to decompose these. Recall that, by Remark 6.4, they are all true characters.

First of all, notice that we don't need to consider the unipotent characters.

Remark 18.1. The unipotent characters have already been computed by Geck and Pfeiffer in [GePf92]. To be precise, they treat the case of a connected reductive group $\tilde{\mathbf{G}}$ of type D_4 with connected centre and such that its derived subgroup is simply connected. However, thanks to Proposition 3.11 their work also gives the unipotent characters in our case ($\mathbf{G}$ simply connected). We just need to specify the regular embedding $\mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$.

For the rest of the characters, we proceed as explained in Section 6. We start by introducing unknowns, i.e. for each character χ_j that decomposes and for each class (belonging to a splitting class of $\mathbf{G}$) in a class type c_i we introduce unknowns $f_{j,i}$ in the table.

Remark 18.2. By Remark 5.55, each character introduces (in theory) only one unknown per class type. In practice, all values on a certain class type differ only by a root of unity (each class type has a representative in the centre of the centralizer of any of its elements).

We know that the constituents of each uniform almost character form a unique $\tilde{\mathbf{G}}^F$-orbit by Clifford theory (Theorem 2.21) and Remark 6.4. Then, we can relate their unknowns on different classes thanks to the conjugation with representatives in $\mathbf{G}$ of $H^1(F, Z(\mathbf{G}))$ (see Remark 1.64).

Finally, we write systems of equations involving the modified Gel'fand–Graev characters and the unknowns $f_{j,i}$ of the Levi subgroups that intersect the class c_i, via Lusztig restriction.

As seen in Figure 4, the Levi subgroup lattice of $\mathrm{Spin}_8^+(q)$ is richer than that of $\mathrm{SL}_4(q)$. This means that some splitting classes might be contained in more than one Levi subgroup. For example those of the form zu with $z \in Z(L_1)$ and $u \in L_1$ unipotent might be contained also in M_1 or M_3. Thus, one needs to be careful that all the computations/labellings are consistent.

We treat in some detail representative examples of the computations. In practice, we decompose a regular character, a semisimple character and a character which is neither of those.

Notation 18.3.

- Recall that we denote by $\mathbf{G}^*$ the dual group of $\mathbf{G}$, with Frobenius F^* corresponding to F. Then, we will write $G^* := \mathbf{G}^{*F^*}$.
- Furthermore, recall that for $g \in \mathbf{G}$, we use the notation $A_{\mathbf{G}}(g) := C_{\mathbf{G}}(g)/C_{\mathbf{G}}(g)^\circ$.
- We denote by $h_{\alpha_1^*}, h_{\alpha_2^*}, h_{\alpha_3^*}, h_{\alpha_4^*}$ the coroots of $\mathbf{G}^*$, whose span is the dual $\mathbf{T}_0^*$ of our reference maximally split torus $\mathbf{T}_0$ of $\mathbf{G}$. Then, in short, for $t_1, t_2, t_3, t_4 \in \bar{\mathbb{F}}_q^\times$, we will use the notation $h^*(t_1, t_2, t_3, t_4) := h_{\alpha_1^*}(t_1) h_{\alpha_2^*}(t_2) h_{\alpha_3^*}(t_3) h_{\alpha_4^*}(t_4)$ for the elements of $\mathbf{T}_0^*$.
- Characters and classes will be indexed as they are in the table provided by Lübeck.
- Recall that, at the end of Section 13.2, we have chosen elements in $\mathbf{G}$ representing every $z \in H^1(F, Z(\mathbf{G}))$. For $z = h_Z(k_a, k_b)$ (notation as in Section 13.2) we denote this representative by $g_{(k_a,k_b)}$, for $k_a, k_b = 0, 1$.
- Finally, recall that we denote the triality automorphism of $\mathbf{G}$ by τ.

18.1 A regular character

In this section, we treat the example of a regular character which is not semisimple.

We consider the Lusztig series $\mathscr{E}(G, s_{15})$ with $s_{15} = h^*(1, 1, -1, 1) \in G^*$. It is easy to see that the centralizer $C_{\mathbf{G}^*}(s_{15})$ is of type A_1^4 and $A_{\mathbf{G}^*}(s_{15}) \cong C_2 \times C_2$ is of order 4. Because the centralizer is not a torus, the element s_{15} is not regular. Therefore, the regular characters and the semisimple characters of $\mathscr{E}(G, s_{15})$ are distinct. Because s_{15} is triality invariant, we expect some triality invariance to appear here and there in the computations below.

We decompose the uniform almost character in the span of $\mathscr{E}(G, s_{15})$ which has non-zero scalar product with the Gel'fand–Graev characters. We denote this character by R_{15}. The character R_{15} has norm equal to 4. Its irreducible constituents are the regular characters $\chi := \chi_{(s_{15}),h_Z(0,0)}$, $\chi^{g(0,1)} = \chi_{(s_{15}),h_Z(0,1)}$, $\chi^{g(1,0)} = \chi_{(s_{15}),h_Z(1,0)}$, and $\chi^{g(1,1)} = \chi_{(s_{15}),h_Z(1,1)}$.

Due to these conjugations, we can write the unknowns of these characters just in terms of those of χ. Moreover, we get some other constraint by imposing $R_{15} = \chi + \chi^{g(0,1)} + \chi^{g(1,0)} + \chi^{g(1,1)}$. Furthermore, by forcing $\langle \Gamma^{\mathbf{G}}_{k_a,k_b}, \chi^{g(k_a,k_b)} \rangle = 1$, for $k_a, k_b = 0, 1$, we can fix the unknowns of 3 unipotent classes in terms of the others. Other equations are given by Theorem 5.37 (a), i.e. $\chi(u) = 0$ for u regular unipotent since χ is not semisimple.

Next, it is easy to check that $\mathbf{t}_z R_{15} = R_{15}$ (for example on the identity element), for all $z \in Z(G)$. Thus, by Proposition 5.54, the constituents of R_{15} belong to $\mathrm{CF}(G)^{1_Z}$, where 1_Z is the trivial character of $Z(G)$. This fixes $\chi(zu) = \chi(u)$ for all $z \in Z(G)$ and z unipotent.

At this point, we used all the information available at the level of G, we turn, then, to the computations in the Levi subgroups.

Notation 18.4. Because we are considering Lusztig restriction of characters only on elements of the form zu (z central and u unipotent), we will encounter many characters that look like the Steinberg character (when restricted to those elements). However, they cannot be it for geometric reasons (the Steinberg character belongs to the Lusztig series associated with 1). We denote these characters by $\mathrm{St}'_{\mathbf{L}}$ to make it apparent that they have values in common with $\mathrm{St}_{\mathbf{L}}$.

Lemma 18.5. *We have the following relations:*

(a) ${}^*R^{\mathbf{G}}_{\mathbf{L}_1}\chi = \mathrm{St}'_{\mathbf{L}_1} + \chi^{\mathbf{L}_1}_{(s_{15}),1} \in \mathrm{CF}(L_1)^1 \oplus \mathrm{CF}(L_1)^{\varphi}$ *with* $\varphi : Z(L_1) \to \mathbb{C}, h_{Z_{L_1}}(k_a, k_b, k_c) \mapsto (-1)^{k_b}$,

(b) ${}^*R^{\mathbf{G}}_{\mathbf{L}_2}\chi = \chi^{\mathbf{L}_2}_{(s_{15}),z_{\mathbf{L}_2}} \in \mathrm{CF}(L_2)^{\varphi}$ *with* $\varphi : Z(L_2) \to \mathbb{C}, h_{Z_{L_2}}(k_a, k_b) \mapsto (-1)^{k_a}$,

(c) ${}^*R^{\mathbf{G}}_{\mathbf{L}_3}\chi = -\mathrm{St}'_{\mathbf{L}_3} \in \mathrm{CF}(L_3)^1$,

(d) ${}^*R^{\mathbf{G}}_{\mathbf{L}_4}\chi = \mathrm{St}'_{\mathbf{L}_4} + \chi^{\mathbf{L}_4}_{(s_{15}),1} \in \mathrm{CF}(L_4)^1 \oplus \mathrm{CF}(L_4)^{\varphi}$ *with* $\varphi : Z(L_4) \to \mathbb{C}, h_{Z_{L_4}}(k_a, k_b, k_c) \mapsto (-1)^{k_b+k_c}$,

(e) ${}^*R^{\mathbf{G}}_{\mathbf{L}_5}\chi = 0$

(f) ${}^*R^{\mathbf{G}}_{\mathbf{M}_1}\chi = \chi^{\mathbf{M}_1}_{(s_{15}),1} \in \mathrm{CF}(M_1)^1$ *and it is not a semisimple character of* M_1,

(g) ${}^*R^{\mathbf{G}}_{\mathbf{M}_2}\chi = \chi^{\mathbf{M}_2}_{(s_{15}),z_{\mathbf{M}_2}} \in \mathrm{CF}(M_2)^1$ *and it is not a semisimple character of* M_2,

(h) ${}^*R^{\mathbf{G}}_{\mathbf{M}_3}\chi = \mathrm{St}'_{\mathbf{M}_3} + \chi^{\mathbf{M}_3}_{(s_{15}),1} \in \mathrm{CF}(M_3)^1 \oplus \mathrm{CF}(M_3)^{\varphi}$ *with*
$\varphi : Z(M_3) \to \mathbb{C}, h_{Z_{M_3}}(k_a, k_b, k_c) \mapsto (-1)^{k_a+k_b+k_c}$,

(i) ${}^*R^{\mathbf{G}}_{\mathbf{M}_4}\chi = -\mathrm{St}'_{\mathbf{M}_4} - \chi^{\mathbf{M}_4}_{(s_{15}),1} \in \mathrm{CF}(M_4)^1 \oplus \mathrm{CF}(M_4)^{\varphi}$ *with*
$\varphi : Z(M_4) \to \mathbb{C}, h_{Z_{M_4}}(k_a, k_b, k_c) \mapsto (-1)^{k_a+k_b+k_c}$,

Also, the same relations hold when applying triality to the Levi subgroups (as expected).

Proof. We start with a preliminary remark. Notice that for all the Levi subgroups $\mathbf{L}$ different from $\mathbf{M}_3$ and $\mathbf{M}_4$ the canonical surjection $\mathfrak{h}_{\mathbf{L}}$ has kernel of order 2 (as seen in Section 15). Then, for those Levi subgroups ${}^*R^{\mathbf{G}}_{\mathbf{L}}R_{15}$ is twice the sum of, possibly different, irreducible characters. Also, since the constituents of R_{15} are regular characters, by Theorem 5.35 the restriction ${}^*R^{\mathbf{G}}_{\mathbf{L}}R_{15}$ is, up to sign, a true character.

(a) We explicitly compute $\langle \Gamma^{\mathbf{L}_1}_{0,0,0;k}, {}^*R^{\mathbf{G}}_{\mathbf{L}_1}R_{15}\rangle = 4$, $\langle \Gamma^{\mathbf{L}_1}_{0,\frac{q-1}{2},0;k}, {}^*R^{\mathbf{G}}_{\mathbf{L}_1}R_{15}\rangle = 2$ for $k = 0, 1$. By inspection of the values ${}^*R^{\mathbf{G}}_{\mathbf{L}_1}R_{15}(zu)$ we can identify the part in common with $\Gamma^{\mathbf{L}_1}_{0,0,0;k}$ and $\Gamma^{\mathbf{L}_1}_{0,\frac{q-1}{2},0;k}$ (where $z \in Z(L_1)$ and u unipotent in L_1). Then, we can write the decomposition ${}^*R^{\mathbf{G}}_{\mathbf{L}_1}R_{15} = 4 \cdot \mathrm{St}'_{\mathbf{L}_1} + 2\sum_{z\in H^1(F,Z(\mathbf{L}_1))} \chi^{\mathbf{L}_1}_{(s_{15}),z}$. Finally, the claim follows by the remark at the beginning of the proof, Theorem 5.35 and Remark 5.53.

The other points are proven in the exact same way. We just give the different decompositions.

(b) We compute $\langle \Gamma^{\mathbf{L}_2}_{\frac{q^2-1}{2},0;k}, {}^*R^{\mathbf{G}}_{\mathbf{L}_2}R_{15}\rangle = 2$ for $k = 0, 1$, which implies the decomposition ${}^*R^{\mathbf{G}}_{\mathbf{L}_2}R_{15} = 2\sum_{z\in H^1(F,Z(\mathbf{L}_2))} \chi^{\mathbf{L}_2}_{(s_{15}),z}$.

(c) We directly see on the character values that ${}^*R^{\mathbf{G}}_{\mathbf{L}_3}R_{15} = -4 \cdot \mathrm{St}'_{\mathbf{L}_3}$.

(d) We compute $\langle \Gamma^{\mathbf{L}_4}_{0,0,0;k}, {}^*R^{\mathbf{G}}_{\mathbf{L}_4}R_{15}\rangle = 4$ and $\langle \Gamma^{\mathbf{L}_4}_{0,\frac{q+1}{2},1;k}, {}^*R^{\mathbf{G}}_{\mathbf{L}_4}R_{15}\rangle = 2$ for $k = 0, 1$, which implies the decomposition ${}^*R^{\mathbf{G}}_{\mathbf{L}_4}R_{15} = 4 \cdot \mathrm{St}'_{\mathbf{L}_4} + 2\sum_{z\in H^1(F,Z(\mathbf{L}_4))} \chi^{\mathbf{L}_4}_{(s_{15}),z}$.

(e) We compute ${}^*R^{\mathbf{G}}_{\mathbf{L}_5}R_{15} = 0$.

(f) We compute $\langle \Gamma^{\mathbf{M}_1}_{0,0;k}, {}^*R^{\mathbf{G}}_{\mathbf{M}_1}R_{15}\rangle = 2$ for $k = 0, 1$, which implies the decomposition ${}^*R^{\mathbf{G}}_{\mathbf{M}_1}R_{15} = 2\sum_{z\in H^1(F,Z(\mathbf{M}_1))} \chi^{\mathbf{M}_1}_{(s_{15}),z}$. By Lemma 18.7 (f), in the next section, the constituents of ${}^*R^{\mathbf{G}}_{\mathbf{M}_1}R_{30}$ are not regular ($R_{30} = D_{\mathbf{G}}R_{15}$ is the sum of semisimple characters). It follows that $\chi^{\mathbf{M}_1}_{(s_{15}),z}$ is not semisimple, for $z \in H^1(F, Z(\mathbf{M}_1))$.

(g) This is exactly like the previous point with $\mathbf{M}_1$ replaced with $\mathbf{M}_2$.

(h) We compute $\langle \Gamma^{\mathbf{M}_3}_{0,0,0;k}, {}^*R^{\mathbf{G}}_{\mathbf{M}_3}R_{15}\rangle = 4$ and $\langle \Gamma^{\mathbf{M}_3}_{\frac{q-1}{2},1,1;k}, {}^*R^{\mathbf{G}}_{\mathbf{M}_3}R_{15}\rangle = 1$ for $k = 0, 1$, which implies the decomposition ${}^*R^{\mathbf{G}}_{\mathbf{M}_3}R_{15} = 4 \cdot \mathrm{St}'_{\mathbf{M}_3} + \sum_{z\in H^1(F,Z(\mathbf{M}_3))} \chi^{\mathbf{M}_3}_{(s_{15}),z}$.

(i) We compute $\langle \Gamma^{\mathbf{M}_4}_{0,0,0;k}, {}^*R^{\mathbf{G}}_{\mathbf{M}_4}R_{15}\rangle = -4$ and $\langle \Gamma^{\mathbf{M}_4}_{\frac{q+1}{2},1,1;k}, {}^*R^{\mathbf{G}}_{\mathbf{M}_4}R_{15}\rangle = -1$ for $k = 0, 1$, which implies the decomposition ${}^*R^{\mathbf{G}}_{\mathbf{M}_4}R_{15} = -4 \cdot \mathrm{St}'_{\mathbf{M}_4} - \sum_{z\in H^1(F,Z(\mathbf{M}_4))} \chi^{\mathbf{M}_4}_{(s_{15}),z}$.

Also, the same relations hold when applying triality to the Levi subgroups, by explicit computation (as expected). □

We can now use Lemma 18.5 to fix the unknown values of χ. By the triality invariance of R_{15} every time we mention a Levi subgroup $\mathbf{L}$, in the discussion that follows, we imply $\mathbf{L}$ or $\tau(\mathbf{L})$ or $\tau^2(\mathbf{L})$.

First of all, we have 12 unknown character values on unipotent classes. We already fixed 3 of them with the scalar products $\langle \chi, \Gamma^{\mathbf{G}}_z\rangle = \delta_{1,z}$, for $z \in H^1(F, Z(\mathbf{G}))$. Then, by points (c), (e) and (f) (or (g)) of the Lemma we get 9 other linearly independent equations between those unknowns, fixing the values completely. In practice, each of those statement of the lemma translates as ${}^*R^{\mathbf{G}}_{\mathbf{L}}\chi(zu) = 0$ for $z \in Z(\mathbf{L}^F)$ and u regular unipotent in $\mathbf{L}^F$, where $\mathbf{L} = \mathbf{L}_3, \mathbf{L}_5, \mathbf{M}_1, \mathbf{M}_2$. For $\mathbf{M}_1$ and $\mathbf{M}_2$ this is a consequence of Theorem 5.37 (a).

The knowledge of $\chi(zu)$ for all $z \in Z(G)$ and $u \in G_{\mathrm{uni}}$ implies the knowledge of $\chi^{\mathbf{L}}_{(s_{15}),1}(zu)$ with $u \in \mathbf{L}^F_{\mathrm{uni}}$, for any F-stable Levi subgroup $\mathbf{L}$. Then, by Lemma 18.5 we get the values more generally for $z \in Z(\mathbf{L}^F)$. In practice, we have $\chi^{\mathbf{L}}_{(s_{15}),1}(zu) = \varphi(z)\chi^{\mathbf{L}}_{(s_{15}),1}(u)$ if $\chi^{\mathbf{L}}_{(s_{15}),1} \in \mathrm{CF}(\mathbf{L}^F)^\varphi$ for a certain $\varphi \in \mathrm{Irr}(Z(\mathbf{L}^F))$. In particular, we know the values $\chi(su)$ for all semisimple $s \in G$ such that $\mathbf{L} = C_{\mathbf{G}}(s)$ is a Levi subgroup, and $u \in C_{\mathbf{G}}(s)^F_{\mathrm{uni}}$. Indeed, in this case we have ${}^*R^{\mathbf{G}}_{\mathbf{L}}\chi(su) = \chi(su)$, by the character formula in Proposition 3.30 (b) and Remark 4.3.

The only problematic cases are the classes intersecting our centralizer $\mathbf{C} = C_{\mathbf{G}}(s)$ of type A_1^4 with $s = h(-1,-1,1,-1)$. There are 24 unknowns on these classes. By Figure 5, we get equations for these unknowns from ${}^*R^{\mathbf{G}}_{\mathbf{L}}\chi$ when $\mathbf{L} = \mathbf{M}_3, \mathbf{M}_4, \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_4$ (and their images under triality). In total this provides 12 linearly independent equations. We can further reduce the number of unknowns by considering the triality symmetry. It is an easy check that $\tau(R_{15}) = R_{15}$. Equivalently, $\mathscr{E}(G, s_{15})$ is triality-invariant and its regular characters are all constituents of R_{15}. Then, since triality fixes the identity and permutes the elements $g_{(k_a,k_b)}$ for $k_a, k_b = 0, 1$ (not both 0), we have $\tau(\chi) = \chi$ while the other characters $\chi^{g_{(k_a,k_b)}}$ are permuted. This fact fixes 8 other unknowns.

In the end, we know $\chi(g)$ for all $g \in G$ up to some cases where g has semisimple part $h(-1,-1,1,-1)$. In total there are 4 unknowns left in the system.

18.2 A semisimple character

In this section. we treat the semisimple characters of the Lusztig series $\mathscr{E}(G, s_{15})$. By inspection of the degree polynomials, we see that $D_{\mathbf{G}}R_{15} = R_{30}$, see Remark 5.42.

Since R_{15} is the sum of regular characters of G, R_{30} is the sum of semisimple characters (by definition). We write $\varrho = D_{\mathbf{G}}\chi$, then $R_{30} = \varrho + \varrho^{g_{(0,1)}} + \varrho^{g_{(1,0)}} + \varrho^{g_{(1,1)}}$.

