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"ON THE COMPUTATION OF STRESS IN STATIONARY LOADED JOURNAL BEARINGS"

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On the Computation of Stress in Stationary Loaded Journal Bearings

by

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Abstract. In this paper, we deal with the problem of computing the stresses in stationary loaded journal bearings. A method to obtain the pressure in the lubrication fluid, which is given as a solution of Reynolds's differential equation, is presented. Furthermore, using the theory of plain stress, the stresses in the bearing shell are described by derivatives of biharmonic functions. A spline interpolation method for computing these functions is developed and an estimate for the error on the boundaries is presented. Finally the described methods are tested theoretically as well as with real examples.

Introduction

This paper is concerned with the computation of stresses in stationary loaded, hydrodynamic journal bearings. This type of bearing is used, wherever highly loaded axes have to be rested almost free of friction. Examples are axes in turbines and generators or crankshafts in combustion engines. The layout of such components with regard to the permissible stresses and the stability is one of the main tasks during the construction. In order to save time and costs, engineers at the department for mechanical engineering of the university in Kaiserslautern try to do simulations to compute the stress distribution in such bearings. Knowing the stresses, they are able to predict the lifetime and the necessary dimensions of their constructions without testing them in reality. A contribution to that aim should be given in this work.

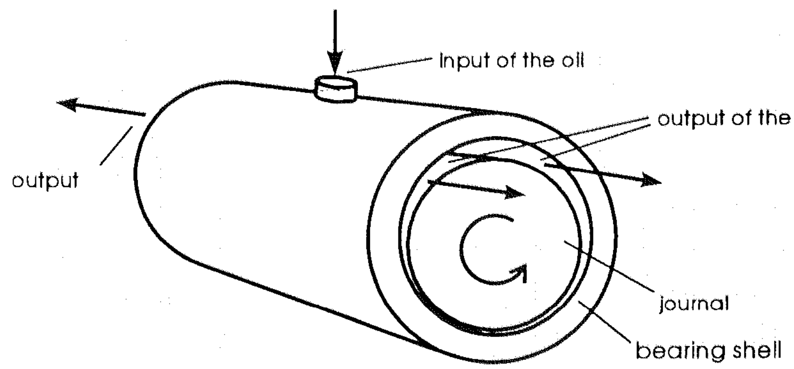
The computation of the stresses is divided into two parts. First the pressure in the lubrication fluid, which consists most times of oil, has to be computed. The necessary physical principles are well-known. Assumed that the pressure over the thickness of the lubrication film is constant, it follows the so called Reynolds's differential equation (cf. Lang[14]). The resulting free boundary value problem can be transformed into a linear complementarity problem (cf. Cimatti[2]), which can be solved with a point SOR-method (cf. Pang[17]). Knowing the pressure distribution in the lubrication fluid, the stress acting on the inner wall of the bearing shell is known. It is used to solve the second problem, which is concerned with the calculation of the stresses in the bearing shell. Because the maximum forces appear in the middle of the element, we can restrict the computation on a section through the bearing, which has the shape of a circular ring. If the material of the bearing shell is homogeneous and isotropic and if only small deformations are considered, Hooke's law can be used and the problem can be solved using the theory of linear elasticity (cf. Goeldner[8]). Mathematically this leads to the problem of finding a solution of the Bi-Laplace equation $\triangle\triangle F = 0$, from which the desired stresses can be obtained by applying certain differential operators (cf. Leipholz[13]).
While in the literature the solutions are only described by Fourier series, where the coefficients are obtained by comparison with the coefficients of the series representations of the boundary functions, in this work the computation will be done with help of a spline interpolation method (cf. Freeden[5]). Discretly given boundary values are interpolated in such a way, that the interpolant minimizes a given norm. Because this norm has some free parameters, it is possible to adapt the interpolation method to the given problem by choosing special norms. Beside this, there exists another significant difference between the spline interpolation and the Fourier method, namely the fact, that in the here used technique all frequencies are represented, whereas otherwise the series representation is cutted after a certain number of terms and the high frequency parts are lost.

The paper is organized as follows:
in the first chapter the principle structure and function of the machine element journal bearing is described. The problem of computing the oil pressure is treated, a solution method is described and demonstrated on two real examples. In Chapter 2 a short introduction to theory of linear elasticity is given. The third chapter is concerned with the main problem of the work, the computation of the stresses in the bearing shell. It is formulated from a physical, as well as from a mathematical point of view. In the fourth chapter basic settings and notations for the spline theory are presented, theorems about the representation of harmonic and biharmonic functions defined on circular areas are proved, the spline interpolation method is described theoretically and an estimate for the error on the boundaries is proved. A survey about the practical aspects of the method is following in the fifth chapter, where the choice of the norm is discussed, the method is tested with some theoretical examples and it is applied to one of the real configurations of Chapter 1. Finally a possibility for smoothing the result by a least square technique is described, where beside the boundary forces some more informations are used.

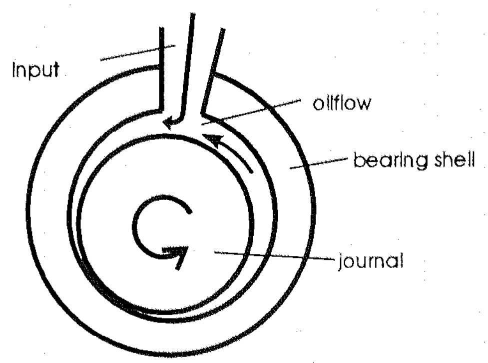
1 Hydrodynamic Lubrication

1.1 Description of the Pressure Build-Up

In order to compute the stresses in journal bearings, it is necessary to understand the principle structure and function of this machine element. The two following pictures are showing a three-dimensional representation and a section through the middle of such a bearing.



Picture 1: three-dimensional representation of a journal bearing



Picture 2: section through the middle of a journal bearing

Hydrodynamic working journal bearings are used to reduce the friction between two rotating machine parts. As shown in the pictures, oil is pumped into the bearing through a hole at the top. Due to the rotating journal, the oil is forced to run with high velocity in angular direction before it leaves the bearing at the endings. By the weight of the axis and additional outer forces, there is an excentricity between the journal and the bearing shell, that means that their rotation axes are parallel but don't coincide. Because of that, in the narrow gap between the journal and the bearing shell the oil is compressed and in the stationary case this pressure compensates the forces acting on the journal. It is 'swimming' on the lubrication fluid and the friction is drastically reduced.
In order to compute the stresses in the bearing shell, it is necessary to know the pressure distribution in the oil, which causes these stresses. As the thickness of the oil film is very small, the pressure can be assumed to be constant in radial direction. Thus it is a

function defined on the inner side of the bearing shell depending on φ and $\bar{z}$, that means

$$P : [0, 2\pi] \times [-1, 1] \to \boldsymbol{R}, \tag{1.1.1}$$

where the width of the bearing has been transformed on the intervall $[-1, 1]$.
The pressure can be described by the so called Reynolds's differential equation (cf. Lang[14]):

$$\frac{\partial}{\partial \varphi}\left(H^3 \frac{\partial \Pi}{\partial \varphi}\right) + \left(\frac{D}{B}\right)^2 \frac{\partial}{\partial \bar{z}}\left(H^3 \frac{\partial \Pi}{\partial \bar{z}}\right) = 6 \frac{\partial H}{\partial \varphi} \quad , \tag{1.1.2}$$

where the following notations have been used: $\Pi = \frac{P\Psi^2}{\eta\omega}$ for the pressure index, $H = \frac{h}{R-r} = 1 + \epsilon\cos(\varphi)$ for the relative height of the gap, $\Psi = \frac{R-r}{R}$ the relative bearing clearance, $\epsilon = \frac{e}{R-r}$ the relative excentricity, R the inner radius of the bearing shell, D the inner diameter of the bearing shell, B the width of the bearing, r the radius of the journal , η the viscosity of the lubrication fluid, ω the effective angular velocity and e the excentricity.
If Π is replaced by $\bar{\Pi} = \Pi H^{\frac{3}{2}}$, we get another form of Reynolds's differential equation, namely:

$$\frac{\partial^2}{\partial \varphi^2}\bar{\Pi} + \left(\frac{D}{B}\right)^2 \frac{\partial^2}{\partial \bar{z}^2}\bar{\Pi} + a(\varphi)\bar{\Pi} + b(\varphi) = 0 \tag{1.1.3}$$

with

$$a(\varphi) = \frac{3}{4}\,\frac{2\epsilon\cos(\varphi) - \epsilon^2 + 3\epsilon^2\cos(\varphi)^2}{(1+\epsilon\cos(\varphi))^2} \tag{1.1.4}$$

and

$$b(\varphi) = \frac{6\epsilon\sin(\varphi)}{(1+\epsilon\cos(\varphi))^{\frac{3}{2}}} \quad . \tag{1.1.5}$$

In order to compute the pressure distribution we have to solve this differential equation, where the solution has to fulfill the following conditions:

1. $P(\varphi, \pm 1) = 0 \qquad \forall \varphi \in \boldsymbol{R}$
2. $P(\varphi, z) = P(\varphi + 2\pi, z) \qquad \forall \varphi \in \boldsymbol{R},\ z \in [-1, 1]$
3. $\frac{\partial}{\partial\varphi}P(\varphi, z)\,|_{\varphi=\bar{\varphi}(z)} = P(\varphi, z)\,|_{\varphi=\bar{\varphi}(z)} = 0$ for a regular curve $\bar{\varphi}(z)$
4. $P(\varphi, z) \geq 0 \qquad \forall \varphi \in \boldsymbol{R},\ \ z \in [-1, 1]$

A complete derivation of this problem can be found in the book of Lang[14], with the only difference that he used the condition

$$P(0, z) = 0 \quad \forall z \in [-1, 1] \tag{1.1.6}$$

instead of the periodicity in angular direction. The condition was changed in this work, because equation (1.1.6) can't be pressumed without loss of generality, and therefore this condition is not representing the most general case.
To solve the problem, a method due to Christoffersen can be used (cf. Cimatti[2]). Coming from a finite difference scheme, we have to solve a linear complementarity problem of the form:

$$b + Ap \geq 0 \tag{1.1.7}$$

$$p \geq 0 \tag{1.1.8}$$

$$p^t(b + Ap) = 0\ , \tag{1.1.9}$$

where A is the matrix and b the right hand side of the difference scheme, and the vector p represents the pressure values in the grid points. This problem can be solved with a modified point-SOR-method (cf. Pang[17]). The vector p is computed iteratively by the following iteration scheme:

$$\hat{p}_i^{k+1} = -\left(b_i + \sum_{j<i} a_{ij} p_j^{k+1} + \sum_{j>i} a_{ij} p_j^k \right) / a_{ii} \tag{1.1.10}$$

$$p_i^{k+1} = max\{0, p_i^k + \tilde{\omega}(\hat{p}_i^{k+1} - p_i^k)\}\quad , \tag{1.1.11}$$

where $\tilde{\omega} \in (0,2)$ is a free parameter. Because of the special structure of the matrix A, it is only necessary to store the diagonal of A. The other not vanishing entries are limited to a few constants, which will be used directly by the specially developed algorithm. The iteration is stopped, if the difference between two vectors p^k and p^{k+1} in the infinity-norm is less than a given number ϵ.

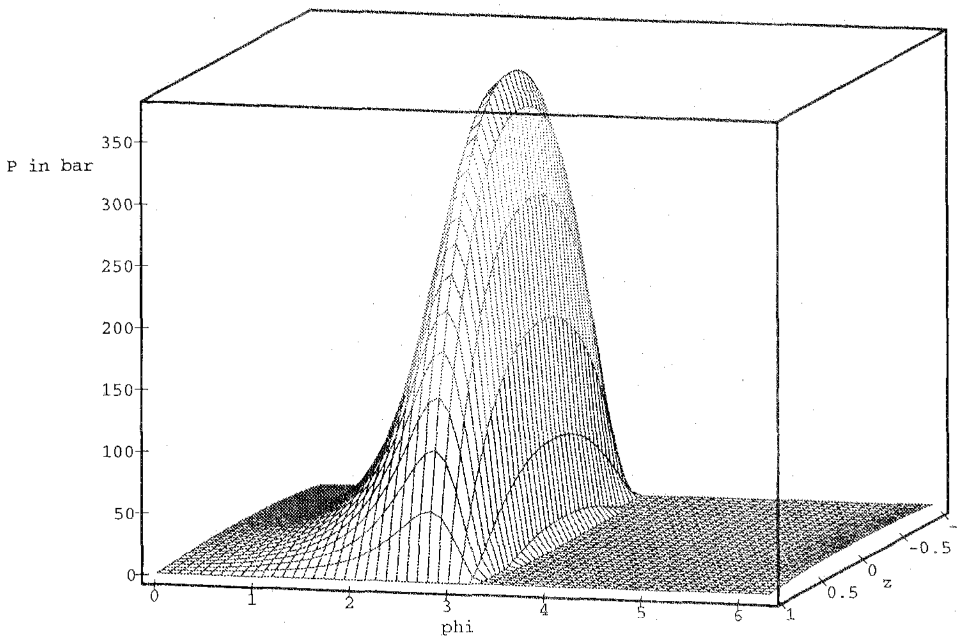
1.2 Examples

In this section, the above described method for computing the pressure distribution in a stationary loaded journal bearing will be applied to the real configurations given by:

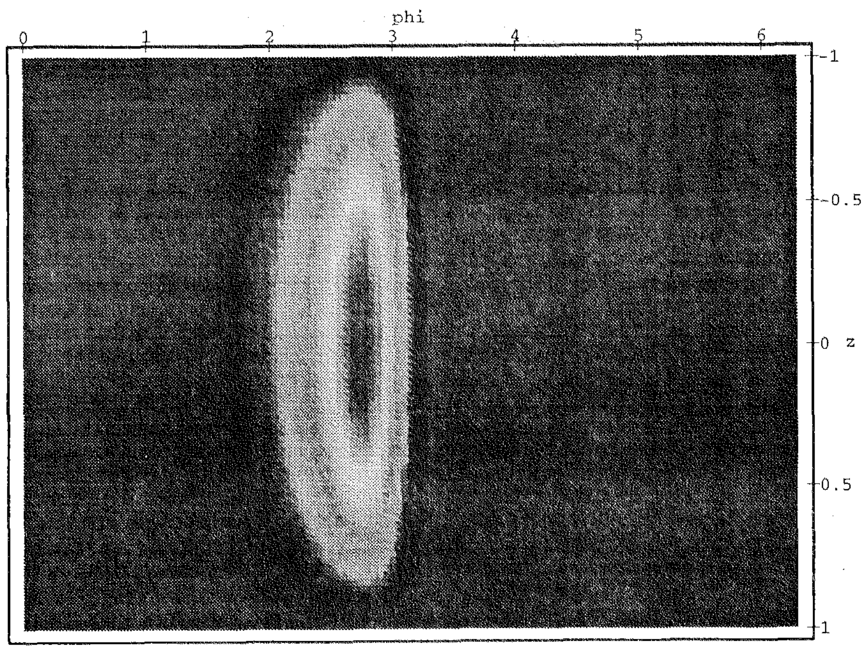
a) $\eta = 0.017\frac{Nsec}{m^2},\quad \omega = 200\frac{1}{sec},\quad \epsilon = 0.8,\quad \Psi = 0.001,\quad \frac{D}{B} = 1.0$

b) $\eta = 0.017\frac{Nsec}{m^2},\quad \omega = 200\frac{1}{sec},\quad \epsilon = 0.6,\quad \Psi = 0.001,\quad \frac{D}{B} = 0.8$

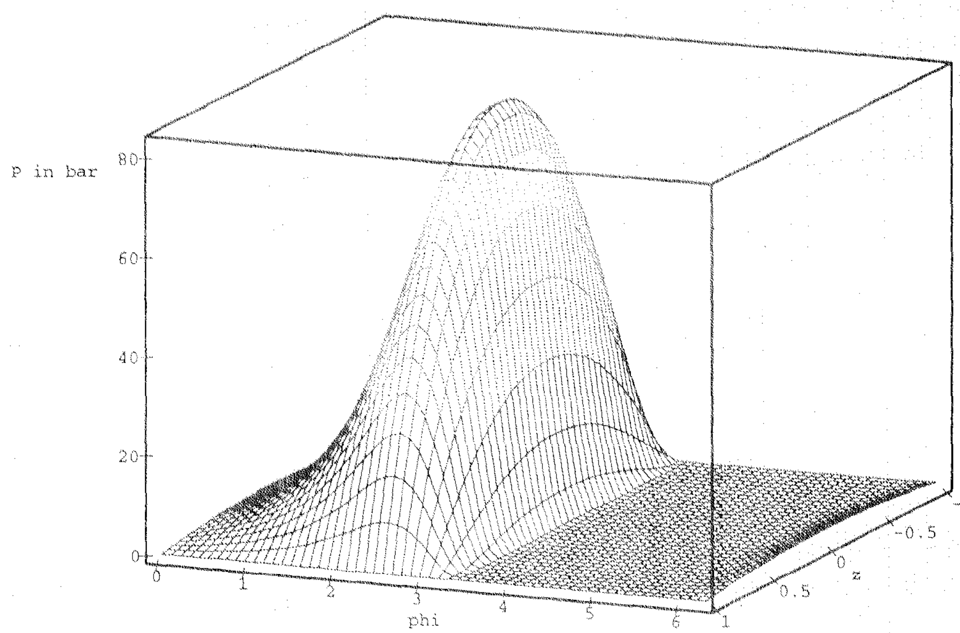
The computation was done with the following parameters:
The grid for the finite difference scheme was divided in angular direction into sixty and in radial direction into forty equidistant intervals, the parameter $\tilde{\omega}$ was set to 1.8. The computation was done with a machine accuracy of 10^{-16} and the iteration was stopped, if the difference between two pressure vectors in the infinity-norm $\|p^k - p^{k+1}\|_\infty$ was less than 10^{-15}. The resulting pressure distributions are illustrated on the next pages as three-dimensional plots as well as contour plots. The results were used in the further work as boundary values for computing the stresses in the bearing shell.



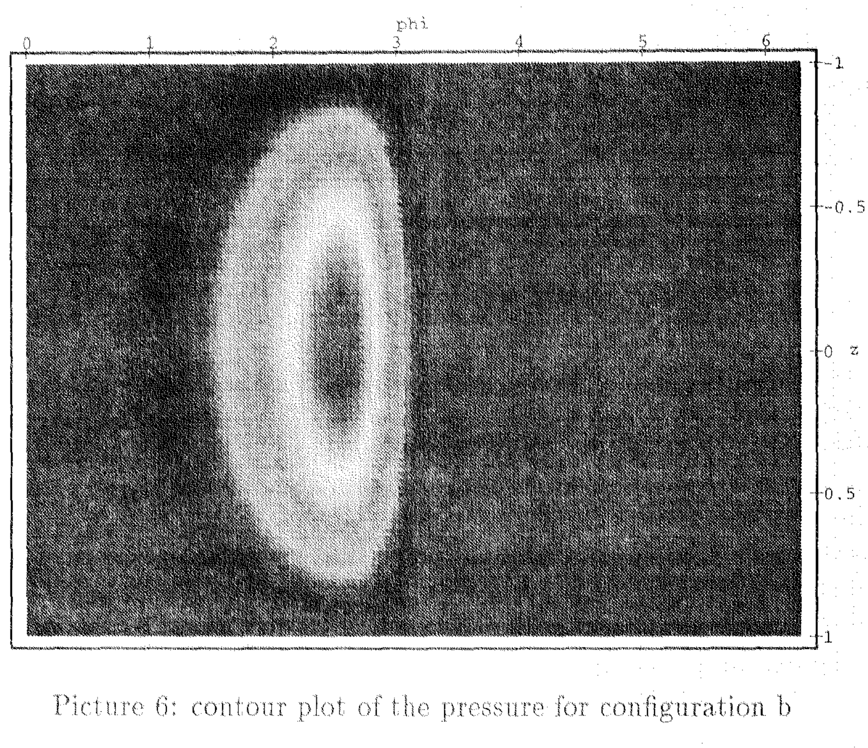
Picture 3: three dimensional plot of the pressure for configuration a



