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Discrete Dividends:

Modeling, Estimation and Portfolio Optimization

Sarah Grün

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1. Gutachter: Prof. Dr. Ralf Korn
2. Gutachter: Prof. Dr. Alexander Szimayer

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Abstract

In this thesis we integrate discrete dividends into the stock model, estimate future outstanding dividend payments and solve different portfolio optimization problems. Therefore, we discuss three well-known stock models, including discrete dividend payments and evolve a model, which also takes early announcement into account.

In order to estimate the future outstanding dividend payments, we develop a general estimation framework. First, we investigate a model-free, no-arbitrage methodology, which is based on the put-call parity for European options. Our approach integrates all available option market data and simultaneously calculates the market-implied discount curve. We illustrate our method using stocks of European blue-chip companies and show within a statistical assessment that the estimate performs well in practice.

As American options are more common, we additionally develop a methodology, which is based on market prices of American at-the-money options. This method relies on a linear combination of no-arbitrage bounds of the dividends, where the corresponding optimal weight is determined via a historical least squares estimation using realized dividends. We demonstrate our method using all Dow Jones Industrial Average constituents and provide a robustness check with respect to the used discount factor. Furthermore, we backtest our results against the method using European options and against a so called simple estimate.

In the last part of the thesis we solve the terminal wealth portfolio optimization problem for a dividend paying stock. In the case of the logarithmic utility function, we show that the optimal strategy is not a constant anymore but connected to the Merton strategy. Additionally, we solve a special optimal consumption problem, where the investor is only allowed to consume dividends. We show that this problem can be reduced to the before solved terminal wealth problem.

Zusammenfassung

In dieser Arbeit geht es um die Integration von diskreten Dividenden Zahlungen in das Aktienmodell, um die Schätzung von zukünftigen Dividenden und um das Lösen verschiedener Portfolio Optimierungsprobleme. Dabei werden schon bekannte Aktienmodelle, die diskrete Dividenden einbinden kritisch untersucht und darauf aufbauend ein Aktienmodell entwickelt, das zudem eine frühzeitige Bekanntgabe der Dividenden ermöglicht.
Um die zukünftigen Dividenden Auszahlungen zu schätzen, haben wir zwei Methoden entwickelt. Die erste No-Arbitrage Methode ist modellfrei und basiert auf der Put-Call Parität für europäische Optionen. Dabei verwenden wir alle vorhandenen Optionsdaten und berechnen die marktspezifischen Discount-Kurven in einem. In der praktischen Umsetzung für europäische Blue-chip Unternehmen weist die Methode eine gute Performance auf, die durch eine statistische Auswertung belegt wird.
Da jedoch amerikanische Optionen weiter verbreitet sind, haben wir im nächsten Schritt eine zweite Methode entwickeln, die at-the-money Optionen verwendet. Diese Methode basiert auf einer Linearkombination zweier No-Arbitrage Schranken für die Dividenden. Dabei wird der optimale Gewichtungsfaktor anhand einer historischen Kleinste Quadrate Schätzung unter Einbindung bereits realisierter Dividenden berechnet. Um diese Methode in der Praxis zu testen, werden Daten der Dow Jones Industrial Average Aktien verwendet. Hier wird wieder eine statistische Analyse durchgeführt und zudem die Eingabe verschiedener Discount-Faktoren getestet. Des Weiteren wird die Performance der Methode mit der sogenannten einfachen Methode und der Methode, die Europäische Optionen verwendet verglichen.
In dem letzten Teil der Arbeit wird das klassische Portfolio Problem für Dividenden zahlende Aktien betrachten und gelöst. Im Beispiel der logarithmischen Nutzenfunktion ist der optimale Portfolio Prozess keine Konstante mehr. Dennoch ist eine Abhängigkeit zur Merton Strategie gegeben. Zusätzlich wird ein spezielles Konsumproblem gelöst, bei dem der Investor nur Dividenden konsumiert darf. Dieses Problem kann gelöst werden in dem es auf das zuvor gelöste Portfolio Problem zurückgeführt wird.

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List of Symbols

$\lambda^{*(\Delta)}$	Historical weight	55
$\mathcal{A}(x_0)$	Admissible set for the initial capital x_0	70
μ	Trend parameter	70
$\pi(t)$	Time-t portfolio process	71
σ	Volatility	70
τ	Time to maturity	14
τ_i	Time between estimation t and payment T_i	54
$\tilde{T}_i$	Maturity corresponding to the payment day T_i	16
$\tilde{\tau}_i$	Time between the estimation t and the maturity corresponding to the payment date T_i	54
$\tilde{\tau}_{j,i}$	Time between the historical date t_j and the maturity corresponding to the payment date T_i	54
$\tilde{S}$	Time-t price of a non-dividend paying stock	69
$\varphi(t)$	Trading strategy	70
$B(t)$	Time-t price of a bond	70
$C(t)$	Price of a European call option at time t with underlying S, strike K, and maturity T	7
$D^*(t, T_i)$	Estimate for $D(t, T_i)$	13
$D^*_{i,t}$	Estimate for the forward price $\mathbb{E}_t^{T_i}[D_i]$ at time t.	17