The main difference with the last section is that we use Remark 5.39 instead of Theorem 5.37 (a) (in G but also in the Levi subgroups). This tells us that

$$\varrho(u_z) = \varepsilon_{C_{\mathbf{G}^*}(s_{15})} \sum_{z' \in Z(\mathbf{G}^F)} \sigma_{zz'^{-1}} \langle \chi, \Gamma^{\mathbf{G}}_{z'} \rangle = \sigma_z = \frac{\Gamma^{\mathbf{G}}_z(u_1)}{|Z(\mathbf{G})^F|}$$

for u_z a regular unipotent element in the class parametrized by $z \in H^1(F, Z(\mathbf{G}))$.

Because ϱ is not regular, we get the equations $\langle \varrho, \Gamma^{\mathbf{G}}_z \rangle = 0$ for all $z \in H^1(F, Z(\mathbf{G}))$. Like before, this fixes 3 more unknowns on the unipotent classes.

We have seen in the previous section that $R_{15} \in \mathrm{CF}(G)^{1_Z}$. Then, by Proposition 5.54 also $\varrho \in \mathrm{CF}(G)^{1_Z}$. This fixes $\varrho(zu) = \varrho(u)$ for all $z \in Z(G)$ and z unipotent.

Next, we consider the Levi subgroups.

Notation 18.6. Because we are considering Lusztig restriction of characters only on elements of the form zu (z central and u unipotent), we will encounter many characters that look like the trivial character (when restricted to those elements). However, they cannot be it for geometric reasons (the trivial character belongs to the Lusztig series associated with 1). We will denote these characters $1'_L$ to make it apparent that they have values in common with 1_L.

Lemma 18.7. *We have the following relations:*

(a) ${}^*R^{\mathbf{G}}_{\mathbf{L}_1}\varrho = 1'_{L_1} + \chi^{\mathbf{L}_1}_{(s_{15}),1} \in \mathrm{CF}(L_1)^1 \oplus \mathrm{CF}(L_1)^\varphi$ *with* $\varphi : Z(L_1) \to \mathbb{C}, h_{Z_{L_1}}(k_a, k_b, k_c) \mapsto (-1)^{k_b}$,

(b) ${}^*R^{\mathbf{G}}_{\mathbf{L}_2}\varrho = \chi^{\mathbf{L}_2}_{(s_{15}),z_{\mathbf{L}_2}} \in \mathrm{CF}(L_2)^\varphi$ *with* $\varphi : Z(L_2) \to \mathbb{C}, h_{Z_{L_2}}(k_a, k_b) \mapsto (-1)^{k_a}$,

(c) ${}^*R^{\mathbf{G}}_{\mathbf{L}_3}\varrho = 1'_{L_3} \in \mathrm{CF}(L_3)^1$,

(d) ${}^*R^{\mathbf{G}}_{\mathbf{L}_4}\varrho = 1'_{L_4} + \chi^{\mathbf{L}_4}_{(s_{15}),1} \in \mathrm{CF}(L_4)^1 \oplus \mathrm{CF}(L_4)^\varphi$ *with* $\varphi : Z(L_4) \to \mathbb{C}, h_{Z_{L_4}}(k_a, k_b, k_c) \mapsto (-1)^{k_b + k_c}$,

(e) ${}^*R^{\mathbf{G}}_{\mathbf{L}_5}\varrho = 0$

(f) $^*R^{\mathbf{G}}_{\mathbf{M}_1}\varrho = \varrho^{\mathbf{M}_1}_{(s_{15}),1} \in \mathrm{CF}(M_1)^1$ *and it is not a regular character of* M_1,

(g) $^*R^{\mathbf{G}}_{\mathbf{M}_2}\varrho = \varrho^{\mathbf{M}_2}_{(s_{15}),z_{\mathbf{M}_2}} \in \mathrm{CF}(M_2)^1$ *and it is not a regular character of* M_2,

(h) $^*R^{\mathbf{G}}_{\mathbf{M}_3}\varrho = 1'_{M_3} + \chi^{\mathbf{M}_3}_{(s_{15}),1} \in \mathrm{CF}(M_3)^1 \oplus \mathrm{CF}(M_3)^{\varphi}$ *with*
$\varphi : Z(M_3) \to \mathbb{C}, h_{Z_{M_3}}(k_a, k_b, k_c) \mapsto (-1)^{k_a+k_b+k_c}$,

(i) $^*R^{\mathbf{G}}_{\mathbf{M}_4}\varrho = 1'_{M_4} - \chi^{\mathbf{M}_4}_{(s_{15}),1} \in \mathrm{CF}(M_4)^1 \oplus \mathrm{CF}(M_4)^{\varphi}$ *with*
$\varphi : Z(M_4) \to \mathbb{C}, h_{Z_{M_4}}(k_a, k_b, k_c) \mapsto (-1)^{k_a+k_b+k_c}$,

Also, the same relations hold when applying triality to the Levi subgroups (as expected).

Proof. The statements follow by Lemma 18.5, the fact that $D_{\mathbf{G}}$ and $^*R^{\mathbf{G}}_{\mathbf{L}}$ commute and by evaluating scalar products of the Lusztig restrictions $^*R^{\mathbf{G}}_{\mathbf{L}}R_{30}$ with the (modified) Gel'fand–Graev characters of $\mathbf{L}$. □

Like in the previous section, we use points (c), (e) and (f) (or (g)) of this lemma to fix the unknown values on the unipotent classes. In practice we have $^*R^{\mathbf{G}}_{\mathbf{L}_3}\varrho(u_z) = 1$, $^*R^{\mathbf{G}}_{\mathbf{L}_5}\varrho(u_z) = 0$, $^*R^{\mathbf{G}}_{\mathbf{M}_1}\varrho(u_z) = \frac{\Gamma^{\mathbf{M}_1}_z(u_1^{M_1})}{|Z(M_1)|}$ and $^*R^{\mathbf{G}}_{\mathbf{M}_2}\varrho(u_z) = \frac{\Gamma^{\mathbf{M}_2}_z(u_1^{M_2})}{|Z(M_2)|}$ for u_z a regular unipotent element of the Levi $\mathbf{L}^F$ in the class parametrized by $z \in H^1(F, Z(\mathbf{L}))$, where $\mathbf{L}$ is respectively $\mathbf{L}_3$, $\mathbf{L}_5$, $\mathbf{M}_1$ or $\mathbf{M}_2$.

Again, thanks to the character values on the unipotent classes, we can use the lemma to find $\varrho(su)$ for $s \in G$ semisimple such that $\mathbf{L} = C_{\mathbf{G}}(s)$ is a Levi subgroup, and $u \in C_{\mathbf{G}}(s)^F_{\mathrm{uni}}$. And, by exactly the same arguments as at the end of the previous section, we can fix almost all the values $\varrho(su)$ for $s = h(-1,-1,1,-1)$ (and $u \in C_{\mathbf{G}}(s)^F_{\mathrm{uni}}$).

In the end, like for χ, we know $\varrho(g)$ for all $g \in G$ up to some cases where g has semisimple part $h(-1,-1,1,-1)$. In total there are 4 unknowns left in the system.

18.3 A third possibility

It is clear that for characters that are both regular and semisimple we can apply a procedure which is a mixture of the last two sections, to find all the character values. And the computations are made even easier by the fact that the Lusztig restriction of these characters may have constituents which are both regular and semisimple.

Therefore, we consider the decomposition of a uniform almost character whose constituents are neither regular nor semisimple. However, we will see that the method to use is similar to the one of the previous cases.

One more time, we work in $\mathbb{Z}\mathscr{E}(G, s_{15})$. We have three uniform almost characters R_{18}, R_{20} and R_{21} that are permuted by triality (by explicit inspection of their values). We choose to treat R_{18}. This character has norm equal to 2. Then, before starting the computations, we have to determine for which $z \in H^1(F, Z(\mathbf{G}))$ the element $g_z \in \mathbf{G}$ fixes the constituents of R_{18}. Assume that $R_{18} = \theta + \theta^{g_z}$ for θ irreducible. Then, if $z \in \ker(\mathfrak{h}_{\mathbf{L}})$, for a certain Levi subgroup $\mathbf{L}$, $^*R^{\mathbf{G}}_{\mathbf{L}}R_{18}$ is twice an irreducible character (recall that $\mathfrak{h}_{\mathbf{L}} : H^1(F, Z(\mathbf{G})) \to H^1(F, Z(\mathbf{L}))$ is the canonical surjection). Thus, we check for which $\mathbf{L} = \mathbf{M}_1, \tau(\mathbf{M}_1), \tau^2(\mathbf{M}_1)$ not all the values of $^*R^{\mathbf{G}}_{\mathbf{L}}R_{18}$ are multiples of 2. In practice, we look for character values equal 1. Since $\frac{1}{2}$ is not an algebraic integer a restriction that takes value 1 must be the sum of two distinct irreducible characters.

In the present case we have that $g_{(0,1)}$ fixes the constituents of R_{18}. Then we have the decomposition $R_{18} = \theta + \theta^{g_{(1,0)}} = \theta + \theta^{g_{(1,1)}}$. Moreover, we can use the relation $\theta^{g_{(0,1)}} = \theta$ to greatly reduce the number of unknowns (to 25).

We proceed similarly to the last two sections to fix the unknown values of θ.

Since θ is not a regular character we can impose $\langle \theta, \Gamma^{\mathbf{G}}_z \rangle = 0$ for all $z \in H^1(F, Z(\mathbf{G}))$ and because it is not semisimple we have $\theta(u) = 0$ for all regular unipotent elements $u \in G$.

The main "difficulty" comes from the fact that we know nothing, a priori, about the Lusztig restriction of θ (namely, its constituents might be regular, semisimple or neither). However, by taking scalar products with the modified Gel'fand–Graev characters of the Levi subgroups and by some ad hoc arguments, we can find the analogue of Lemmas 18.5 and 18.7.

Lemma 18.8. *We have the following relations:*

(a) $^*R^{\mathbf{G}}_{\mathbf{L}_1}\theta = \mathrm{St}'_{\mathbf{L}_1} + 1'_{L_1} + 2\chi^{\mathbf{L}_1}_{(s_{15}),1} \in \mathrm{CF}(L_1)^1 \oplus \mathrm{CF}(L_1)^{\varphi}$ *with* $\varphi : Z(L_1) \to \mathbb{C}, h_{Z_{L_1}}(k_a, k_b, k_c) \mapsto (-1)^{k_b}$*,*

(b) $^*R^{\mathbf{G}}_{\mathbf{L}_2}\theta = \sum_{z \in H^1(F, Z(\mathbf{L}_1))} \chi^{\mathbf{L}_2}_{(s_{15}),z} \in \mathrm{CF}(L_2)^{\varphi}$ *with* $\varphi : Z(L_2) \to \mathbb{C}, h_{Z_{L_2}}(k_a, k_b) \mapsto (-1)^{k_a}$*,*

(c) $^*R^{\mathbf{G}}_{\mathbf{L}_3}\theta = \mathrm{St}'_{\mathbf{L}_1} - 1'_{L_3} \in \mathrm{CF}(L_3)^1$*,*

(d) $^*R^{\mathbf{G}}_{\mathbf{L}_4}\theta = \mathrm{St}'_{\mathbf{L}_1} + 1'_{L_4} + 2\chi^{\mathbf{L}_4}_{(s_{15}),z} \in \mathrm{CF}(L_4)^1 \oplus \mathrm{CF}(L_4)^{\varphi}$ *where* $1 \neq z \in H^1(F, Z(\mathbf{L}_4))$ *and with* $\varphi : Z(L_4) \to \mathbb{C}, h_{Z_{L_4}}(k_a, k_b, k_c) \mapsto (-1)^{k_b + k_c}$*,*

(e) $^*R^{\mathbf{G}}_{\mathbf{L}_5}\theta = 0$

(f) $^*R^{\mathbf{G}}_{\mathbf{M}_1}\theta = \chi^{\mathbf{M}_1}_{(s_{15}),1} + \varrho^{\mathbf{M}_1}_{(s_{15}),1} \in \mathrm{CF}(M_1)^1$*,*

(g) $^*R^{\mathbf{G}}_{\mathbf{M}_2}\theta \in \mathrm{CF}(M_2)^1$ *,*

(h) $^*R^{\mathbf{G}}_{\mathbf{M}_3}\theta - \chi^{\mathbf{M}_3}_{(s_{15}),1} - \chi^{\mathbf{M}_3}_{(s_{15}),z} \in \mathrm{CF}(M_3)^1$ *with* $z = h(1, -1, 1, 1) \in H^1(F, Z(\mathbf{M}_3))$*,*

(i) $^*R^{\mathbf{G}}_{\mathbf{M}_4}\theta + \chi^{\mathbf{M}_4}_{(s_{15}),1} + \chi^{\mathbf{M}_4}_{(s_{15}),z} \in \mathrm{CF}(M_4)^1$ *with* $z = h(1, -1, 1, 1) \in H^1(F, Z(\mathbf{M}_4))$*.*

Proof. We first prove points (c), (e) and (f), then use them to prove the rest.

(c) By examining its values, we see that $^*R^{\mathbf{G}}_{\mathbf{L}_3}R_{18} \in \mathrm{CF}(\mathbf{G}^F)^1$. Next, we explicitly compute $\langle \Gamma^{\mathbf{L}_3}_{0,0,0;k}, {}^*R^{\mathbf{G}}_{\mathbf{L}_3}R_{18} \rangle = 2$ for $k = 0, 1$. We have seen in Lemma 18.5 (c) that the only regular character in

$$\mathrm{CF}(L_3)^1 \cap \bigcup_{(t) \subseteq (s_{15})} \mathscr{E}(L_3, (t))$$

is $\mathrm{St}'_{\mathbf{L}_3}$, where the union is on the semisimple classes (t) of $\mathbf{L}_3^{*F^*}$ contained in the class (s_{15}) of $\mathbf{G}^{*F^*}$. Finally, we check easily that $^*R^{\mathbf{G}}_{\mathbf{L}_3}R_{18} - 2 \cdot \mathrm{St}'_{\mathbf{L}_3} = 2 \cdot 1'_{L_3}$.

(e) We compute $^*R^{\mathbf{G}}_{\mathbf{L}_5}R_{18} = 0$.

(f) We compute $\langle \Gamma^{\mathbf{M}_1}_{0,0,0;k}, {}^*R^{\mathbf{G}}_{\mathbf{M}_1}R_{18} \rangle = 1$ for $k = 0, 1$. Then, both $\chi^{\mathbf{M}_1}_{(s_{15}),1}$ and $\chi^{\mathbf{M}_1}_{(s_{15}),z}$ are constituents of $^*R^{\mathbf{G}}_{\mathbf{M}_1}R_{18}$, where $1 \neq z \in H^1(F, Z(\mathbf{M}_1))$. Moreover, we see on the values that $^*R^{\mathbf{G}}_{\mathbf{M}_1}R_{18} - \chi^{\mathbf{M}_1}_{(s_{15}),1} - \chi^{\mathbf{M}_1}_{(s_{15}),z} = \varrho^{\mathbf{M}_1}_{(s_{15}),1} + \varrho^{\mathbf{M}_1}_{(s_{15}),z}$. Therefore, there are four possibilities for the restriction of θ, $^*R^{\mathbf{G}}_{\mathbf{M}_1}\theta = \chi^{\mathbf{M}_1}_{(s_{15}),a} + \varrho^{\mathbf{M}_1}_{(s_{15}),b}$ for a, b being any combination of $1, z$. For the time being we still have some degree of freedom for choosing in which case we are. Ultimately, it is the semisimple part that will give us a relation to fix the values of θ on unipotent elements (remember that only the semisimple characters have non-zero value on regular unipotent classes). We choose $^*R^{\mathbf{G}}_{\mathbf{M}_1}\theta = \chi^{\mathbf{M}_1}_{(s_{15}),a} + \varrho^{\mathbf{M}_1}_{(s_{15}),1}$ ($a = 1$ or z). Before fixing a, notice that we have enough information to fix all the values of θ on unipotent elements of G. From point (c) we have the equation $^*R^{\mathbf{G}}_{\mathbf{L}_3}\theta(u) = -1$ for u regular unipotent in L_3. From point (e) we get the equation $^*R^{\mathbf{G}}_{\mathbf{L}_5}\theta(u) = 0$ for u regular unipotent in L_5. And from the discussion above we have ${}^*R^{\mathbf{G}}_{\mathbf{M}_1}\theta(u) = \sigma_1^{\mathbf{M}_1}$ for u in the regular unipotent class of M_1 parametrized by $1 \in H^1(F, Z(\mathbf{M}_1))$. With these equations, we can fix the unknowns of θ on unipotent classes. Finally, we can explicitly compute ${}^*R^{\mathbf{G}}_{\mathbf{M}_1}\theta$ on unipotent classes fixing $a = 1$.

As already mentioned in point (f), the relations that we just proved are enough to fix all the values $\theta(u)$ for $u \in G$ unipotent. This allows us to explicitly compute the scalar product between ${}^*R^{\mathbf{G}}_{\mathbf{L}}\theta$ and the Gel'fand–Graev characters of $\mathbf{L}^F$, for $\mathbf{L} = \mathbf{L}_2, \mathbf{L}_4, \mathbf{L}_5, \mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4$.

(a) We compute $\langle \Gamma^{\mathbf{L}_1}_{0,0,0;k}, {}^*R^{\mathbf{G}}_{\mathbf{L}} R_{18} \rangle = 2$ and $\langle \Gamma^{\mathbf{L}_1}_{0,\frac{q-1}{2},0;k}, {}^*R^{\mathbf{G}}_{\mathbf{L}} R_{18} \rangle = 2$. Thanks to Lemma 18.5 we know what regular characters of L_1 can appear here. Therefore, we deduce that ${}^*R^{\mathbf{G}}_{\mathbf{L}} R_{18} = 2 \cdot \mathrm{St}'_{\mathbf{L}_1} + 2 \sum_{z \in H^1(F, Z(\mathbf{L}_1))} \chi_{(s_{15}),z} + \dots$. The non-regular part is easily determined to be $2 \cdot 1'_{L_1}$. Moreover, we compute the following scalar products with the (usual) Gel'fand–Graev characters: $\langle \Gamma^{\mathbf{L}_1}_0, {}^*R^{\mathbf{G}}_{\mathbf{L}}\theta \rangle = 1$ and $\langle \Gamma^{\mathbf{L}_1}_1, {}^*R^{\mathbf{G}}_{\mathbf{L}}\theta \rangle = 3$. The result follows.

(b) We compute the scalar products with the Gel'fand–Graev characters: $\langle \Gamma^{\mathbf{L}_2}_k, {}^*R^{\mathbf{G}}_{\mathbf{L}}\theta \rangle = 1$. From Lemma 18.5, we know the only possible regular characters that can be constituents of ${}^*R^{\mathbf{G}}_{\mathbf{L}}\theta$. And we check the stated result.

(d) The proof happens to be exactly like the one for point (a).

(g,h,i) Although it might be possible to give more precise information on the Lusztig restriction of θ to $\mathbf{L} = \mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4$, it is enough for us to know the decomposition of ${}^*R^{\mathbf{G}}_{\mathbf{L}}\theta$ in $\bigoplus_{\varphi \in \mathrm{Irr}(Z(\mathbf{L}^F))} \mathrm{CF}(\mathbf{L}^F)^{\varphi}$. We see directly that ${}^*R^{\mathbf{G}}_{\mathbf{M}_2} R_{18} \in \mathrm{CF}(M_2)^1$; then by Lemma 5.57 all its constituents have the same property. For $\mathbf{L} = \mathbf{M}_3, \mathbf{M}_4$ the restriction ${}^*R^{\mathbf{G}}_{\mathbf{L}} R_{18}$ has constituents not in $\mathrm{CF}(\mathbf{L}^F)^1$. We identify them thanks to scalar products with the modified Gel'fand–Graev characters of $\mathbf{L}^F$ and Lemma 18.5. Then, we see, by subtracting these constituents from ${}^*R^{\mathbf{G}}_{\mathbf{L}} R_{18}$, that we obtain a class function in $\mathrm{CF}(\mathbf{L}^F)^1$. And, by Lemma 5.57 each constituent also lies in $\mathrm{CF}(\mathbf{L}^F)^1$.

□

As already stated in the proof, we can use points (c), (e) and (f) to fix all the remaining unknown character values of θ on unipotent classes of G. And then, exactly like in the two previous sections we use all of Lemma 18.8 to find the values $\theta(su)$ for $s \in G$ semisimple such that $\mathbf{L} = C_{\mathbf{G}}(s)$ is a Levi subgroup, and $u \in C_{\mathbf{G}}(s)^F_{\mathrm{uni}}$. Again, when $C_{\mathbf{G}}(s)$ is not a Levi subgroup, we can find all but 4 unknowns, exactly like in the last two sections.