Picture 4: contour plot of the pressure for configuration a



Picture 5: three dimensional plot of the pressure for configuration b



Picture 6: contour plot of the pressure for configuration b

2 Basic Theory of Linear Elasticity

Linear elasticity is concerned with the deformations of solids under the influence of inner and outer forces. For simplicity we presume, that the material of the body is homogeneous and isotropic, and that only small deformations are considered.
In this chapter some basic results of linear elasticity are given, which will be used in this work. For a detailed explanation see the books of Leipholz[13] or Goeldner[8].
If a body is rested without motion, the sum of all acting forces has to be zero. Hence, if no outer forces are occuring, the so called equilibrium conditions

$$\frac{\partial}{\partial r} t_{rr} + \frac{1}{r}\frac{\partial}{\partial \varphi} t_{r\varphi} + \frac{\partial}{\partial z} t_{rz} + \frac{t_{rr} + t_{\varphi\varphi}}{r} = 0 \tag{2.1}$$

$$\frac{\partial}{\partial r} t_{r\varphi} + \frac{1}{r}\frac{\partial}{\partial \varphi} t_{\varphi\varphi} + \frac{\partial}{\partial z} t_{\varphi z} + \frac{2\, t_{r\varphi}}{r} = 0 \tag{2.2}$$

$$\frac{\partial}{\partial r} t_{rz} + \frac{1}{r}\frac{\partial}{\partial \varphi} t_{\varphi z} + \frac{\partial}{\partial z} t_{zz} + \frac{t_{rz}}{r} = 0 \tag{2.3}$$

can be deduced, where t denote the stresses. Analising the displacements v, the distortions ϵ and the angular deformations γ, the following geometric relations can be shown:

$$\epsilon_r = \frac{\partial}{\partial r} v_r \tag{2.4}$$

$$\epsilon_\varphi = \frac{v_r}{r} + \frac{1}{r}\frac{\partial}{\partial \varphi} v_\varphi \tag{2.5}$$

$$\epsilon_z = \frac{\partial}{\partial z} v_z \tag{2.6}$$

$$\gamma_{r\varphi} = \frac{1}{r}\frac{\partial}{\partial \varphi} v_r + \frac{\partial}{\partial r} v_\varphi - \frac{v_\varphi}{r} \tag{2.7}$$

$$\gamma_{\varphi z} = \frac{\partial}{\partial z} v_\varphi + \frac{1}{r}\frac{\partial}{\partial \varphi} v_z \tag{2.8}$$

$$\gamma_{zr} = \frac{\partial}{\partial z} v_r + \frac{\partial}{\partial r} v_z\,, \tag{2.9}$$

which are connected with the equilibrium conditions by Hooke's law:

$$\epsilon_r = \frac{1}{E}(\sigma_r - \nu(\sigma_\varphi + \sigma_z)) \tag{2.10}$$

$$\epsilon_\varphi = \frac{1}{E}(\sigma_\varphi - \nu(\sigma_r + \sigma_z)) \tag{2.11}$$

$$\epsilon_z = \frac{1}{E}(\sigma_z - \nu(\sigma_r + \sigma_\varphi)) \tag{2.12}$$

$$\gamma_{r\varphi} = \frac{1}{G}\tau_{r\varphi} \tag{2.13}$$

$$\gamma_{\varphi z} = \frac{1}{G}\tau_{\varphi z} \tag{2.14}$$

$$\gamma_{zr} = \frac{1}{G}\tau_{zr}\,. \tag{2.15}$$

E denotes the modulus of elasticity, $G = \frac{E}{2(1+\nu)}$ the shear modulus and ν the number of lateral deformation. It is possible to determine the elastic state of a body with this fifteen equations, but it can be shown, that in some cases the number of unknowns and equations can be reduced. This leads to the theory of plain stress, which is characterised by the assumption that σ_z, τ_{rz} and $\tau_{\varphi z}$ are identically zero, and that the other components are independent of z. This can be assumed if the body has the shape of a thin plate and the forces are acting only at the boundaries, parallel to the plate, as shown in Picture 7. In this case the system of equations can be simplified, and we have to solve the following equations:

$$\frac{\partial}{\partial r}\sigma_r + \frac{1}{r}\frac{\partial}{\partial \varphi}\tau_{r\varphi} + \frac{\sigma_r - \sigma_\varphi}{r} = 0 \tag{2.16}$$

$$\frac{\partial}{\partial r}\,\tau_{r\varphi} + \frac{1}{r}\,\frac{\partial}{\partial\varphi}\,\sigma_\varphi + \frac{2\,\tau_{r\varphi}}{r} \;=\; 0 \tag{2.17}$$

$$\epsilon_r \;=\; \frac{1}{E}(\sigma_r - \nu\sigma_\varphi) \tag{2.18}$$

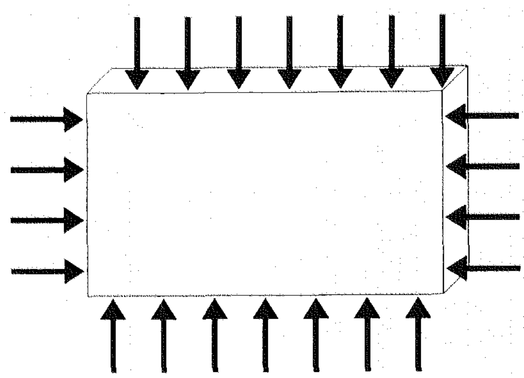
$$\epsilon_\varphi \;=\; \frac{1}{E}(\sigma_\varphi - \nu\sigma_r) \tag{2.19}$$

$$\gamma_{r\varphi} \;=\; \frac{\tau_{r\varphi}}{G} \tag{2.20}$$

$$\epsilon_r \;=\; \frac{\partial}{\partial r}\,v_r \tag{2.21}$$

$$\epsilon_\varphi \;=\; \frac{v_r}{r} + \frac{1}{r}\,\frac{\partial}{\partial\varphi}\,v_\varphi \tag{2.22}$$

$$\gamma_{r\varphi} \;=\; \frac{1}{r}\,\frac{\partial}{\partial\varphi}\,v_r + \frac{\partial}{\partial r}\,v_\varphi - \frac{v_\varphi}{r}\,. \tag{2.23}$$



Picture 7: forces in the case of plain stress

Given the functions σ_r and $\tau_{r\varphi}$ on the boundaries, it can be proved that this system has a unique solution. To find this solution, a so called Airy stress function F is introduced by the following equations:

$$\sigma_r \;=\; \left(\frac{1}{r}\,\frac{\partial}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2}{\partial\varphi^2}\right) F \tag{2.24}$$

$$\sigma_\varphi \;=\; \frac{\partial^2}{\partial r^2}\,F \tag{2.25}$$

$$\tau_{r\varphi} \;=\; -\frac{\partial}{\partial r}\left(\frac{1}{r}\,\frac{\partial}{\partial\varphi}\,F\right)\,. \tag{2.26}$$

It is easy to see, that these equations fulfill the equilibrium conditions identically. In view of Hooke's law and the geometric conditions, it can be shown that F has to be biharmonic, that means:

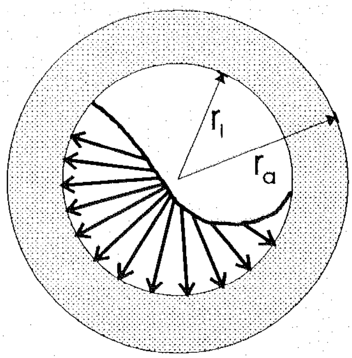
$$\triangle\triangle F = 0. \tag{2.27}$$

Consequently the elastic state of a body can be determined by solving the following problem: Find a solution F of the Bi-Laplace equation, from which the stress functions can be obtained by equations (2.24) to (2.26) and the distortions and displacements by equations (2.18) to (2.23). If the functions σ_r and $\tau_{r\varphi}$ are given on the boundaries, the solution of this problem is not unique. To achieve uniqeness it is necessary, that every function of the elastic state is 2π-periodic and that the angular deformations $\gamma_{r\varphi}$ given by equation (2.20) and (2.23) are the same.

3 Formulation of the Problem

3.1 Physical Formulation of the Problem

The theory of plane stress, mentioned in the last chapter, is now used in order to compute the stresses in the shell of a journal bearing. The shell has the shape of a thick-walled pipe and in the easiest case it consists of a single homogeneous and isotropic material. From the pressure distribution of the oil, which is given by the solution of Reynolds's differential equation, as shown in Chapter 1, the normal resp. the radial stress σ_r at the inner side of the bearing shell is known. The shearing stress $\tau = \tau_{r\varphi}$ at the inner side is zero, because the adherence of the oil at the surface can be neglected. The stresses at the outer side of the bearing are unknown. Sometimes these stresses are set to zero, in other papers a linear dependence to the stresses at the inner side is used, that means $\tau(r_a, \varphi) = 0$ and $\sigma_r(r_a, \varphi) = c\, \sigma_r(r_i, \varphi)$, with a constant c.
From the pressure distribution in the oil it can be deduced, that the maximum forces appear in the middle of the bearing. Therefore in order to estimate the stress in the bearing shell, it is sufficient to compute the stresses in a section through the middle of the machine element. Such a section is shown in the following picture, where the distribution of the oil pressure is illustrated with arrows.



Picture 8: distribution of the oil pressure

Because of the symmetric form of the bearing, the stresses vanish in axial direction and the requirements for using the plain stress theory are fulfilled. The stresses can be derived from a solution F of the Bi-Laplace equation $\triangle\triangle F = 0$, where the resulting stresses σ_r and τ have to fulfill the above described boundary conditions.

3.2 Mathematical Formulation of the Problem

Let r_i and r_a be two positive and real numbers greater than zero with $r_i < r_a$. Let the circles Γ_{r_i} and Γ_{r_a} be defined by

$$\Gamma_{r_i} = \{x \in \boldsymbol{R}^2 \,, \quad \|x\| = r_i\} \quad \text{and} \quad \Gamma_{r_a} = \{x \in \boldsymbol{R}^2 \,, \quad \|x\| = r_a\} \,, \tag{3.2.1}$$

the circular ring Ω by

$$\Omega = \{x \in \boldsymbol{R}^2 \,, \quad r_i \leq \|x\| \leq r_a\} \,, \tag{3.2.2}$$

and the operators B_1 and B_2 by

$$B_1 F(r,\varphi) = \frac{1}{r}\frac{\partial F}{\partial r}(r,\varphi) + \frac{1}{r^2}\frac{\partial^2 F}{\partial \varphi^2}(r,\varphi) \tag{3.2.3}$$

and

$$B_2 F(r,\varphi) = -\frac{1}{r}\frac{\partial^2 F}{\partial r \partial \varphi}(r,\varphi) + \frac{1}{r^2}\frac{\partial F}{\partial \varphi}(r,\varphi) \tag{3.2.4}$$

for any sufficiently often differentiable function $F : \Omega \to \boldsymbol{R}$.

Further let $\sigma^{(int)}(x'_j)$, $\tau^{(int)}(x''_j)$, $\sigma^{(ext)}(x'''_j)$ and $\tau^{(ext)}(x''''_j)$ be some known values from the unknown functions $\sigma^{(int)}, \tau^{(int)}: \Gamma_{r_i} \to \boldsymbol{R}$ and $\sigma^{(ext)}, \tau^{(ext)}: \Gamma_{r_a} \to \boldsymbol{R}$ with $x'_j \in X' \subset \Gamma_{r_i}$, $x''_j \in X'' \subset \Gamma_{r_i}$, $x'''_j \in X''' \subset \Gamma_{r_a}$ and $x''''_j \in X'''' \subset \Gamma_{r_a}$ for some finite sets X', X'', X''' and X'''' of points.

Our task is to find an approximation S_X of a function $F : \Omega \to \boldsymbol{R}$, with:

1. $\triangle\triangle F = 0$ in Ω
2. $B_1 F|_{\Gamma_{r_i}} = \sigma^{(int)}$, $B_1 F|_{\Gamma_{r_a}} = \sigma^{(ext)}$, $B_2 F|_{\Gamma_{r_i}} = \tau^{(int)}$, $B_2 F|_{\Gamma_{r_a}} = \tau^{(ext)}$
3. F fulfills the conditions of 2π-periodicity and of unique angular deformations.

The function S_X should be biharmonic, it should deliver unique angular deformations and 2π-periodic functions and it should interpolate the boundary values:

$$\begin{array}{lllllll} B_1 S_X(x_j) & = & \sigma^{(int)}(x_j) & \forall x_j \in X' & B_1 S_X(x_j) & = & \sigma^{(ext)}(x_j) \quad \forall x_j \in X''' \\ B_2 S_X(x_j) & = & \tau^{(int)}(x_j) & \forall x_j \in X'' & B_2 S_X(x_j) & = & \tau^{(ext)}(x_j) \quad \forall x_j \in X''''. \end{array}$$

4 The Spline Interpolation Method

4.1 Basic Settings and Notations

Let x and $\bar{x}$ be points of the two-dimensional Euclidean space. In cylindrical coordinates they have the representations:

$$x = (r\ \cos(\varphi)\,,\ r\ \sin(\varphi))^t \tag{4.1.1}$$

and

$$\bar{x} = (\bar{r}\ \cos(\bar{\varphi})\,,\ \bar{r}\ \sin(\bar{\varphi}))^t\ . \tag{4.1.2}$$

For simplicity in the sequel the cartesian as well as the cylindrical coordinates will be used. As usual every polynomial $P_n : \boldsymbol{R}^2 \to \boldsymbol{R}$ of the form

$$P_n(x) = \sum_{[\alpha]=n} c_\alpha x^\alpha \tag{4.1.3}$$

is called homogeneous polynomial of degree n. $\alpha = (\alpha_1, \alpha_2)^t$ denotes a multiindex, that means a pair of non-negative integers α_1 and α_2. $[\alpha] = \alpha_1 + \alpha_2$ is the absolute value of the multiindex and x^α is defined by $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}$.
Obviously the set of monomials of degree n is a basis for the space of homogeneous polynomials of degree n. The number of such monomials is precisely the number of ways a pair of numbers α_1 and α_2 can be chosen with $[\alpha] = n$, namely $n + 1$.
In the last chapter the Laplace operator $\triangle$ was introduced. In cartesian coordinates it has the representation:

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{4.1.4}$$

in cylindrical coordinates:

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\triangle^*, \tag{4.1.5}$$

where $\triangle^* = \partial^2/\partial\varphi^2$ defines the Beltrami-operator of the unit circle. We say a homogeneous polynomial P_n is harmonic, if it fulfills Laplace's differential equation $\triangle P_n = 0$. The restriction of a harmonic polynomial of degree n to a circle Γ_ω is called circular function of degree n and is denoted with Y_n, the set of all circular functions of degree n by $\mathcal{Y}_n$. As it is well known for $n > 0$ there exist precisely two linearly independent circular functions $Y_{n\,1}$ and $Y_{n\,2}$ of degree n. If the system $Y_{n\,j}$ is orthonormalized in the sense:

$$\int_{\Gamma_\omega} Y_{n\,i}(x)\, Y_{n\,j}(x)\, dx = \delta_{ij}, \tag{4.1.6}$$

they have the following representation:

$$\begin{aligned}
Y_0(\varphi) &= \frac{1}{\sqrt{2\pi\omega}} & & \\
Y_{11}(\varphi) &= \frac{1}{\sqrt{\pi\omega}}\sin(\varphi) & Y_{12}(\varphi) &= \frac{1}{\sqrt{\pi\omega}}\cos(\varphi) \\
Y_{21}(\varphi) &= \frac{1}{\sqrt{\pi\omega}}\sin(2\varphi) & Y_{22}(\varphi) &= \frac{1}{\sqrt{\pi\omega}}\cos(2\varphi) \\
&\vdots & &\vdots \\
Y_{n1}(\varphi) &= \frac{1}{\sqrt{\pi\omega}}\sin(n\varphi) & Y_{n2}(\varphi) &= \frac{1}{\sqrt{\pi\omega}}\cos(n\varphi)
\end{aligned}$$

The circular functions of degree n have the property, that they are the infinitely often differentiable eigenfunctions of the Beltrami-operator $\triangle^*$ to the eigenvalues $\lambda_n = n^2 \quad (n=0,1,2,\ldots)$, i.e.

$$\triangle^* Y_n(\varphi) + n^2 Y_n(\varphi) = 0 \qquad \forall\, \varphi \in [0, 2\pi],\ n \geq 0. \tag{4.1.7}$$

As it is commonly known, for every $n > 0$ the Y_{nj} satisfy the addition theorem

$$\sum_{j=1}^{2} Y_{nj}(\varphi)\, Y_{nj}(\bar{\varphi}) = \frac{1}{\pi\omega} \cos\left(n(\varphi - \bar{\varphi})\right). \tag{4.1.8}$$

Using the circular functions the outer harmonics H^o_{nj} are defined by

$$H^o_{nj}(r,\varphi) = \left(\frac{\omega_1}{r}\right)^n Y^{(1)}_{nj}(\varphi) \quad \text{for } r \geq \alpha > w_1\,, \quad \varphi \in [0, 2\pi]\,, \tag{4.1.9}$$

where the $Y^{(1)}_{nj}$ are orthonormalized in the sense of $\mathcal{L}^2(\Gamma_{\omega_1})$. They are dense in the space of all harmonic functions u which are defined on $\Omega_e = \{x \in \boldsymbol{R}^2\,,\ \alpha \leq \|x\| \leq \infty\}$ and fulfill some conditions describing the behaviour at infinity. This means for every $\epsilon > 0$ there exist coefficients C^o_{nj} and a $N = N(\epsilon)$ with

$$\sup_{x \in \Omega_e} \left| u(x) - \sum_{n=0}^{N} \sum_{j} C^o_{nj} H^o_{nj}(x) \right| \leq \epsilon. \tag{4.1.10}$$

Furthermore, inner harmonics H^i_{nj} are defined with $\mathcal{L}^2(\Gamma_{\omega_2})$-orthonormalized circular functions $Y^{(2)}_{nj}$ by

$$H^i_{nj}(r,\varphi) = \left(\frac{r}{\omega_2}\right)^n Y^{(2)}_{nj}(\varphi) \quad \text{for } r \leq \beta < w_2\,, \quad \varphi \in [0, 2\pi]\,. \tag{4.1.11}$$

They are dense in the space of all in $\Omega_i = \{x \in \boldsymbol{R}^2\,,\ \|x\| \leq \beta\}$ harmonic functions u, that means for every $\epsilon > 0$ there exist coefficients C^i_{nj} and a $N = N(\epsilon)$ with

$$\sup_{x \in \Omega_i} \left| u(x) - \sum_{n=0}^{N} \sum_{j} C^i_{nj} H^i_{nj}(x) \right| \leq \epsilon. \tag{4.1.12}$$

For the representation of harmonic and biharmonic functions on a circular ring we have the following theorems:

Theorem.

Let $\Omega = \{x \in \boldsymbol{R}^2\,,\ \alpha \leq |x| \leq \beta\}$ be a circular ring and $F : \Omega \to \boldsymbol{R}$ a harmonic function. Then there exist constants C^i_0, C^o_0, C^i_{nj} and C^o_{nj}, such that:

$$F(x) = C^i_0 + C^o_0 \ln(r) + \sum_{n=1}^{\infty} \sum_{j=1}^{2} C^i_{nj} H^i_{nj}(x) + \sum_{n=1}^{\infty} \sum_{j=1}^{2} C^o_{nj} H^o_{nj}(x). \tag{4.1.13}$$

Proof.

In view of the maximum principle it remains to show the above representation for the boundaries $|x| = \alpha$ and $|x| = \beta$. As it is known, the circular functions are dense in the space of all continuous functions defined on a circle. Thus there are constants d_{nj} and e_{nj} such that

$$F(\alpha, \varphi) = \sum_{n=1}^{\infty} \sum_{j=1}^{2} d_{nj} Y_{nj}(\varphi) + d_0 \tag{4.1.14}$$

and

$$F(\beta, \varphi) = \sum_{n=1}^{\infty} \sum_{j=1}^{2} e_{nj} Y_{nj}(\varphi) + e_0 \,. \tag{4.1.15}$$

Neglecting orthonormalization constants and comparing the coefficients of the equations (4.1.13), (4.1.14) and (4.1.15) we obtain:

$$C_0^i + C_0^o \ln(\alpha) = d_0 \qquad C_{nj}^i \frac{\alpha^n}{\omega_2^n} + C_{nj}^o \frac{\omega_1^n}{\alpha^n} = d_{nj}$$

$$C_0^i + C_0^o \ln(\beta) = e_0 \qquad C_{nj}^i \frac{\beta^n}{\omega_2^n} + C_{nj}^o \frac{\omega_1^n}{\beta^n} = e_{nj} \,.$$

From these equations the coefficients C_0^i, C_0^o, C_{nj}^i and C_{nj}^o can be uniquely determined.

■

Theorem.

Let $\Omega = \{x \in \boldsymbol{R}^2 \,,\ \alpha \leq |x| \leq \beta\}$ be a circular ring and $F : \Omega \to \boldsymbol{R}$ a biharmonic function. Then there exist harmonic functions f and g as well as constants c_1 and c_2, such that

$$F(x) = r^2 f(x) + g(x) + \frac{c_1 r}{\sqrt{\pi \omega_1}} \cos(\varphi) \ln(r) + \frac{c_2 r}{\sqrt{\pi \omega_1}} \sin(\varphi) \ln(r). \tag{4.1.16}$$

Proof.

Because F is biharmonic, $\triangle F$ is harmonic and there exist constants C_0^i, C_0^o, C_{nj}^i and C_{nj}^o such that

$$\triangle F(x) = C_0^i + C_0^o \ln(r) + \sum_{n=1}^{\infty} \sum_{j=1}^{2} C_{nj}^i H_{nj}^i(x) + \sum_{n=1}^{\infty} \sum_{j=1}^{2} C_{nj}^o H_{nj}^o(x) \,. \tag{4.1.17}$$

Replacing $r^2 f(x)$ by

$$r^2 f(x) = r^2 d_0^i + r^2 \ln(r) d_0^o + r^2 \left(\sum_{n=1}^{\infty} \sum_{j=1}^{2} d_{nj}^i H_{nj}^i(x) + \sum_{n=1}^{\infty} \sum_{j=1}^{2} d_{nj}^o H_{nj}^o(x) \right) \tag{4.1.18}$$

we get:

$$\triangle \left(r^2 f + g + \frac{c_1 r}{\sqrt{\pi \omega_1}} \cos(\varphi) \ln(r) + \frac{c_2 r}{\sqrt{\pi \omega_1}} \sin(\varphi) ln(r) \right) =$$

$$4 d_0^i + d_0^o (4 \ln(r) + 4) + \sum_{n=1}^{\infty} \sum_{j=1}^{2} (4 + 4n) d_{nj}^i H_{nj}^i$$

$$+ \sum_{n=1}^{\infty} \sum_{j=1}^{2} (4 - 4n) d_{nj}^o H_{nj}^o + \frac{2 c_1}{r \sqrt{\pi \omega_1}} \cos(\varphi) + \frac{2 c_2}{r \sqrt{\pi \omega_1}} \sin(\varphi) \,.$$

Comparison of the coefficients yields:

$$c_1 = \frac{C_{11}^o}{2} \omega_1 \qquad c_2 = \frac{C_{12}^o}{2} \omega_1$$

$$
\begin{aligned}
d_0^o &= \frac{C_0^o}{4} & d_0^i &= \frac{C_0^i - C_0^o}{4} \\
d_{nj}^i &= \frac{C_{nj}^i}{4+4n} & &\text{for } n = 1,2,3,\ldots \\
d_{nj}^o &= \frac{C_{nj}^o}{4-4n} & &\text{for } n = 2,3,4,\ldots \\
d_{1j}^o &= 0
\end{aligned}
$$

With these equations the function f and the constants c_1 and c_2 are fixed and

$$g(x) = F(x) - r^2 f(x) - c_1 r\cos(\varphi)\ln(r) - c_2 r\sin(\varphi)\ln(r) \tag{4.1.19}$$

is a harmonic function.

■

4.2 Restriction of the Space of Solutions

As mentioned in Chapter 2, the biharmonic solutions of our problem have to deliver unique angular deformations and 2π-periodic functions. Inserting the representation (4.1.16) in the equations (2.18) to (2.26) we find, that all harmonic functions and all harmonic functions times r^2, apart from $r^2\ln(r)$, which gives no 2π-periodic distortions in angular direction, satisfy these properties. As the functions $r\cos(\varphi)\ln(r)$ and $r\sin(\varphi)\ln(r)$ are in contradiction to the uniqueness of the angular deformation, our problem has to be solved in the space of all functions $F(x) = f(x) + r^2 g(x)$, where $f(x)$ and $g(x)$ are harmonic functions and $g(x)$ consists of non term of the form $\ln(r)$.