Notice that, as already stated, the other non-irreducible uniform almost characters in the span of $\mathscr{E}(G, s_{15})$ are obtained by applying triality to R_{18}. Then by applying τ to θ (or by reproducing the same computations as above), we complete the determination of the irreducible characters in the Lusztig series $\mathscr{E}(G, s_{15})$, up to four unknowns per character that we decomposed.

18.4 Closing remarks

What we have seen in Section 18 can be summarized as follow.

Assuming that:

- we know the values of the irreducible characters of G on the unipotent classes,
- we know the uniform almost characters of G, and
- we know the 2-parameter Green function $Q^{\mathbf{G}}_{\mathbf{L}}$ for all F-stable Levi subgroup $\mathbf{L}$, then

we can compute the character values on all the elements of the form su where s is semisimple such that $C_{\mathbf{G}}(s)$ is a Levi subgroup of $\mathbf{G}$ and $u \in C_{\mathbf{L}}(s)_{\text{uni}}$, including the case $s \in Z(G)$.

When $C_{\mathbf{G}}(s)$ is not a Levi subgroup then we still get informations on the character values for elements su if moreover

- we know $Q_{\mathbf{L}}^{C_{\mathbf{G}}(s)}$ for all the F-stable Levi subgroups of $\mathbf{G}$ contained in $C_{\mathbf{G}}(s)$, and
- we find other relations by ad hoc methods.

To compute the character values on unipotent classes, we need:

- the values of the uniform almost characters of G on unipotent classes,
- the values of the Gel'fand–Graev characters of $\mathbf{G}^F$ and of its Levi subgroups,
- the 2-parameter Green functions $Q_{\mathbf{L}}^{\mathbf{G}}$ for all Levi subgroup $\mathbf{L}$.

We have seen in the rest of the work that to compute the Gel'fand–Graev characters and the 2-parameter Green functions the main tool needed is the fusion of unipotent classes from the maximal unipotent subgroup U_0 to G. This can, in principle, always be computed by the method described in Section 6.3. Notice that, at the moment of the writing, it doesn't seem to be possible to implement the algorithm of Section 6.3 in a computer program in general for the treatment of finite groups of Lie type, without specifying the characteristic of $\mathbf{G}$. At our knowledge there is no computer program that solves equations over a finite field $\mathbb{F}_q$ where q is specified only up to congruence.

Summarizing, the computation of the generic character table can be achieved when it is possible to find the values on unipotent elements and on elements whose semisimple parts have centralizers which are not Levi subgroup. We are convinced that the first part is possible for $\text{Spin}_8^+(q)$. In fact, every character has at most the number of unknowns that appeared in the case of χ or ϱ. For other characters in a $\tilde{\mathbf{G}}^F$-orbit of order 4 this number is equal. In this case these characters are either regular or semisimple. While for characters with only one distinct $\tilde{\mathbf{G}}^F$-conjugate this number is lower. This is due to the fact that there exist elements of $\tilde{\mathbf{G}}^F$ fixing the character, thus removing unknowns, like in the previous section. The other reason is that for the cases that we treated, we never used more than a third of the equations available. In practice, on unipotent elements, each character has at most 15 unknowns, and we get at least one equation involving them from each of 23 Levi subgroups (counting also the images under triality of $\mathbf{M}_1, \mathbf{M}_2, \mathbf{L}_1, ..., \mathbf{L}_5$). Moreover, notice that even if we treat distinct characters we always encounter the same equations between the unknowns, up to the character values of the uniform almost character that is being decomposed. This is obvious since the Lusztig restriction functor ${}^*R_{\mathbf{L}}^{\mathbf{G}}$ only depends on the Levi subgroup $\mathbf{L}$ and not on the character being restricted. In other words, the left-hand side of any equation of Lemma 18.5 is always the same independently of the character that we are considering. The only dependency on the uniform almost characters comes from the right-hand side of the equation, and we have shown how this can be determined. Furthermore, we know that the equations that we obtain (like in Lemma 18.5) are all solvable since their solution yields (portions of) the character table, which exists.

On the other side, the number of unknowns left in the system is so small that we believe that with ad-hoc methods it is possible to complete the character table. For example, we did not use the fact that irreducible characters have norm equal one, that the scalar product between different irreducible characters is zero, or the second orthogonality relation.

Appendices

A Tables

A.1 Unipotent classes representatives

Table 48: Unipotent classes of $\mathrm{Spin}_8^+(q)$. The table shows the Jordan normal form, the numbering of the class, a representative in Steinberg presentation ($\langle \mu \rangle = \mathbb{F}_q^\times$) and the image under the triality automorphism. On the classes with Jordan form 53 the triality automorphism acts differently depending on the congruence of q. The action for $q \equiv 1\,(4)$ is given first.

1^8	u_1	1	u_1
$2^2 1^4$	u_2	$u_1(1)$	u_2
2^4+	u_3	$u_1(1)u_2(1)$	u_7
	u_4	$u_1(\mu)u_2(1)$	u_8
2^4-	u_5	$u_1(1)u_4(1)$	u_3
	u_6	$u_1(\mu)u_4(1)$	u_4
31^5	u_7	$u_2(1)u_4(1)$	u_5
	u_8	$u_2(\mu)u_4(1)$	u_6
$32^2 1$	u_9	$u_1(1)u_2(1)u_4(1)$	u_9
	u_{10}	$u_1(1)u_2(\mu)u_4(1)$	u_{12}
	u_{11}	$u_1(\mu)u_2(1)u_4(1)$	u_{10}
	u_{12}	$u_1(\mu)u_2(\mu)u_4(1)$	u_{11}
$3^2 1^2$	u_{13}	$u_1(1)u_3(1)$	u_{13}
$3^2 1^2$	u_{14}	$u_1(\mu)u_2(1)u_4(1)u_{12}(1)$	u_{14}
4^2+	u_{15}	$u_1(1)u_2(1)u_3(1)$	u_{19}
	u_{16}	$u_1(\mu)u_2(1)u_3(1)$	u_{20}
4^2-	u_{17}	$u_1(1)u_3(1)u_4(1)$	u_{15}
	u_{18}	$u_1(\mu)u_3(1)u_4(1)$	u_{16}
51^3	u_{19}	$u_2(1)u_3(1)u_4(1)$	u_{17}
	u_{20}	$u_2(\mu)u_3(1)u_4(1)$	u_{18}
53	u_{21}	$u_1(1)u_2(1)u_3(1)u_{10}(1)$	u_{21}/u_{22}
	u_{22}	$u_1(1)u_2(1)u_3(\mu)u_{10}(1)$	u_{23}/u_{24}
	u_{23}	$u_1(\mu)u_2(1)u_3(1)u_{10}(1)$	u_{24}/u_{23}
	u_{24}	$u_1(\mu)u_2(1)u_3(\mu)u_{10}(1)$	u_{22}/u_{21}
71	u_{25}	$u_1(1)u_2(1)u_3(1)u_4(1)$	u_{25}
	u_{26}	$u_1(1)u_2(\mu)u_3(1)u_4(1)$	u_{28}
	u_{27}	$u_1(\mu)u_2(1)u_3(1)u_4(1)$	u_{26}
	u_{28}	$u_1(\mu)u_2(\mu)u_3(1)u_4(1)$	u_{27}

Table 49: Representatives of the unipotent classes of the Levi subgroups $A_1(q)^3(q-1)$ and $A_1(q)^3(q+1)$ of $\mathrm{Spin}_8^+(q)$ with the action of triality, where $\langle\mu\rangle = \mathbb{F}_q^\times$.

v_1	1	v_1
v_2	$u_1(1)$	v_4
v_3	$u_2(1)$	v_2
v_4	$u_4(1)$	v_3
v_5	$u_1(1)u_2(1)$	v_9
v_6	$u_1(\mu)u_2(1)$	v_{10}
v_7	$u_1(1)u_4(1)$	v_5
v_8	$u_1(\mu)u_4(1)$	v_6
v_9	$u_2(1)u_4(1)$	v_7
v_{10}	$u_2(\mu)u_4(1)$	v_8
v_{11}	$u_1(1)u_2(1)u_4(1)$	v_{11}
v_{12}	$u_1(1)u_2(\mu)u_4(1)$	v_{14}
v_{13}	$u_1(\mu)u_2(1)u_4(1)$	v_{12}
v_{14}	$u_1(\mu)u_2(\mu)u_4(1)$	v_{13}

A.2 2-parameter Green functions

Table 50: $\tilde{Q}_{\mathbf{L}}^{\mathbf{G}}(u,v)$ for the split Levi subgroup $A_3(q)(q-1)$ of $\mathrm{Spin}_8^+(q)$.

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}
v_1	$\Phi_2^2\Phi_4\Phi_6$	Φ_2	.	.	.	.	1	1	.	.	.	.
v_2	.	$q^2\Phi_2^2$	$\Phi_2\Phi_4$	$\Phi_2\Phi_4$	$\Phi_2\Phi_4$	$\Phi_2\Phi_4$	.	.	1	1	1	1
v_3	.	.	.	.	.	.	$q\Phi_2\Phi_4$	.	$q\Phi_2$	.	$q\Phi_2$	.
v_4	.	.	.	.	.	.	.	$q\Phi_2\Phi_4$	.	$q\Phi_2$	.	$q\Phi_2$
v_5	.	.	.	.	.	.	.	.	.	.	.	.
v_6	.	.	.	.	.	.	.	.	.	.	.	.
v_7	.	.	.	.	.	.	.	.	.	.	.	.

	u_{13}	u_{14}	u_{15}	u_{16}	u_{17}	u_{18}	u_{19}	u_{20}	u_{21}	u_{22}	u_{23}	u_{24}	u_{25}	u_{26}	u_{27}	u_{28}
v_1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
v_2	2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
v_3	$\frac{\Phi_1}{2}$	$\frac{\Phi_2}{2}$	.	.	.	.	1	.	.	.	.	.	.	.	.	.
v_4	$\frac{\Phi_1}{2}$	$\frac{\Phi_2}{2}$	.	.	.	.	.	1	.	.	.	.	.	.	.	.
v_5	$2q^2$	.	Φ_2	Φ_2	Φ_2	Φ_2	.	.	1	1	1	1	.	.	.	.
v_6	.	.	.	.	.	.	$q\Phi_2$	.	$\frac{1+\varepsilon_q}{2}q$	$\frac{1-\varepsilon_q}{2}q$	$\frac{1+\varepsilon_q}{2}q$	$\frac{1-\varepsilon_q}{2}q$	1	.	1	.
v_7	.	.	.	.	.	.	.	$q\Phi_2$	$\frac{1-\varepsilon_q}{2}q$	$\frac{1+\varepsilon_q}{2}q$	$\frac{1-\varepsilon_q}{2}q$	$\frac{1+\varepsilon_q}{2}q$	.	1	.	1

$$\varepsilon_q = (-1)^{\frac{q-1}{2}}$$

Table 51: $\tilde{Q}_{\mathbf{L}}^{\mathbf{G}}(u,v)$ for the twisted Levi subgroup ${}^2A_3(q)(q+1)$ of $\mathrm{Spin}_8^+(q)$.

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}
v_1	$\Phi_1^2\Phi_3\Phi_4$	$-\Phi_1$	.	.	.	.	1	1	.	.	.	.
v_2	.	$q^2\Phi_1^2$	$-\Phi_1\Phi_4$	$-\Phi_1\Phi_4$	$-\Phi_1\Phi_4$	$-\Phi_1\Phi_4$	.	.	1	1	1	1
v_3	.	.	.	.	.	.	.	$q\Phi_1\Phi_4$	.	$q\Phi_1$	.	$q\Phi_1$
v_4	.	.	.	.	.	.	$q\Phi_1\Phi_4$	.	$q\Phi_1$	.	$q\Phi_1$	.
v_5	.	.	.	.	.	.	.	.	.	.	.	.
v_6	.	.	.	.	.	.	.	.	.	.	.	.
v_7	.	.	.	.	.	.	.	.	.	.	.	.

	u_{13}	u_{14}	u_{15}	u_{16}	u_{17}	u_{18}	u_{19}	u_{20}
v_1	.	.	.	.	.	.	.	.
v_2	.	2	.	.	.	.	.	.
v_3	$-\frac{\Phi_1}{2}$	$-\frac{\Phi_2}{2}$	.	.	.	.	1	.
v_4	$-\frac{\Phi_1}{2}$	$-\frac{\Phi_2}{2}$	.	.	.	.	.	1
v_5	.	$2q^2$	$-\Phi_1$	$-\Phi_1$	$-\Phi_1$	$-\Phi_1$	.	.
v_6	.	.	.	.	.	.	.	$q\Phi_1$
v_7	.	.	.	.	.	.	$q\Phi_1$	.

	u_{21}	u_{22}	u_{23}	u_{24}	u_{25}	u_{26}	u_{27}	u_{28}
v_1	.	.	.	.	.	.	.	.
v_2	.	.	.	.	.	.	.	.
v_3	.	.	.	.	.	.	.	.
v_4	.	.	.	.	.	.	.	.
v_5	1	1	1	1	.	.	.	.
v_6	$\frac{\varepsilon_q-1}{2}q$	$-\frac{\varepsilon_q+1}{2}q$	$\frac{\varepsilon_q-1}{2}q$	$-\frac{\varepsilon_q+1}{2}q$	1	.	1	.
v_7	$-\frac{\varepsilon_q+1}{2}q$	$\frac{\varepsilon_q-1}{2}q$	$-\frac{\varepsilon_q+1}{2}q$	$\frac{\varepsilon_q-1}{2}q$	.	1	.	1

$$\varepsilon_q = (-1)^{\frac{q-1}{2}}$$

Table 52: $\tilde{Q}_{\mathbf{L}}^{\mathbf{G}}(u,v)$ for the split Levi subgroup $A_1(q)^3(q-1)$ of $\mathrm{Spin}_8^+(q)$.

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}
v_1	$\Phi_2\Phi_3\Phi_4^2\Phi_6$	$*$	$\Phi_2\Phi_4$	$\Phi_2\Phi_4$	$\Phi_2\Phi_4$	$\Phi_2\Phi_4$	$\Phi_2\Phi_4$	$\Phi_2\Phi_4$	Φ_2	Φ_2	Φ_2	Φ_2
v_2	.	$q^4\Phi_2$	.	.	.	.	$q^2\Phi_4$	$q^2\Phi_4$	.	.	.	.
v_3	.	$q^4\Phi_2$	.	.	$q^2\Phi_4$	$q^2\Phi_4$	.	.	.	.	.	.
v_4	.	$q^4\Phi_2$	$q^2\Phi_4$	$q^2\Phi_4$	.	.	.	.	.	.	.	.
v_5	.	.	$q^2\Phi_2\Phi_4$	.	.	.	.	.	q^2	.	.	q^2
v_6	.	.	.	$q^2\Phi_2\Phi_4$	.	.	.	.	.	q^2	q^2	.
v_7	.	.	.	.	$q^2\Phi_2\Phi_4$	.	.	.	q^2	q^2	.	.
v_8	.	.	.	.	.	$q^2\Phi_2\Phi_4$	.	.	.	.	q^2	q^2
v_9	.	.	.	.	.	.	$q^2\Phi_2\Phi_4$	.	q^2	.	q^2	.
v_{10}	.	.	.	.	.	.	.	$q^2\Phi_2\Phi_4$	.	q^2	.	q^2
v_{11}	.	.	.	.	.	.	.	.	$q^3\Phi_2$	.	.	.
v_{12}	.	.	.	.	.	.	.	.	.	$q^3\Phi_2$	.	.
v_{13}	.	.	.	.	.	.	.	.	.	.	$q^3\Phi_2$	.
v_{14}	.	.	.	.	.	.	.	.	.	.	.	$q^3\Phi_2$

	u_{13}	u_{14}	u_{15}	u_{16}	u_{17}	u_{18}	u_{19}	u_{20}	u_{21}	u_{22}	u_{23}	u_{24}	u_{25}	u_{26}	u_{27}	u_{28}
v_1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
v_2	$2q$	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.
v_3	$2q$	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.
v_4	$2q$	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.
v_5	$q\Phi_1$	.	Φ_2	.	.	.	.	.	1	1	.	.	.	.	.	.
v_6	$q\Phi_1$	.	.	Φ_2	.	.	.	.	.	.	1	1	.	.	.	.
v_7	$q\Phi_1$	.	.	.	Φ_2	.	.	.	1	.	.	1	.	.	.	.
v_8	$q\Phi_1$	.	.	.	.	Φ_2	.	.	.	1	1	.	.	.	.	.
v_9	$q\Phi_1$	.	.	.	.	.	Φ_2	.	$\frac{1+\varepsilon_q}{2}$	$\frac{1-\varepsilon_q}{2}$	$\frac{1+\varepsilon_q}{2}$	$\frac{1-\varepsilon_q}{2}$	.	.	.	.
v_{10}	$q\Phi_1$	.	.	.	.	.	.	Φ_2	$\frac{1-\varepsilon_q}{2}$	$\frac{1+\varepsilon_q}{2}$	$\frac{1-\varepsilon_q}{2}$	$\frac{1+\varepsilon_q}{2}$	.	.	.	.
v_{11}	$\frac{q\Phi_1^2}{4}$	$\frac{q\Phi_2^2}{4}$	$\frac{\Phi_1\Phi_2}{2}$	.	$\frac{\Phi_1\Phi_2}{2}$	.	$\frac{\Phi_1\Phi_2}{2}$	.	$\frac{q-4-\varepsilon_q}{4}$	$\frac{q-2+\varepsilon_q}{4}$	$\frac{q-\varepsilon_q}{4}$	$\frac{q-2+\varepsilon_q}{4}$	1	.	.	.
v_{12}	$\frac{q\Phi_1^2}{4}$	$\frac{q\Phi_2^2}{4}$	.	$\frac{\Phi_1\Phi_2}{2}$	$\frac{\Phi_1\Phi_2}{2}$	.	.	$\frac{\Phi_1\Phi_1}{2}$	$\frac{q-2+\varepsilon_q}{4}$	$\frac{q-\varepsilon_q}{4}$	$\frac{q-2+\varepsilon_q}{4}$	$\frac{q-4-\varepsilon_q}{4}$	.	1	.	.
v_{13}	$\frac{q\Phi_1^2}{4}$	$\frac{q\Phi_2^2}{4}$	.	$\frac{\Phi_1\Phi_2}{2}$	.	$\frac{\Phi_1\Phi_2}{2}$	$\frac{\Phi_1\Phi_2}{2}$	.	$\frac{q-\varepsilon_q}{4}$	$\frac{q-2+\varepsilon_q}{4}$	$\frac{q-4-\varepsilon_q}{4}$	$\frac{q-2+\varepsilon_q}{4}$	.	.	1	.
v_{14}	$\frac{q\Phi_1^2}{4}$	$\frac{q\Phi_2^2}{4}$	$\frac{\Phi_1\Phi_2}{2}$	.	.	$\frac{\Phi_1\Phi_2}{2}$	.	$\frac{\Phi_1\Phi_2}{2}$	$\frac{q-2+\varepsilon_q}{4}$	$\frac{q-4-\varepsilon_q}{4}$	$\frac{q-2+\varepsilon_q}{4}$	$\frac{q-\varepsilon_q}{4}$	.	.	.	1

$$* = q^4 + 3q^3 + 3q^2 + q + 1$$

$$\varepsilon_q = (-1)^{\frac{q-1}{2}}$$

Table 53: $\tilde{Q}_{\mathbf{L}}^{\mathbf{G}}(u,v)$ for the twisted Levi subgroup $A_1(q)^3(q+1)$ of $\mathrm{Spin}_8^+(q)$.