4.3 Spline Interpolation

The purpose of this section is to develop a spline interpolation method, based on the space described in the last section.
Let $q^{(1)} : I\!N \rightarrow \boldsymbol{R}\ n \mapsto q_n^{(1)}$ and $q^{(2)} : I\!N \rightarrow \boldsymbol{R}\ n \mapsto q_n^{(2)}$ be two positive sequences, which satisfy for every $0 < s < 1$ the following inequalities:

$$\sum_{n=1}^{\infty} \frac{1}{q_n^{(1)}} s^n < \infty \qquad \text{resp.} \qquad \sum_{n=1}^{\infty} \frac{1}{q_n^{(2)}} s^n < \infty\,. \tag{4.3.1}$$

By $Q^{(1)}Y_{nj} = q_n^{(1)}Y_{nj}$ resp. $Q^{(2)}Y_{nj} = q_n^{(2)}Y_{nj}$ $\forall\, n \in I\!N$, $j = 1,2$, two invariant pseudo-differential operators $Q^{(1)}$ and $Q^{(2)}$ are defined on the circles Γ_{ω_1} resp. Γ_{ω_2} . Furthermore let

$$h^{(1)} = \{f \in l' \mid Q^{(1)\frac{1}{2}} f \in \mathcal{L}^2(\Gamma_{\omega_1})\} \tag{4.3.2}$$

and

$$h^{(2)} = \{f \in l' \mid Q^{(2)\frac{1}{2}} f \in \mathcal{L}^2(\Gamma_{\omega_2})\} \tag{4.3.3}$$

be the spaces of all distributions f for which $Q^{(1)\frac{1}{2}} f$ resp. $Q^{(2)\frac{1}{2}} f$ are square-integrable with respect to Γ_{ω_1} resp. Γ_{ω_2}. $h^{(1)}$ and $h^{(2)}$ equipped with the inner products

$$(f,g)_{h^{(1)}} = \int_{\Gamma_{\omega_1}} |Q^{(1)\frac{1}{2}} f||Q^{(1)\frac{1}{2}} g|\,dx \tag{4.3.4}$$

and

$$(f,g)_{h^{(2)}} = \int_{\Gamma_{\omega_2}} |Q^{(2)^{\frac{1}{2}}} f||Q^{(2)^{\frac{1}{2}}} g| dx \tag{4.3.5}$$

define two Sobolev-like Hilbert spaces and it can be easily shown, that a circular function

$$F(\varphi) = \sum_{n=1}^{\infty}\sum_{j=1}^{2} C_{nj}^{F} Y_{nj}(\varphi) \tag{4.3.6}$$

is in $h^{(1)}$ resp. in $h^{(2)}$, if and only if the constants C_{nj}^{F} satisfy

$$\sum_{n=1}^{\infty}\sum_{j=1}^{2} |C_{nj}^{F}|^2 q_n^{(1)} < \infty \qquad \text{resp.} \qquad \sum_{n=1}^{\infty}\sum_{j=1}^{2} |C_{nj}^{F}|^2 q_n^{(2)} < \infty. \tag{4.3.7}$$

For numerical reasons we will now define separable spaces, based on the Hilbert spaces $h^{(1)}$ and $h^{(2)}$.

Definition.

Let $q^{(1)}$ and $q^{(2)}$ be two sequences with the properties described above and let α and β be two real numbers with $\omega_1 < \alpha < \beta < \omega_2$. By

$$\mathcal{H}_1 = \left\{ \mathrm{F(x)} = \sum_{n=1}^{\infty}\sum_{j=1}^{2} \mathrm{C_{nj}} \left(\frac{\omega_1}{\mathrm{r}}\right)^{\mathrm{n}} \mathrm{Y_{nj}^{(1)}}(\varphi),\ \alpha \le \mathrm{r} \le \beta,\ \sum_{n=1}^{\infty}\sum_{j=1}^{2} \mathrm{C_{nj}^2}\ \mathrm{q_n^{(1)}} < \infty \right\}$$

$$(F,G)_{\mathcal{H}_1} = \sum_{n=1}^{\infty}\sum_{j=1}^{2} q_n^{(1)} \left(\frac{\alpha}{\omega_1}\right)^{2n-2} (F, Y_{nj}^{(1)})_{\mathcal{L}^2(\Gamma_\alpha)} (G, Y_{nj}^{(1)})_{\mathcal{L}^2(\Gamma_\alpha)}$$

$$\mathcal{H}_2 = \left\{ \mathrm{F(x)} = \sum_{n=1}^{\infty}\sum_{j=1}^{2} \mathrm{C_{nj}} \left(\frac{\omega_1}{\mathrm{r}}\right)^{\mathrm{n-2}} \mathrm{Y_{nj}^{(1)}}(\varphi),\ \alpha \le \mathrm{r} \le \beta,\ \sum_{n=1}^{\infty}\sum_{j=1}^{2} \mathrm{C_{nj}^2}\ \mathrm{q_n^{(1)}} < \infty \right\}$$

$$(F,G)_{\mathcal{H}_2} = \sum_{n=1}^{\infty}\sum_{j=1}^{2} q_n^{(1)} \left(\frac{\alpha}{\omega_1}\right)^{2n-6} (F, Y_{nj}^{(1)})_{\mathcal{L}^2(\Gamma_\alpha)} (G, Y_{nj}^{(1)})_{\mathcal{L}^2(\Gamma_\alpha)}$$

$$\mathcal{H}_3 = \left\{ \mathrm{F(x)} = \sum_{n=1}^{\infty}\sum_{j=1}^{2} \mathrm{C_{nj}} \left(\frac{\mathrm{r}}{\omega_2}\right)^{\mathrm{n}} \mathrm{Y_{nj}^{(2)}}(\varphi),\ \alpha \le \mathrm{r} \le \beta,\ \sum_{n=1}^{\infty}\sum_{j=1}^{2} \mathrm{C_{nj}^2}\ \mathrm{q_n^{(2)}} < \infty \right\}$$

$$(F,G)_{\mathcal{H}_3} = \sum_{n=1}^{\infty}\sum_{j=1}^{2} q_n^{(2)} \left(\frac{\omega_2}{\beta}\right)^{2n+2} (F, Y_{nj}^{(2)})_{\mathcal{L}^2(\Gamma_\beta)} (G, Y_{nj}^{(2)})_{\mathcal{L}^2(\Gamma_\beta)}$$

$$\mathcal{H}_4 = \left\{ \mathrm{F(x)} = \sum_{n=1}^{\infty}\sum_{j=1}^{2} \mathrm{C_{nj}} \left(\frac{\mathrm{r}}{\omega_2}\right)^{\mathrm{n+2}} \mathrm{Y_{nj}^{(2)}}(\varphi),\ \alpha \le \mathrm{r} \le \beta,\ \sum_{n=1}^{\infty}\sum_{j=1}^{2} \mathrm{C_{nj}^2}\ \mathrm{q_n^{(2)}} < \infty \right\}$$

$$(F,G)_{\mathcal{H}_4} = \sum_{n=1}^{\infty}\sum_{j=1}^{2} q_n^{(2)} \left(\frac{\omega_2}{\beta}\right)^{2n+6} (F, Y_{nj}^{(2)})_{\mathcal{L}^2(\Gamma_\beta)} (G, Y_{nj}^{(2)})_{\mathcal{L}^2(\Gamma_\beta)}$$

$$\mathcal{H}_5 = \left\{ \mathrm{F(x) = a_0 + a_1 r^2 + a_2 \ln(r), \quad a_i \in \boldsymbol{R}, \quad a_i < \infty, \quad \alpha \le r \le \beta} \right\}$$

$$(F,G)_{\mathcal{H}_5} = \sum_{i=0}^{2} a_i\, b_i$$

the spaces $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4$ and $\mathcal{H}_5$ with their corresponding inner products are defined. Again $Y_{nj}^{(1)}$ resp. $Y_{nj}^{(2)}$ denote the circular functions orthonormalized in the sense of $\mathcal{L}^2(\Gamma_{\omega_1})$ and $\mathcal{L}^2(\Gamma_{\omega_2})$.
It is easy to see, that the $h^{(1)}$-norm of a circular function corresponds to the $\mathcal{H}_1$- resp. the $\mathcal{H}_2$-norm of this function. The same is true for $h^{(2)}$-norm and the $\mathcal{H}_3$- resp. the $\mathcal{H}_4$-norm. Furthermore we obtain the following theorem:

Theorem.

$\mathcal{H}$ defined by $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \mathcal{H}_5$ and the inner product

$$(F,G)_{\mathcal{H}} = \sum_{i=1}^{5} (F_{\mathcal{H}_i}, G_{\mathcal{H}_i})_{\mathcal{H}_i},$$

where $F_{\mathcal{H}_i}$ and $G_{\mathcal{H}_i}$ denote the projections of F and G into the space $\mathcal{H}_i$ is a separable Hilbert space with the reproducing kernel

$$\mathcal{K}(\mathrm{x},\bar{\mathrm{x}}) = \sum_{\mathrm{n}=1}^{\infty}\sum_{\mathrm{j}=1}^{2} \frac{1}{\mathrm{q}_\mathrm{n}^{(1)}} \left(\frac{\omega_1}{\mathrm{r}}\right)^{\mathrm{n}} \left(\frac{\omega_1}{\bar{\mathrm{r}}}\right)^{\mathrm{n}} \mathrm{Y}_{\mathrm{nj}}^{(1)}(\varphi)\mathrm{Y}_{\mathrm{nj}}^{(1)}(\bar{\varphi}) \left(1 + \frac{\mathrm{r}^2\bar{\mathrm{r}}^2}{\omega_1^4}\right) +$$

$$+ \sum_{n=1}^{\infty}\sum_{j=1}^{2} \frac{1}{q_n^{(2)}} \left(\frac{r}{\omega_2}\right)^{n} \left(\frac{\bar{r}}{\omega_2}\right)^{n} Y_{nj}^{(2)}(\varphi)Y_{nj}^{(2)}(\bar{\varphi}) \left(1 + \frac{r^2\bar{r}^2}{\omega_2^4}\right)$$

$$+ \quad 1 + r^2\bar{r}^2 + \ln(r)\ln(\bar{r})\,. \tag{4.3.8}$$

Proof.

a) $\mathcal{K}(\mathrm{x},\cdot) \in \mathcal{H}$:

This follows easily by the representation of the kernel function.

b) $(F(\cdot), \mathcal{K}(\mathrm{x},\cdot))_{\mathcal{H}} = \mathrm{F(x)}$:

$$(F(\cdot),\mathcal{K}(\mathrm{x},\cdot))_{\mathcal{H}}$$

$$= \sum_{n,j} q_n^{(1)} \left(\frac{\alpha}{\omega_1}\right)^{2n-2} \left(\sum_{n',j'} \frac{1}{q_{n'}^{(1)}} \left(\frac{\omega_1}{r}\right)^{n'} \left(\frac{\omega_1}{\bar{r}}\right)^{n'} Y_{n'j'}^{(1)}(\varphi)Y_{n'j'}^{(1)}(\bar{\varphi}), Y_{nj}^{(1)}(\bar{\varphi}) \right)_{\mathcal{L}^2(\Gamma_\alpha)}$$

$$\left(\sum_{n',j'} C_{n'j'}^{F_1} \left(\frac{\omega_1}{\bar{r}}\right)^{n'} Y_{n'j'}^{(1)}(\bar{\varphi}), Y_{nj}^{(1)}(\bar{\varphi}) \right)_{\mathcal{L}^2(\Gamma_\alpha)} +$$

$$\sum_{n,j} q_n^{(1)} \left(\frac{\alpha}{\omega_1}\right)^{2n-6} \left(\sum_{n',j'} \frac{1}{q_{n'}^{(1)}} \left(\frac{\omega_1}{r}\right)^{n'} \left(\frac{\omega_1}{\bar{r}}\right)^{n'} Y_{n'j'}^{(1)}(\varphi)Y_{n'j'}^{(1)}(\bar{\varphi}), Y_{nj}^{(1)}(\bar{\varphi}) \right)_{\mathcal{L}^2(\Gamma_\alpha)}$$

$$\left(\sum_{n',j'} C^{F_2}_{n'j'} \left(\frac{\omega_1}{\bar r}\right)^{n'} \left(\frac{\bar r}{\omega_1}\right)^2 Y^{(1)}_{n'j'}(\bar\varphi), Y^{(1)}_{nj}(\bar\varphi)\right)_{\mathcal{L}^2(\Gamma_\alpha)} +$$

$$\sum_{n,j} q^{(2)}_n \left(\frac{\omega_2}{\beta}\right)^{2n+2} \left(\sum_{n',j'} \frac{1}{q^{(2)}_{n'}} \left(\frac{r}{\omega_2}\right)^{n'} \left(\frac{\bar r}{\omega_2}\right)^{n'} Y^{(2)}_{n'j'}(\varphi) Y^{(2)}_{n'j'}(\bar\varphi), Y^{(2)}_{nj}(\bar\varphi)\right)_{\mathcal{L}^2(\Gamma_\beta)}$$

$$\left(\sum_{n',j'} C^{F_3}_{n'j'} \left(\frac{\bar r}{\omega_2}\right)^{n'} Y^{(2)}_{n'j'}(\bar\varphi), Y^{(2)}_{nj}(\bar\varphi)\right)_{\mathcal{L}^2(\Gamma_\beta)} +$$

$$\sum_{n,j} q^{(2)}_n \left(\frac{\omega_2}{\beta}\right)^{2n+6} \left(\sum_{n',j'} \frac{1}{q^{(2)}_{n'}} \left(\frac{r}{\omega_2}\right)^{n'} \left(\frac{\bar r}{\omega_2}\right)^{n'} Y^{(2)}_{n'j'}(\varphi) Y^{(2)}_{n'j'}(\bar\varphi), Y^{(2)}_{nj}(\bar\varphi)\right)_{\mathcal{L}^2(\Gamma_\beta)}$$

$$\left(\sum_{n',j'} C^{F_4}_{n'j'} \left(\frac{\bar r}{\omega_2}\right)^{n'} \left(\frac{\bar r}{\omega_2}\right)^2 Y^{(2)}_{n'j'}(\bar\varphi), Y^{(2)}_{nj}(\bar\varphi)\right)_{\mathcal{L}^2(\Gamma_\beta)} +$$

$$\left(a_0 + a_1 \bar r^2 + a_2 \ln(\bar r), 1 + r^2 \bar r^2 + \ln(r)\ln(\bar r)\right)_{\mathcal{H}_5}$$

$$= \sum_{n,j} C^{F_1}_{nj} \left(\frac{\omega_1}{r}\right)^{n} Y^{(1)}_{nj}(\varphi) + \sum_{n,j} C^{F_2}_{nj} \left(\frac{\omega_1}{r}\right)^{n-2} Y^{(1)}_{nj}(\varphi) +$$

$$\sum_{n,j} C^{F_3}_{nj} \left(\frac{r}{\omega_2}\right)^{n} Y^{(2)}_{nj}(\varphi) \sum_{n,j} C^{F_4}_{nj} \left(\frac{r}{\omega_2}\right)^{n+2} Y^{(2)}_{nj}(\varphi) +$$

$$a_0 + a_1 r^2 + a_2 \ln(r) = F(x)$$

c) $|F| < \infty \ \forall F \in \mathcal{H}$:

Applying Cauchy's inequality we get: $|F|^2 = (F(\cdot), \mathcal{K}(\mathrm{x}, \cdot))^2_{\mathcal{H}} \le \|\mathrm{F}\|^2_{\mathcal{H}} \|\mathcal{K}\|^{\in}_{\mathcal{H}}$

In view of $\|F\|^2_{\mathcal{H}} < \infty$, it remains to prove that $\|K\|^2_{\mathcal{H}}$ is bounded.

$$\|\mathcal{K}\|^2_{\mathcal{H}} = (\mathcal{K}, \mathcal{K})_{\mathcal{H}} =$$

$$\sum_{n,j} q^{(1)}_n \left(\frac{\alpha}{\omega_1}\right)^{2n-2} \left(\sum_{n',j'} \frac{1}{q^{(1)}_{n'}} \left(\frac{\omega_1}{r}\right)^{n'} \left(\frac{\omega_1}{\bar r}\right)^{n'} Y^{(1)}_{n'j'}(\varphi) Y^{(1)}_{n'j'}(\bar\varphi), Y^{(1)}_{nj}(\bar\varphi)\right)^2_{\mathcal{L}^2(\Gamma_\alpha)} +$$

$$\sum_{n,j} q^{(1)}_n \left(\frac{\alpha}{\omega_1}\right)^{2n-6} \left(\sum_{n',j'} \frac{1}{q^{(1)}_{n'}} \left(\frac{\omega_1}{r}\right)^{n'} \left(\frac{\omega_1}{\bar r}\right)^{n'} \frac{r^2 \bar r^2}{\omega_1^4} Y^{(1)}_{n'j'}(\varphi) Y^{(1)}_{n'j'}(\bar\varphi), Y^{(1)}_{nj}(\bar\varphi)\right)^2_{\mathcal{L}^2(\Gamma_\alpha)} +$$

$$\sum_{n,j} q^{(2)}_n \left(\frac{\omega_2}{\beta}\right)^{2n+2} \left(\sum_{n',j'} \frac{1}{q^{(2)}_{n'}} \left(\frac{r}{\omega_2}\right)^{n'} \left(\frac{\bar r}{\omega_2}\right)^{n'} Y^{(2)}_{n'j'}(\varphi) Y^{(2)}_{n'j'}(\bar\varphi), Y^{(2)}_{nj}(\bar\varphi)\right)^2_{\mathcal{L}^2(\Gamma_\beta)} +$$

$$\sum_{n,j} q^{(2)}_n \left(\frac{\omega_1}{\beta}\right)^{2n+6} \left(\sum_{n',j'} \frac{1}{q^{(2)}_{n'}} \left(\frac{r}{\omega_2}\right)^{n'} \left(\frac{\bar r}{\omega_2}\right)^{n'} \frac{r^2 \bar r^2}{\omega_2^4} Y^{(2)}_{n'j'}(\varphi) Y^{(2)}_{n'j'}(\bar\varphi), Y^{(2)}_{nj}(\bar\varphi)\right)^2_{\mathcal{L}^2(\Gamma_\beta)} +$$

$1 + r^4 + \ln^2(r) =$

$$\sum_{n,j} \frac{1}{q_n^{(1)}} \left(\frac{\omega_1}{r}\right)^{2n} Y_{nj}^{(1)^2}(\varphi) + \sum_{n,j} \frac{1}{q_n^{(1)}} \left(\frac{\omega_1}{r}\right)^{2n} \left(\frac{r}{\omega_1}\right)^4 Y_{nj}^{(1)^2}(\varphi) +$$
$$\sum_{n,j} \frac{1}{q_n^{(2)}} \left(\frac{r}{\omega_2}\right)^{2n} Y_{nj}^{(2)^2}(\varphi) + \sum_{n,j} \frac{1}{q_n^{(2)}} \left(\frac{r}{\omega_2}\right)^{2n} \left(\frac{r}{\omega_2}\right)^4 Y_{nj}^{(2)^2}(\varphi) +$$
$$1 + r^4 + \ln^2(r)$$

Observing that the circular functions $Y_{nj}^{(1)}$ and $Y_{nj}^{(2)}$, as well as the series

$$\sum_n \frac{1}{q_n^{(1)}} \left(\frac{\omega_1}{r}\right)^{2n} \quad \text{and} \quad \sum_n \frac{1}{q_n^{(2)}} \left(\frac{r}{\omega_2}\right)^{2n}$$

are bounded and that $1+r^4+\ln^2(r) < \infty$ for $\alpha \le r \le \beta$, we have proved that $\|\mathcal{K}\|_{\mathcal{H}} < \infty$ and consequently $|F|$ is bounded. This is a necessary condition on the space $\mathcal{H}$ to be separable (cf. Davis[3]). ■

In view of Section 4.1 and 4.2 we have now a separable Hilbert space $\mathcal{H}$, consisting of all biharmonic functions, which fulfill the conditions of 2π-periodicity and unique angular deformation.

For simplicity let K_1 and K_2 be the derivatives of $\mathcal{K}$ given by $K_1(x,\bar{x}) = B_1\mathcal{K}(\mathrm{x},\bar{\mathrm{x}})$ and $K_2(x,\bar{x}) = B_2\mathcal{K}(\mathrm{x},\bar{\mathrm{x}})$, where the operators B_1 and B_2 are applied to the second variable $\bar{x}$.

Lemma.

For any function F of the Hilbert space $\mathcal{H}$ we have the following representations:

$$B_1 F(\bar{x}) = (K_1(\bar{x},\cdot), F(\cdot))_{\mathcal{H}} \tag{4.3.9}$$
$$B_2 F(\bar{x}) = (K_2(\bar{x},\cdot), F(\cdot))_{\mathcal{H}}\,. \tag{4.3.10}$$

Proof.

The functions $K_1(x,\bar{x})$ and $K_2(x,\bar{x})$ can be written as

$$\begin{aligned} K_1(x,\bar{x}) = & \sum_{n=1}^{\infty} \frac{1}{q_n^{(1)}} \left(\frac{\omega_1}{r}\right)^n \left(\frac{\omega_1}{\bar{r}}\right)^n \frac{1}{\bar{r}^2} \cos(n(\varphi-\bar{\varphi}))(-n-n^2)\frac{1}{\pi\omega_1} + \\ & \sum_{n=1}^{\infty} \frac{1}{q_n^{(1)}} \left(\frac{\omega_1}{r}\right)^n \left(\frac{\omega_1}{\bar{r}}\right)^n \frac{r^2}{\omega_1^4} \cos(n(\varphi-\bar{\varphi}))(2-n-n^2)\frac{1}{\pi\omega_1} + \\ & \sum_{n=1}^{\infty} \frac{1}{q_n^{(2)}} \left(\frac{r}{\omega_2}\right)^n \left(\frac{\bar{r}}{\omega_2}\right)^n \frac{1}{\bar{r}^2} \cos(n(\varphi-\bar{\varphi}))(n-n^2)\frac{1}{\pi\omega_2} + \\ & \sum_{n=1}^{\infty} \frac{1}{q_n^{(2)}} \left(\frac{r}{\omega_2}\right)^n \left(\frac{\bar{r}}{\omega_2}\right)^n \frac{r^2}{\omega_2^4} \cos(n(\varphi-\bar{\varphi}))(2+n-n^2)\frac{1}{\pi\omega_2} + \\ & 2r^2 + \frac{1}{\bar{r}^2}\ln(r) \end{aligned}$$

and

$$
\begin{aligned}
K_2(x,\bar{x}) &= \sum_{n=1}^{\infty} \frac{1}{q_n^{(1)}} \left(\frac{\omega_1}{r}\right)^n \left(\frac{\omega_1}{\bar{r}}\right)^n \frac{1}{\bar{r}^2} \sin(n(\varphi-\bar{\varphi}))(n+n^2)\frac{1}{\pi\omega_1} + \\
&\sum_{n=1}^{\infty} \frac{1}{q_n^{(1)}} \left(\frac{\omega_1}{r}\right)^n \left(\frac{\omega_1}{\bar{r}}\right)^n \frac{r^2}{\omega_1^4} \sin(n(\varphi-\bar{\varphi}))(-n(2-n)+n)\frac{1}{\pi\omega_1} + \\
&\sum_{n=1}^{\infty} \frac{1}{q_n^{(2)}} \left(\frac{r}{\omega_2}\right)^n \left(\frac{\bar{r}}{\omega_2}\right)^n \frac{1}{\bar{r}^2} \sin(n(\varphi-\bar{\varphi}))(n-n^2)\frac{1}{\pi\omega_2} + \\
&\sum_{n=1}^{\infty} \frac{1}{q_n^{(2)}} \left(\frac{r}{\omega_2}\right)^n \left(\frac{\bar{r}}{\omega_2}\right)^n \frac{r^2}{\omega_2^4} \sin(n(\varphi-\bar{\varphi}))(-n(2+n)+n)\frac{1}{\pi\omega_2} .
\end{aligned}
$$

With these representations it is easy to see that the functions are elements of the Hilbert space $\mathcal{H}$. In view of

$$
\begin{aligned}
B_1F(\bar{x}) &= \sum_{n=1}^{\infty} C_{nj}^{F_1} \left(\frac{\omega_1}{\bar{r}}\right)^n Y_{nj}^{(1)}(\bar{\varphi})\frac{1}{\bar{r}^2}(-n-n^2) + \\
&\sum_{n=1}^{\infty} C_{nj}^{F_2} \left(\frac{\omega_1}{\bar{r}}\right)^n Y_{nj}^{(1)}(\bar{\varphi})\frac{1}{\omega_1^2}(2-n-n^2) + \\
&\sum_{n=1}^{\infty} C_{nj}^{F_3} \left(\frac{\bar{r}}{\omega_2}\right)^n Y_{nj}^{(2)}(\bar{\varphi})\frac{1}{\bar{r}^2}(n-n^2) + \\
&\sum_{n=1}^{\infty} C_{nj}^{F_4} \left(\frac{\bar{r}}{\omega_2}\right)^n Y_{nj}^{(2)}(\bar{\varphi})\frac{1}{\omega_2^2}(2+n-n^2) + \\
&a_1 2 + a_2\frac{1}{\bar{r}^2}
\end{aligned}
$$

and

$$
\begin{aligned}
B_2F(\bar{x}) &= \sum_{n=1}^{\infty} C_{nj}^{F_1} \left(\frac{\omega_1}{\bar{r}}\right)^n Y_{nj}^{(1)}(\bar{\varphi})\frac{1}{\bar{r}^2}(n+n^2)(-1)^j + \\
&\sum_{n=1}^{\infty} C_{nj}^{F_2} \left(\frac{\omega_1}{\bar{r}}\right)^n Y_{nj}^{(1)}(\bar{\varphi})\frac{1}{\omega_1^2}(-n+n^2)(-1)^j + \\
&\sum_{n=1}^{\infty} C_{nj}^{F_3} \left(\frac{\bar{r}}{\omega_2}\right)^n Y_{nj}^{(2)}(\bar{\varphi})\frac{1}{\bar{r}^2}(n-n^2)(-1)^j + \\
&\sum_{n=1}^{\infty} C_{nj}^{F_4} \left(\frac{\bar{r}}{\omega_2}\right)^n Y_{nj}^{(2)}(\bar{\varphi})\frac{1}{\omega_2^2}(-n-n^2)(-1)^j ,
\end{aligned}
$$

the proof follows easily by a simple calculation. ■

In the sequel we use $K_1^{(ext)}$ resp. $K_2^{(ext)}$ for the outer and $K_1^{(int)}$ resp. $K_2^{(int)}$ for the inner part of K_1 resp. K_2. In detail this means:

$$
\begin{aligned}
K_1^{(int)} &= \sum_{n=1}^{\infty} \frac{1}{q_n^{(2)}} \left(\frac{r}{\omega_2}\right)^n \left(\frac{\bar{r}}{\omega_2}\right)^n \frac{1}{\bar{r}^2} \cos(n(\varphi-\bar{\varphi}))(n-n^2)\frac{1}{\pi\omega_2} + \\
&\sum_{n=1}^{\infty} \frac{1}{q_n^{(2)}} \left(\frac{r}{\omega_2}\right)^n \left(\frac{\bar{r}}{\omega_2}\right)^n \frac{r^2}{\omega_2^4} \cos(n(\varphi-\bar{\varphi}))(2+n-n^2)\frac{1}{\pi\omega_2} +
\end{aligned}
$$

$$2r^2 + \frac{1}{\bar{r}^2}\ln(r) \tag{4.3.11}$$

$$\begin{aligned} K_2^{(int)} &= \sum_{n=1}^{\infty} \frac{1}{q_n^{(2)}} \left(\frac{r}{\omega_2}\right)^n \left(\frac{\bar{r}}{\omega_2}\right)^n \frac{1}{\bar{r}^2} \sin(n(\varphi-\bar{\varphi}))(n-n^2)\frac{1}{\pi\omega_2} + \\ & \sum_{n=1}^{\infty} \frac{1}{q_n^{(2)}} \left(\frac{r}{\omega_2}\right)^n \left(\frac{\bar{r}}{\omega_2}\right)^n \frac{r^2}{\omega_2^4} \sin(n(\varphi-\bar{\varphi}))(-n^2-n)\frac{1}{\pi\omega_2} \end{aligned} \tag{4.3.12}$$

$$\begin{aligned} K_1^{(ext)} &= \sum_{n=1}^{\infty} \frac{1}{q_n^{(1)}} \left(\frac{\omega_1}{r}\right)^n \left(\frac{\omega_1}{\bar{r}}\right)^n \frac{1}{\bar{r}^2} \cos(n(\varphi-\bar{\varphi}))(-n-n^2)\frac{1}{\pi\omega_1} + \\ & \sum_{n=1}^{\infty} \frac{1}{q_n^{(1)}} \left(\frac{\omega_1}{r}\right)^n \left(\frac{\omega_1}{\bar{r}}\right)^n \frac{r^2}{\omega_1^4} \cos(n(\varphi-\bar{\varphi}))(2-n-n^2)\frac{1}{\pi\omega_1} \end{aligned} \tag{4.3.13}$$

$$\begin{aligned} K_2^{(ext)} &= \sum_{n=1}^{\infty} \frac{1}{q_n^{(1)}} \left(\frac{\omega_1}{r}\right)^n \left(\frac{\omega_1}{\bar{r}}\right)^n \frac{1}{\bar{r}^2} \sin(n(\varphi-\bar{\varphi}))(n+n^2)\frac{1}{\pi\omega_1} + \\ & \sum_{n=1}^{\infty} \frac{1}{q_n^{(1)}} \left(\frac{\omega_1}{r}\right)^n \left(\frac{\omega_1}{\bar{r}}\right)^n \frac{r^2}{\omega_1^4} \sin(n(\varphi-\bar{\varphi}))(n^2-n)\frac{1}{\pi\omega_1} \ . \end{aligned} \tag{4.3.14}$$

Using these notations we define spline functions as follows:

Definition.

1. Let $X = \{x_i' \in \Gamma_{r_i}\ ,\ i = 1,\ldots,N'\ ;\ \ x_i'' \in \Gamma_{r_i}\ ,\ i = 1,\ldots,N''\ ;$

   $x_i''' \in \Gamma_{r_a}\ ,\ i = 1,\ldots,N'''\ ;\ \ x_i'''' \in \Gamma_{r_a}\ ,\ i = 1,\ldots,N''''\}$

   be a system of points. X is called admissible, if the matrix

$$\begin{pmatrix} \left(B_1K_1^{(ext)}(x_i',x_j')\right)_{ji} & \left(B_1K_2^{(ext)}(x_i'',x_j')\right)_{ji} & \left(B_1K_1^{(int)}(x_i''',x_j')\right)_{ji} & \left(B_1K_2^{(int)}(x_i'''',x_j')\right)_{ji} \\ \left(B_2K_1^{(ext)}(x_i',x_j'')\right)_{ji} & \left(B_2K_2^{(ext)}(x_i'',x_j'')\right)_{ji} & \left(B_2K_1^{(int)}(x_i''',x_j'')\right)_{ji} & \left(B_2K_2^{(int)}(x_i'''',x_j'')\right)_{ji} \\ \left(B_1K_1^{(ext)}(x_i',x_j''')\right)_{ji} & \left(B_1K_2^{(ext)}(x_i'',x_j''')\right)_{ji} & \left(B_1K_1^{(int)}(x_i''',x_j''')\right)_{ji} & \left(B_1K_2^{(int)}(x_i'''',x_j''')\right)_{ji} \\ \left(B_2K_1^{(ext)}(x_i',x_j'''')\right)_{ji} & \left(B_2K_2^{(ext)}(x_i'',x_j'''')\right)_{ji} & \left(B_2K_1^{(int)}(x_i''',x_j'''')\right)_{ji} & \left(B_2K_2^{(int)}(x_i'''',x_j'''')\right)_{ji} \end{pmatrix}$$

   is invertible.

2. A function of the form

$$\begin{aligned} S(x) &= \sum_{i=1}^{N'} a_i' K_1^{(ext)}(x_i',x) + \sum_{i=1}^{N''} a_i'' K_2^{(ext)}(x_i'',x) + \\ & \sum_{i=1}^{N'''} a_i''' K_1^{(int)}(x_i''',x) + \sum_{i=1}^{N''''} a_i'''' K_2^{(int)}(x_i'''',x) \end{aligned} \tag{4.3.15}$$

   is called spline function with respect to X. The space of all spline functions with respect to X is denoted by $\mathcal{S}_X$.

For spline functions we obtain:

Lemma.

Let $S \in \mathcal{S}_X$ and $F \in \mathcal{H}$. Furthermore let F be splitted into a inner space part $F^{(int)}$ and a outer space part $F^{(ext)}$. Then we have:

$$(S,F)_{\mathcal{H}} = \sum_{i=1}^{N'} a_i' B_1 F^{(ext)}(x_i') + \sum_{i=1}^{N''} a_i'' B_2 F^{(ext)}(x_i'') +$$

$$\sum_{i=1}^{N'''} a_i''' B_1 F^{(int)}(x_i''') + \sum_{i=1}^{N''''} a_i'''' B_2 F^{(int)}(x_i'''') \, , \tag{4.3.16}$$

Proof.

$$\begin{aligned}
(S,F)_{\mathcal{H}} &= \Bigg(\sum_{i=1}^{N'} a_i' K_1^{(ext)}(x_i',\cdot) + \sum_{i=1}^{N''} a_i'' K_2^{(ext)}(x_i'',\cdot) + \\
&\quad \sum_{i=1}^{N'''} a_i''' K_1^{(int)}(x_i''',\cdot) + \sum_{i=1}^{N''''} a_i'''' K_2^{(int)}(x_i'''',\cdot), F(\cdot) \Bigg)_{\mathcal{H}} \\
&= \sum_{i=1}^{N'} a_i' \left(K_1^{(ext)}(x_i',\cdot), F(\cdot) \right)_{\mathcal{H}} + \sum_{i=1}^{N''} a_i'' \left(K_2^{(ext)}(x_i'',\cdot), F(\cdot) \right)_{\mathcal{H}} + \\
&\quad \sum_{i=1}^{N'''} a_i''' \left(K_1^{(int)}(x_i''',\cdot), F(\cdot) \right)_{\mathcal{H}} + \sum_{i=1}^{N''''} a_i'''' \left(K_2^{(int)}(x_i'''',\cdot), F(\cdot) \right)_{\mathcal{H}} \\
&= \sum_{i=1}^{N'} a_i' B_1 F^{(ext)}(x_i') + \sum_{i=1}^{N''} a_i'' B_2 F^{(ext)}(x_i'') + \\
&\quad \sum_{i=1}^{N'''} a_i''' B_1 F^{(int)}(x_i''') + \sum_{i=1}^{N''''} a_i'''' B_2 F^{(int)}(x_i'''')
\end{aligned}$$

■

As usual for a system of points X the set of all interpolants $\mathcal{I}_{\mathrm{X}}$ is defined by

$$\begin{aligned}
\mathcal{I}_{\mathrm{X}} = \{\mathrm{F} \in \mathcal{H} \text{ mit } \quad & B_1 F(x_i') = \sigma_i^{(int)} && \text{for } i = 1,\ldots,N' \\
& B_2 F(x_i'') = \tau_i^{(int)} && \text{for } i = 1,\ldots,N'' \\
& B_1 F(x_i''') = \sigma_i^{(ext)} && \text{for } i = 1,\ldots,N''' \\
& B_2 F(x_i'''') = \tau_i^{(ext)} && \text{for } i = 1,\ldots,N''''\} \, ,
\end{aligned} \tag{4.3.17}$$

with given real numbers $\sigma_i^{(int)}$, $\tau_i^{(int)}$, $\sigma_i^{(ext)}$, $\tau_i^{(ext)}$. As shown in the next lemma, there exists a unique interpolant in the space $\mathcal{S}_{\mathcal{X}}$.

Lemma.

Let X be an admissible system, $\mathcal{I}_{\mathrm{X}}$ the set of all interpolants and $\mathcal{S}_{\mathrm{X}}$ the space of all spline functions relative to X. Then there exists a unique $S_X \in \mathcal{I}_{\mathrm{X}} \cap \mathcal{S}_{\mathrm{X}}$.

Proof.

Every spline function contains a total of $N = N' + N'' + N''' + N''''$ free coefficients. We have N' conditions of the form

$$\begin{aligned}
B_1 S_X(x_j') &= \sum_{i=1}^{N'} a_i' B_1 K_1^{(ext)}(x_i', x_j') + \sum_{i=1}^{N''} a_i'' B_1 K_2^{(ext)}(x_i'', x_j') + \\
&\quad \sum_{i=1}^{N'''} a_i''' B_1 K_1^{(int)}(x_i''', x_j') + \sum_{i=1}^{N''''} a_i'''' B_1 K_2^{(int)}(x_i'''', x_j') = \sigma_j^{(int)}
\end{aligned}$$

N'' conditions of the form

$$\begin{aligned} B_2 S_X(x_j'') &= \sum_{i=1}^{N'} a_i' B_2 K_1^{(ext)}(x_i', x_j'') + \sum_{i=1}^{N''} a_i'' B_2 K_2^{(ext)}(x_i'', x_j'') + \\ & \sum_{i=1}^{N'''} a_i''' B_2 K_1^{(int)}(x_i''', x_j'') + \sum_{i=1}^{N''''} a_i'''' B_2 K_2^{(int)}(x_i'''', x_j'') = \tau_j^{(int)} \end{aligned}$$

N''' conditions of the form

$$\begin{aligned} B_1 S_X(x_j''') &= \sum_{i=1}^{N'} a_i' B_1 K_1^{(ext)}(x_i', x_j''') + \sum_{i=1}^{N''} a_i'' B_1 K_2^{(ext)}(x_i'', x_j''') + \\ & \sum_{i=1}^{N'''} a_i''' B_1 K_1^{(int)}(x_i''', x_j''') + \sum_{i=1}^{N''''} a_i'''' B_1 K_2^{(int)}(x_i'''', x_j''') = \sigma_j^{(ext)} \end{aligned}$$

and N'''' conditions of the form

$$\begin{aligned} B_2 S_X(x_j'''') &= \sum_{i=1}^{N'} a_i' B_2 K_1^{(ext)}(x_i', x_j'''') + \sum_{i=1}^{N''} a_i'' B_2 K_2^{(ext)}(x_i'', x_j'''') + \\ & \sum_{i=1}^{N'''} a_i''' B_2 K_1^{(int)}(x_i''', x_j'''') + \sum_{i=1}^{N''''} a_i'''' B_2 K_2^{(int)}(x_i'''', x_j'''') = \tau_j^{(ext)} . \end{aligned}$$

These conditions can be written in the form of a system of linear equations $Aa = b$ with a coefficient matrix $A =$

$$\begin{pmatrix} \left(B_1 K_1^{(ext)}(x_i', x_j')\right)_{ji} & \left(B_1 K_2^{(ext)}(x_i'', x_j')\right)_{ji} & \left(B_1 K_1^{(int)}(x_i''', x_j')\right)_{ji} & \left(B_1 K_2^{(int)}(x_i'''', x_j')\right)_{ji} \\ \left(B_2 K_1^{(ext)}(x_i', x_j'')\right)_{ji} & \left(B_2 K_2^{(ext)}(x_i'', x_j'')\right)_{ji} & \left(B_2 K_1^{(int)}(x_i''', x_j'')\right)_{ji} & \left(B_2 K_2^{(int)}(x_i'''', x_j'')\right)_{ji} \\ \left(B_1 K_1^{(ext)}(x_i', x_j''')\right)_{ji} & \left(B_1 K_2^{(ext)}(x_i'', x_j''')\right)_{ji} & \left(B_1 K_1^{(int)}(x_i''', x_j''')\right)_{ji} & \left(B_1 K_2^{(int)}(x_i'''', x_j''')\right)_{ji} \\ \left(B_2 K_1^{(ext)}(x_i', x_j'''')\right)_{ji} & \left(B_2 K_2^{(ext)}(x_i'', x_j'''')\right)_{ji} & \left(B_2 K_1^{(int)}(x_i''', x_j'''')\right)_{ji} & \left(B_2 K_2^{(int)}(x_i'''', x_j'''')\right)_{ji} \end{pmatrix}$$

and a vector

$$b = (\sigma_1^{(int)}, \ldots, \sigma_{N'}^{(int)}, \tau_1^{(int)}, \ldots, \sigma_{N''}^{(int)}, \sigma_1^{(ext)}, \ldots, \sigma_{N'''}^{(ext)}, \tau_1^{(ext)}, \ldots, \tau_{N''''}^{(ext)})^t.$$

Because X is an admissible system, the matrix A is invertible and the system of equations can be solved uniquely.

■

If for any function $G \in \mathcal{I}_X$ the set $\mathcal{I}_X^G$ is defined by

$$\begin{aligned} \mathcal{I}_X^G = \{F \in \mathcal{H} \text{ with} & \\ & B_1 F^{(int)}(x_i') = B_1 G^{(int)}(x_i') \quad B_1 F^{(ext)}(x_i') = B_1 G^{(ext)}(x_i') \quad \forall i = 1, \ldots, N' \\ & B_2 F^{(int)}(x_i'') = B_2 G^{(int)}(x_i'') \quad B_2 F^{(ext)}(x_i'') = B_2 G^{(ext)}(x_i'') \quad \forall i = 1, \ldots, N'' \\ & B_1 F^{(int)}(x_i''') = B_1 G^{(int)}(x_i''') \quad B_1 F^{(ext)}(x_i''') = B_1 G^{(ext)}(x_i''') \quad \forall i = 1, \ldots, N''' \\ & B_2 F^{(int)}(x_i'''') = B_2 G^{(int)}(x_i'''') \quad B_2 F^{(ext)}(x_i'''') = B_2 G^{(ext)}(x_i'''') \quad \forall i = 1, \ldots, N''''\}, \end{aligned}$$

we obtain the following lemmata:

Lemma.

Let S_X be the unique interpolating spline and F any interpolating function of the space $\mathcal{I}_X^{S_X}$. Then we get:

$$(S_X - F, S_X)_{\mathcal{H}} = 0 \,. \tag{4.3.18}$$

Proof.

$$\begin{aligned}
(S_X - F, S_X)_{\mathcal{H}} &= \sum_{i=1}^{N'} a_i'(S_X(\cdot), K_1^{(ext)}(x_i', \cdot))_{\mathcal{H}} - \sum_{i=1}^{N'} a_i'(F(\cdot), K_1^{(ext)}(x_i', \cdot))_{\mathcal{H}} + \\
&\sum_{i=1}^{N''} a_i''(S_X(\cdot), K_2^{(ext)}(x_i'', \cdot))_{\mathcal{H}} - \sum_{i=1}^{N''} a_i''(F(\cdot), K_2^{(ext)}(x_i'', \cdot))_{\mathcal{H}} + \\
&\sum_{i=1}^{N'''} a_i'''(S_X(\cdot), K_1^{(int)}(x_i''', \cdot))_{\mathcal{H}} - \sum_{i=1}^{N'''} a_i'''(F(\cdot), K_1^{(int)}(x_i''', \cdot))_{\mathcal{H}} + \\
&\sum_{i=1}^{N''''} a_i''''(S_X(\cdot), K_2^{(int)}(x_i'''', \cdot))_{\mathcal{H}} - \sum_{i=1}^{N''''} a_i''''(F(\cdot), K_2^{(int)}(x_i'''', \cdot))_{\mathcal{H}} \\
&= \sum_{i=1}^{N'} a_i' B_1 S_X^{(ext)}(x_i') - \sum_{i=1}^{N'} a_i' B_1 F^{(ext)}(x_i') + \\
&\sum_{i=1}^{N''} a_i'' B_2 S_X^{(ext)}(x_i'') - \sum_{i=1}^{N''} a_i'' B_2 F^{(ext)}(x_i'') + \\
&\sum_{i=1}^{N'''} a_i''' B_1 S_X^{(int)}(x_i''') - \sum_{i=1}^{N'''} a_i''' B_1 F^{(int)}(x_i''') + \\
&\sum_{i=1}^{N''''} a_i'''' B_2 S_X^{(int)}(x_i'''') - \sum_{i=1}^{N''''} a_i'''' B_2 F^{(int)}(x_i'''') = 0
\end{aligned}$$

■

Lemma.

Let S_X and F be defined as above. Then: $\|F\|_{\mathcal{H}}^2 = \|S_X\|_{\mathcal{H}}^2 + \|S_X - F\|_{\mathcal{H}}^2$.

Proof.

$$\begin{aligned}
\|F\|_{\mathcal{H}}^2 = (F(\cdot), F(\cdot))_{\mathcal{H}} &= (S_X(\cdot) - (S_X(\cdot) - F(\cdot)), S_X(\cdot) - (S_X(\cdot) - F(\cdot)))_{\mathcal{H}} \\
&= (S_X(\cdot), S_X(\cdot))_{\mathcal{H}} + (S_X(\cdot) - F(\cdot), S_X(\cdot) - F(\cdot))_{\mathcal{H}} \\
&= \|S_X\|^2 + \|S_X - F\|_{\mathcal{H}}^2
\end{aligned}$$

■

Summarizing our results we find the following theorem:

Theorem.

The spline interpolation problem: Find a function $S_X \in \mathcal{I}_X \cap \mathcal{S}_X$ with

$$\|S_X\|_{\mathcal{H}} = \inf_{F \in \mathcal{I}_X^{S_X}} \|F\|_{\mathcal{H}} \tag{4.3.19}$$

is well-posed in the sense that its solution exists, is unique and depends continuously on the given data.