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
v_1	$-\Phi_1\Phi_3\Phi_4^2\Phi_6$	$*$	$-\Phi_1\Phi_4$	$-\Phi_1\Phi_4$	$-\Phi_1\Phi_4$	$-\Phi_1\Phi_4$	$-\Phi_1\Phi_4$	$-\Phi_1\Phi_4$
v_2	.	$-q^4\Phi_1$	.	.	.	.	$q^2\Phi_4$	$q^2\Phi_4$
v_3	.	$-q^4\Phi_1$	.	.	$q^2\Phi_4$	$q^2\Phi_4$	.	.
v_4	.	$-q^4\Phi_1$	$q^2\Phi_4$	$q^2\Phi_4$	.	.	.	.
v_5	.	.	$-q^2\Phi_1\Phi_4$	.	.	.	.	.
v_6	.	.	.	$-q^2\Phi_1\Phi_4$	.	.	.	.
v_7	.	.	.	.	$-q^2\Phi_1\Phi_4$	.	.	.
v_8	.	.	.	.	.	$-q^2\Phi_1\Phi_4$	.	.
v_9	.	.	.	.	.	.	$-q^2\Phi_1\Phi_4$	.
v_{10}	.	.	.	.	.	.	.	$-q^2\Phi_1\Phi_4$
v_{11}	.	.	.	.	.	.	.	.
v_{12}	.	.	.	.	.	.	.	.
v_{13}	.	.	.	.	.	.	.	.
v_{14}	.	.	.	.	.	.	.	.

	u_9	u_{10}	u_{11}	u_{12}	u_{13}	u_{14}	u_{15}	u_{16}	u_{17}	u_{18}	u_{19}	u_{20}
v_1	$-\Phi_1$	$-\Phi_1$	$-\Phi_1$	$-\Phi_1$	1	1	.	.	.	.	.	.
v_2	.	.	.	.	.	$-2q$	.	.	.	.	1	1
v_3	.	.	.	.	.	$-2q$	.	.	1	1	.	.
v_4	.	.	.	.	.	$-2q$	1	1	.	.	.	.
v_5	q^2	.	.	q^2	.	$q\Phi_2$	.	$-\Phi_1$	.	.	.	.
v_6	.	q^2	q^2	.	.	$q\Phi_2$	$-\Phi_1$	.	.	.	.	.
v_7	q^2	q^2	.	.	.	$q\Phi_2$	.	.	.	$-\Phi_1$	.	.
v_8	.	.	q^2	q^2	.	$q\Phi_2$	.	.	$-\Phi_1$	.	.	.
v_9	q^2	.	q^2	.	.	$q\Phi_2$	.	.	.	.	.	$-\Phi_1$
v_{10}	.	q^2	.	q^2	.	$q\Phi_2$	.	.	.	.	$-\Phi_1$	.
v_{11}	$q^3\Phi_1$	.	.	.	$-\frac{q\Phi_1^2}{4}$	$-\frac{q\Phi_2^2}{4}$	.	$\frac{\Phi_1\Phi_2}{2}$	.	$\frac{\Phi_1\Phi_2}{2}$	.	$\frac{\Phi_1\Phi_2}{2}$
v_{12}	.	$q^3\Phi_1$	.	.	$-\frac{q\Phi_1^2}{4}$	$-\frac{q\Phi_2^2}{4}$	$\frac{\Phi_1\Phi_2}{2}$	.	.	$\frac{\Phi_1\Phi_2}{2}$	$\frac{\Phi_1\Phi_2}{2}$	.
v_{13}	.	.	$q^3\Phi_1$	.	$-\frac{q\Phi_1^2}{4}$	$-\frac{q\Phi_2^2}{4}$	$\frac{\Phi_1\Phi_2}{2}$	.	$\frac{\Phi_1\Phi_2}{2}$	.	.	$\frac{\Phi_1\Phi_2}{2}$
v_{14}	.	.	.	$q^3\Phi_1$	$-\frac{q\Phi_1^2}{4}$	$-\frac{q\Phi_2^2}{4}$	.	$\frac{\Phi_1\Phi_2}{2}$	$\frac{\Phi_1\Phi_2}{2}$	.	$\frac{\Phi_1\Phi_2}{2}$	.

	u_{21}	u_{22}	u_{23}	u_{24}	u_{25}	u_{26}	u_{27}	u_{28}
v_1	.	.	.	.	.	.	.	.
v_2	.	.	.	.	.	.	.	.
v_3	.	.	.	.	.	.	.	.
v_4	.	.	.	.	.	.	.	.
v_5	.	.	1	1	.	.	.	.
v_6	1	1	.	.	.	.	.	.
v_7	.	1	1	.	.	.	.	.
v_8	1	.	.	1	.	.	.	.
v_9	$\frac{1-\varepsilon_q}{2}$	$\frac{1+\varepsilon_q}{2}$	$\frac{1-\varepsilon_q}{2}$	$\frac{1+\varepsilon_q}{2}$	.	.	.	.
v_{10}	$\frac{1+\varepsilon_q}{2}$	$\frac{1-\varepsilon_q}{2}$	$\frac{1+\varepsilon_q}{2}$	$\frac{1-\varepsilon_q}{2}$	.	.	.	.
v_{11}	$-\frac{q-\varepsilon_q}{4}$	$-\frac{q+2+\varepsilon_q}{4}$	$-\frac{q+4-\varepsilon_q}{4}$	$-\frac{q+2+\varepsilon_q}{4}$	1	.	.	.
v_{12}	$-\frac{q+2+\varepsilon_q}{4}$	$-\frac{q+4-\varepsilon_q}{4}$	$-\frac{q+2+\varepsilon_q}{4}$	$-\frac{q-\varepsilon_q}{4}$	.	1	.	.
v_{13}	$-\frac{q+4-\varepsilon_q}{4}$	$-\frac{q+2+\varepsilon_q}{4}$	$-\frac{q-\varepsilon_q}{4}$	$-\frac{q+2+\varepsilon_q}{4}$	.	.	1	.
v_{14}	$-\frac{q+2+\varepsilon_q}{4}$	$-\frac{q-\varepsilon_q}{4}$	$-\frac{q+2+\varepsilon_q}{4}$	$-\frac{q+4-\varepsilon_q}{4}$	.	.	.	1

$$* = q^4 - 3q^3 + 3q^2 - q + 1$$

$$\varepsilon_q = (-1)^{\frac{q-1}{2}}$$

Table 54: $\tilde{Q}_{\mathbf{L}}^{\mathbf{G}}(u,v)$ for the split Levi subgroup $A_1(q)^2(q-1)^2$ of $\mathrm{Spin}_8^+(q)$.

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
v_1	$\Phi_2^2\Phi_3\Phi_4^2\Phi_6$	$*$	$\Phi_2^2\Phi_4$	$\Phi_2^2\Phi_4$	$\Phi_2^2\Phi_4$	$\Phi_2^2\Phi_4$	$(2q^2+2q+1)\Phi_4$	$(2q^2+2q+1)\Phi_4$
v_2	.	$q^4\Phi_2^2$	$q^2\Phi_2\Phi_4$	$q^2\Phi_2\Phi_4$	$q^2\Phi_2\Phi_4$	$q^2\Phi_2\Phi_4$	.	.
v_3	.	$q^4\Phi_2^2$	$q^2\Phi_2\Phi_4$	$q^2\Phi_2\Phi_4$	$q^2\Phi_2\Phi_4$	$q^2\Phi_2\Phi_4$	.	.
v_4	.	.	.	.	.	.	$q^2\Phi_2^2\Phi_4$	.
v_5	.	.	.	.	.	.	.	$q^2\Phi_2^2\Phi_4$

	u_9	u_{10}	u_{11}	u_{12}	u_{13}	u_{14}	u_{15}	u_{16}	u_{17}	u_{18}	u_{19}	u_{20}
v_1	Φ_2^2	Φ_2^2	Φ_2^2	Φ_2^2	$3q+1$	Φ_2	.	.	.	.	1	1
v_2	q^2	q^2	q^2	q^2	$4q^2$	.	Φ_2	Φ_2	Φ_2	Φ_2	.	.
v_3	q^2	q^2	q^2	q^2	$4q^2$	.	Φ_2	Φ_2	Φ_2	Φ_2	.	.
v_4	$q^2\Phi_2^2$	.	$q^2\Phi_2^2$	.	$q(3q+1)\frac{\Phi_1}{2}$	$q\frac{\Phi_2^2}{2}$	$\frac{\Phi_1\Phi_2}{2}$	$\frac{\Phi_1\Phi_2}{2}$	$\frac{\Phi_1\Phi_2}{2}$	$\frac{\Phi_1\Phi_2}{2}$	$2q\Phi_2$	.
v_5	.	$q^2\Phi_2^2$	.	$q^2\Phi_2^2$	$q(3q+1)\frac{\Phi_1}{2}$	$q\frac{\Phi_2^2}{2}$	$\frac{\Phi_1\Phi_2}{2}$	$\frac{\Phi_1\Phi_2}{2}$	$\frac{\Phi_1\Phi_2}{2}$	$\frac{\Phi_1\Phi_2}{2}$	.	$2q\Phi_2$

	u_{21}	u_{22}	u_{23}	u_{24}	u_{25}	u_{26}	u_{27}	u_{28}
v_1	.	.	.	.	.	.	.	.
v_2	1	1	1	1	.	.	.	.
v_3	1	1	1	1	.	.	.	.
v_4	$\frac{(2+\varepsilon_q)q-1}{2}$	$\frac{(2-\varepsilon_q)q-1}{2}$	$\frac{(2+\varepsilon_q)q-1}{2}$	$\frac{(2-\varepsilon_q)q-1}{2}$	1	.	1	.
v_5	$\frac{(2-\varepsilon_q)q-1}{2}$	$\frac{(2+\varepsilon_q)q-1}{2}$	$\frac{(2-\varepsilon_q)q-1}{2}$	$\frac{(2+\varepsilon_q)q-1}{2}$	.	1	.	1

$$* = (2q^4+3q^3+3q^2+q+1)\Phi_2$$

$$\varepsilon_q = (-1)^{\frac{q-1}{2}}$$

Table 55: $\tilde{Q}_{\mathbf{L}}^{\mathbf{G}}(u,v)$ for the twisted Levi subgroup $A_1(q^2)(q^2-1)$ of $\mathrm{Spin}_8^+(q)$.

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
v_1	$\Phi_1^2\Phi_2^2\Phi_3\Phi_4\Phi_6$	$-\Phi_1\Phi_2\Phi_4$	$-\Phi_1\Phi_2\Phi_4$	$-\Phi_1\Phi_2\Phi_4$	$-\Phi_1\Phi_2\Phi_4$	$-\Phi_1\Phi_2\Phi_4$	$2q^4+q^2+1$	$2q^4+q^2+1$
v_2	.	.	.	.	.	.	.	$q^2\Phi_1\Phi_2\Phi_4$
v_3	.	.	.	.	.	.	$q^2\Phi_1\Phi_2\Phi_4$	.

	u_9	u_{10}	u_{11}	u_{12}	u_{13}	u_{14}	u_{15}	u_{16}	u_{17}	u_{18}	u_{19}	u_{20}
v_1	Φ_4	Φ_4	Φ_4	Φ_4	$-\Phi_1$	Φ_2	.	.	.	.	1	1
v_2	.	$q^2\Phi_1\Phi_2$	.	$q^2\Phi_1\Phi_2$	$-q\frac{\Phi_1\Phi_2}{2}$	$q\frac{\Phi_1\Phi_2}{2}$	$-\frac{\Phi_1\Phi_2}{2}$	$-\frac{\Phi_1\Phi_2}{2}$	$-\frac{\Phi_1\Phi_2}{2}$	$-\frac{\Phi_1\Phi_2}{2}$	$q\Phi_2$	$q\Phi_1$
v_3	$q^2\Phi_1\Phi_2$	.	$q^2\Phi_1\Phi_2$	.	$-q\frac{\Phi_1\Phi_2}{2}$	$q\frac{\Phi_1\Phi_2}{2}$	$-\frac{\Phi_1\Phi_2}{2}$	$-\frac{\Phi_1\Phi_2}{2}$	$-\frac{\Phi_1\Phi_2}{2}$	$-\frac{\Phi_1\Phi_2}{2}$	$q\Phi_1$	$q\Phi_2$

	u_{21}	u_{22}	u_{23}	u_{24}	u_{25}	u_{26}	u_{27}	u_{28}
v_1	.	.	.	.	.	.	.	.
v_2	$\frac{1+\varepsilon_q q}{2}$	$\frac{1-\varepsilon_q q}{2}$	$\frac{1+\varepsilon_q q}{2}$	$\frac{1-\varepsilon_q q}{2}$	1	.	1	.
v_3	$\frac{1-\varepsilon_q q}{2}$	$\frac{1+\varepsilon_q q}{2}$	$\frac{1-\varepsilon_q q}{2}$	$\frac{1+\varepsilon_q q}{2}$	.	1	.	1

$$\varepsilon_q = (-1)^{\frac{q-1}{2}}$$

Table 56: $\tilde{Q}_{\mathbf{L}}^{\mathbf{G}}(u,v)$ for the twisted Levi subgroup $A_1(q)^2(q^2-1)$ of $\mathrm{Spin}_8^+(q)$.

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
v_1	$-\Phi_1\Phi_2\Phi_3\Phi_4^2\Phi_6$	$-q^4+2q^2+1$	$-\Phi_1\Phi_2\Phi_4$	$-\Phi_1\Phi_2\Phi_4$	$-\Phi_1\Phi_2\Phi_4$	$-\Phi_1\Phi_2\Phi_4$	Φ_4	Φ_4
v_2	.	$-q^4\Phi_1\Phi_2$	$-q^2\Phi_1\Phi_4$	$-q^2\Phi_1\Phi_4$	$q^2\Phi_2\Phi_4$	$q^2\Phi_2\Phi_4$	.	.
v_3	.	$-q^4\Phi_1\Phi_2$	$q^2\Phi_2\Phi_4$	$q^2\Phi_2\Phi_4$	$-q^2\Phi_1\Phi_4$	$-q^2\Phi_1\Phi_4$	.	.
v_4	.	.	.	.	.	.	$-q^2\Phi_1\Phi_2\Phi_4$	.
v_5	.	.	.	.	.	.	.	$-q^2\Phi_1\Phi_2\Phi_4$

	u_9	u_{10}	u_{11}	u_{12}	u_{13}	u_{14}	u_{15}	u_{16}	u_{17}	u_{18}	u_{19}	u_{20}
v_1	$-\Phi_1\Phi_2$	$-\Phi_1\Phi_2$	$-\Phi_1\Phi_2$	$-\Phi_1\Phi_2$	Φ_2	$-\Phi_1$	.	.	.	.	1	1
v_2	q^2	q^2	q^2	q^2	.	.	$-\Phi_1$	$-\Phi_1$	Φ_2	Φ_2	.	.
v_3	q^2	q^2	q^2	q^2	.	.	Φ_2	Φ_2	$-\Phi_1$	$-\Phi_1$	.	.
v_4	$q^2\Phi_4$	.	$q^2\Phi_4$	.	$-q\frac{\Phi_1^2}{2}$	$q\frac{\Phi_2^2}{2}$	$\frac{\Phi_1\Phi_2}{2}$	$\frac{\Phi_1\Phi_2}{2}$	$\frac{\Phi_1\Phi_2}{2}$	$\frac{\Phi_1\Phi_2}{2}$	.	.
v_5	.	$q^2\Phi_4$	.	$q^2\Phi_4$	$-q\frac{\Phi_1^2}{2}$	$q\frac{\Phi_2^2}{2}$	$\frac{\Phi_1\Phi_2}{2}$	$\frac{\Phi_1\Phi_2}{2}$	$\frac{\Phi_1\Phi_2}{2}$	$\frac{\Phi_1\Phi_2}{2}$	.	.

	u_{21}	u_{22}	u_{23}	u_{24}	u_{25}	u_{26}	u_{27}	u_{28}
v_1	.	.	.	.	.	.	.	.
v_2	1	1	1	1	.	.	.	.
v_3	1	1	1	1	.	.	.	.
v_4	$-\frac{\varepsilon_q q+1}{2}$	$\frac{\varepsilon_q q-1}{2}$	$-\frac{\varepsilon_q q+1}{2}$	$\frac{\varepsilon_q q-1}{2}$	1	.	1	.
v_5	$\frac{\varepsilon_q q-1}{2}$	$-\frac{\varepsilon_q q+1}{2}$	$\frac{\varepsilon_q q-1}{2}$	$-\frac{\varepsilon_q q+1}{2}$	.	1	.	1

$$\varepsilon_q = (-1)^{\frac{q-1}{2}}$$

Table 57: $\tilde{Q}_{\mathbf{L}}^{\mathbf{G}}(u,v)$ for the twisted Levi subgroup $A_1(q)^2(q+1)^2$ of $\mathrm{Spin}_8^+(q)$.

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
v_1	$\Phi_1^2\Phi_3\Phi_4^2\Phi_6$	$*$	$\Phi_1^2\Phi_4$	$\Phi_1^2\Phi_4$	$\Phi_1^2\Phi_4$	$\Phi_1^2\Phi_4$	$(2q^2-2q+1)\Phi_4$	$(2q^2-2q+1)\Phi_4$
v_2	.	$q^4\Phi_1^2$	$-q^2\Phi_1\Phi_4$	$-q^2\Phi_1\Phi_4$	$-q^2\Phi_1\Phi_4$	$-q^2\Phi_1\Phi_4$	.	.
v_3	.	$q^4\Phi_1^2$	$-q^2\Phi_1\Phi_4$	$-q^2\Phi_1\Phi_4$	$-q^2\Phi_1\Phi_4$	$-q^2\Phi_1\Phi_4$	.	.
v_4	.	.	.	.	.	.	$q^2\Phi_1^2\Phi_4$	.
v_5	.	.	.	.	.	.	.	$q^2\Phi_1^2\Phi_4$

	u_9	u_{10}	u_{11}	u_{12}	u_{13}	u_{14}	u_{15}	u_{16}	u_{17}	u_{18}	u_{19}	u_{20}
v_1	Φ_1^2	Φ_1^2	Φ_1^2	Φ_1^2	$-\Phi_1$	$1-3q$	.	.	.	.	1	1
v_2	q^2	q^2	q^2	q^2	.	$4q^2$	$-\Phi_1$	$-\Phi_1$	$-\Phi_1$	$-\Phi_1$	.	.
v_3	q^2	q^2	q^2	q^2	.	$4q^2$	$-\Phi_1$	$-\Phi_1$	$-\Phi_1$	$-\Phi_1$	.	.
v_4	$q^2\Phi_1^2$	.	$q^2\Phi_1^2$	.	$-q\frac{\Phi_1^2}{2}$	$q(1-3q)\frac{\Phi_2}{2}$	$\frac{\Phi_1\Phi_2}{2}$	$\frac{\Phi_1\Phi_2}{2}$	$\frac{\Phi_1\Phi_2}{2}$	$\frac{\Phi_1\Phi_2}{2}$	.	$2q\Phi_1$
v_5	.	$q^2\Phi_1^2$	.	$q^2\Phi_1^2$	$-q\frac{\Phi_1^2}{2}$	$q(1-3q)\frac{\Phi_2}{2}$	$\frac{\Phi_1\Phi_2}{2}$	$\frac{\Phi_1\Phi_2}{2}$	$\frac{\Phi_1\Phi_2}{2}$	$\frac{\Phi_1\Phi_2}{2}$	$2q\Phi_1$	.

	u_{21}	u_{22}	u_{23}	u_{24}	u_{25}	u_{26}	u_{27}	u_{28}
v_1	.	.	.	.	.	.	.	.
v_2	1	1	1	1	.	.	.	.
v_3	1	1	1	1	.	.	.	.
v_4	$\frac{(\varepsilon_q-2)q-1}{2}$	$-\frac{(\varepsilon_q+2)q+1}{2}$	$\frac{(\varepsilon_q-2)q-1}{2}$	$-\frac{(\varepsilon_q+2)q+1}{2}$	1	.	1	.
v_5	$-\frac{(\varepsilon_q+2)q+1}{2}$	$\frac{(\varepsilon_q-2)q-1}{2}$	$-\frac{(\varepsilon_q+2)q+1}{2}$	$\frac{(\varepsilon_q-2)q-1}{2}$	.	1	.	1

$$* = -(2q^4-3q^3+3q^2-q+1)\Phi_1$$

$$\varepsilon_q = (-1)^{\frac{q-1}{2}}$$

Table 58: $\tilde{Q}_{\mathbf{L}}^{\mathbf{G}}(u,v)$ for the twisted Levi subgroup $A_1(q^2)(q^2+1)$ of $\mathrm{Spin}_8^+(q)$.

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
v_1	$-\Phi_1^3\Phi_2^3\Phi_3\Phi_6$	$\Phi_1^2\Phi_2^2$	$\Phi_1^2\Phi_2^2$	$\Phi_1^2\Phi_2^2$	$\Phi_1^2\Phi_2^2$	$\Phi_1^2\Phi_2^2$	$-\Phi_1\Phi_2$	$-\Phi_1\Phi_2$
v_2	.	.	.	.	.	.	.	$-q^2\Phi_1^2\Phi_2^2$
v_3	.	.	.	.	.	.	$-q^2\Phi_1^2\Phi_2^2$	.

	u_9	u_{10}	u_{11}	u_{12}	u_{13}	u_{14}	u_{15}	u_{16}	u_{17}	u_{18}	u_{19}	u_{20}
v_1	$-\Phi_1\Phi_2$	$-\Phi_1\Phi_2$	$-\Phi_1\Phi_2$	$-\Phi_1\Phi_2$	$-\Phi_1$	Φ_2	.	.	.	.	1	1
v_2	.	$q^2\Phi_1\Phi_2$	.	$q^2\Phi_1\Phi_2$	$\frac{q\Phi_1^2}{2}$	$-\frac{q\Phi_2^2}{2}$	$-\frac{\Phi_1\Phi_2}{2}$	$-\frac{\Phi_1\Phi_2}{2}$	$-\frac{\Phi_1\Phi_2}{2}$	$-\frac{\Phi_1\Phi_2}{2}$	.	.
v_3	$q^2\Phi_1\Phi_2$	.	$q^2\Phi_1\Phi_2$	.	$\frac{q\Phi_1^2}{2}$	$-\frac{q\Phi_2^2}{2}$	$-\frac{\Phi_1\Phi_2}{2}$	$-\frac{\Phi_1\Phi_2}{2}$	$-\frac{\Phi_1\Phi_2}{2}$	$-\frac{\Phi_1\Phi_2}{2}$	.	.

	u_{21}	u_{22}	u_{23}	u_{24}	u_{25}	u_{26}	u_{27}	u_{28}
v_1	.	.	.	.	.	.	.	.
v_2	$\frac{1-\varepsilon_q q}{2}$	$\frac{1+\varepsilon_q q}{2}$	$\frac{1-\varepsilon_q q}{2}$	$\frac{1+\varepsilon_q q}{2}$	1	.	1	.
v_3	$\frac{1+\varepsilon_q q}{2}$	$\frac{1-\varepsilon_q q}{2}$	$\frac{1+\varepsilon_q q}{2}$	$\frac{1-\varepsilon_q q}{2}$	.	1	.	1

$$\varepsilon_q = (-1)^{\frac{q-1}{2}}$$

B Unipotent classes of the unipotent subgroup

Table 59: Representatives of the unipotent conjugacy classes of U_0, with generic elements in their U_0-orbit. $r_1, ..., r_6 \in \mathbb{F}_q^\times$, $t_1, ..., t_6 \in \mathbb{F}_q$.

u	${}^x u$	# classes	order	$\lvert C_{U_0}(u) \rvert$
$u_6(r_6)$	$u_6(r_6)$	$q-1$	1	q^6
$u_5(r_5)$	$u_5(r_5)u_6(t_1r_5)$	$q-1$	q	q^5
$u_4(r_4)$	$u_4(r_4)u_6(-t_3r_4)$	$q-1$	q	q^5
$u_4(r_4)u_5(r_5)$	$u_4(r_4)u_5(r_5)u_6(t_1r_5 - t_3r_4)$	$(q-1)^2$	q	q^5
$u_3(r_3)$	$u_3(r_3)u_5(t_2r_3)u_6(t_1t_2r_3 + t_4r_3)$	$q-1$	q^2	q^4
$u_3(r_3)u_4(r_4)$	$u_3(r_3)u_4(r_4)u_5(t_2r_3)u_6(t_1t_2r_3 - t_3r_4 + t_4r_3)$	$(q-1)^2$	q^2	q^4
$u_2(r_2)$	$u_2(r_2)u_4(t_1r_2)u_5(-t_3r_2)u_6(-t_1t_3r_2)$	$q-1$	q^2	q^4
$u_2(r_2)u_3(r_3)$	$u_2(r_2)u_3(r_3)u_4(t_1r_2)u_5(t_2r_3 - t_3r_2)u_6(t_1r_3r_2 + t_1t_2r_3 - t_1t_3r_2 + t_4r_3)$	$(q-1)^2$	q^3	q^3
$u_2(r_2)u_6(r_6)$	$u_2(r_2)u_4(t_1r_2)u_5(-t_3r_2)u_6(-t_1t_3r_2 + r_6)$	$(q-1)^2$	q^2	q^4
$u_1(r_1)$	$u_1(r_1)u_4(-t_2r_1)u_6(-t_5r_1)$	$q-1$	q^2	q^4
$u_1(r_1)u_2(r_2)$	$u_1(r_1)u_2(r_2)u_4(t_1r_2 - t_2r_1)u_5(-t_3r_2)u_6(-t_1t_3r_2 - t_5r_1)$	$(q-1)^2$	q^3	q^3
$u_1(r_1)u_3(r_3)$	$u_1(r_1)u_3(r_3)u_4(-t_2r_1)u_5(t_2r_3)u_6(t_1t_2r_3 - t_2r_3r_1 + t_4r_3 - t_5r_1)$	$(q-1)^2$	q^2	q^4
$u_1(r_1)u_5(r_5)$	$u_1(r_1)u_4(-t_2r_1)u_5(r_5)u_6(t_1r_5 - t_5r_1)$	$(q-1)^2$	q^2	q^4
$u_1(r_1)u_2(r_2)u_3(r_3)$	$u_1(r_1)u_2(r_2)u_3(r_3)u_4(t_1r_2 - t_2r_1)u_5(t_2r_3 - t_3r_2)$ $u_6(t_1r_3r_2 + t_1t_2r_3 - t_1t_3r_2 - t_2r_3r_1 + t_4r_3 - t_5r_1)$	$(q-1)^3$	q^3	q^3
$u_1(r_1)u_3(r_3)u_5(r_5)$	$u_1(r_1)u_3(r_3)u_4(-t_2r_1)u_5(t_2r_3 + r_5)$ $u_6(t_1t_2r_3 + t_1r_5 - t_2r_3r_1 + t_4r_3 - t_5r_1)$	$(q-1)^3$	q^2	q^4

Table 60: Representatives of the unipotent conjugacy classes of U_0 in $\mathrm{Spin}_8^+(q)$, with generic elements in their U_0-orbit. $r_1, ..., r_{12} \in \mathbb{F}_q^\times$, $t_1, ..., t_{12} \in \mathbb{F}_q$. (Continues on the following pages)

u	${}^x u$	# classes	order	$\lvert C_{U_0}(u)\rvert$
$u_{12}(r_{12})$	$u_{12}(r_{12})$	$q-1$	1	q^{12}
$u_{11}(r_{11})$	$u_{11}(r_{11})u_{12}(t_3r_{11})$	$q-1$	q	q^{11}
$u_{10}(r_{10})$	$u_{10}(r_{10})u_{11}(t_1r_{10})u_{12}(-t_5r_{10})$	$q-1$	q^2	q^{10}
$u_9(r_9)$	$u_9(r_9)u_{11}(t_2r_9)u_{12}(-t_6r_9)$	$q-1$	q^2	q^{10}
$u_9(r_9)u_{10}(r_{10})$	$u_9(r_9)u_{10}(r_{10})u_{11}(t_1r_{10}+t_2r_9)u_{12}(-t_5r_{10}-t_6r_9)$	$(q-1)^2$	q^2	q^{10}
$u_8(r_8)$	$u_8(r_8)u_{11}(t_4r_8)u_{12}(t_3t_4r_8-t_7r_8)$	$q-1$	q^2	q^{10}
$u_8(r_8)u_9(r_9)$	$u_8(r_8)u_9(r_9)u_{11}(t_2r_9+t_4r_8)u_{12}(t_3t_4r_8-t_6r_9-t_7r_8)$	$(q-1)^2$	q^2	q^{10}
$u_8(r_8)u_{10}(r_{10})$	$u_8(r_8)u_{10}(r_{10})u_{11}(t_1r_{10}+t_4r_8)u_{12}(t_3t_4r_8-t_5r_{10}-t_7r_8)$	$(q-1)^2$	q^2	q^{10}
$u_8(r_8)u_9(r_9)u_{10}(r_{10})$	$u_8(r_8)u_9(r_9)u_{10}(r_{10})u_{11}(t_1r_{10}+t_2r_9+t_4r_8)u_{12}(t_3t_4r_8-t_5r_{10}-t_6r_9-t_7r_8)$	$(q-1)^3$	q^2	q^{10}
$u_7(r_7)$	$u_7(r_7)u_9(t_1r_7)u_{10}(t_2r_7)u_{11}(t_1t_2r_7)u_{12}(t_8r_7)$	$q-1$	q^3	q^9
$u_7(r_7)u_8(r_8)$	$u_7(r_7)u_8(r_8)u_9(t_1r_7)u_{10}(t_2r_7)u_{11}(t_1t_2r_7+t_4r_8)u_{12}(t_3t_4r_8-t_7r_8+t_8r_7)$	$(q-1)^2$	q^4	q^8
$u_7(r_7)u_{11}(r_{11})$	$u_7(r_7)u_9(t_1r_7)u_{10}(t_2r_7)u_{11}(t_1t_2r_7+r_{11})u_{12}(t_3r_{11}+t_8r_7)$	$(q-1)^2$	q^3	q^9
$u_6(r_6)$	$u_6(r_6)u_8(t_1r_6)u_{10}(t_4r_6)u_{11}(t_1t_4r_6)u_{12}(t_9r_6)$	$q-1$	q^3	q^9
$u_6(r_6)u_7(r_7)$	$u_6(r_6)u_7(r_7)u_8(t_1r_6)u_9(t_1r_7)u_{10}(t_2r_7+t_4r_6)u_{11}(t_1t_2r_7+t_1t_4r_6)u_{12}(t_1r_7r_6+t_8r_7+t_9r_6)$	$(q-1)^2$	q^3	q^9
$u_6(r_6)u_9(r_9)$	$u_6(r_6)u_8(t_1r_6)u_9(r_9)u_{10}(t_4r_6)u_{11}(t_1t_4r_6+t_2r_9)u_{12}(-t_6r_9+t_9r_6)$	$(q-1)^2$	q^4	q^8
$u_6(r_6)u_{11}(r_{11})$	$u_6(r_6)u_8(t_1r_6)u_{10}(t_4r_6)u_{11}(t_1t_4r_6+r_{11})u_{12}(t_3r_{11}+t_9r_6)$	$(q-1)^2$	q^3	q^9
$u_6(r_6)u_7(r_7)u_9(r_9)$	$u_6(r_6)u_7(r_7)u_8(t_1r_6)u_9(t_1r_7+r_9)u_{10}(t_2r_7+t_4r_6)u_{11}(t_1t_2r_7+t_1t_4r_6+t_2r_9)u_{12}(t_1r_7r_6-t_6r_9+t_8r_7+t_9r_6)$	$(q-1)^3$	q^4	q^8
$u_6(r_6)u_7(r_7)u_{11}(r_{11})$	$u_6(r_6)u_7(r_7)u_8(t_1r_6)u_9(t_1r_7)u_{10}(t_2r_7+t_4r_6)u_{11}(t_1t_2r_7+t_1t_4r_6+r_{11})u_{12}(t_1r_7r_6+t_3r_{11}+t_8r_7+t_9r_6)$	$(q-1)^3$	q^3	q^9
$u_5(r_5)$	$u_5(r_5)u_8(t_2r_5)u_9(t_4r_5)u_{11}(t_2t_4r_5)u_{12}(t_{10}r_5)$	$q-1$	q^3	q^9
$u_5(r_5)u_6(r_6)$	$u_5(r_5)u_6(r_6)u_8(t_1r_6+t_2r_5)u_9(t_4r_5)u_{10}(t_4r_6)u_{11}(t_1t_4r_6+t_2t_4r_5)u_{12}(t_4r_6r_5+t_9r_6+t_{10}r_5)$	$(q-1)^2$	q^3	q^9
$u_5(r_5)u_7(r_7)$	$u_5(r_5)u_7(r_7)u_8(t_2r_5)u_9(t_1r_7+t_4r_5)u_{10}(t_2r_7)u_{11}(t_1t_2r_7+t_2t_4r_5)u_{12}(t_2r_7r_5+t_8r_7+t_{10}r_5)$	$(q-1)^2$	q^3	q^9
$u_5(r_5)u_{10}(r_{10})$	$u_5(r_5)u_8(t_2r_5)u_9(t_4r_5)u_{10}(r_{10})u_{11}(t_1r_{10}+t_2t_4r_5)u_{12}(-t_5r_{10}+t_{10}r_5)$	$(q-1)^2$	q^4	q^8
$u_5(r_5)u_{11}(r_{11})$	$u_5(r_5)u_8(t_2r_5)u_9(t_4r_5)u_{11}(t_2t_4r_5+r_{11})u_{12}(t_3r_{11}+t_{10}r_5)$	$(q-1)^2$	q^3	q^9
$u_5(r_5)u_6(r_6)u_7(r_7)$	$u_5(r_5)u_6(r_6)u_7(r_7)u_8(t_1r_6+t_2r_5)u_9(t_1r_7+t_4r_5)u_{10}(t_2r_7+t_4r_6)u_{11}(t_1t_2r_7+t_1t_4r_6+t_2t_4r_5)$ $u_{12}(t_1r_7r_6+t_2r_7r_5+t_4r_6r_5+t_8r_7+t_9r_6+t_{10}r_5)$	$(q-1)^3$	q^4	q^8
$u_5(r_5)u_6(r_6)u_{10}(r_{10})$	$u_5(r_5)u_6(r_6)u_8(t_1r_6+t_2r_5)u_9(t_4r_5)u_{10}(t_4r_6+r_{10})u_{11}(t_1t_4r_6+t_1r_{10}+t_2t_4r_5)u_{12}(t_4r_6r_5-t_5r_{10}+t_9r_6+t_{10}r_5)$	$(q-1)^3$	q^4	q^8
$u_5(r_5)u_6(r_6)u_{11}(r_{11})$	$u_5(r_5)u_6(r_6)u_8(t_1r_6+t_2r_5)u_9(t_4r_5)u_{10}(t_4r_6)u_{11}(t_1t_4r_6+t_2t_4r_5+r_{11})u_{12}(t_3r_{11}+t_4r_6r_5+t_9r_6+t_{10}r_5)$	$(q-1)^3$	q^3	q^9
$u_5(r_5)u_7(r_7)u_{10}(r_{10})$	$u_5(r_5)u_7(r_7)u_8(t_2r_5)u_9(t_1r_7+t_4r_5)u_{10}(t_2r_7+r_{10})u_{11}(t_1t_2r_7+t_1r_{10}+t_2t_4r_5)u_{12}(t_2r_7r_5-t_5r_{10}+t_8r_7+t_{10}r_5)$	$(q-1)^3$	q^4	q^8
$u_5(r_5)u_7(r_7)u_{11}(r_{11})$	$u_5(r_5)u_7(r_7)u_8(t_2r_5)u_9(t_1r_7+t_4r_5)u_{10}(t_2r_7)u_{11}(t_1t_2r_7+t_2t_4r_5+r_{11})u_{12}(t_2r_7r_5+t_3r_{11}+t_8r_7+t_{10}r_5)$	$(q-1)^3$	q^3	q^9
$u_5(r_5)u_6(r_6)u_7(r_7)u_{11}(r_{11})$	$u_5(r_5)u_6(r_6)u_7(r_7)u_8(t_1r_6+t_2r_5)u_9(t_1r_7+t_4r_5)u_{10}(t_2r_7+t_4r_6)u_{11}(t_1t_2r_7+t_1t_4r_6+t_2t_4r_5+r_{11})$ $u_{12}(t_1r_7r_6+t_2r_7r_5+t_3r_{11}+t_4r_6r_5+t_8r_7+t_9r_6+t_{10}r_5)$	$(q-1)^4$	q^4	q^8