For this solution an estimate for the error on the boundaries Γ_{r_i} and Γ_{r_a} can be proved.

Lemma.

Let X be an admissible system and $S_X \in \mathcal{I}_X \cap \mathcal{S}_X$ the solution of the spline interpolation problem with the interpolation values:

$$\begin{array}{llll} \sigma_i^{(int)} & \text{for } i = 1, \ldots, N' & ; \quad \tau_i^{(int)} & \text{for } i = 1, \ldots, N'' \\ \sigma_i^{(ext)} & \text{for } i = 1, \ldots, N''' & ; \quad \tau_i^{(ext)} & \text{for } i = 1, \ldots, N'''' . \end{array}$$

For any $F \in \mathcal{I}_X^{S_X}$ we have:

$$\sup_{x \in \Gamma_{r_a} \text{ resp. } \Gamma_{r_i}} |B_1 F(x) - B_1 S_X(x)| \leq A_1 \sqrt{\theta}\, \|F\|_{\mathcal{H}} \tag{4.3.20}$$

$$\sup_{x \in \Gamma_{r_a} \text{ resp. } \Gamma_{r_i}} |B_2 F(x) - B_2 S_X(x)| \leq A_2 \sqrt{\theta}\, \|F\|_{\mathcal{H}} \tag{4.3.21}$$

with

$$\theta = \max_{\varphi} \min_{\tilde{\varphi}} |\varphi - \tilde{\varphi}| , \tag{4.3.22}$$

$$\begin{aligned} A_1 \;=\; & \Bigg| 8 \sum_{n=1}^{\infty} \left(\frac{1}{q_n^{(1)}} \left(\frac{\omega_1}{r}\right)^{2n} \frac{1}{\pi \omega_1} \left(\frac{-n-n^2}{r^2} + \frac{r^2(2-n-n^2)}{\omega_1^4} \right) + \right. \\ & \left. \frac{1}{q_n^{(2)}} \left(\frac{r}{\omega_2}\right)^{2n} \frac{1}{\pi \omega_2} \left(\frac{n-n^2}{r^2} + \frac{r^2(2+n-n^2)}{\omega_2^4} \right) \right) \Bigg|^{\frac{1}{2}} \end{aligned} \tag{4.3.23}$$

and

$$\begin{aligned} A_2 \;=\; & \Bigg| 8 \sum_{n=1}^{\infty} \left(\frac{1}{q_n^{(1)}} \left(\frac{\omega_1}{r}\right)^{2n} \frac{1}{\pi \omega_1} \left(\frac{n+n^2}{r^2} + \frac{r^2(n^2-n)}{\omega_1^4} \right) + \right. \\ & \left. \frac{1}{q_n^{(2)}} \left(\frac{r}{\omega_2}\right)^{2n} \frac{1}{\pi \omega_2} \left(\frac{n-n^2}{r^2} + \frac{r^2(-n-n^2)}{\omega_2^4} \right) \right) \Bigg|^{\frac{1}{2}} , \end{aligned} \tag{4.3.24}$$

where $\varphi \in [0, 2\pi]$ and $\tilde{\varphi}$ is the angle corresponding to a point $\tilde{x}$ of X', X'', X''' resp. X'''', depending on the boundary operator.

Proof.

For any $x \in \Gamma_{r_i}$ resp. Γ_{r_a} we get:

$$\begin{aligned} |B_1 F(x) - B_1 S_X(x)| &= |B_1 F(x) - B_1 F(\tilde{x}) + B_1 S_X(\tilde{x}) - B_1 S_X(x)| \\ &\leq |(K_1(x,\cdot) - K_1(\tilde{x},\cdot), F(\cdot))_{\mathcal{H}}| + |(K_1(x,\cdot) - K_1(\tilde{x},\cdot), S_X(\cdot))_{\mathcal{H}}| \\ &\leq |(K_1(x,\cdot) - K_1(\tilde{x},\cdot), K_1(x,\cdot) - K_1(\tilde{x},\cdot))_{\mathcal{H}}|^{\frac{1}{2}} (\|S_X\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}) , \end{aligned}$$

where the point $\tilde{x} \in X'$ resp. X''' is chosen such that $|\varphi - \tilde{\varphi}| \leq \theta$.

As $\|S_X\|_{\mathcal{H}} \leq \|F\|_{\mathcal{H}}$, we obtain:

$$|B_1 F(x) - B_1 S_X(x)| \leq 2 |K_1(x,x) - 2K_1(x,\tilde{x}) + K_1(\tilde{x},\tilde{x})|^{\frac{1}{2}} \|F\|_{\mathcal{H}} .$$

The proof of the first estimate follows by observing

$$
\begin{aligned}
&|K_1(x,x) - 2K_1(x,\tilde{x}) + K_1(\tilde{x},\tilde{x})| \\
&= \left|\sum_{n=1}^{\infty} 2(1-\cos(n(\varphi-\tilde{\varphi}))\left(\frac{1}{q_n^{(1)}}\left(\frac{\omega_1}{r}\right)^{2n}\frac{1}{\pi\omega_1}\left(\frac{-n-n^2}{r^2}+\frac{r^2(2-n-n^2)}{\omega_1^4}\right)\right)+\right. \\
&\qquad \left.\sum_{n=1}^{\infty} 2(1-\cos(n(\varphi-\tilde{\varphi}))\left(\frac{1}{q_n^{(2)}}\left(\frac{r}{\omega_2}\right)^{2n}\frac{1}{\pi\omega_2}\left(\frac{n-n^2}{r^2}+\frac{r^2(2+n-n^2)}{\omega_2^4}\right)\right)\right| \\
&\leq 2\theta\left|\sum_{n=1}^{\infty}\left(\frac{1}{q_n^{(1)}}\left(\frac{\omega_1}{r}\right)^{2n}\frac{1}{\pi\omega_1}\left(\frac{-n-n^2}{r^2}+\frac{r^2(2-n-n^2)}{\omega_1^4}\right)\right)+\right. \\
&\qquad \left.\frac{1}{q_n^{(2)}}\left(\frac{r}{\omega_2}\right)^{2n}\frac{1}{\pi\omega_2}\left(\frac{n-n^2}{r^2}+\frac{r^2(2+n-n^2)}{\omega_2^4}\right)\right)\right|,
\end{aligned}
$$

where $|1-\cos(n(\varphi-\tilde{\varphi}))| \leq |n(\varphi-\tilde{\varphi})|$ is used. The second estimate can be shown analogously. ■

5 Application

In this chapter a solution of the boundary value problem described in Section 3.2 will be computed using the spline interpolation method presented in the last chapter. First of all we have to choose a Hilbert space $\mathcal{H}$ with an associated inner product $(\cdot,\cdot)_{\mathcal{H}}$.

5.1 Choice of the Norm

The inner product $(\cdot,\cdot)_{\mathcal{H}}$ of the space $\mathcal{H}$ is defined by the sequences $q^{(1)}$ and $q^{(2)}$, as well as by the free parameters ω_1 and ω_2. From the theoretical point of view one may assume, that for every admissible ω_1, ω_2, $q^{(1)}$ and $q^{(2)}$ the desired result can be achieved, because the absolute error on the boundaries tends to zero as the number of interpolation points increases. From a practical point of view, that means from numerical reasons, the result depends essentially on the choice of the norm. This leads to the problem to adapt the norm to the physical problem. For example, in view of

$$n^2\, Y_{nj} = -\frac{\partial^2}{\partial\varphi^2} Y_{nj}, \tag{5.1.1}$$

the sequence q with $q_n = n^2$ for $n \in I\!N$ symbolizes the negative derivative of a function. As this is a measure for the curvature of the belonging curve, by the norm-minimalisation-property of the spline interpolation, the integral over the second derivative, i.e. the total curvature, is minimized.
Another important criterion, which should be used for choosing the sequences $q^{(1)}$ and $q^{(2)}$ should be, that the kernel functions

$$
\begin{aligned}
K^{(int)}(x,\bar{x}) &= \sum_{n=1}^{\infty}\sum_{j=1}^{2}\frac{1}{q_n^{(2)}}\left(\frac{r}{\omega_2}\right)^n\left(\frac{\bar{r}}{\omega_2}\right)^n Y_{nj}^{(2)}(\varphi)Y_{nj}^{(2)}(\bar{\varphi}))\left(1+\frac{r^2\bar{r}^2}{\omega_2^4}\right)+ \\
&\quad 1 + r^2\bar{r}^2 + \ln(r)\ln(\bar{r})
\end{aligned}
\tag{5.1.2}
$$

and

$$K^{(ext)}(x,\bar{x}) = \sum_{n=1}^{\infty}\sum_{j=1}^{2} \frac{1}{q_n^{(1)}} \left(\frac{\omega_1}{r}\right)^n \left(\frac{\omega_1}{\bar{r}}\right)^n Y_{nj}^{(1)}(\varphi) Y_{nj}^{(1)}(\bar{\varphi})) \left(1 + \frac{r^2\bar{r}^2}{\omega_1^4}\right) \qquad (5.1.3)$$

can be represented as elementary functions. This is the case for the sequences $q_n = n$, $q_n = 1/n$, $q_n = n!$ and $q_n = 1$ for all $n \in I\!N$. The first sequence gives the representation

$$\begin{aligned} K^{(int)}(x,\bar{x}) &= -\frac{1}{2\pi\omega_2} \ln\left(1 + \frac{r^2\bar{r}^2}{w_2^4} - 2\frac{r\bar{r}}{w_2^2}\cos(\varphi-\bar{\varphi})\right)\left(1 + \frac{r^2\bar{r}^2}{\omega_2^4}\right) + \\ & \quad 1 + r^2\bar{r}^2 + \ln(r)\ln(\bar{r}) \\ K^{(ext)}(x,\bar{x}) &= -\frac{1}{2\pi\omega_1} \ln\left(1 + \frac{w_1^4}{r^2\bar{r}^2} - 2\frac{w_1^2}{r\bar{r}}\cos(\varphi-\bar{\varphi})\right)\left(1 + \frac{r^2\bar{r}^2}{\omega_1^4}\right), \end{aligned}$$

the second one

$$\begin{aligned} K^{(int)}(x,\bar{x}) &= \frac{1}{\pi\omega_2} \frac{\frac{r\bar{r}}{\omega_2^2}(\cos(\varphi-\bar{\varphi}) + \frac{r^2\bar{r}^2}{\omega_2^4}\cos(\varphi-\bar{\varphi}) - 2\frac{r\bar{r}}{\omega_2^2})}{(1 - 2\frac{r\bar{r}}{\omega_2^2}\cos(\varphi-\bar{\varphi}) + \frac{r^2\bar{r}^2}{\omega_2^4})^2}\left(1 + \frac{r^2\bar{r}^2}{\omega_2^4}\right) + \\ & \quad 1 + r^2\bar{r}^2 + \ln(r)\ln(\bar{r}) \\ K^{(ext)}(x,\bar{x} &= \frac{1}{\pi\omega_1} \frac{\frac{\omega_1^2}{r\bar{r}}(\cos(\varphi-\bar{\varphi}) + \frac{\omega_1^4}{r^2\bar{r}^2}\cos(\varphi-\bar{\varphi}) - 2\frac{\omega_1^2}{r\bar{r}})}{(1 - 2\frac{\omega_1^2}{r\bar{r}}\cos(\varphi-\bar{\varphi}) + \frac{\omega_1^4}{r^2\bar{r}^2})^2}\left(1 + \frac{r^2\bar{r}^2}{\omega_1^4}\right), \end{aligned}$$

the third one

$$\begin{aligned} K^{(int)}(x,\bar{x}) &= -\frac{1}{\pi\omega_2} e^{\frac{r\bar{r}}{\omega_2^2}\cos(\varphi-\bar{\varphi})} \cos\left(\frac{r\bar{r}}{\omega_2^2}\sin(\varphi-\bar{\varphi})\right)\left(1 + \frac{r^2\bar{r}^2}{\omega_2^4}\right) + \\ & \quad r^2\bar{r}^2 + \ln(r)\ln(\bar{r}) \\ K^{(ext)}(x,\bar{x}) &= -\frac{1}{\pi\omega_1} e^{\frac{\omega_1^2}{r\bar{r}}\cos(\varphi-\bar{\varphi})} \cos\left(\frac{\omega_1^2}{r\bar{r}}\sin(\varphi-\bar{\varphi})\right)\left(1 + \frac{r^2\bar{r}^2}{\omega_1^4}\right) \end{aligned}$$

and the fourth one

$$\begin{aligned} K^{(int)}(x,\bar{x}) &= \frac{1}{\pi\omega_2} \frac{1 - \frac{r\bar{r}}{\omega_2^2}\cos(\varphi-\bar{\varphi})}{1 + \frac{r^2\bar{r}^2}{w_2^4} - 2\frac{r\bar{r}}{w_2^2}\cos(\varphi-\bar{\varphi})}\left(1 + \frac{r^2\bar{r}^2}{w_2^4}\right) + \\ & \quad r^2\bar{r}^2 + \ln(r)\ln(\bar{r}) \\ K^{(ext)}(x,\bar{x}) &= \frac{1}{\pi\omega_1} \frac{1 - \frac{\omega_1^2}{r\bar{r}}\cos(\varphi-\bar{\varphi})}{1 + \frac{w_1^4}{r^2\bar{r}^2} - 2\frac{w_1^2}{r\bar{r}}\cos(\varphi-\bar{\varphi})}\left(1 + \frac{r^2\bar{r}^2}{w_1^4}\right). \end{aligned}$$

Of course it is also possible to use other sequences. In this case it is necessary to abort the series representation after a sufficiently big number of terms, what results in an inevitable error. Furthermore, because of the enormous work for computing the kernel functions, one has to put up with long running times of the program. For this reasons it is better to work with kernels, which can be represented by elementary functions.

5.2 Implementation of the Method

The necessary derivatives have been computed with the programming package *Mathematica*. In the case $q_n^{(1)} = q_n^{(2)} = 1 \ \forall n \in I\!N$ we obtain the following functions:

$$
\begin{aligned}
&B_1 K_1^{(int)}(x,\bar{x}) = \\
&\quad (-140r^6\bar{r}^6\omega_2^4 + 40r^4\bar{r}^4\omega_2^8 + 120r^2\bar{r}^2\omega_2^{12} + 4\omega_2^{16} + \\
&\quad \cos(\varphi-\bar{\varphi})(76r^7\bar{r}^7\omega_2^2 - 140r^3\bar{r}^3\omega_2^{10} - 56r\bar{r}\omega_2^{14}) + \\
&\quad \cos(2(\varphi-\bar{\varphi}))(-16r^8\bar{r}^8 + 84r^6\bar{r}^6\omega_2^4 + 140r^4\bar{r}^4\omega_2^8 - 20r^2\bar{r}^2\omega_2^{12} + 4\omega_2^{16}) + \\
&\quad \cos(3(\varphi-\bar{\varphi}))(-20r^7\bar{r}^7\omega_2^2 - 84r^5\bar{r}^5\omega_2^6 - 44r^3\bar{r}^3\omega_2^{10} + 16r\bar{r}\omega_2^{14}) + \\
&\quad \cos(4(\varphi-\bar{\varphi}))(16r^6\bar{r}^6\omega_2^4 + 20r^4\bar{r}^4\omega_2^8 + 4r^2\bar{r}^2\omega_2^{12}) - \\
&\quad \cos(5(\varphi-\bar{\varphi}))4r^5\bar{r}^5\omega_2^6)\ /(r^2\bar{r}^2 + \omega_2^4 - 2r\bar{r}\omega_2^2\cos(\varphi-\bar{\varphi}))^5 \ + 4 + \frac{1}{r^2\bar{r}^2}
\end{aligned}
$$

$$
\begin{aligned}
&B_1 K_1^{(ext)}(x,\bar{x}) = \\
&\quad (-4r^{10}\bar{r}^{10} - 100r^8\bar{r}^8\omega_1^4 - 540r^6\bar{r}^6\omega_1^8 - 360r^4\bar{r}^4\omega_1^{12} + 20r^2\bar{r}^2\omega_1^{16} + \\
&\quad \cos(\varphi-\bar{\varphi})(40r^9\bar{r}^9\omega_1^2 + 476r^7\bar{r}^7\omega_1^6 + 800r^5\bar{r}^5\omega_1^{10} + 260r^3\bar{r}^3\omega_1^{14} - 16r\bar{r}\omega_1^{18}) + \\
&\quad \cos(2(\varphi-\bar{\varphi}))(-96r^8\bar{r}^8\omega_1^4 - 316r^6\bar{r}^6\omega_1^8 - 260r^4\bar{r}^4\omega_1^{12} - 100r^2\bar{r}^2\omega_1^{16} + 4\omega_1^{20} + \\
&\quad \cos(3(\varphi-\bar{\varphi}))(60r^7\bar{r}^7\omega_1^6 + 116r^5\bar{r}^5\omega_1^{10} + 36r^3\bar{r}^3\omega_1^{14} + 16r\bar{r}\omega_1^{18}) - \\
&\quad \cos(4(\varphi-\bar{\varphi}))(-24r^6\bar{r}^6\omega_1^8 - 20r^4\bar{r}^4\omega_1^{12} + 4r^2\bar{r}^2\omega_1^{16}) + \\
&\quad \cos(5(\varphi-\bar{\varphi}))4r^5\bar{r}^5\omega_1^{10})\ / \\
&\quad (\omega_1^4(-r^2\bar{r}^2 - \omega_1^4 + 2r\bar{r}\omega_1^2\cos(\varphi-\bar{\varphi}))^5)
\end{aligned}
$$

$$
\begin{aligned}
&B_1 K_2^{(int)}(x,\bar{x}) = \\
&\quad (-16r^7\bar{r}^7\omega_2^2 + 184r^5\bar{r}^5\omega_2^6 + 128r^3\bar{r}^3\omega_2^{10} - 8r\bar{r}\omega_2^{14} + \\
&\quad \cos(\varphi-\bar{\varphi})(16r^8\bar{r}^8 - 176r^6\bar{r}^6\omega_2^4 - 120r^4\bar{r}^4\omega_2^8 - 112r^2\bar{r}^2\omega_2^{12} + 8\omega_2^{16}) + \\
&\quad \cos(2(\varphi-\bar{\varphi}))(40r^7\bar{r}^7\omega_2^2 + 48r^5\bar{r}^5\omega_2^6 - 24r^3\bar{r}^3\omega_2^{10} + 32r\bar{r}\omega_2^{14}) + \\
&\quad \cos(3(\varphi-\bar{\varphi}))(-8r^6\bar{r}^6\omega_2^4 + 8r^2\bar{r}^2\omega_2^{12})\sin(\varphi-\bar{\varphi}))\ / \\
&\quad (r^2\bar{r}^2 + \omega_2^4 - 2r\bar{r}\omega_2^2\cos(\varphi-\bar{\varphi}))^5
\end{aligned}
$$

$B_1K_2^{(ext)}(x,\bar{x}) =$

$$(16r^7\bar{r}^7\omega_1^2 - 184r^5\bar{r}^5\omega_1^6 - 128r^3\bar{r}3\omega_1^{10} + 8r\bar{r}\omega_1^{14}+$$
$$\cos(\varphi-\bar{\varphi})(-16r^8\bar{r}^8 + 176r^6\bar{r}^6\omega_1^4 + 120r^4\bar{r}^4\omega_1^8 + 112r^2\bar{r}^2\omega_1^{12} - 8\omega_1^{16})+$$
$$\cos(2(\varphi-\bar{\varphi}))(-40r^7\bar{r}^7\omega_1^2 - 48r^5\bar{r}^5\omega_1^6 + 24r^3\bar{r}^3\omega_1^{10} - 32r\bar{r}\omega_1^{14})+$$
$$\cos(3(\varphi-\bar{\varphi}))(8r^6\bar{r}^6\omega_1^4 - 8r^2\bar{r}^2\omega_1^{12})\sin(\varphi-\bar{\varphi}))\ /$$
$$(r^2\bar{r}^2 + \omega_1^4 - 2r\bar{r}\omega_1^2\cos(\varphi-\bar{\varphi}))^5$$

$B_2K_1^{(int)}(x,\bar{x}) =$

$$(16r^7\bar{r}^7\omega_2^2 - 184r^5\bar{r}^5\omega_2^6 - 128r^3\bar{r}3\omega_2^{10} + 8r\bar{r}\omega_2^{14}+$$
$$\cos(\varphi-\bar{\varphi})(-16r^8\bar{r}^8 + 176r^6\bar{r}^6\omega_2^4 + 120r^4\bar{r}^4\omega_2^8 + 112r^2\bar{r}^2\omega_1^{12} - 8\omega_2^{16})+$$
$$\cos(2(\varphi-\bar{\varphi}))(-40r^7\bar{r}^7\omega_2^2 - 48r^5\bar{r}^5\omega_2^5 + 24r^3\bar{r}^3\omega_2^{10} - 32r\bar{r}\omega_2^{14})+$$
$$\cos(3(\varphi-\bar{\varphi}))(8r^6\bar{r}^6\omega_2^4 - 8r^2\bar{r}^2\omega_2^{12})\sin(\varphi-\bar{\varphi}))\ /$$
$$(r^2\bar{r}^2 + \omega_2^4 - 2r\bar{r}\omega_2^2\cos(\varphi-\bar{\varphi}))^5$$

$B_2K_1^{(ext)}(x,\bar{x}) =$

$$(-16r^7\bar{r}^7\omega_1^2 + 184r^5\bar{r}^5\omega_1^6 + 128r^3\bar{r}3\omega_1^{10} - 8r\bar{r}\omega_1^{14}+$$
$$\cos(\varphi-\bar{\varphi})(16r^8\bar{r}^8 - 176r^6\bar{r}^6\omega_1^4 - 120r^4\bar{r}^4\omega_1^8 - 112r^2\bar{r}^2\omega_1^{12} + 8\omega_1^{16})+$$
$$\cos(2(\varphi-\bar{\varphi}))(40r^7\bar{r}^7\omega_1^2 + 48r^5\bar{r}^5\omega_2^5 - 24r^3\bar{r}^3\omega_1^{10} + 32r\bar{r}\omega_1^{14})+$$
$$\cos(3(\varphi-\bar{\varphi}))(-8r^6\bar{r}^6\omega_1^4 + 8r^2\bar{r}^2\omega_1^{12})\sin(\varphi-\bar{\varphi}))\ /$$
$$(r^2\bar{r}^2 + \omega_1^4 - 2r\bar{r}\omega_1^2\cos(\varphi-\bar{\varphi}))^5$$