u	${}^{x}u$	# classes	order	$\lvert C_{U_0}(u)\rvert$
$u_4(r_4)$	$u_4(r_4)u_7(-t_3r_4)u_9(-t_1t_3r_4-t_5r_4)u_{10}(-t_2t_3r_4-t_6r_4)u_{11}(-t_1t_2t_3r_4-t_1t_6r_4-t_2t_5r_4-t_8r_4)u_{12}(-t_3t_8r_4+t_5t_6r_4)$	$q-1$	q^4	q^8
$u_4(r_4)u_5(r_5)$	$u_4(r_4)u_5(r_5)u_7(-t_3r_4)u_8(t_2r_5)u_9(-t_1t_3r_4+t_4r_5-t_5r_4)u_{10}(-t_2t_3r_4-t_6r_4)$ $u_{11}(-t_1t_2t_3r_4-t_1t_6r_4+t_2t_4r_5-t_2t_5r_4-t_8r_4)u_{12}(-t_2t_3r_5r_4-t_3t_8r_4+t_5t_6r_4-t_6r_5r_4+t_{10}r_5)$	$(q-1)^2$	q^6	q^6
$u_4(r_4)u_6(r_6)$	$u_4(r_4)u_6(r_6)u_7(-t_3r_4)u_8(t_1r_6)u_9(-t_1t_3r_4-t_5r_4)u_{10}(-t_2t_3r_4+t_4r_6-t_6r_4)$ $u_{11}(-t_1t_2t_3r_4+t_1t_4r_6-t_1t_6r_4-t_2t_5r_4-t_8r_4)u_{12}(-t_1t_3r_6r_4-t_3t_8r_4+t_5t_6r_4-t_5r_6r_4+t_9r_6)$	$(q-1)^2$	q^6	q^6
$u_4(r_4)u_8(r_8)$	$u_4(r_4)u_7(-t_3r_4)u_8(r_8)u_9(-t_1t_3r_4-t_5r_4)u_{10}(-t_2t_3r_4-t_6r_4)u_{11}(-t_1t_2t_3r_4-t_1t_6r_4-t_2t_5r_4+t_4r_8-t_8r_4)$ $u_{12}(t_3t_4r_8-t_3t_8r_4+t_5t_6r_4-t_7r_8)$	$(q-1)^2$	q^5	q^7
$u_4(r_4)u_{12}(r_{12})$	$u_4(r_4)u_7(-t_3r_4)u_9(-t_1t_3r_4-t_5r_4)u_{10}(-t_2t_3r_4-t_6r_4)u_{11}(-t_1t_2t_3r_4-t_1t_6r_4-t_2t_5r_4-t_8r_4)$ $u_{12}(-t_3t_8r_4+t_5t_6r_4+r_{12})$	$(q-1)^2$	q^4	q^8
$u_4(r_4)u_5(r_5)u_6(r_6)$	$u_4(r_4)u_5(r_5)u_6(r_6)u_7(-t_3r_4)u_8(t_1r_6+t_2r_5)u_9(-t_1t_3r_4+t_4r_5-t_5r_4)u_{10}(-t_2t_3r_4+t_4r_6-t_6r_4)$ $u_{11}(-t_1t_2t_3r_4+t_1t_4r_6-t_1t_6r_4+t_2t_4r_5-t_2t_5r_4-t_8r_4)$ $u_{12}(-t_1t_3r_6r_4-t_2t_3r_5r_4-t_3t_8r_4+t_4r_6r_5+t_5t_6r_4-t_5r_6r_4-t_6r_5r_4+t_9r_6+t_{10}r_5)$	$(q-1)^3$	q^6	q^6
$u_3(r_3)$	$u_3(r_3)u_5(t_1r_3)u_6(t_2r_3)u_7(t_4r_3)u_8(t_1t_2r_3)u_9(t_1t_4r_3)u_{10}(t_2t_4r_3)u_{11}(t_1t_2t_4r_3)u_{12}(t_1t_2t_4r_3^2-t_{11}r_3)$	$q-1$	q^4	q^8
$u_3(r_3)u_4(r_4)$	$u_3(r_3)u_4(r_4)u_5(t_1r_3)u_6(t_2r_3)u_7(-t_3r_4+t_4r_3)u_8(t_1t_2r_3)u_9(-t_1r_4r_3-t_1t_3r_4+t_1t_4r_3-t_5r_4)$ $u_{10}(-t_2r_4r_3-t_2t_3r_4+t_2t_4r_3-t_6r_4)u_{11}(-t_1t_2r_4r_3-t_1t_2t_3r_4+t_1t_2t_4r_3-t_1t_6r_4-t_2t_5r_4-t_8r_4)$ $u_{12}(-t_1t_2r_4r_3^2-t_1t_2t_3r_4r_3+t_1t_2t_4r_3^2-t_3t_8r_4+t_5t_6r_4-t_{11}r_3)$	$(q-1)^2$	q^7	q^5
$u_3(r_3)u_8(r_8)$	$u_3(r_3)u_5(t_1r_3)u_6(t_2r_3)u_7(t_4r_3)u_8(t_1t_2r_3+r_8)u_9(t_1t_4r_3)u_{10}(t_2t_4r_3)u_{11}(t_1t_2t_4r_3+t_4r_8)$ $u_{12}(t_1t_2t_4r_3^2+t_3t_4r_8-t_7r_8-t_{11}r_3)$	$(q-1)^2$	q^4	q^8
$u_3(r_3)u_9(r_9)$	$u_3(r_3)u_5(t_1r_3)u_6(t_2r_3)u_7(t_4r_3)u_8(t_1t_2r_3)u_9(t_1t_4r_3+r_9)u_{10}(t_2t_4r_3)u_{11}(t_1t_2t_4r_3+t_2r_9)u_{12}(t_1t_2t_4r_3^2-t_6r_9-t_{11}r_3)$	$(q-1)^2$	q^4	q^8
$u_3(r_3)u_{10}(r_{10})$	$u_3(r_3)u_5(t_1r_3)u_6(t_2r_3)u_7(t_4r_3)u_8(t_1t_2r_3)u_9(t_1t_4r_3)u_{10}(t_2t_4r_3+r_{10})u_{11}(t_1t_2t_4r_3+t_1r_{10})$ $u_{12}(t_1t_2t_4r_3^2-t_5r_{10}-t_{11}r_3)$	$(q-1)^2$	q^4	q^8
$u_3(r_3)u_{11}(r_{11})$	$u_3(r_3)u_5(t_1r_3)u_6(t_2r_3)u_7(t_4r_3)u_8(t_1t_2r_3)u_9(t_1t_4r_3)u_{10}(t_2t_4r_3)u_{11}(t_1t_2t_4r_3+r_{11})u_{12}(t_1t_2t_4r_3^2+t_3r_{11}-t_{11}r_3)$	$(q-1)^2$	q^4	q^8
$u_3(r_3)u_4(r_4)u_8(r_8)$	$u_3(r_3)u_4(r_4)u_5(t_1r_3)u_6(t_2r_3)u_7(-t_3r_4+t_4r_3)u_8(t_1t_2r_3+r_8)u_9(-t_1r_4r_3-t_1t_3r_4+t_1t_4r_3-t_5r_4)$ $u_{10}(-t_2r_4r_3-t_2t_3r_4+t_2t_4r_3-t_6r_4)u_{11}(-t_1t_2r_4r_3-t_1t_2t_3r_4+t_1t_2t_4r_3-t_1t_6r_4-t_2t_5r_4+t_4r_8-t_8r_4)$ $u_{12}(-t_1t_2r_4r_3^2-t_1t_2t_3r_4r_3+t_1t_2t_4r_3^2+t_3t_4r_8-t_3t_8r_4+t_5t_6r_4-t_7r_8-t_{11}r_3)$	$(q-1)^3$	q^7	q^5
$u_3(r_3)u_8(r_8)u_9(r_9)$	$u_3(r_3)u_5(t_1r_3)u_6(t_2r_3)u_7(t_4r_3)u_8(t_1t_2r_3+r_8)u_9(t_1t_4r_3+r_9)u_{10}(t_2t_4r_3)u_{11}(t_1t_2t_4r_3+t_2r_9+t_4r_8)$ $u_{12}(t_1t_2t_4r_3^2+t_3t_4r_8-t_6r_9-t_7r_8-t_{11}r_3)$	$(q-1)^3$	q^4	q^8
$u_3(r_3)u_8(r_8)u_{10}(r_{10})$	$u_3(r_3)u_5(t_1r_3)u_6(t_2r_3)u_7(t_4r_3)u_8(t_1t_2r_3+r_8)u_9(t_1t_4r_3)u_{10}(t_2t_4r_3+r_{10})u_{11}(t_1t_2t_4r_3+t_1r_{10}+t_4r_8)$ $u_{12}(t_1t_2t_4r_3^2+t_3t_4r_8-t_5r_{10}-t_7r_8-t_{11}r_3)$	$(q-1)^3$	q^4	q^8
$u_3(r_3)u_8(r_8)u_{11}(r_{11})$	$u_3(r_3)u_5(t_1r_3)u_6(t_2r_3)u_7(t_4r_3)u_8(t_1t_2r_3+r_8)u_9(t_1t_4r_3)u_{10}(t_2t_4r_3)u_{11}(t_1t_2t_4r_3+t_4r_8+r_{11})$ $u_{12}(t_1t_2t_4r_3^2+t_3t_4r_8+t_3r_{11}-t_7r_8-t_{11}r_3)$	$(q-1)^3$	q^4	q^8
$u_3(r_3)u_9(r_9)u_{10}(r_{10})$	$u_3(r_3)u_5(t_1r_3)u_6(t_2r_3)u_7(t_4r_3)u_8(t_1t_2r_3)u_9(t_1t_4r_3+r_9)u_{10}(t_2t_4r_3+r_{10})u_{11}(t_1t_2t_4r_3+t_1r_{10}+t_2r_9)$ $u_{12}(t_1t_2t_4r_3^2-t_5r_{10}-t_6r_9-t_{11}r_3)$	$(q-1)^3$	q^4	q^8
$u_3(r_3)u_9(r_9)u_{11}(r_{11})$	$u_3(r_3)u_5(t_1r_3)u_6(t_2r_3)u_7(t_4r_3)u_8(t_1t_2r_3)u_9(t_1t_4r_3+r_9)u_{10}(t_2t_4r_3)u_{11}(t_1t_2t_4r_3+t_2r_9+r_{11})$ $u_{12}(t_1t_2t_4r_3^2+t_3r_{11}-t_6r_9-t_{11}r_3)$	$(q-1)^3$	q^4	q^8
$u_3(r_3)u_{10}(r_{10})u_{11}(r_{11})$	$u_3(r_3)u_5(t_1r_3)u_6(t_2r_3)u_7(t_4r_3)u_8(t_1t_2r_3)u_9(t_1t_4r_3)u_{10}(t_2t_4r_3+r_{10})u_{11}(t_1t_2t_4r_3+t_1r_{10}+r_{11})$ $u_{12}(t_1t_2t_4r_3^2+t_3r_{11}-t_5r_{10}-t_{11}r_3)$	$(q-1)^3$	q^4	q^8
$u_3(r_3)u_8(r_8)u_9(r_9)u_{10}(r_{10})$	$u_3(r_3)u_5(t_1r_3)u_6(t_2r_3)u_7(t_4r_3)u_8(t_1t_2r_3+r_8)u_9(t_1t_4r_3+r_9)u_{10}(t_2t_4r_3+r_{10})u_{11}(t_1t_2t_4r_3+t_1r_{10}+t_2r_9+t_4r_8)$ $u_{12}(t_1t_2t_4r_3^2+t_3t_4r_8-t_5r_{10}-t_6r_9-t_7r_8-t_{11}r_3)$	$(q-1)^4$	q^4	q^8
$u_3(r_3)u_8(r_8)u_9(r_9)u_{11}(r_{11})$	$u_3(r_3)u_5(t_1r_3)u_6(t_2r_3)u_7(t_4r_3)u_8(t_1t_2r_3+r_8)u_9(t_1t_4r_3+r_9)u_{10}(t_2t_4r_3)u_{11}(t_1t_2t_4r_3+t_2r_9+t_4r_8+r_{11})$ $u_{12}(t_1t_2t_4r_3^2+t_3t_4r_8+t_3r_{11}-t_6r_9-t_7r_8-t_{11}r_3)$	$(q-1)^4$	q^4	q^8
$u_3(r_3)u_8(r_8)u_{10}(r_{10})u_{11}(r_{11})$	$u_3(r_3)u_5(t_1r_3)u_6(t_2r_3)u_7(t_4r_3)u_8(t_1t_2r_3+r_8)u_9(t_1t_4r_3)u_{10}(t_2t_4r_3+r_{10})u_{11}(t_1t_2t_4r_3+t_1r_{10}+t_4r_8+r_{11})$ $u_{12}(t_1t_2t_4r_3^2+t_3t_4r_8+t_3r_{11}-t_5r_{10}-t_7r_8-t_{11}r_3)$	$(q-1)^4$	q^4	q^8
$u_3(r_3)u_9(r_9)u_{10}(r_{10})u_{11}(r_{11})$	$u_3(r_3)u_5(t_1r_3)u_6(t_2r_3)u_7(t_4r_3)u_8(t_1t_2r_3)u_9(t_1t_4r_3+r_9)u_{10}(t_2t_4r_3+r_{10})u_{11}(t_1t_2t_4r_3+t_1r_{10}+t_2r_9+r_{11})$ $u_{12}(t_1t_2t_4r_3^2+t_3r_{11}-t_5r_{10}-t_6r_9-t_{11}r_3)$	$(q-1)^4$	q^4	q^8
$u_3(r_3)u_8(r_8)u_9(r_9)u_{10}(r_{10})u_{11}(r_{11})$	$u_3(r_3)u_5(t_1r_3)u_6(t_2r_3)u_7(t_4r_3)u_8(t_1t_2r_3+r_8)u_9(t_1t_4r_3+r_9)u_{10}(t_2t_4r_3+r_{10})u_{11}(t_1t_2t_4r_3+t_1r_{10}+t_2r_9+t_4r_8+r_{11})$ $u_{12}(t_1t_2t_4r_3^2+t_3t_4r_8+t_3r_{11}-t_5r_{10}-t_6r_9-t_7r_8-t_{11}r_3)$	$(q-1)^5$	q^4	q^8

u	${}^{x}u$	# classes	order	$\lvert C_{U_0}(u)\rvert$
$u_2(r_2)$	$u_2(r_2)u_6(-t_3r_2)u_8(-t_1t_3r_2-t_5r_2)u_{10}(-t_7r_2)u_{11}(-t_1t_7r_2-t_4t_5r_2-t_9r_2)u_{12}(-t_3t_4t_5r_2-t_3t_9r_2+t_5t_7r_2)$	$q-1$	q^4	q^8
$u_2(r_2)u_3(r_3)$	$u_2(r_2)u_3(r_3)u_5(t_1r_3)u_6(t_2r_3-t_3r_2)u_7(t_4r_3)u_8(t_1t_2r_3-t_1t_3r_2-t_5r_2)u_9(t_1t_4r_3)u_{10}(t_2t_4r_3-t_7r_2)u_{11}(t_1t_2t_4r_3-t_1t_7r_2-t_4t_5r_2-t_9r_2)u_{12}(t_1t_2t_4r_3^2-t_1t_3t_4r_3r_2-t_3t_4t_5r_2-t_3t_9r_2+t_5t_7r_2+t_9r_3r_2-t_{11}r_3)$	$(q-1)^2$	q^7	q^5
$u_2(r_2)u_4(r_4)$	$u_2(r_2)u_4(r_4)u_6(-t_3r_2)u_7(-t_3r_4)u_8(-t_1t_3r_2-t_5r_2)u_9(-t_1t_3r_4-t_5r_4)u_{10}(-t_2t_3r_4+t_3r_4r_2-t_6r_4-t_7r_2)u_{11}(-t_1t_2t_3r_4+t_1t_3r_4r_2-t_1t_6r_4-t_1t_7r_2-t_2t_5r_4-t_4t_5r_2+t_5r_4r_2-t_8r_4-t_9r_2)u_{12}(t_1t_3^2r_4r_2-t_3t_4t_5r_2+t_3t_5r_4r_2-t_3t_8r_4-t_3t_9r_2+t_5t_6r_4+t_5t_7r_2)$	$(q-1)^2$	q^4	q^8
$u_2(r_2)u_5(r_5)$	$u_2(r_2)u_5(r_5)u_6(-t_3r_2)u_8(-t_1t_3r_2+t_2r_5-t_5r_2)u_9(t_4r_5)u_{10}(-t_7r_2)u_{11}(-t_1t_7r_2+t_2t_4r_5-t_4t_5r_2-t_9r_2)u_{12}(-t_3t_4t_5r_2-t_3t_9r_2+t_5t_7r_2-t_7r_5r_2+t_{10}r_5)$	$(q-1)^2$	q^6	q^6
$u_2(r_2)u_7(r_7)$	$u_2(r_2)u_6(-t_3r_2)u_7(r_7)u_8(-t_1t_3r_2-t_5r_2)u_9(t_1r_7)u_{10}(t_2r_7-t_7r_2)u_{11}(t_1t_2r_7-t_1t_7r_2-t_4t_5r_2-t_9r_2)u_{12}(-t_1t_3r_7r_2-t_3t_4t_5r_2-t_3t_9r_2+t_5t_7r_2-t_5r_7r_2+t_8r_7)$	$(q-1)^2$	q^6	q^6
$u_2(r_2)u_9(r_9)$	$u_2(r_2)u_6(-t_3r_2)u_8(-t_1t_3r_2-t_5r_2)u_9(r_9)u_{10}(-t_7r_2)u_{11}(-t_1t_7r_2+t_2r_9-t_4t_5r_2-t_9r_2)u_{12}(-t_3t_4t_5r_2-t_3t_9r_2+t_5t_7r_2-t_6r_9)$	$(q-1)^2$	q^5	q^7
$u_2(r_2)u_{12}(r_{12})$	$u_2(r_2)u_6(-t_3r_2)u_8(-t_1t_3r_2-t_5r_2)u_{10}(-t_7r_2)u_{11}(-t_1t_7r_2-t_4t_5r_2-t_9r_2)u_{12}(-t_3t_4t_5r_2-t_3t_9r_2+t_5t_7r_2+r_{12})$	$(q-1)^2$	q^4	q^8
$u_2(r_2)u_3(r_3)u_4(r_4)$	$u_2(r_2)u_3(r_3)u_4(r_4)u_5(t_1r_3)u_6(t_2r_3-t_3r_2)u_7(-t_3r_4+t_4r_3)u_8(t_1t_2r_3-t_1t_3r_2-t_5r_2)u_9(-t_1r_4r_3-t_1t_3r_4+t_1t_4r_3-t_5r_4)u_{10}(-t_2r_4r_3-t_2t_3r_4+t_2t_4r_3+t_3r_4r_2-t_6r_4-t_7r_2)u_{11}(-t_1t_2r_4r_3-t_1t_2t_3r_4+t_1t_2t_4r_3+t_1t_3r_4r_2-t_1t_6r_4-t_1t_7r_2-t_2t_5r_4-t_4t_5r_2+t_5r_4r_2-t_8r_4-t_9r_2)u_{12}(-t_1t_2r_4r_3^2+t_1t_3r_4r_3r_2-t_1t_2t_3r_4r_3+t_1t_3^2r_4r_2+t_1t_2t_4r_3^2-t_1t_3t_4r_3r_2-t_3t_4t_5r_2+t_3t_5r_4r_2-t_3t_8r_4-t_3t_9r_2+t_5t_6r_4+t_5t_7r_2+t_9r_3r_2-t_{11}r_3)$	$(q-1)^3$	q^7	q^5
$u_2(r_2)u_3(r_3)u_9(r_9)$	$u_2(r_2)u_3(r_3)u_5(t_1r_3)u_6(t_2r_3-t_3r_2)u_7(t_4r_3)u_8(t_1t_2r_3-t_1t_3r_2-t_5r_2)u_9(t_1t_4r_3+r_9)u_{10}(t_2t_4r_3-t_7r_2)u_{11}(t_1t_2t_4r_3-t_1t_7r_2+t_2r_9-t_4t_5r_2-t_9r_2)u_{12}(t_1t_2t_4r_3^2-t_1t_3t_4r_3r_2-t_3t_4t_5r_2-t_3t_9r_2+t_5t_7r_2-t_6r_9+t_9r_3r_2-t_{11}r_3)$	$(q-1)^3$	q^7	q^5
$u_2(r_2)u_4(r_4)u_5(r_5)$	$u_2(r_2)u_4(r_4)u_5(r_5)u_6(-t_3r_2)u_7(-t_3r_4)u_8(-t_1t_3r_2+t_2r_5-t_5r_2)u_9(-t_1t_3r_4+t_4r_5-t_5r_4)$ $u_{10}(-t_2t_3r_4+t_3r_4r_2-t_6r_4-t_7r_2)u_{11}(-t_1t_2t_3r_4+t_1t_3r_4r_2-t_1t_6r_4-t_1t_7r_2+t_2t_4r_5-t_2t_5r_4-t_4t_5r_2+t_5r_4r_2-t_8r_4-t_9r_2)u_{12}(t_1t_3^2r_4r_2-t_2t_3r_5r_4-t_3t_4t_5r_2+t_3t_5r_4r_2-t_3t_8r_4-t_3t_9r_2+t_5t_6r_4+t_5t_7r_2+t_3r_5r_4r_2-t_6r_5r_4-t_7r_5r_2+t_{10}r_5)$	$(q-1)^3$	q^6	q^6
$u_2(r_2)u_4(r_4)u_7(r_7)$	$u_2(r_2)u_4(r_4)u_6(-t_3r_2)u_7(-t_3r_4+r_7)u_8(-t_1t_3r_2-t_5r_2)u_9(-t_1t_3r_4+t_1r_7-t_5r_4)$ $u_{10}(-t_2t_3r_4+t_2r_7+t_3r_4r_2-t_6r_4-t_7r_2)u_{11}(-t_1t_2t_3r_4+t_1t_2r_7+t_1t_3r_4r_2-t_1t_6r_4-t_1t_7r_2-t_2t_5r_4-t_4t_5r_2+t_5r_4r_2-t_8r_4-t_9r_2)u_{12}(t_1t_3^2r_4r_2-t_1t_3r_7r_2-t_3t_4t_5r_2+t_3t_5r_4r_2-t_3t_8r_4-t_3t_9r_2+t_5t_6r_4+t_5t_7r_2-t_5r_7r_2+t_8r_7)$	$(q-1)^3$	q^6	q^6
$u_2(r_2)u_4(r_4)u_9(r_9)$	$u_2(r_2)u_4(r_4)u_6(-t_3r_2)u_7(-t_3r_4)u_8(-t_1t_3r_2-t_5r_2)u_9(-t_1t_3r_4-t_5r_4+r_9)u_{10}(-t_2t_3r_4+t_3r_4r_2-t_6r_4-t_7r_2)u_{11}(-t_1t_2t_3r_4+t_1t_3r_4r_2-t_1t_6r_4-t_1t_7r_2-t_2t_5r_4+t_2r_9-t_4t_5r_2+t_5r_4r_2-t_8r_4-t_9r_2)u_{12}(t_1t_3^2r_4r_2-t_3t_4t_5r_2+t_3t_5r_4r_2-t_3t_8r_4-t_3t_9r_2+t_5t_6r_4+t_5t_7r_2-t_6r_9)$	$(q-1)^3$	q^5	q^7
$u_2(r_2)u_4(r_4)u_{12}(r_{12})$	$u_2(r_2)u_4(r_4)u_6(-t_3r_2)u_7(-t_3r_4)u_8(-t_1t_3r_2-t_5r_2)u_9(-t_1t_3r_4-t_5r_4)u_{10}(-t_2t_3r_4+t_3r_4r_2-t_6r_4-t_7r_2)u_{11}(-t_1t_2t_3r_4+t_1t_3r_4r_2-t_1t_6r_4-t_1t_7r_2-t_2t_5r_4-t_4t_5r_2+t_5r_4r_2-t_8r_4-t_9r_2)u_{12}(t_1t_3^2r_4r_2-t_3t_4t_5r_2+t_3t_5r_4r_2-t_3t_8r_4-t_3t_9r_2+t_5t_6r_4+t_5t_7r_2+r_{12})$	$(q-1)^3$	q^4	q^8
$u_2(r_2)u_5(r_5)u_7(r_7)$	$u_2(r_2)u_5(r_5)u_6(-t_3r_2)u_7(r_7)u_8(-t_1t_3r_2+t_2r_5-t_5r_2)u_9(t_1r_7+t_4r_5)u_{10}(t_2r_7-t_7r_2)u_{11}(t_1t_2r_7-t_1t_7r_2+t_2t_4r_5-t_4t_5r_2-t_9r_2)u_{12}(-t_1t_3r_7r_2+t_2r_7r_5-t_3t_4t_5r_2-t_3t_9r_2+t_5t_7r_2-t_5r_7r_2+t_8r_7-t_7r_5r_2+t_{10}r_5)$	$(q-1)^3$	q^6	q^6
$u_2(r_2)u_3(r_3)u_4(r_4)u_9(r_9)$	$u_2(r_2)u_3(r_3)u_4(r_4)u_5(t_1r_3)u_6(t_2r_3-t_3r_2)u_7(-t_3r_4+t_4r_3)u_8(t_1t_2r_3-t_1t_3r_2-t_5r_2)u_9(-t_1r_4r_3-t_1t_3r_4+t_1t_4r_3-t_5r_4+r_9)u_{10}(-t_2r_4r_3-t_2t_3r_4+t_2t_4r_3+t_3r_4r_2-t_6r_4-t_7r_2)u_{11}(-t_1t_2r_4r_3-t_1t_2t_3r_4+t_1t_2t_4r_3+t_1t_3r_4r_2-t_1t_6r_4-t_1t_7r_2-t_2t_5r_4+t_2r_9-t_4t_5r_2+t_5r_4r_2-t_8r_4-t_9r_2)u_{12}(-t_1t_2r_4r_3^2+t_1t_3r_4r_3r_2-t_1t_2t_3r_4r_3+t_1t_3^2r_4r_2+t_1t_2t_4r_3^2-t_1t_3t_4r_3r_2-t_3t_4t_5r_2+t_3t_5r_4r_2-t_3t_8r_4-t_3t_9r_2+t_5t_6r_4+t_5t_7r_2-t_6r_9+t_9r_3r_2-t_{11}r_3)$	$(q-1)^4$	q^7	q^5
$u_2(r_2)u_4(r_4)u_5(r_5)u_7(r_7)$	$u_2(r_2)u_4(r_4)u_5(r_5)u_6(-t_3r_2)u_7(-t_3r_4+r_7)u_8(-t_1t_3r_2+t_2r_5-t_5r_2)u_9(-t_1t_3r_4+t_1r_7+t_4r_5-t_5r_4)u_{10}(-t_2t_3r_4+t_2r_7+t_3r_4r_2-t_6r_4-t_7r_2)u_{11}(-t_1t_2t_3r_4+t_1t_2r_7+t_1t_3r_4r_2-t_1t_6r_4-t_1t_7r_2+t_2t_4r_5-t_2t_5r_4-t_4t_5r_2+t_5r_4r_2-t_8r_4-t_9r_2)u_{12}(t_1t_3^2r_4r_2-t_1t_3r_7r_2-t_2t_3r_5r_4+t_2r_7r_5-t_3t_4t_5r_2+t_3t_5r_4r_2-t_3t_8r_4-t_3t_9r_2+t_5t_6r_4+t_5t_7r_2-t_5r_7r_2+t_8r_7+t_3r_5r_4r_2-t_6r_5r_4-t_7r_5r_2+t_{10}r_5)$	$(q-1)^4$	q^6	q^6

u	${}^{x}u$	# classes	order	$\lvert C_{U_0}(u)\rvert$
$u_1(r_1)$	$u_1(r_1)u_5(-t_3r_1)u_8(-t_2t_3r_1-t_6r_1)u_9(-t_7r_1)u_{11}(-t_2t_7r_1-t_4t_6r_1-t_{10}r_1)u_{12}(-t_3t_4t_6r_1-t_3t_{10}r_1+t_6t_7r_1)$	$q-1$	q^4	q^8
$u_1(r_1)u_2(r_2)$	$u_1(r_1)u_2(r_2)u_5(-t_3r_1)u_6(-t_3r_2)u_8(-t_1t_3r_2-t_2t_3r_1+t_3r_2r_1-t_5r_2-t_6r_1)u_9(-t_7r_1)u_{10}(-t_7r_2)u_{11}(-t_1t_7r_2-t_2t_7r_1-t_4t_5r_2-t_4t_6r_1+t_7r_2r_1-t_9r_2-t_{10}r_1)u_{12}(-t_3t_4t_5r_2-t_3t_4t_6r_1+t_3t_7r_2r_1-t_3t_9r_2-t_3t_{10}r_1+t_5t_7r_2+t_6t_7r_1)$	$(q-1)^2$	q^4	q^8
$u_1(r_1)u_3(r_3)$	$u_1(r_1)u_3(r_3)u_5(t_1r_3-t_3r_1)u_6(t_2r_3)u_7(t_4r_3)u_8(t_1t_2r_3-t_2t_3r_1-t_6r_1)u_9(t_1t_4r_3-t_7r_1)u_{10}(t_2t_4r_3)u_{11}(t_1t_2t_4r_3-t_2t_7r_1-t_4t_6r_1-t_{10}r_1)u_{12}(t_1t_2t_4r_3^2-t_2t_3t_4r_3r_1-t_3t_4t_6r_1-t_3t_{10}r_1+t_6t_7r_1+t_{10}r_3r_1-t_{11}r_3)$	$(q-1)^2$	q^7	q^5
$u_1(r_1)u_4(r_4)$	$u_1(r_1)u_4(r_4)u_5(-t_3r_1)u_7(-t_3r_4)u_8(-t_2t_3r_1-t_6r_1)u_9(-t_1t_3r_4+t_3r_4r_1-t_5r_4-t_7r_1)u_{10}(-t_2t_3r_4-t_6r_4)u_{11}(-t_1t_2t_3r_4-t_1t_6r_4+t_2t_3r_4r_1-t_2t_5r_4-t_2t_7r_1-t_4t_6r_1+t_6r_4r_1-t_8r_4-t_{10}r_1)u_{12}(t_2t_3^2r_4r_1-t_3t_4t_6r_1+t_3t_6r_4r_1-t_3t_8r_4-t_3t_{10}r_1+t_5t_6r_4+t_6t_7r_1)$	$(q-1)^2$	q^4	q^8
$u_1(r_1)u_6(r_6)$	$u_1(r_1)u_5(-t_3r_1)u_6(r_6)u_8(t_1r_6-t_2t_3r_1-t_6r_1)u_9(-t_7r_1)u_{10}(t_4r_6)u_{11}(t_1t_4r_6-t_2t_7r_1-t_4t_6r_1-t_{10}r_1)u_{12}(-t_3t_4t_6r_1-t_3t_{10}r_1+t_6t_7r_1-t_7r_6r_1+t_9r_6)$	$(q-1)^2$	q^6	q^6
$u_1(r_1)u_7(r_7)$	$u_1(r_1)u_5(-t_3r_1)u_7(r_7)u_8(-t_2t_3r_1-t_6r_1)u_9(t_1r_7-t_7r_1)u_{10}(t_2r_7)u_{11}(t_1t_2r_7-t_2t_7r_1-t_4t_6r_1-t_{10}r_1)u_{12}(-t_2t_3r_7r_1-t_3t_4t_6r_1-t_3t_{10}r_1+t_6t_7r_1-t_6r_7r_1+t_8r_7)$	$(q-1)^2$	q^6	q^6
$u_1(r_1)u_{10}(r_{10})$	$u_1(r_1)u_5(-t_3r_1)u_8(-t_2t_3r_1-t_6r_1)u_9(-t_7r_1)u_{10}(r_{10})u_{11}(t_1r_{10}-t_2t_7r_1-t_4t_6r_1-t_{10}r_1)u_{12}(-t_3t_4t_6r_1-t_3t_{10}r_1-t_5r_{10}+t_6t_7r_1)$	$(q-1)^2$	q^5	q^7
$u_1(r_1)u_{12}(r_{12})$	$u_1(r_1)u_5(-t_3r_1)u_8(-t_2t_3r_1-t_6r_1)u_9(-t_7r_1)u_{11}(-t_2t_7r_1-t_4t_6r_1-t_{10}r_1)u_{12}(-t_3t_4t_6r_1-t_3t_{10}r_1+t_6t_7r_1+r_{12})$	$(q-1)^2$	q^4	q^8
$u_1(r_1)u_2(r_2)u_3(r_3)$	$u_1(r_1)u_2(r_2)u_3(r_3)u_5(t_1r_3-t_3r_1)u_6(t_2r_3-t_3r_2)u_7(t_4r_3)u_8(t_1t_2r_3-t_1t_3r_2-t_2t_3r_1+t_3r_2r_1-t_5r_2-t_6r_1)u_9(t_1t_4r_3-t_7r_1)u_{10}(t_2t_4r_3-t_7r_2)u_{11}(t_1t_2t_4r_3-t_1t_7r_2-t_2t_7r_1-t_4t_5r_2-t_4t_6r_1+t_7r_2r_1-t_9r_2-t_{10}r_1)u_{12}(t_1t_2t_4r_3^2-t_1t_3t_4r_3r_2-t_2t_3t_4r_3r_1-t_3t_4t_5r_2-t_3t_4t_6r_1+t_3t_7r_2r_1-t_3t_9r_2-t_3t_{10}r_1+t_5t_7r_2+t_6t_7r_1+t_3t_4r_3r_2r_1-t_7r_3r_2r_1+t_9r_3r_2+t_{10}r_3r_1-t_{11}r_3)$	$(q-1)^3$	q^7	q^5
$u_1(r_1)u_2(r_2)u_4(r_4)$	$u_1(r_1)u_2(r_2)u_4(r_4)u_5(-t_3r_1)u_6(-t_3r_2)u_7(-t_3r_4)u_8(-t_1t_3r_2-t_2t_3r_1+t_3r_2r_1-t_5r_2-t_6r_1)$ $u_9(-t_1t_3r_4+t_3r_4r_1-t_5r_4-t_7r_1)u_{10}(-t_2t_3r_4+t_3r_4r_2-t_6r_4-t_7r_2)u_{11}(-t_1t_2t_3r_4+t_1t_3r_4r_2-t_1t_6r_4-t_1t_7r_2+t_2t_3r_4r_1-t_2t_5r_4-t_2t_7r_1-t_4t_5r_2-t_4t_6r_1-t_3r_4r_2r_1+t_5r_4r_2+t_6r_4r_1-t_8r_4+t_7r_2r_1-t_9r_2-t_{10}r_1)u_{12}(t_1t_3^2r_4r_2+t_2t_3^2r_4r_1-t_3t_4t_5r_2-t_3t_4t_6r_1-2t_3^2r_4r_2r_1+t_3t_5r_4r_2+t_3t_6r_4r_1-t_3t_8r_4+t_3t_7r_2r_1-t_3t_9r_2-t_3t_{10}r_1+t_5t_6r_4+t_5t_7r_2+t_6t_7r_1)$	$(q-1)^3$	q^5	q^7
$u_1(r_1)u_2(r_2)u_6(r_6)$	$u_1(r_1)u_2(r_2)u_5(-t_3r_1)u_6(-t_3r_2+r_6)u_8(-t_1t_3r_2+t_1r_6-t_2t_3r_1+t_3r_2r_1-t_5r_2-t_6r_1)u_9(-t_7r_1)u_{10}(t_4r_6-t_7r_2)u_{11}(t_1t_4r_6-t_1t_7r_2-t_2t_7r_1-t_4t_5r_2-t_4t_6r_1+t_7r_2r_1-t_9r_2-t_{10}r_1)u_{12}(-t_3t_4t_5r_2-t_3t_4t_6r_1+t_3t_7r_2r_1-t_3t_9r_2-t_3t_{10}r_1+t_5t_7r_2+t_6t_7r_1-t_7r_6r_1+t_9r_6)$	$(q-1)^3$	q^6	q^6
$u_1(r_1)u_2(r_2)u_7(r_7)$	$u_1(r_1)u_2(r_2)u_5(-t_3r_1)u_6(-t_3r_2)u_7(r_7)u_8(-t_1t_3r_2-t_2t_3r_1+t_3r_2r_1-t_5r_2-t_6r_1)u_9(t_1r_7-t_7r_1)u_{10}(t_2r_7-t_7r_2)u_{11}(t_1t_2r_7-t_1t_7r_2-t_2t_7r_1-t_4t_5r_2-t_4t_6r_1+t_7r_2r_1-t_9r_2-t_{10}r_1)u_{12}(-t_1t_3r_7r_2-t_2t_3r_7r_1-t_3t_4t_5r_2-t_3t_4t_6r_1+t_3t_7r_2r_1-t_3t_9r_2-t_3t_{10}r_1+t_5t_7r_2+t_6t_7r_1+t_3r_7r_2r_1-t_5r_7r_2-t_6r_7r_1+t_8r_7)$	$(q-1)^3$	q^6	q^6
$u_1(r_1)u_2(r_2)u_{10}(r_{10})$	$u_1(r_1)u_2(r_2)u_5(-t_3r_1)u_6(-t_3r_2)u_8(-t_1t_3r_2-t_2t_3r_1+t_3r_2r_1-t_5r_2-t_6r_1)u_9(-t_7r_1)u_{10}(-t_7r_2+r_{10})u_{11}(-t_1t_7r_2+t_1r_{10}-t_2t_7r_1-t_4t_5r_2-t_4t_6r_1+t_7r_2r_1-t_9r_2-t_{10}r_1)u_{12}(-t_3t_4t_5r_2-t_3t_4t_6r_1+t_3t_7r_2r_1-t_3t_9r_2-t_3t_{10}r_1+t_5t_7r_2-t_5r_{10}+t_6t_7r_1)$	$(q-1)^3$	q^5	q^7
$u_1(r_1)u_2(r_2)u_{12}(r_{12})$	$u_1(r_1)u_2(r_2)u_5(-t_3r_1)u_6(-t_3r_2)u_8(-t_1t_3r_2-t_2t_3r_1+t_3r_2r_1-t_5r_2-t_6r_1)u_9(-t_7r_1)u_{10}(-t_7r_2)u_{11}(-t_1t_7r_2-t_2t_7r_1-t_4t_5r_2-t_4t_6r_1+t_7r_2r_1-t_9r_2-t_{10}r_1)u_{12}(-t_3t_4t_5r_2-t_3t_4t_6r_1+t_3t_7r_2r_1-t_3t_9r_2-t_3t_{10}r_1+t_5t_7r_2+t_6t_7r_1+r_{12})$	$(q-1)^3$	q^4	q^8
$u_1(r_1)u_3(r_3)u_4(r_4)$	$u_1(r_1)u_3(r_3)u_4(r_4)u_5(t_1r_3-t_3r_1)u_6(t_2r_3)u_7(-t_3r_4+t_4r_3)u_8(t_1t_2r_3-t_2t_3r_1-t_6r_1)u_9(-t_1r_4r_3-t_1t_3r_4+t_1t_4r_3+t_3r_4r_1-t_5r_4-t_7r_1)u_{10}(-t_2r_4r_3-t_2t_3r_4+t_2t_4r_3-t_6r_4)u_{11}(-t_1t_2r_4r_3-t_1t_2t_3r_4+t_1t_2t_4r_3-t_1t_6r_4+t_2t_3r_4r_1-t_2t_5r_4-t_2t_7r_1-t_4t_6r_1+t_6r_4r_1-t_8r_4-t_{10}r_1)u_{12}(-t_1t_2r_4r_3^2-t_1t_2t_3r_4r_3+t_1t_2t_4r_3^2+t_2t_3r_4r_3r_1+t_2t_3^2r_4r_1-t_2t_3t_4r_3r_1-t_3t_4t_6r_1+t_3t_6r_4r_1-t_3t_8r_4-t_3t_{10}r_1+t_5t_6r_4+t_6t_7r_1+t_{10}r_3r_1-t_{11}r_3)$	$(q-1)^3$	q^7	q^5
$u_1(r_1)u_3(r_3)u_{10}(r_{10})$	$u_1(r_1)u_3(r_3)u_5(t_1r_3-t_3r_1)u_6(t_2r_3)u_7(t_4r_3)u_8(t_1t_2r_3-t_2t_3r_1-t_6r_1)u_9(t_1t_4r_3-t_7r_1)u_{10}(t_2t_4r_3+r_{10})u_{11}(t_1t_2t_4r_3+t_1r_{10}-t_2t_7r_1-t_4t_6r_1-t_{10}r_1)u_{12}(t_1t_2t_4r_3^2-t_2t_3t_4r_3r_1-t_3t_4t_6r_1-t_3t_{10}r_1-t_5r_{10}+t_6t_7r_1+t_{10}r_3r_1-t_{11}r_3)$	$(q-1)^3$	q^7	q^5
$u_1(r_1)u_4(r_4)u_6(r_6)$	$u_1(r_1)u_4(r_4)u_5(-t_3r_1)u_6(r_6)u_7(-t_3r_4)u_8(t_1r_6-t_2t_3r_1-t_6r_1)u_9(-t_1t_3r_4+t_3r_4r_1-t_5r_4-t_7r_1)$ $u_{10}(-t_2t_3r_4+t_4r_6-t_6r_4)u_{11}(-t_1t_2t_3r_4+t_1t_4r_6-t_1t_6r_4+t_2t_3r_4r_1-t_2t_5r_4-t_2t_7r_1-t_4t_6r_1+t_6r_4r_1-t_8r_4-t_{10}r_1)u_{12}(-t_1t_3r_6r_4+t_2t_3^2r_4r_1-t_3t_4t_6r_1+t_3t_6r_4r_1-t_3t_8r_4-t_3t_{10}r_1+t_5t_6r_4+t_6t_7r_1+t_3r_6r_4r_1-t_5r_6r_4-t_7r_6r_1+t_9r_6)$	$(q-1)^3$	q^6	q^6
$u_1(r_1)u_4(r_4)u_7(r_7)$	$u_1(r_1)u_4(r_4)u_5(-t_3r_1)u_7(-t_3r_4+r_7)u_8(-t_2t_3r_1-t_6r_1)u_9(-t_1t_3r_4+t_1r_7+t_3r_4r_1-t_5r_4-t_7r_1)$ $u_{10}(-t_2t_3r_4+t_2r_7-t_6r_4)u_{11}(-t_1t_2t_3r_4+t_1t_2r_7-t_1t_6r_4+t_2t_3r_4r_1-t_2t_5r_4-t_2t_7r_1-t_4t_6r_1+t_6r_4r_1-t_8r_4-t_{10}r_1)u_{12}(t_2t_3^2r_4r_1-t_2t_3r_7r_1-t_3t_4t_6r_1+t_3t_6r_4r_1-t_3t_8r_4-t_3t_{10}r_1+t_5t_6r_4+t_6t_7r_1-t_6r_7r_1+t_8r_7)$	$(q-1)^3$	q^6	q^6
$u_1(r_1)u_4(r_4)u_{10}(r_{10})$	$u_1(r_1)u_4(r_4)u_5(-t_3r_1)u_7(-t_3r_4)u_8(-t_2t_3r_1-t_6r_1)u_9(-t_1t_3r_4+t_3r_4r_1-t_5r_4-t_7r_1)u_{10}(-t_2t_3r_4-t_6r_4+r_{10})u_{11}(-t_1t_2t_3r_4-t_1t_6r_4+t_1r_{10}+t_2t_3r_4r_1-t_2t_5r_4-t_2t_7r_1-t_4t_6r_1+t_6r_4r_1-t_8r_4-t_{10}r_1)u_{12}(t_2t_3^2r_4r_1-t_3t_4t_6r_1+t_3t_6r_4r_1-t_3t_8r_4-t_3t_{10}r_1+t_5t_6r_4-t_5r_{10}+t_6t_7r_1)$	$(q-1)^3$	q^5	q^7
$u_1(r_1)u_4(r_4)u_{12}(r_{12})$	$u_1(r_1)u_4(r_4)u_5(-t_3r_1)u_7(-t_3r_4)u_8(-t_2t_3r_1-t_6r_1)u_9(-t_1t_3r_4+t_3r_4r_1-t_5r_4-t_7r_1)u_{10}(-t_2t_3r_4-t_6r_4)u_{11}(-t_1t_2t_3r_4-t_1t_6r_4+t_2t_3r_4r_1-t_2t_5r_4-t_2t_7r_1-t_4t_6r_1+t_6r_4r_1-t_8r_4-t_{10}r_1)u_{12}(t_2t_3^2r_4r_1-t_3t_4t_6r_1+t_3t_6r_4r_1-t_3t_8r_4-t_3t_{10}r_1+t_5t_6r_4+t_6t_7r_1+r_{12})$	$(q-1)^3$	q^4	q^8
$u_1(r_1)u_6(r_6)u_7(r_7)$	$u_1(r_1)u_5(-t_3r_1)u_6(r_6)u_7(r_7)u_8(t_1r_6-t_2t_3r_1-t_6r_1)u_9(t_1r_7-t_7r_1)u_{10}(t_2r_7+t_4r_6)u_{11}(t_1t_2r_7+t_1t_4r_6-t_2t_7r_1-t_4t_6r_1-t_{10}r_1)u_{12}(t_1r_7r_6-t_2t_3r_7r_1-t_3t_4t_6r_1-t_3t_{10}r_1+t_6t_7r_1-t_6r_7r_1+t_8r_7-t_7r_6r_1+t_9r_6)$	$(q-1)^3$	q^6	q^6