$B_2K_2^{(int)}(x,\bar{x}) =$

$$(4r^2\bar{r}^2 - 4\omega_2^4)\,(5r^4\bar{r}^4\omega_2^4 + 5r^2\bar{r}^2\omega_2^8+$$
$$\cos(\varphi-\bar{\varphi})(-4r^5\bar{r}^5\omega_2^2 + 16r^3\bar{r}^3\omega_2^6 - 4r\bar{r}\omega_2^{10})+$$
$$\cos(2(\varphi-\bar{\varphi}))(r^6\bar{r}^6 - 15r^4\bar{r}^4\omega_2^4 - 15r^2\bar{r}^2\omega_2^8 + \omega_2^{12})+$$
$$\cos(3(\varphi-\bar{\varphi}))(4r^5\bar{r}^5\omega_2^2 + 4r\bar{r}\omega_2^{10}) + \cos(4(\varphi-\bar{\varphi}))(r^4\bar{r}^4\omega_2^4 + r^2\bar{r}^2\omega_2^8)\ /$$
$$(-r^2\bar{r}^2 - w_2^4 + 2r\bar{r}\omega_2^2\cos(\varphi-\bar{\varphi}))^5$$

$$B_2K_2^{(ext)}(x,\bar{x}) =$$

$$(4r^2\bar{r}^2 - 4\omega_1^4)\,(5r^4\bar{r}^4\omega_1^4 + 5r^2\bar{r}^2\omega_1^8 +$$

$$\cos(\varphi - \bar{\varphi})(-4r^5\bar{r}^5\omega_1^2 + 16r^3\bar{r}^3\omega_1^6 - 4r\bar{r}\omega_1^{10}) +$$

$$\cos(2(\varphi - \bar{\varphi}))(r^6\bar{r}^6 - 15r^4\bar{r}^4\omega_1^4 - 15r^2\bar{r}^2\omega_1^8 + \omega_2^{12}) +$$

$$\cos(3(\varphi - \bar{\varphi}))(4r^5\bar{r}^5\omega_1^2 + 4r\bar{r}\omega_1^{10}) + \cos(4(\varphi - \bar{\varphi}))(r^4\bar{r}^4\omega_1^4 + r^2\bar{r}^2\omega_1^8)\,/$$

$$(r^2\bar{r}^2 + w_1^4 - 2r\bar{r}\omega_1^2\cos(\varphi - \bar{\varphi}))^5$$

Using these functions, the spline interpolation method has been implemented on a PC in the programming language PASCAL. First of all, the matrix $A=$

$$\begin{pmatrix}
\left(B_1K_1^{(ext)}(x_i',x_j')\right)_{ji} & \left(B_1K_2^{(ext)}(x_i'',x_j')\right)_{ji} & \left(B_1K_1^{(int)}(x_i''',x_j')\right)_{ji} & \left(B_1K_2^{(int)}(x_i'''',x_j')\right)_{ji} \\
\left(B_2K_1^{(ext)}(x_i',x_j'')\right)_{ji} & \left(B_2K_2^{(ext)}(x_i'',x_j'')\right)_{ji} & \left(B_2K_1^{(int)}(x_i''',x_j'')\right)_{ji} & \left(B_2K_2^{(int)}(x_i'''',x_j'')\right)_{ji} \\
\left(B_1K_1^{(ext)}(x_i',x_j''')\right)_{ji} & \left(B_1K_2^{(ext)}(x_i'',x_j''')\right)_{ji} & \left(B_1K_1^{(int)}(x_i''',x_j''')\right)_{ji} & \left(B_1K_2^{(int)}(x_i'''',x_j''')\right)_{ji} \\
\left(B_2K_1^{(ext)}(x_i',x_j'''')\right)_{ji} & \left(B_2K_2^{(ext)}(x_i'',x_j'''')\right)_{ji} & \left(B_2K_1^{(int)}(x_i''',x_j'''')\right)_{ji} & \left(B_2K_2^{(int)}(x_i'''',x_j'''')\right)_{ji}
\end{pmatrix}$$

and the vector

$$b = (\sigma_1^{(int)},\ldots,\sigma_{N'}^{(int)},\tau_1^{(int)},\ldots,\sigma_{N''}^{(int)},\sigma_1^{(ext)},\ldots,\sigma_{N'''}^{(ext)},\tau_1^{(ext)},\ldots,\tau_{N''''}^{(ext)})^t\,,$$

are built up. For solving the linear system of equations, an elimination method with pivoting or an orthogonalisation method due to Householder can be used. Several other procedures have been tested, but because of the ill conditioned matrices, they turned out to be unsuitable.

With help of the solution vector a, the stresses σ_r and τ can be computed at any point x by

$$\begin{aligned}
\sigma_r(x) \;=\;& \sum_{i=1}^{N'} a_i' B_1K_1^{(ext)}(x_i',x) + \sum_{i=1}^{N''} a_i'' B_1K_2^{(ext)}(x_i'',x) + \\
& \sum_{i=1}^{N'''} a_i''' B_1K_1^{(int)}(x_i''',x) + \sum_{i=1}^{N''''} a_i'''' B_1K_2^{(int)}(x_i'''',x) \qquad (5.2.1)
\end{aligned}$$

and

$$\begin{aligned}
\tau(x) \;=\;& \sum_{i=1}^{N'} a_i' B_2K_1^{(ext)}(x_i',x) + \sum_{i=1}^{N''} a_i'' B_2K_2^{(ext)}(x_i'',x) + \\
& \sum_{i=1}^{N'''} a_i''' B_2K_1^{(int)}(x_i''',x) + \sum_{i=1}^{N''''} a_i'''' B_2K_2^{(int)}(x_i'''',x). \qquad (5.2.2)
\end{aligned}$$

Testing the program, one observes, that the parameters ω_1 and ω_2 have a strong influence on the result. The quality of the interpolation on the inner boundary is essentially determined by ω_1, the quality on the outer boundary by ω_2. Regarding the residuum $r = Aa - b$, it turns out, that the linear system becomes more stable, if the parameters ω_1 and ω_2 are chosen close to the boundaries of the circular ring, but that at the same time oscillations in the solution functions are increasing. In the reverse case, that means with increasing distance of the parameters from the boundaries, the solution becomes more and more smoother, but the system of equations is very instable, and for some values ω_1 and ω_2, because of the big residuum, the solution is useless. Consequently, in the practice we have to find a compromise between a stable system of linear equations with small residuums on one side, and a smooth solution on the other side.

5.3 Examples

In order to assess the quality of the method, in this section we apply it to three examples. For $1 \leq r \leq 2$ and $0 \leq \varphi \leq 2\pi$ we define the following biharmonic functions:

$$f_1(r,\varphi) = r^3\cos^3(\varphi) \qquad (5.3.1)$$

$$f_2(r,\varphi) = \frac{\sin(4\varphi)}{r^2} + \frac{\cos(3\varphi)}{r} \qquad (5.3.2)$$

$$f_3(r,\varphi) = r^4\left(\cos(\varphi) - 3\cos^2(\varphi)\sin^2(\varphi)\right). \qquad (5.3.3)$$

Applying the operators B_1 and B_2, the corresponding radial stresses are given by:

$$B_1 f_1(r,\varphi) = 6r\cos(\varphi)\sin^2(\varphi) \qquad (5.3.4)$$

$$B_1 f_2(r,\varphi) = -\frac{10\cos(3\varphi)}{r^3} - \frac{18\sin(4\varphi)}{r^4} \qquad (5.3.5)$$

$$B_1 f_3(r,\varphi) = -6r^2\cos(4\varphi) \qquad (5.3.6)$$

and the shearing stresses by:

$$B_2 f_1(r,\varphi) = 6r\cos^2(\varphi)\sin(\varphi) \qquad (5.3.7)$$

$$B_2 f_2(r,\varphi) = \frac{12\cos(4\varphi)}{r^4} - \frac{6\sin(3\varphi)}{r^3} \qquad (5.3.8)$$

$$B_2 f_3(r,\varphi) = 3r^2(\sin(2\varphi) + 2\sin(4\varphi)). \qquad (5.3.9)$$

These stresses have been evaluated at n equidistant distributed points on the circles given by $\|x\| = r_i = 1$ and $\|x\| = r_a = 2$ to get some interpolation values. After that, the interpolation functions have been computed with several parameters ω_1 and ω_2. As kernel functions the sequences $q^{(1)} = q^{(2)} = q$ with $q_n = 1$ for $n = 1, 2, 3, \ldots$ have been used, which are described in the last section. Finally the resulting interpolants have been compared with the original functions on a grid T, which was divided in angular direction in 27 and in radial direction in 17 points, i.e. on a grid consisting of a total of 459 points. Tables of the maximum error are shown on the next pages, where the following notations are used:

$$\epsilon_1^{\Gamma_{ra}} = \max_{x \in \Gamma_{ra} \cap T} |B_1 S_X(x) - B_1 f(x)| \tag{5.3.10}$$

$$\epsilon_1^{\Gamma_{ri}} = \max_{x \in \Gamma_{ri} \cap T} |B_1 S_X(x) - B_1 f(x)| \tag{5.3.11}$$

$$\epsilon_1^{\Omega} = \max_{x \in \Omega \cap T} |B_1 S_X(x) - B_1 f(x)| \tag{5.3.12}$$

for the maximum error of the radial stress on the inner boundary, the outer boundary and the whole circular ring and

$$\epsilon_2^{\Gamma_{ra}} = \max_{x \in \Gamma_{ra} \cap T} |B_2 S_X(x) - B_2 f(x)| \tag{5.3.13}$$

$$\epsilon_2^{\Gamma_{ri}} = \max_{x \in \Gamma_{ri} \cap T} |B_2 S_X(x) - B_2 f(x)| \tag{5.3.14}$$

$$\epsilon_2^{\Omega} = \max_{x \in \Omega \cap T} |B_2 S_X(x) - B_2 f(x)| \tag{5.3.15}$$

for the corresponding errors of the shearing stress.

From the tables can be deduced, that the maximum error occures most times on the boundaries, but that there are also some counterexamples as function f_2 with $\omega_1 = 0.4$, $\omega_2 = 7$ and $n = 35$ (cf. table 1). Using more interpolation points ($n > 40$) didn't improve the result. The reason for this is the mentioned increasing condition number of the matrices and the corresponding problem solving the system of linear equations.

	$\epsilon \setminus n$	15	20	25	30	35	40
f_1	$\epsilon_1^{\Gamma_{ra}}$	$2.75\,10^{-4}$	$4.80\,10^{-9}$	$5.46\,10^{-14}$	$1.75\,10^{-16}$	$1.61\,10^{-16}$	$4.88\,10^{-16}$
	$\epsilon_1^{\Gamma_{ri}}$	$1.79\,10^{-5}$	$4.47\,10^{-13}$	$2.34\,10^{-16}$	$1.42\,10^{-16}$	$1.20\,10^{-16}$	$3.00\,10^{-16}$
	ϵ_1^{Ω}	$2.75\,10^{-4}$	$4.80\,10^{-9}$	$5.46\,10^{-14}$	$2.43\,10^{-16}$	$1.62\,10^{-16}$	$6.05\,10^{-16}$
	$\epsilon_2^{\Gamma_{ra}}$	$2.85\,10^{-4}$	$4.70\,10^{-9}$	$5.43\,10^{-14}$	$1.84\,10^{-16}$	$1.25\,10^{-16}$	$3.89\,10^{-16}$
	$\epsilon_2^{\Gamma_{ri}}$	$1.44\,10^{-5}$	$1.50\,10^{-13}$	$2.20\,10^{-16}$	$1.56\,10^{-16}$	$1.08\,10^{-16}$	$2.01\,10^{-16}$
	ϵ_2^{Ω}	$2.85\,10^{-4}$	$4.70\,10^{-9}$	$5.43\,10^{-14}$	$2.60\,10^{-16}$	$1.33\,10^{-16}$	$3.89\,10^{-16}$
f_2	$\epsilon_1^{\Gamma_{ra}}$	$2.33\,10^{-5}$	$1.86\,10^{-10}$	$1.84\,10^{-15}$	$1.30\,10^{-16}$	$1.15\,10^{-16}$	$1.32\,10^{-16}$
	$\epsilon_1^{\Gamma_{ri}}$	$2.59\,10^{-2}$	$1.25\,10^{-5}$	$4.05\,10^{-9}$	$1.00\,10^{-12}$	$3.12\,10^{-16}$	$1.42\,10^{-16}$
	ϵ_1^{Ω}	$2.59\,10^{-2}$	$1.25\,10^{-5}$	$4.05\,10^{-9}$	$1.00\,10^{-12}$	$1.18\,10^{-14}$	$1.16\,10^{-14}$
	$\epsilon_2^{\Gamma_{ra}}$	$1.40\,10^{-6}$	$1.69\,10^{-10}$	$1.71\,10^{-15}$	$5.09\,10^{-17}$	$2.34\,10^{-17}$	$1.29\,10^{-16}$
	$\epsilon_2^{\Gamma_{ri}}$	$2.13\,10^{-2}$	$1.16\,10^{-5}$	$3.74\,10^{-9}$	$8.85\,10^{-13}$	$2.86\,10^{-17}$	$7.89\,10^{-17}$
	ϵ_2^{Ω}	$2.13\,10^{-2}$	$1.16\,10^{-5}$	$3.74\,10^{-9}$	$8.85\,10^{-13}$	$1.28\,10^{-14}$	$1.25\,10^{-14}$
f_3	$\epsilon_1^{\Gamma_{ra}}$	$8.31\,10^{-7}$	$1.16\,10^{-11}$	$1.46\,10^{-16}$	$1.17\,10^{-17}$	$1.73\,10^{-17}$	$1.52\,10^{-17}$
	$\epsilon_1^{\Gamma_{ri}}$	$6.96\,10^{-9}$	$3.77\,10^{-16}$	$6.50\,10^{-18}$	$1.44\,10^{-17}$	$1.43\,10^{-17}$	$1.74\,10^{-17}$
	ϵ_1^{Ω}	$8.31\,10^{-7}$	$1.16\,10^{-11}$	$1.46\,10^{-16}$	$1.60\,10^{-17}$	$1.73\,10^{-17}$	$2.14\,10^{-17}$
	$\epsilon_2^{\Gamma_{ra}}$	$8.59\,10^{-7}$	$1.40\,10^{-11}$	$1.58\,10^{-16}$	$1.30\,10^{-17}$	$1.56\,10^{-17}$	$1.19\,10^{-17}$
	$\epsilon_2^{\Gamma_{ri}}$	$4.61\,10^{-9}$	$3.93\,10^{-16}$	$6.45\,10^{-18}$	$2.08\,10^{-17}$	$9.54\,10^{-18}$	$1.32\,10^{-17}$
	ϵ_2^{Ω}	$8.59\,10^{-7}$	$1.40\,10^{-11}$	$1.58\,10^{-16}$	$2.60\,10^{-17}$	$1.56\,10^{-17}$	$1.90\,10^{-17}$

Table 1: error for $\omega_1 = 0.4$ and $\omega_2 = 7$

	$\epsilon \setminus n$	15	20	25	30	35	40
f_1	$\epsilon_1^{\Gamma ra}$	$5.29\,10^{-1}$	$3.58\,10^{-3}$	$1.17\,10^{-5}$	$2.12\,10^{-8}$	$4.21\,10^{-12}$	$8.88\,10^{-14}$
	$\epsilon_1^{\Gamma ri}$	$1.51\,10^{-1}$	$1.09\,10^{-5}$	$1.14\,10^{-10}$	$1.82\,10^{-15}$	$1.56\,10^{-17}$	$9.10\,10^{-18}$
	ϵ_1^{Ω}	$5.29\,10^{-1}$	$3.58\,10^{-3}$	$1.17\,10^{-5}$	$2.12\,10^{-8}$	$2.64\,10^{-11}$	$8.88\,10^{-14}$
	$\epsilon_2^{\Gamma ra}$	$5.65\,10^{-1}$	$3.39\,10^{-3}$	$1.14\,10^{-5}$	$2.69\,10^{-8}$	$5.49\,10^{-11}$	$9.02\,10^{-14}$
	$\epsilon_2^{\Gamma ri}$	$1.38\,10^{-1}$	$9.22\,10^{-6}$	$6.70\,10^{-11}$	$9.61\,10^{-16}$	$1.34\,10^{-17}$	$1.25\,10^{-17}$
	ϵ_2^{Ω}	$5.65\,10^{-1}$	$3.39\,10^{-3}$	$1.14\,10^{-5}$	$2.69\,10^{-8}$	$5.49\,10^{-11}$	$9.02\,10^{-14}$
f_2	$\epsilon_1^{\Gamma ra}$	$2.90\,10^{-2}$	$1.88\,10^{-6}$	$2.05\,10^{-11}$	$3.99\,10^{-16}$	$2.60\,10^{-17}$	$2.28\,10^{-17}$
	$\epsilon_1^{\Gamma ri}$	$3.66\,10^{-1}$	$2.77\,10^{-3}$	$8.79\,10^{-6}$	$2.02\,10^{-8}$	$4.10\,10^{-11}$	$6.70\,10^{-14}$
	ϵ_1^{Ω}	$3.66\,10^{-1}$	$2.77\,10^{-3}$	$8.79\,10^{-6}$	$2.02\,10^{-8}$	$4.10\,10^{-11}$	$6.70\,10^{-14}$
	$\epsilon_2^{\Gamma ra}$	$2.77\,10^{-2}$	$1.88\,10^{-6}$	$1.34\,10^{-11}$	$1.56\,10^{-16}$	$3.36\,10^{-18}$	$4.58\,10^{-18}$
	$\epsilon_2^{\Gamma ri}$	$2.87\,10^{-1}$	$2.70\,10^{-3}$	$8.19\,10^{-6}$	$1.71\,10^{-8}$	$3.48\,10^{-12}$	$6.60\,10^{-14}$
	ϵ_2^{Ω}	$2.87\,10^{-1}$	$2.70\,10^{-3}$	$8.19\,10^{-6}$	$1.71\,10^{-8}$	$1.29\,10^{-11}$	$6.60\,10^{-14}$
f_3	$\epsilon_1^{\Gamma ra}$	$2.50\,10^{-2}$	$1.05\,10^{-4}$	$3.26\,10^{-7}$	$7.28\,10^{-10}$	$1.32\,10^{-12}$	$2.15\,10^{-15}$
	$\epsilon_1^{\Gamma ri}$	$2.91\,10^{-3}$	$6.05\,10^{-8}$	$7.78\,10^{-13}$	$2.25\,10^{-17}$	$2.16\,10^{-18}$	$4.11\,10^{-18}$
	ϵ_1^{Ω}	$2.50\,10^{-2}$	$1.05\,10^{-4}$	$3.26\,10^{-7}$	$7.28\,10^{-10}$	$1.32\,10^{-12}$	$2.15\,10^{-15}$
	$\epsilon_2^{\Gamma ra}$	$2.89\,10^{-2}$	$1.34\,10^{-4}$	$3.78\,10^{-7}$	$4.25\,10^{-10}$	$9.67\,10^{-13}$	$2.25\,10^{-15}$
	$\epsilon_2^{\Gamma ri}$	$2.53\,10^{-3}$	$6.49\,10^{-8}$	$2.50\,10^{-13}$	$8.99\,10^{-18}$	$3.36\,10^{-18}$	$3.03\,10^{-18}$
	ϵ_2^{Ω}	$2.89\,10^{-2}$	$1.34\,10^{-4}$	$3.78\,10^{-7}$	$4.25\,10^{-10}$	$9.67\,10^{-13}$	$2.25\,10^{-15}$

Table 2: error for $\omega_1 = 0.5$ and $\omega_2 = 4$

	$\epsilon \setminus n$	15	20	25	30	35	40
f_1	$\epsilon_1^{\Gamma ra}$	$1.87\,10^{-6}$	$9.14\,10^{-13}$	$1.55\,10^{-15}$	$1.28\,10^{-15}$	$6.42\,10^{-15}$	$5.32\,10^{-16}$
	$\epsilon_1^{\Gamma ri}$	$3.77\,10^{-9}$	$2.85\,10^{-15}$	$2.16\,10^{-15}$	$1.86\,10^{-15}$	$7.44\,10^{-15}$	$7.50\,10^{-16}$
	ϵ_1^{Ω}	$1.87\,10^{-6}$	$9.14\,10^{-13}$	$2.38\,10^{-15}$	$2.01\,10^{-15}$	$7.44\,10^{-15}$	$8.53\,10^{-16}$
	$\epsilon_2^{\Gamma ra}$	$1.90\,10^{-6}$	$9.06\,10^{-13}$	$2.07\,10^{-15}$	$1.25\,10^{-15}$	$1.89\,10^{-15}$	$4.02\,10^{-16}$
	$\epsilon_2^{\Gamma ri}$	$3.61\,10^{-9}$	$2.10\,10^{-15}$	$1.17\,10^{-15}$	$8.15\,10^{-16}$	$2.05\,10^{-15}$	$3.30\,10^{-16}$
	ϵ_2^{Ω}	$1.90\,10^{-6}$	$9.06\,10^{-13}$	$2.43\,10^{-15}$	$1.25\,10^{-15}$	$3.07\,10^{-15}$	$4.16\,10^{-16}$
f_2	$\epsilon_1^{\Gamma ra}$	$8.19\,10^{-10}$	$6.01\,10^{-14}$	$4.12\,10^{-14}$	$2.43\,10^{-13}$	$1.09\,10^{-13}$	$2.78\,10^{-13}$
	$\epsilon_1^{\Gamma ri}$	$1.64\,10^{-6}$	$7.46\,10^{-13}$	$5.21\,10^{-14}$	$2.10\,10^{-13}$	$1.57\,10^{-13}$	$3.34\,10^{-13}$
	ϵ_1^{Ω}	$1.64\,10^{-6}$	$7.46\,10^{-13}$	$7.49\,10^{-14}$	$2.68\,10^{-13}$	$1.79\,10^{-13}$	$3.90\,10^{-13}$
	$\epsilon_2^{\Gamma ra}$	$6.44\,10^{-10}$	$3.67\,10^{-14}$	$3.93\,10^{-14}$	$1.38\,10^{-13}$	$3.53\,10^{-14}$	$1.67\,10^{-13}$
	$\epsilon_2^{\Gamma ri}$	$1.36\,10^{-6}$	$6.57\,10^{-13}$	$2.87\,10^{-14}$	$1.84\,10^{-13}$	$2.60\,10^{-14}$	$1.47\,10^{-13}$
	ϵ_2^{Ω}	$1.36\,10^{-6}$	$6.57\,10^{-13}$	$5.56\,10^{-14}$	$2.03\,10^{-13}$	$8.32\,10^{-14}$	$1.69\,10^{-13}$
f_3	$\epsilon_1^{\Gamma ra}$	$1.31\,10^{-9}$	$5.45\,10^{-16}$	$4.87\,10^{-17}$	$1.35\,10^{-17}$	$1.83\,10^{-16}$	$4.57\,10^{-17}$
	$\epsilon_1^{\Gamma ri}$	$1.27\,10^{-12}$	$1.29\,10^{-16}$	$5.96\,10^{-17}$	$1.65\,10^{-17}$	$2.54\,10^{-16}$	$7.01\,10^{-17}$
	ϵ_1^{Ω}	$1.31\,10^{-9}$	$5.45\,10^{-16}$	$6.79\,10^{-17}$	$1.65\,10^{-17}$	$2.54\,10^{-16}$	$7.01\,10^{-17}$
	$\epsilon_2^{\Gamma ra}$	$1.33\,10^{-9}$	$6.89\,10^{-16}$	$4.85\,10^{-17}$	$9.32\,10^{-18}$	$4.47\,10^{-17}$	$2.08\,10^{-17}$
	$\epsilon_2^{\Gamma ri}$	$1.29\,10^{-12}$	$9.08\,10^{-17}$	$3.20\,10^{-17}$	$1.18\,10^{-17}$	$7.24\,10^{-17}$	$2.40\,10^{-17}$
	ϵ_2^{Ω}	$1.33\,10^{-9}$	$6.89\,10^{-16}$	$6.41\,10^{-17}$	$1.18\,10^{-17}$	$7.62\,10^{-17}$	$3.23\,10^{-17}$