u	${}^{x}u$	# classes	order	$\lvert C_{U_0}(u)\rvert$
$u_1(r_1)u_2(r_2)u_3(r_3)u_4(r_4)$	$u_1(r_1)u_2(r_2)u_3(r_3)u_4(r_4)u_5(t_1r_3 - t_3r_1)u_6(t_2r_3 - t_3r_2)u_7(-t_3r_4 + t_4r_3)u_8(t_1t_2r_3 - t_1t_3r_2 - t_2t_3r_1 + t_3r_2r_1 - t_5r_2 - t_6r_1)u_9(-t_1r_4r_3 - t_1t_3r_4 + t_1t_4r_3 + t_3r_4r_1 - t_5r_4 - t_7r_1)u_{10}(-t_2r_4r_3 - t_2t_3r_4 + t_2t_4r_3 + t_3r_4r_2 - t_6r_4 - t_7r_2)u_{11}(-t_1t_2r_4r_3 - t_1t_2t_3r_4 + t_1t_2t_4r_3 + t_1t_3r_4r_2 - t_1t_6r_4 - t_1t_7r_2 + t_2t_3r_4r_1 - t_2t_5r_4 - t_2t_7r_1 - t_4t_5r_2 - t_4t_6r_1 - t_3r_4r_2r_1 + t_5r_4r_2 + t_6r_4r_1 - t_8r_4 + t_7r_2r_1 - t_9r_2 - t_{10}r_1)u_{12}(-t_1t_2r_4r_3^2 + t_1t_3r_4r_3r_2 - t_1t_2t_3r_4r_3 + t_1t_3^2r_4r_2 + t_1t_2t_4r_3^2 - t_1t_3t_4r_3r_2 + t_2t_3r_4r_3r_1 + t_2t_3^2r_4r_1 - t_2t_3t_4r_3r_1 - t_3t_4t_5r_2 - t_3t_4t_6r_1 - 2t_3^2r_4r_2r_1 + t_3t_5r_4r_2 + t_3t_6r_4r_1 - t_3t_8r_4 + t_3t_7r_2r_1 - t_3t_9r_2 - t_3t_{10}r_1 + t_5t_6r_4 + t_5t_7r_2 + t_6t_7r_1 + t_3t_4r_3r_2r_1 - t_7r_3r_2r_1 + t_9r_3r_2 + t_{10}r_3r_1 - t_{11}r_3)$	$(q-1)^4$	q^8	q^4
$u_1(r_1)u_2(r_2)u_3(r_3)u_{10}(r_{10})$	$u_1(r_1)u_2(r_2)u_3(r_3)u_5(t_1r_3 - t_3r_1)u_6(t_2r_3 - t_3r_2)u_7(t_4r_3)u_8(t_1t_2r_3 - t_1t_3r_2 - t_2t_3r_1 + t_3r_2r_1 - t_5r_2 - t_6r_1)u_9(t_1t_4r_3 - t_7r_1)u_{10}(t_2t_4r_3 - t_7r_2 + r_{10})u_{11}(t_1t_2t_4r_3 - t_1t_7r_2 + t_1r_{10} - t_2t_7r_1 - t_4t_5r_2 - t_4t_6r_1 + t_7r_2r_1 - t_9r_2 - t_{10}r_1)u_{12}(t_1t_2t_4r_3^2 - t_1t_3t_4r_3r_2 - t_2t_3t_4r_3r_1 - t_3t_4t_5r_2 - t_3t_4t_6r_1 + t_3t_7r_2r_1 - t_3t_9r_2 - t_3t_{10}r_1 + t_5t_7r_2 - t_5r_{10} + t_6t_7r_1 + t_3t_4r_3r_2r_1 - t_7r_3r_2r_1 + t_9r_3r_2 + t_{10}r_3r_1 - t_{11}r_3)$	$(q-1)^4$	q^7	q^5
$u_1(r_1)u_2(r_2)u_4(r_4)u_6(r_6)$	$u_1(r_1)u_2(r_2)u_4(r_4)u_5(-t_3r_1)u_6(-t_3r_2 + r_6)u_7(-t_3r_4)u_8(-t_1t_3r_2 + t_1r_6 - t_2t_3r_1 + t_3r_2r_1 - t_5r_2 - t_6r_1)u_9(-t_1t_3r_4 + t_3r_4r_1 - t_5r_4 - t_7r_1)u_{10}(-t_2t_3r_4 + t_4r_6 + t_3r_4r_2 - t_6r_4 - t_7r_2)u_{11}(-t_1t_2t_3r_4 + t_1t_4r_6 + t_1t_3r_4r_2 - t_1t_6r_4 - t_1t_7r_2 + t_2t_3r_4r_1 - t_2t_5r_4 - t_2t_7r_1 - t_4t_5r_2 - t_4t_6r_1 - t_3r_4r_2r_1 + t_5r_4r_2 + t_6r_4r_1 - t_8r_4 + t_7r_2r_1 - t_9r_2 - t_{10}r_1)u_{12}(t_1t_3^2r_4r_2 - t_1t_3r_6r_4 + t_2t_3^2r_4r_1 - t_3t_4t_5r_2 - t_3t_4t_6r_1 - 2t_3^2r_4r_2r_1 + t_3t_5r_4r_2 + t_3t_6r_4r_1 - t_3t_8r_4 + t_3t_7r_2r_1 - t_3t_9r_2 - t_3t_{10}r_1 + t_5t_6r_4 + t_5t_7r_2 + t_6t_7r_1 + t_3r_6r_4r_1 - t_5r_6r_4 - t_7r_6r_1 + t_9r_6)$	$(q-1)^4$	q^6	q^6
$u_1(r_1)u_2(r_2)u_4(r_4)u_7(r_7)$	$u_1(r_1)u_2(r_2)u_4(r_4)u_5(-t_3r_1)u_6(-t_3r_2)u_7(-t_3r_4 + r_7)u_8(-t_1t_3r_2 - t_2t_3r_1 + t_3r_2r_1 - t_5r_2 - t_6r_1)u_9(-t_1t_3r_4 + t_1r_7 + t_3r_4r_1 - t_5r_4 - t_7r_1)u_{10}(-t_2t_3r_4 + t_2r_7 + t_3r_4r_2 - t_6r_4 - t_7r_2)u_{11}(-t_1t_2t_3r_4 + t_1t_2r_7 + t_1t_3r_4r_2 - t_1t_6r_4 - t_1t_7r_2 + t_2t_3r_4r_1 - t_2t_5r_4 - t_2t_7r_1 - t_4t_5r_2 - t_4t_6r_1 - t_3r_4r_2r_1 + t_5r_4r_2 + t_6r_4r_1 - t_8r_4 + t_7r_2r_1 - t_9r_2 - t_{10}r_1)u_{12}(t_1t_3^2r_4r_2 - t_1t_3r_7r_2 + t_2t_3^2r_4r_1 - t_2t_3r_7r_1 - t_3t_4t_5r_2 - t_3t_4t_6r_1 - 2t_3^2r_4r_2r_1 + t_3t_5r_4r_2 + t_3t_6r_4r_1 - t_3t_8r_4 + t_3t_7r_2r_1 - t_3t_9r_2 - t_3t_{10}r_1 + t_5t_6r_4 + t_5t_7r_2 + t_6t_7r_1 + t_3r_7r_2r_1 - t_5r_7r_2 - t_6r_7r_1 + t_8r_7)$	$(q-1)^4$	q^6	q^6
$u_1(r_1)u_2(r_2)u_4(r_4)u_{12}(r_{12})$	$u_1(r_1)u_2(r_2)u_4(r_4)u_5(-t_3r_1)u_6(-t_3r_2)u_7(-t_3r_4)u_8(-t_1t_3r_2 - t_2t_3r_1 + t_3r_2r_1 - t_5r_2 - t_6r_1)u_9(-t_1t_3r_4 + t_3r_4r_1 - t_5r_4 - t_7r_1)u_{10}(-t_2t_3r_4 + t_3r_4r_2 - t_6r_4 - t_7r_2)u_{11}(-t_1t_2t_3r_4 + t_1t_3r_4r_2 - t_1t_6r_4 - t_1t_7r_2 + t_2t_3r_4r_1 - t_2t_5r_4 - t_2t_7r_1 - t_4t_5r_2 - t_4t_6r_1 - t_3r_4r_2r_1 + t_5r_4r_2 + t_6r_4r_1 - t_8r_4 + t_7r_2r_1 - t_9r_2 - t_{10}r_1)u_{12}(t_1t_3^2r_4r_2 + t_2t_3^2r_4r_1 - t_3t_4t_5r_2 - t_3t_4t_6r_1 - 2t_3^2r_4r_2r_1 + t_3t_5r_4r_2 + t_3t_6r_4r_1 - t_3t_8r_4 + t_3t_7r_2r_1 - t_3t_9r_2 - t_3t_{10}r_1 + t_5t_6r_4 + t_5t_7r_2 + t_6t_7r_1 + r_{12})$	$(q-1)^4$	q^5	q^7
$u_1(r_1)u_2(r_2)u_6(r_6)u_7(r_7)$	$u_1(r_1)u_2(r_2)u_5(-t_3r_1)u_6(-t_3r_2 + r_6)u_7(r_7)u_8(-t_1t_3r_2 + t_1r_6 - t_2t_3r_1 + t_3r_2r_1 - t_5r_2 - t_6r_1)u_9(t_1r_7 - t_7r_1)u_{10}(t_2r_7 + t_4r_6 - t_7r_2)u_{11}(t_1t_2r_7 + t_1t_4r_6 - t_1t_7r_2 - t_2t_7r_1 - t_4t_5r_2 - t_4t_6r_1 + t_7r_2r_1 - t_9r_2 - t_{10}r_1)u_{12}(-t_1t_3r_7r_2 + t_1r_7r_6 - t_2t_3r_7r_1 - t_3t_4t_5r_2 - t_3t_4t_6r_1 + t_3t_7r_2r_1 - t_3t_9r_2 - t_3t_{10}r_1 + t_5t_7r_2 + t_6t_7r_1 + t_3r_7r_2r_1 - t_5r_7r_2 - t_6r_7r_1 + t_8r_7 - t_7r_6r_1 + t_9r_6)$	$(q-1)^4$	q^6	q^6
$u_1(r_1)u_3(r_3)u_4(r_4)u_{10}(r_{10})$	$u_1(r_1)u_3(r_3)u_4(r_4)u_5(t_1r_3 - t_3r_1)u_6(t_2r_3)u_7(-t_3r_4 + t_4r_3)u_8(t_1t_2r_3 - t_2t_3r_1 - t_6r_1)u_9(-t_1r_4r_3 - t_1t_3r_4 + t_1t_4r_3 + t_3r_4r_1 - t_5r_4 - t_7r_1)u_{10}(-t_2r_4r_3 - t_2t_3r_4 + t_2t_4r_3 - t_6r_4 + r_{10})u_{11}(-t_1t_2r_4r_3 - t_1t_2t_3r_4 + t_1t_2t_4r_3 - t_1t_6r_4 + t_1r_{10} + t_2t_3r_4r_1 - t_2t_5r_4 - t_2t_7r_1 - t_4t_6r_1 + t_6r_4r_1 - t_8r_4 - t_{10}r_1)u_{12}(-t_1t_2r_4r_3^2 - t_1t_2t_3r_4r_3 + t_1t_2t_4r_3^2 + t_2t_3r_4r_3r_1 + t_2t_3^2r_4r_1 - t_2t_3t_4r_3r_1 - t_3t_4t_6r_1 + t_3t_6r_4r_1 - t_3t_8r_4 - t_3t_{10}r_1 + t_5t_6r_4 - t_5r_{10} + t_6t_7r_1 + t_{10}r_3r_1 - t_{11}r_3)$	$(q-1)^4$	q^7	q^5
$u_1(r_1)u_4(r_4)u_6(r_6)u_7(r_7)$	$u_1(r_1)u_4(r_4)u_5(-t_3r_1)u_6(r_6)u_7(-t_3r_4 + r_7)u_8(t_1r_6 - t_2t_3r_1 - t_6r_1)u_9(-t_1t_3r_4 + t_1r_7 + t_3r_4r_1 - t_5r_4 - t_7r_1)u_{10}(-t_2t_3r_4 + t_2r_7 + t_4r_6 - t_6r_4)u_{11}(-t_1t_2t_3r_4 + t_1t_2r_7 + t_1t_4r_6 - t_1t_6r_4 + t_2t_3r_4r_1 - t_2t_5r_4 - t_2t_7r_1 - t_4t_6r_1 + t_6r_4r_1 - t_8r_4 - t_{10}r_1)u_{12}(-t_1t_3r_6r_4 + t_1r_7r_6 + t_2t_3^2r_4r_1 - t_2t_3r_7r_1 - t_3t_4t_6r_1 + t_3t_6r_4r_1 - t_3t_8r_4 - t_3t_{10}r_1 + t_5t_6r_4 + t_6t_7r_1 + t_3r_6r_4r_1 - t_5r_6r_4 - t_6r_7r_1 + t_8r_7 - t_7r_6r_1 + t_9r_6)$	$(q-1)^4$	q^6	q^6
$u_1(r_1)u_2(r_2)u_4(r_4)u_6(r_6)u_7(r_7)$	$u_1(r_1)u_2(r_2)u_4(r_4)u_5(-t_3r_1)u_6(-t_3r_2 + r_6)u_7(-t_3r_4 + r_7)u_8(-t_1t_3r_2 + t_1r_6 - t_2t_3r_1 + t_3r_2r_1 - t_5r_2 - t_6r_1)u_9(-t_1t_3r_4 + t_1r_7 + t_3r_4r_1 - t_5r_4 - t_7r_1)u_{10}(-t_2t_3r_4 + t_2r_7 + t_4r_6 + t_3r_4r_2 - t_6r_4 - t_7r_2)u_{11}(-t_1t_2t_3r_4 + t_1t_2r_7 + t_1t_4r_6 + t_1t_3r_4r_2 - t_1t_6r_4 - t_1t_7r_2 + t_2t_3r_4r_1 - t_2t_5r_4 - t_2t_7r_1 - t_4t_5r_2 - t_4t_6r_1 - t_3r_4r_2r_1 + t_5r_4r_2 + t_6r_4r_1 - t_8r_4 + t_7r_2r_1 - t_9r_2 - t_{10}r_1)u_{12}(t_1t_3^2r_4r_2 - t_1t_3r_6r_4 - t_1t_3r_7r_2 + t_1r_7r_6 + t_2t_3^2r_4r_1 - t_2t_3r_7r_1 - t_3t_4t_5r_2 - t_3t_4t_6r_1 - 2t_3^2r_4r_2r_1 + t_3t_5r_4r_2 + t_3t_6r_4r_1 - t_3t_8r_4 + t_3t_7r_2r_1 - t_3t_9r_2 - t_3t_{10}r_1 + t_5t_6r_4 + t_5t_7r_2 + t_6t_7r_1 + t_3r_7r_2r_1 - t_5r_7r_2 - t_6r_7r_1 + t_8r_7 + t_3r_6r_4r_1 - t_5r_6r_4 - t_7r_6r_1 + t_9r_6)$	$(q-1)^5$	q^6	q^6

References

[Bo00] C. Bonnafé, Mackey formula in type *A*. *Proc. London Math. Soc.* **80** (2000), 545–574; Erratum, *ibid.* **86** (2003), 435–442.

[Bo04] C. Bonnafé, Éléments unipotents réguliers des sous-groupes de Levi. *Canad. J. Math.* **56** (2004), 246–276.

[Bo05] C. Bonnafé, Actions of relative Weyl groups II. *J. Group Theory* **8** (2005), no. 3, 351–387.

[Bo11] C. Bonnafé, *Representations of* $\mathrm{SL}_2(\mathbb{F}_q)$. Algebra and Applications, 13. Springer-Verlag, London, 2011.

[BrLue13] O. Brunat, F. Lübeck, On defining characteristic representations of finite reductive groups. *J. Algebra* **395** (2013), 121–141.

[Ca72] R. W. Carter, *Simple groups of Lie type.* Wiley, New York, 1972; reprinted 1989 as Wiley Classics Library Edition.

[Ca85] R. W. Carter, *Finite groups of Lie type: conjugacy classes and complex characters.* Wiley, New York, 1985.

[DLM92] F. Digne, G. I. Lehrer, J. Michel, The characters of the group of rational points of a reductive group with nonconnected centre. *J. Reine Angew. Math.* **425** (1992), 155–192.

[DLM97] F. Digne, G. I. Lehrer, J. Michel, On Gel'fand-Graev characters of reductive groups with disconnected centre. *J. Reine Angew. Math.* **491** (1997), 131–147.

[DiMi83] F. Digne, J. Michel, Foncteur de Lusztig et fonctions de Green généralisées. *C. R. Acad. Sci. Paris Sér. I Math.* **297** (1983), 89–92.

[DiMi87] F. Digne, J. Michel, Foncteurs de Lusztig et caractères des groupes linéaires et unitaires sur un corps fini. *J. Algebra* **107** (1987), 217–255.

[DiMi91] F. Digne, J. Michel, *Representations of finite groups of Lie type.* London Mathematical Society Student Texts, 21. Cambridge University Press, Cambridge, 1991.

[DiMi20] F. Digne, J. Michel, *Representations of finite groups of Lie type.* 2nd Edition. London Mathematical Society Student Texts, 95. Cambridge University Press, Cambridge, 2020.

[Ge17] M. Geck, Minuscule weights and Chevalley groups. *Finite simple groups: thirty years of the atlas and beyond*, 159–176, Contemp. Math., **694**, Amer. Math. Soc., Providence, RI, (2017).

[Ge18] M. Geck, *A first guide to the character theory of finite groups of Lie type.* Local representation theory and simple groups (eds. R. Kessar, G. Malle, D. Testerman), pp. 63–106, EMS Lecture Notes Series, EMS Publ. House, 2018.

[GHLMP] M. Geck, G. Hiss, F. Lübeck, G. Malle and G. Pfeiffer, CHEVIE–A system for computing and processing generic character tables for finite groups of Lie type, Weyl groups and Hecke algebras. *Appl. Algebra Engrg. Comm. Comput.* **7** (1996), 175–210; See also `www.math.rwth-aachen.de/~CHEVIE`.

[GeMa20] M. Geck, G. Malle, *The character theory of finite groups of Lie type: a guided tour*. Cambridge Studies in Advanced Mathematics, 187. Cambridge University Press, Cambridge, 2020.

[GePf92] M. Geck, G. Pfeiffer, Unipotent characters of the Chevalley groups $D_4(q)$, q odd. *Manuscripta Math.* **76** (1992), 281–304.

[GoRoe09] S. M. Goodwin, G. Röhrle, Calculating conjugacy classes in Sylow p-subgroups of finite Chevalley groups. *J. Algebra* **321** (2009), 3321–3334.

[GLS98] D. Gorenstein, R. Lyons, R. Solomon, *The classification of the finite simple groups*. Math. Surveys and Monographs vol. 40, no. 3, Amer. Math. Soc., 1998.

[Hu75] J. E. Humphreys, *Linear algebraic groups*. Graduate Texts in Mathematics vol. 21, Springer Verlag, Berlin–Heidelberg–New York, 1975.

[JaLi01] G. James, M. Liebeck, *Representations and characters of groups*. Second Edition, Cambridge University Press, New York, 2001.

[Ka85] N. Kawanaka, Generalized Gel'fand-Graev representations and Ennola duality. *Algebraic Groups and Related Topics* (Kyoto/Nagoya, 1983), 175–206, Adv. Stud. Pure Math., 6, North-Holland, Amsterdam, 1985.

[Is76] I. M. Isaacs, *Character theory of finite groups*. Academic Press, New York, 1976; corrected reprint by Dover Publ., New York, 1994.

[LiNi97] R. Lidl, H. Niederreiter, *Finite fields*. 2nd Edition. Encyclopedia of Mathematics and its Applications, vol. 20. Cambridge University Press, Cambridge, 1997.

[LiSe12] M. W. Liebeck, G. M. Seitz, *Unipotent and nilpotent classes in simple algebraic groups and Lie algebras*. Math. Surveys and Monographs, 180, Amer. Math. Soc., Providence, RI, 2012.

[Lue93] F. Lübeck, *Charaktertafeln für die Gruppen* $\mathrm{CSp}_6(q)$ *mit ungeradem* q *und* $\mathrm{Sp}_6(q)$ *mit geradem* q. Dissertation, Universität Heidelberg, 1993.

[Lue20] F. Lübeck, *Remarks on computing Green functions*. arXiv:2004.01686, 2020.

[Lu85] G. Lusztig, Character sheaves. II. *Adv. Math.* **57** (1985), 226–265.

[Lu86a] G. Lusztig, Character sheaves. IV. *Adv. Math.* **59** (1986), 1–63.

[Lu86b] G. Lusztig, Character sheaves. V. *Adv. Math.* **61** (1986), 103–155.

[LuSp79] G. Lusztig, N. Spaltenstein, Induced unipotent classes. *J. London Math. Soc. (2)* **19** (1979), 41–52.

[MaRo20] G. Malle, E. Rotilio, *The 2-parameter Green functions for 8-dimensional spin groups*. arXiv:2003.14231, 2020.

[MaTe11] G. Malle, D. Testerman, *Linear algebraic groups and finite groups of Lie type*. Cambridge Studies in Advanced Mathematics, 133. Cambridge University Press, Cambridge, 2011.

[MiChv] J. Michel, The development version of the CHEVIE package of GAP3. *J. Algebra* **435** (2015), 308–336;
See also `https://webusers.imj-prg.fr/~jean.michel/gap3/chevie.html`.

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