Table 3: error for $\omega_1 = 0.2$ and $\omega_2 = 10$

	$\epsilon \setminus n$	15	20	25	30	35	40
f_1	$\epsilon_1^{\Gamma ra}$	$2.37\,10^{-3}$	$1.96\,10^{-7}$	$1.04\,10^{-11}$	$4.44\,10^{-16}$	$4.92\,10^{-16}$	$6.59\,10^{-17}$
	$\epsilon_1^{\Gamma ri}$	$1.35\,10^{-5}$	$1.18\,10^{-11}$	$4.94\,10^{-17}$	$4.85\,10^{-17}$	$3.09\,10^{-16}$	$3.23\,10^{-17}$
	ϵ_1^{Ω}	$2.37\,10^{-3}$	$1.96\,10^{-7}$	$1.04\,10^{-11}$	$4.44\,10^{-16}$	$4.92\,10^{-16}$	$7.37\,10^{-17}$
	$\epsilon_2^{\Gamma ra}$	$2.49\,10^{-3}$	$1.89\,10^{-7}$	$1.03\,10^{-11}$	$4.73\,10^{-16}$	$2.30\,10^{-16}$	$8.50\,10^{-17}$
	$\epsilon_2^{\Gamma ri}$	$2.94\,10^{-6}$	$1.16\,10^{-11}$	$6.76\,10^{-17}$	$5.03\,10^{-17}$	$4.41\,10^{-16}$	$4.29\,10^{-17}$
	ϵ_2^{Ω}	$2.49\,10^{-3}$	$1.89\,10^{-7}$	$1.03\,10^{-11}$	$4.73\,10^{-16}$	$4.80\,10^{-16}$	$8.50\,10^{-17}$
f_2	$\epsilon_1^{\Gamma ra}$	$1.26\,10^{-6}$	$1.88\,10^{-13}$	$1.88\,10^{-15}$	$2.20\,10^{-15}$	$1.76\,10^{-15}$	$1.42\,10^{-15}$
	$\epsilon_1^{\Gamma ri}$	$4.85\,10^{-4}$	$1.22\,10^{-8}$	$2.18\,10^{-13}$	$1.98\,10^{-15}$	$1.85\,10^{-15}$	$1.06\,10^{-15}$
	ϵ_1^{Ω}	$4.85\,10^{-4}$	$1.22\,10^{-8}$	$2.18\,10^{-13}$	$4.65\,10^{-15}$	$3.81\,10^{-15}$	$4.06\,10^{-15}$
	$\epsilon_2^{\Gamma ra}$	$8.19\,10^{-7}$	$1.65\,10^{-13}$	$3.49\,10^{-16}$	$1.07\,10^{-15}$	$1.14\,10^{-15}$	$5.35\,10^{-16}$
	$\epsilon_2^{\Gamma ri}$	$4.02\,10^{-4}$	$1.11\,10^{-8}$	$2.01\,10^{-13}$	$8.18\,10^{-16}$	$6.79\,10^{-16}$	$3.33\,10^{-16}$
	ϵ_2^{Ω}	$4.02\,10^{-4}$	$1.11\,10^{-8}$	$2.01\,10^{-13}$	$5.06\,10^{-15}$	$4.09\,10^{-16}$	$3.92\,10^{-15}$
f_3	$\epsilon_1^{\Gamma ra}$	$1.38\,10^{-5}$	$9.05\,10^{-10}$	$5.17\,10^{-14}$	$1.08\,10^{-17}$	$1.18\,10^{-16}$	$7.80\,10^{-18}$
	$\epsilon_1^{\Gamma ri}$	$1.44\,10^{-8}$	$2.58\,10^{-14}$	$3.46\,10^{-18}$	$6.61\,10^{-18}$	$9.15\,10^{-17}$	$4.22\,10^{-18}$
	ϵ_1^{Ω}	$1.38\,10^{-5}$	$9.05\,10^{-10}$	$5.17\,10^{-14}$	$1.17\,10^{-17}$	$1.66\,10^{-16}$	$9.54\,10^{-18}$
	$\epsilon_2^{\Gamma ra}$	$1.44\,10^{-5}$	$1.10\,10^{-9}$	$5.60\,10^{-14}$	$6.28\,10^{-18}$	$1.65\,10^{-16}$	$7.20\,10^{-18}$
	$\epsilon_2^{\Gamma ri}$	$1.15\,10^{-8}$	$3.23\,10^{-14}$	$4.77\,10^{-18}$	$5.96\,10^{-18}$	$1.13\,10^{-18}$	$4.98\,10^{-18}$
	ϵ_2^{Ω}	$1.44\,10^{-5}$	$1.10\,10^{-9}$	$5.60\,10^{-14}$	$6.50\,10^{-18}$	$2.55\,10^{-18}$	$7.20\,10^{-18}$

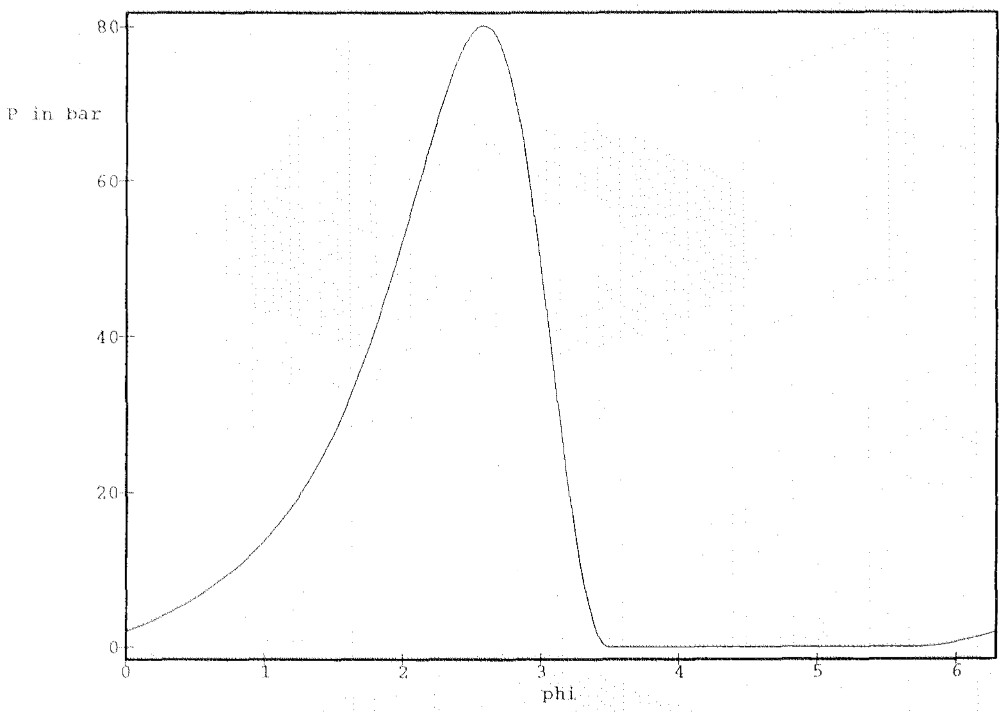
Table 4: error for $\omega_1 = 0.3$ and $\omega_2 = 6$

	$\epsilon \setminus n$	15	20	25	30	35	40
f_1	$\epsilon_1^{\Gamma ra}$	$3.96\,10^{-4}$	$1.63\,10^{-7}$	$5.66\,10^{-11}$	$1.40\,10^{-14}$	$6.93\,10^{-18}$	$3.19\,10^{-17}$
	$\epsilon_1^{\Gamma ri}$	$2.21\,10^{-4}$	$9.56\,10^{-9}$	$6.35\,10^{-14}$	$3.80\,10^{-18}$	$2.38\,10^{-18}$	$2.39\,10^{-17}$
	ϵ_1^{Ω}	$3.96\,10^{-4}$	$1.63\,10^{-7}$	$5.66\,10^{-11}$	$1.40\,10^{-14}$	$7.15\,10^{-18}$	$4.95\,10^{-17}$
	$\epsilon_2^{\Gamma ra}$	$4.28\,10^{-4}$	$2.02\,10^{-7}$	$6.27\,10^{-11}$	$7.49\,10^{-15}$	$6.28\,10^{-18}$	$5.44\,10^{-17}$
	$\epsilon_2^{\Gamma ri}$	$1.96\,10^{-4}$	$1.12\,10^{-8}$	$6.23\,10^{-14}$	$2.92\,10^{-18}$	$3.03\,10^{-18}$	$3.29\,10^{-17}$
	ϵ_2^{Ω}	$4.28\,10^{-4}$	$2.02\,10^{-7}$	$6.27\,10^{-11}$	$7.86\,10^{-15}$	$6.99\,10^{-18}$	$5.64\,10^{-17}$
f_2	$\epsilon_1^{\Gamma ra}$	$1.33\,10^{-2}$	$3.97\,10^{-6}$	$2.09\,10^{-9}$	$8.97\,10^{-13}$	$3.49\,10^{-16}$	$3.68\,10^{-17}$
	$\epsilon_1^{\Gamma ri}$	$1.36\,10^{0}$	$2.14\,10^{-1}$	$5.06\,10^{-3}$	$7.29\,10^{-5}$	$9.03\,10^{-7}$	$9.05\,10^{-9}$
	ϵ_1^{Ω}	$1.36\,10^{0}$	$2.14\,10^{-1}$	$5.06\,10^{-3}$	$7.29\,10^{-5}$	$9.03\,10^{-7}$	$9.05\,10^{-9}$
	$\epsilon_2^{\Gamma ra}$	$1.19\,10^{-2}$	$1.22\,10^{-6}$	$1.85\,10^{-9}$	$7.53\,10^{-13}$	$3.16\,10^{-17}$	$3.22\,10^{-17}$
	$\epsilon_2^{\Gamma ri}$	$1.09\,10^{0}$	$2.10\,10^{-1}$	$4.78\,10^{-3}$	$5.70\,10^{-5}$	$1.45\,10^{-7}$	$8.91\,10^{-9}$
	ϵ_2^{Ω}	$1.09\,10^{0}$	$2.10\,10^{-1}$	$4.78\,10^{-3}$	$5.70\,10^{-5}$	$2.97\,10^{-7}$	$8.91\,10^{-9}$
f_3	$\epsilon_1^{\Gamma ra}$	$2.92\,10^{-2}$	$1.59\,10^{-5}$	$5.37\,10^{-9}$	$1.20\,10^{-12}$	$4.33\,10^{-17}$	$1.50\,10^{-16}$
	$\epsilon_1^{\Gamma ri}$	$1.58\,10^{-2}$	$1.80\,10^{-6}$	$1.31\,10^{-11}$	$6.20\,10^{-17}$	$2.86\,10^{-17}$	$9.32\,10^{-17}$
	ϵ_1^{Ω}	$2.92\,10^{-2}$	$1.59\,10^{-5}$	$5.37\,10^{-9}$	$1.20\,10^{-12}$	$1.68\,10^{-16}$	$1.90\,10^{-16}$
	$\epsilon_2^{\Gamma ra}$	$3.19\,10^{-2}$	$1.48\,10^{-5}$	$5.29\,10^{-9}$	$1.33\,10^{-12}$	$3.33\,10^{-16}$	$2.13\,10^{-16}$
	$\epsilon_2^{\Gamma ri}$	$1.49\,10^{-2}$	$1.56\,10^{-6}$	$1.19\,10^{-11}$	$4.85\,10^{-17}$	$3.64\,10^{-17}$	$1.01\,10^{-16}$
	ϵ_2^{Ω}	$3.19\,10^{-2}$	$1.48\,10^{-5}$	$5.29\,10^{-9}$	$1.33\,10^{-12}$	$3.33\,10^{-16}$	$2.41\,10^{-16}$

Table 5: error for $\omega_1 = 0.6$ and $\omega_2 = 5$

5.4 Application to the Journal Bearing

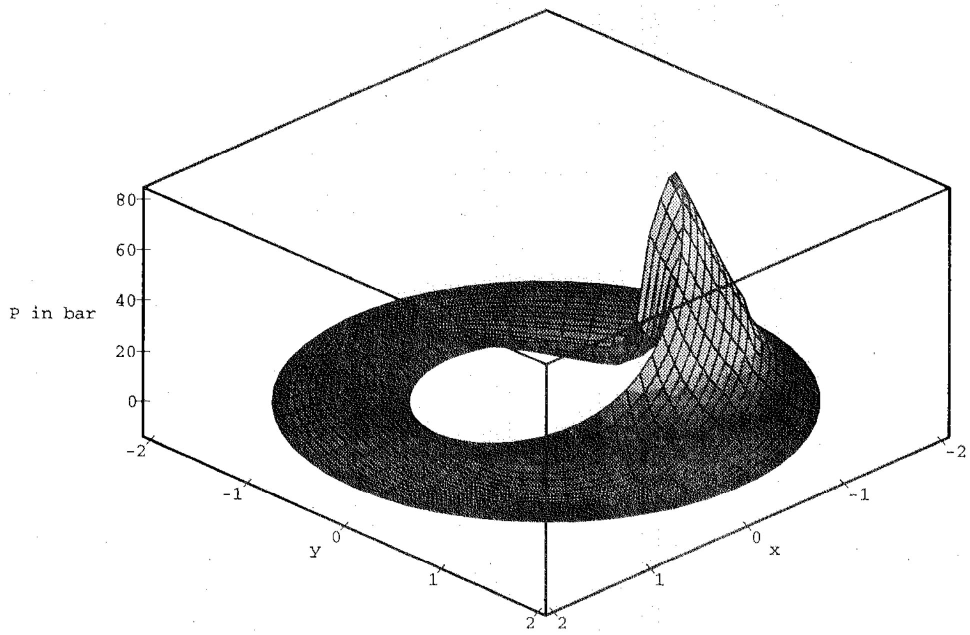
The spline interpolation method is now applied to the second of the bearing configurations of Section 1.2. The necessary radial stress function on the inner boundary ($r_i = 1\ cm$) of the bearing shell is illustrated in the next picture (cf. page 7).



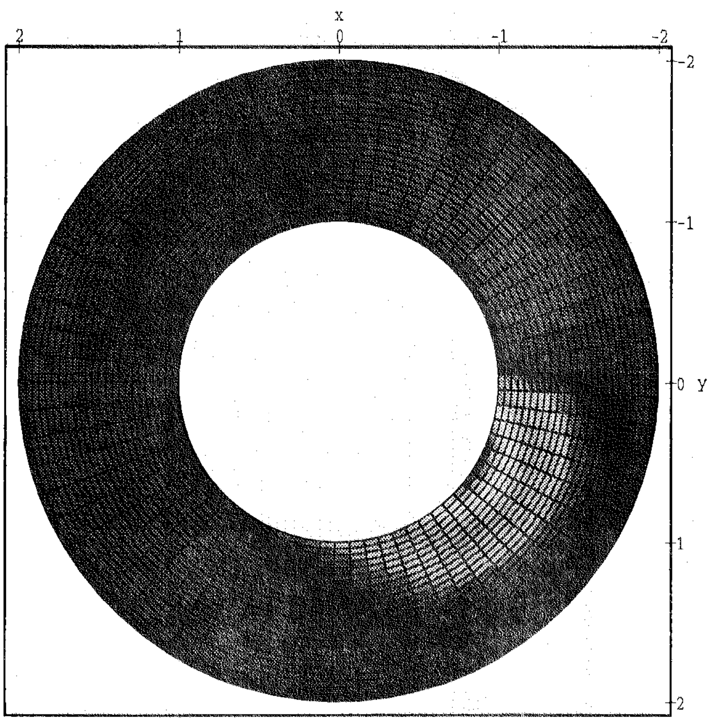
Picture 9: radial stress on the inner side of the bearing shell

For the radial stress on the outer boundary ($r_a = 2\ cm$) we consider two cases (cf. Section 3.1). On one side the stress is assumed to be zero (case 1) and on the other side it is set to a third of the inner pressure (case 2). As described in Section 3.1, the shearing stress is zero on both boundaries. For the interpolation we used the following parameters: As in Section 5.3, we took the sequences $q_n^{(1)} = q_n^{(2)} = 1\ \forall n \in I\!N$ for the Hilbert space, because compared with the other elementary representable kernels, they delivered the best results. On the inner side of the bearing shell we used 60 equidistand distributed points for the radial stress and 40 points for the shearing stress, on the outer side 30 resp. 25 points. The parameters ω_1 and ω_2 had to be chosen very carefully as the results react very sensitive to modifications. The best results have been achieved with the values $\omega_1 = 0.309$ and $\omega_2 = 11.4$. Using them, the maximum error on the boundaries was about 0.1 bar, resp. one thousandth of the maximum stress of the boundaries.

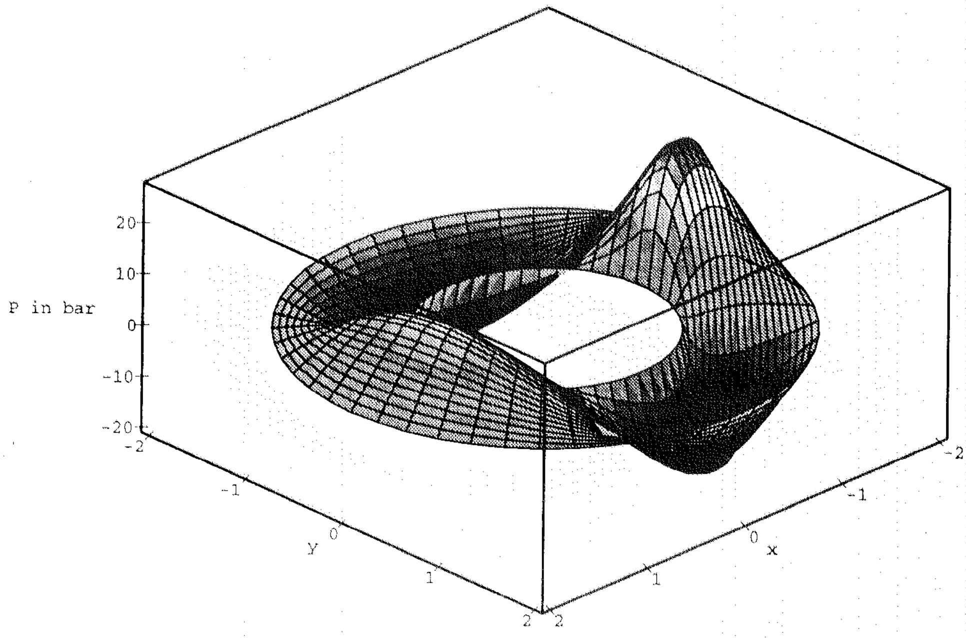
The computed radial and shearing stresses are illustrated on the following pages as three-dimensional as well as contour plots. Especially at the places signed by arrows, stresses can be observed, which can not be interpreted physically. Thus it can be assumed, that there are undesired oscillations in the solution, which could not been eliminated, neither by changing the parameters ω_1 or ω_2, nor by using other kernel functions.



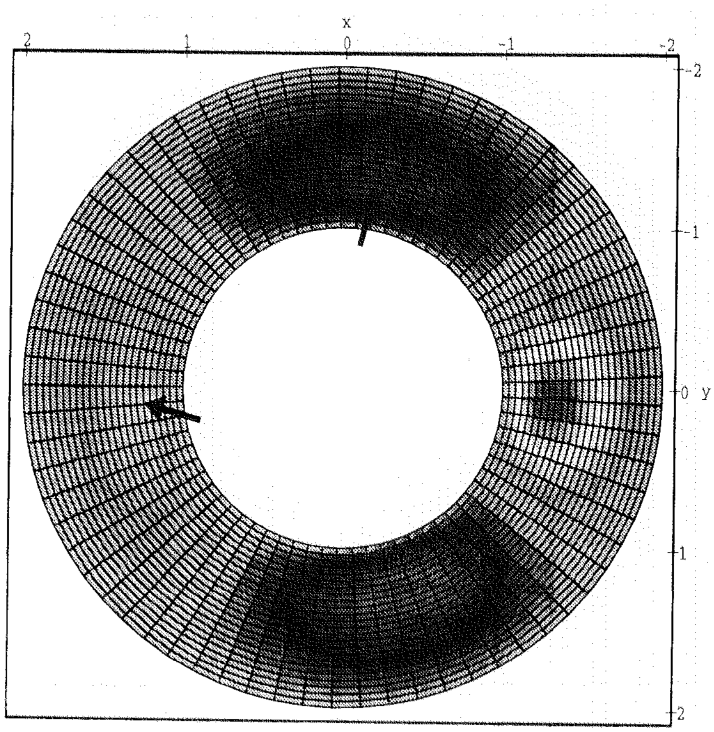
Picture 10: radial stress for case 1



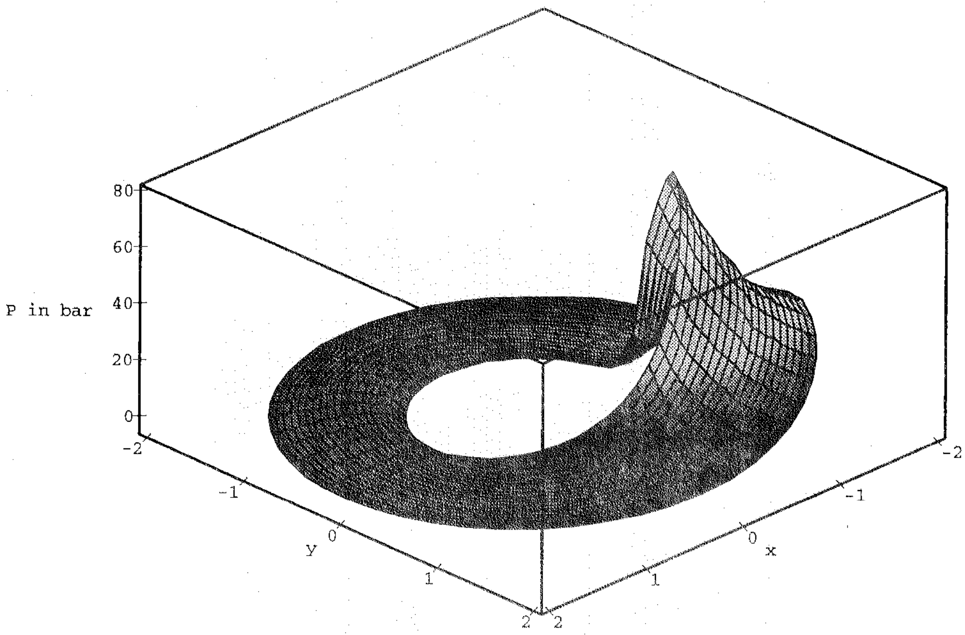
Picture 11: radial stress for case 1



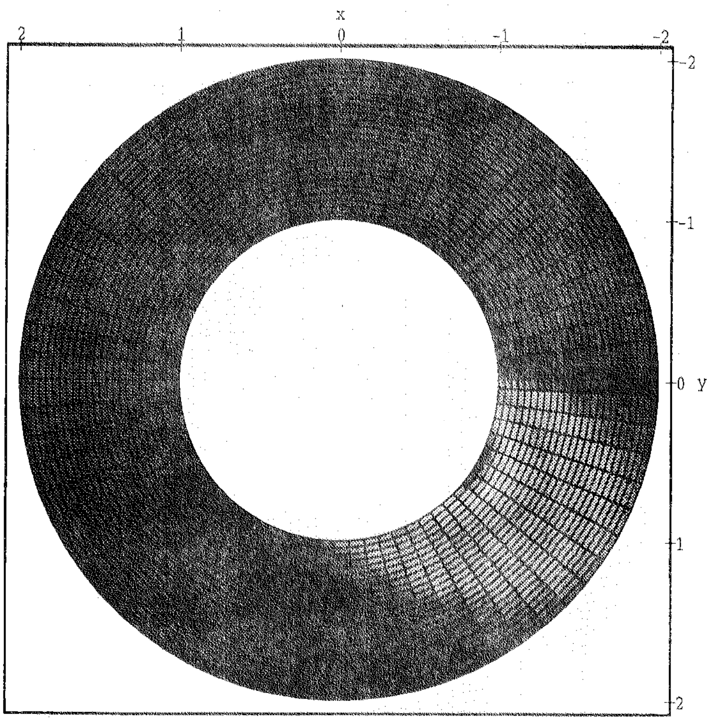
Picture 12: shearing stress for case 1



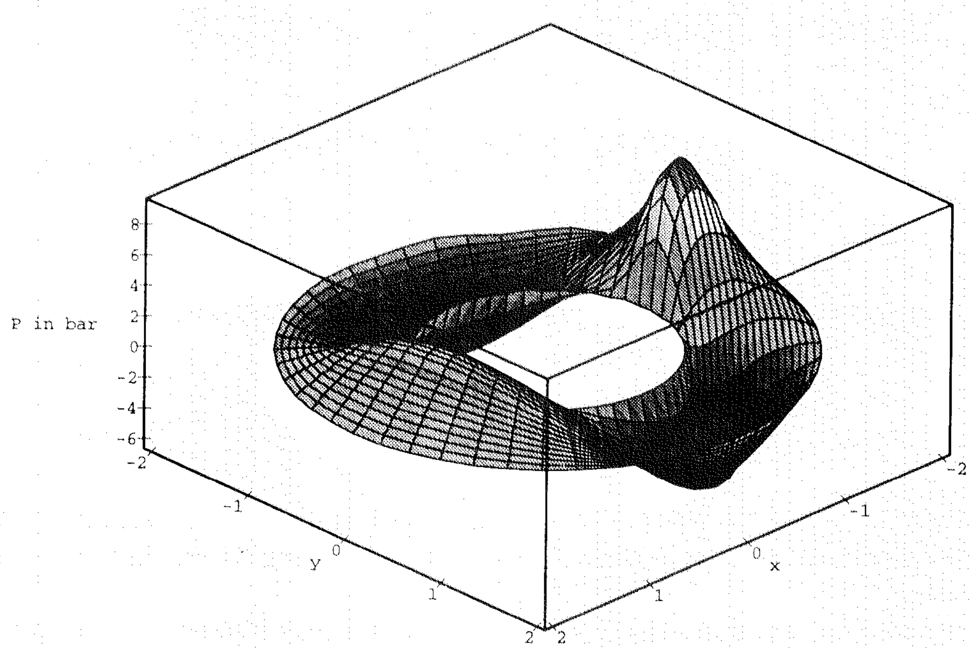
Picture 13: shearing stress for case 1



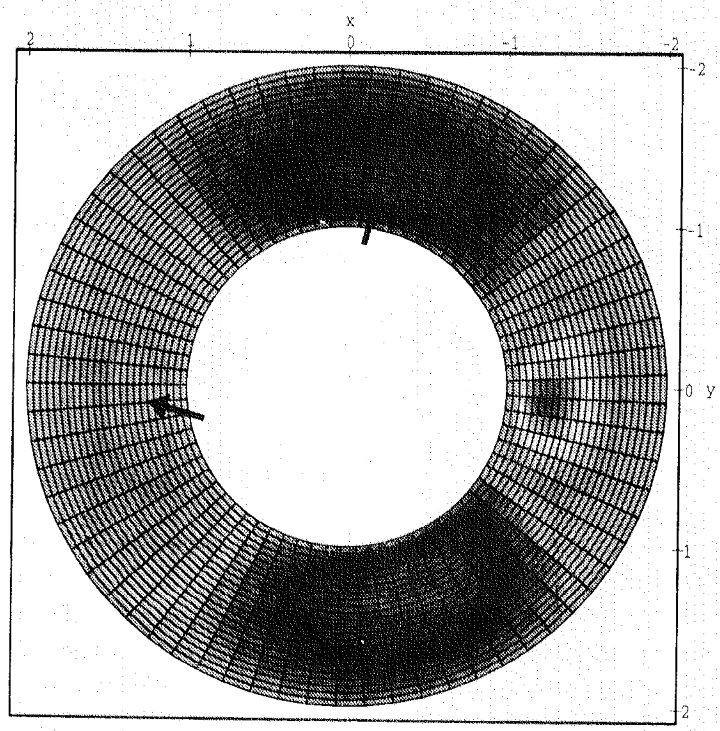
Picture 14: radial stress for case 2



Picture 15: radial stress for case 2



Picture 16: shearing stress for case 2



Picture 17: shearing stress for case 2

A possibility to reduce these oscillations is to force the elimination of the stresses at the noted places explicitly in the computation. This can be done by adding equations of the form

$$\begin{aligned} B_1 S_X(\hat{x}) &= \sum_{i=1}^{N'} a_i' B_1 K_1^{(ext)}(x_i', \hat{x}) + \sum_{i=1}^{N''} a_i'' B_1 K_2^{(ext)}(x_i'', \hat{x}) + \\ &\quad \sum_{i=1}^{N'''} a_i''' B_1 K_1^{(int)}(x_i''', \hat{x}) + \sum_{i=1}^{N''''} a_i'''' B_1 K_2^{(int)}(x_i'''', \hat{x}) = 0 \end{aligned}$$

and

$$\begin{aligned} B_2 S_X(\bar{x}) &= \sum_{i=1}^{N'} a_i' B_2 K_1^{(ext)}(x_i', \bar{x}) + \sum_{i=1}^{N''} a_i'' B_2 K_2^{(ext)}(x_i'', \bar{x}) + \\ &\quad \sum_{i=1}^{N'''} a_i''' B_2 K_1^{(int)}(x_i''', \bar{x}) + \sum_{i=1}^{N''''} a_i'''' B_2 K_2^{(int)}(x_i'''', \bar{x}) = 0 \end{aligned}$$

to the N ($N = N' + N'' + N''' + N''''$) equations used up to now (cf. page 23). They assure that the radial stresses in the points $\hat{x}$, and the shearing stresses in the points $\bar{x}$ are vanishing.
With this additional equations the $N \times N$ matrix A of the pure interpolation is replaced by a $M \times N$ matrix $\bar{A}$ with $M \geq N$. As the resulting system of linear equations is not solvable in general, it is desired to get a solution as good as possible. This means for example the minimization of the Euclidean norm of the residuum vector $\|b - \bar{A}a\|$. One way to solve such a least square problem is to apply Householder transformations $P_1, P_2, \ldots, P_N$ to the matrix $\bar{A}$ in order to get the form

$$P_1 P_2 \cdots P_N \bar{A} = \begin{pmatrix} R \\ 0 \end{pmatrix},$$

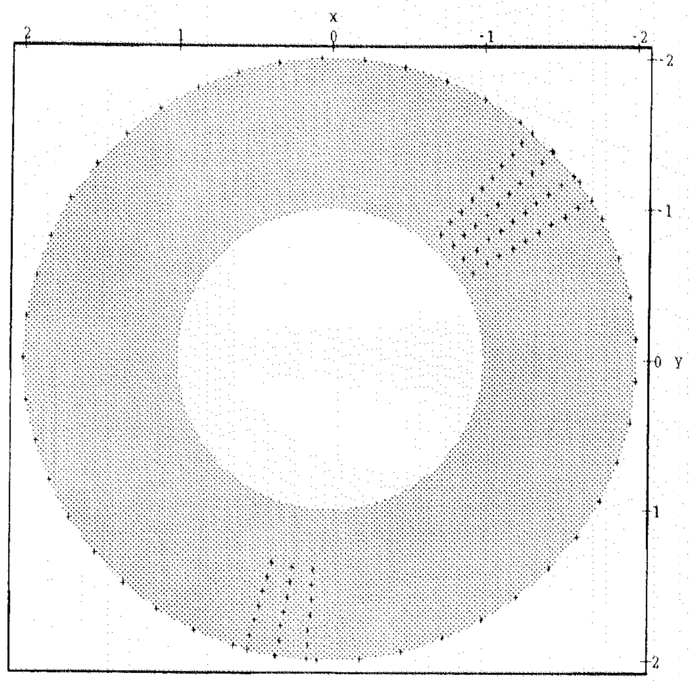
where R denotes a right upper triangular matrix (cf. Stoer[19]). Because the matrices $P_1, P_2, \ldots, P_N$ are unitary transformations, which keep the length $\|u\|$ of a vector u invariant, we get

$$\|b - \bar{A}a\| = \left\| \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} - \begin{pmatrix} R \\ 0 \end{pmatrix} x \right\|,$$

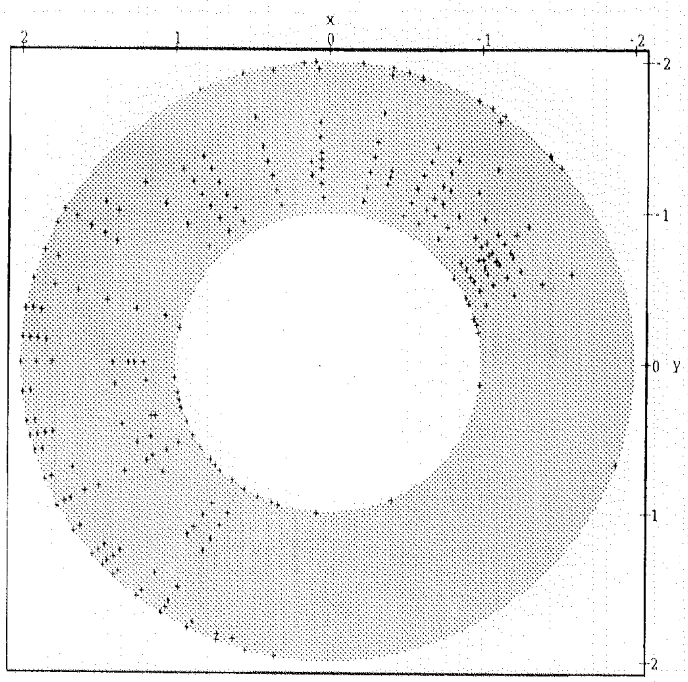
where

$$h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = P_1 P_2 \cdots P_N b.$$

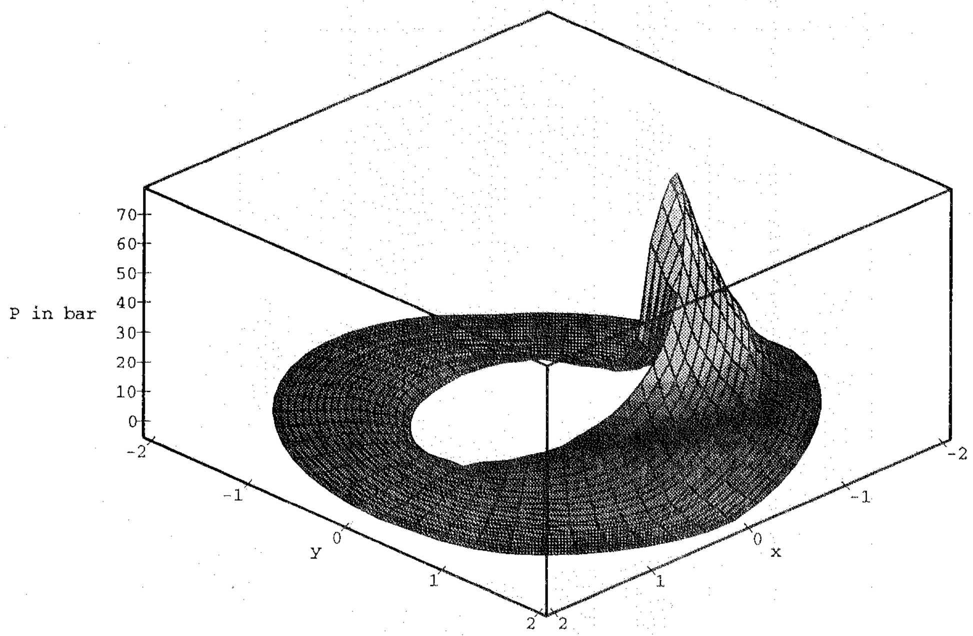
h_1 represents the first N and h_2 the remaining components of h. Consequently the norm $\|b - \bar{A}a\|$ is minimized, if and only if a is chosen so, that h_1 is equal to Ra. This method has been integrated into the program and the stresses of the first example of this section have been computed once more. The distribution of the additionally used points (106 for the radial and 226 for the shearing stress) is shown on the next page. The undesired oscillations are reduced, as can be taken from the pictures one the pages 42 and 43, where the newly computed stresses are presented. On the other hand this method has of course the disadvantage, that the errors on the boundaries are bigger then before, because the given values are now approximated and not interpolated.



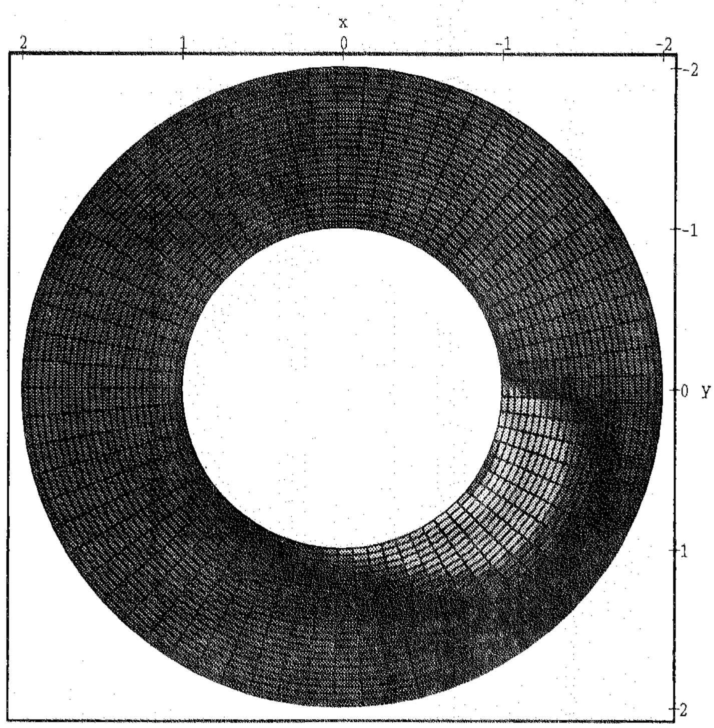
Picture 18: additional points for the radial stress



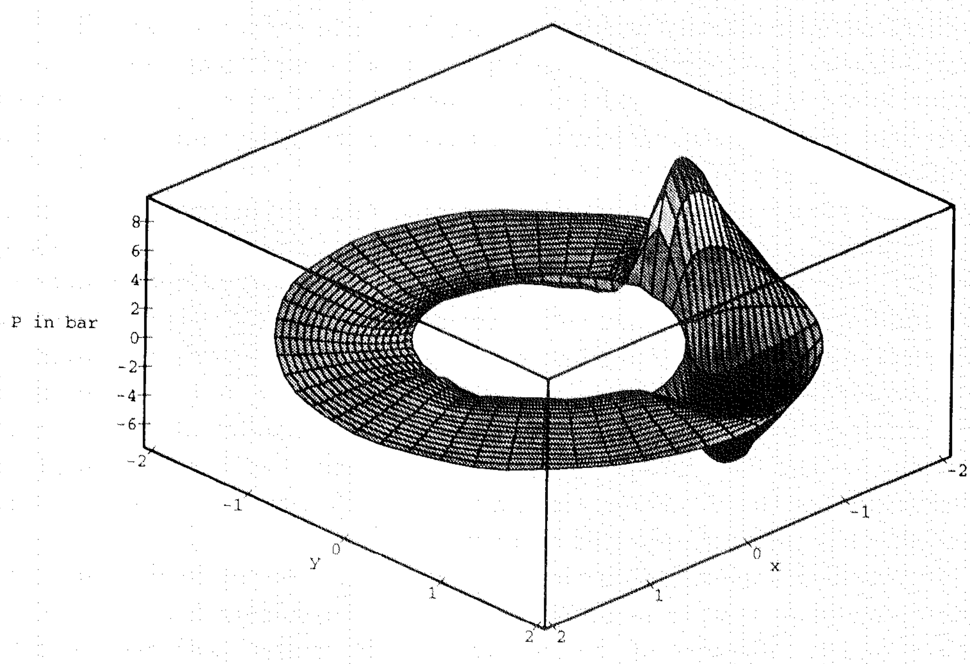
Picture 19: additional points for the shearing stress



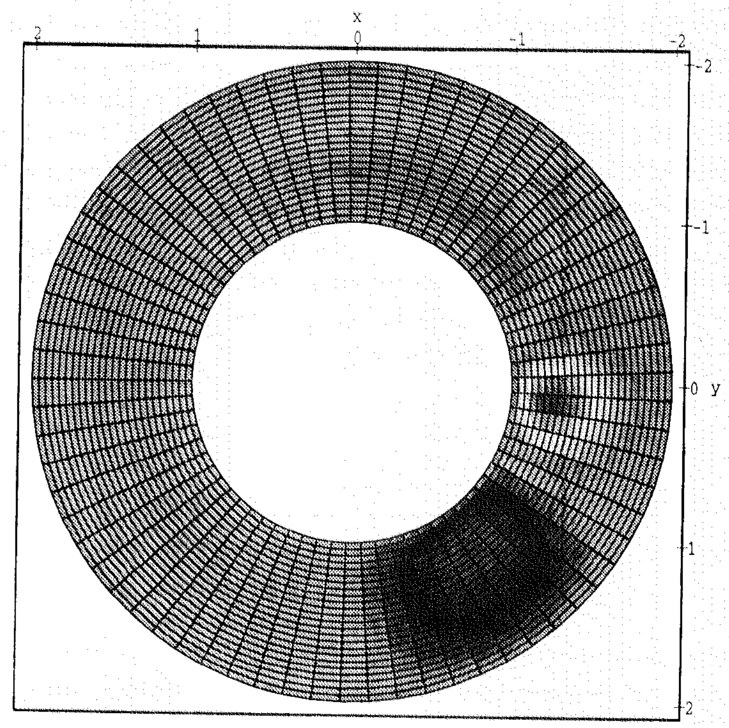
Picture 20: radial stress for case 1



Picture 21: radial stress for case 1



Picture 22: shearing stress for case 1



Picture 23: shearing stress for case 1

6 Summary

In this work methods have been developed to compute the stresses in stationary loaded, hydrodynamic journal bearings. In the first part a special point-SOR-algorithm for solving linear complementarity problems, which appear in connection with Reynolds's differential equation, has been described. With help of its implementation the pressure distribution in the lubrication fluid can be computed. The program has approved at the department of mechanical engineering in Kaiserslautern, the desired results can be quickly achieved on any PC.
The second and main task of the work was concerned with the computing of the stresses in the bearing shell, using the theory of linear elasticity. A solution of the Bi-Laplace equation has to be found, which has to fulfill certain boundary, as well as some additionally conditions. Because of these extra conditions, the problem is not a boundary value problem in the classical sense, and standard methods like series expansions with multipoles or fundamental solutions don't work. The solution method presented here uses a Hilbert space in which the extra conditions are fulfilled automatically. The spline interpolation has been choosen instead of a Fourier method based on the same Hilbert space, because it has the advantage, that here the high frequency parts in the solution are included, while they are neglected otherwise.
About the practical application one has to remark, that the optimal values for the parameters ω_1 and ω_2 have to be found anew for every configuration, that means for any radii r_i and r_a. Because of the sensitivity of the method with regard to these parameters, this may be a tricky work. But on the other hand, if parameters for a configuration have been found, they can be used for many different boundary functions, if the dimensions of the bearing are constant. A better accuracy of the method may be obtained by improving the condition of the matrix, what could be done using kernel functions with smaller support, so that the number of the essential entries in the matrix is reduced. But the construction of such Hilbert spaces will again lead to the problem of integrating the extra conditions, which have to be fulfilled by the solution.
Finally one has to remark, that the model of a journal bearing described in this work is a simplification of the presently in mechanical engineering used bearings. These are consisting of several layers of different materials. In this case, for every layer a special stress function has to be used and the boundary conditions have to be replaced by some transition conditions. A treatment of this general case was left undone, because it would have broken up the frame of this work.

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