D_i	Dividend payment, payable at time T_i	3
$D_l^*(t,T_i)$	Lower bound for $D(t,T_i)$	44
$D_u^*(t,T_i)$	Upper bound for $D(t,T_i)$	44
$F(t,T)$	Forward rate agreement	58
$L(t,T)$	LIBOR rate	58
$P(t)$	Price of a European put option at time t with underlying S, strike K, and maturity T	7
$p(t,T)$	Time-t discount factor for cash flows at time T	6
r	Riskless interest rate	40
$S(t)$	Time-t price of a stock	3
T_i^*	Announcement time of dividend D_i	80
$t_j^{(k)}$	Historical dates of request corresponding to t_k	58
$U(x)$	Utility function	70
$W(t)$	Brownian motion	69
$X^{\varphi}(t)$	Time-t wealth process for trading strategy φ	70
x_0	Initial capital	70
Δ	Estimation period	26
$\mathbb{E}_t^{T_i}[\cdot]$	The time-t conditional expectation under the T_i - forward measure	8
$\tau_{k,i}$	Time between the estimation day t_k and the payment day T_i	26
$C^A(t)$	Price of an American call option at time t with underlying S, strike K, and maturity T	38
$D(t,T)$	Time-t present value of expected future dividend payments up to time T	8
N	Total number of request dates	26

$P^A(t)$	Price of an American put option at time t with underlying S, strike K, and maturity T	38
s_K	Empirical volatility	26
T_i	Payment date of dividend D_i	3
t_k	Data request date/ spot date	22
ATM	At-the-money	52

Chapter 1

Introduction

A *dividend* is a portion of the company's profit, which is payed to its shareholders. The board of directors debate which part of the profit is used for investment purposes and which one they share with the shareholders, i.e. they decide on the size of the dividend. In the classical financial mathematics theory dividends are often neglected. However, including dividends makes a difference. For example, the classic put-call parity or the fact that European and American call option coincide as well as the famous option pricing formula from Black and Scholes are not valid anymore and need to be adapted. Moreover, in reality many companies distribute dividends, as for example all constituents in the Dow Jones Industrial Average (Dow Jones). Only some companies in particular start-ups, do not pay dividends, as they use their earnings for new investments or to repay their liabilities.

As a dividend is a distribution of the company's profit, one can make an inference from the size of the dividend to the profitability of the company. In literature a lot of articles deal with this inference. Among others, Arnott and Asness (2003) find that higher dividends result in a higher future earning growth. Hence, the dividend can help with an investment decision. Besides, many investors follow a strategy to make longterm investments in order to get regular pay offs and do not really care about the purchasing price of the investments. Thus, in the current low (or even negative) interest rate scenario combined with the fact that dividends give an income security, investing in equity markets gets even more attractive. Especially, as they nowadays often outperform a corresponding riskless bond investment.

That a lot of investors are interested in dividend paying stocks is also reflected in the availability of special stock indices as S&P 500 dividend aristocrats in the US or the DivDax in Germany. The constituents of the S&P 500 dividend aristocrats index are part of the S&P 500 and have raised their dividends every year for at least 25 years. In Germany, no stock is such a dividend aristocrat, instead, the DivDax contains the 15 stocks with the biggest dividend yield.

Overall, including dividends into the theory and especially estimating outstanding future dividend payments are important actual tasks. Thus, we focus on these tasks in this thesis. A lot of articles, which deal with dividend paying stocks suppose either, that the dividends are deterministic or they include a stochastic, continuous dividend yield process. However, in real world, all stocks pay dividends at discrete times, such that we consider discrete stochastic dividend payments in this thesis.

1.1 Outline of the Thesis

In the forthcoming section, we provide some basics concerning the concept of dividend paying stocks that are important for the remainder of this thesis. Therefore, we clarify important dates connected to the dividend payment and explain the behavior of the stock price.

In Chapter 2, we develop a no-arbitrage methodology to estimate outstanding dividend payments. The method is based on market data of European call and put options. Therefore, we first proof the put-call parity with dividends and then clarify how we can use it for the dividend estimation. In order to bootstrap the discount factor concerning to a payment, we investigate a so called box spread method and end up with including a linear regression into our method. This approach enables us to simultaneously estimate the amount of the dividend payment and how the market evaluates it. Finally, we illustrate our method using stocks of European blue-chip companies and provide a statistical assessment of the obtained estimates. Additionally, we benchmark the estimate with a commercial forecast.

As options of European type are not available for every stock, we evolve a method in Chapter 3, which relies on options of American type. We first transfer the before developed method to the new setting, i.e. we introduce the no-arbitrage boundaries for American options and examine their usability for an estimate. Out of this, we derive a new method which is based on a linear combination of an upper and a lower bound for the dividends, where the corresponding optimal weight is determined via historical least squares estimation using realized dividends. After developing the method we demonstrate it using all stocks constituent in the Dow Jones Industrial Average and again provide a statistical assessment of the obtained estimates. Furthermore, we make a robustness check with respect to the used discount factor and backtest our method against the one from Chapter 2 as well as against a simple estimate.

In the last chapter, we release ourselves from estimating outstanding dividends. Instead, we focus on including discrete dividend payments in the stock price model and solve different portfolio optimization problems. Therefore, we first recap the classical terminal wealth portfolio problem and its solution for a non-dividend paying stock and introduce important notations. Afterwards, we contemplate three well-known discrete dividend paying stock models, which are used in praxis and solve the concerning terminal wealth problem. Then, we include an early announcement of the dividend, as this results in a certain payout, which should be reflected in the model. We derive two different models and also solve the terminal wealth problem for one of them. Finally, we also consider an optimal consumption problem, which restricts the investor to only consume the dividend. We give a solution to that problem and deal with an important question, that can appear concerning the optimal trading strategy.

1.2 Basics: Dividend Paying Stocks

In this section we explain the concept of a stock going ex-dividend. Therefore, $S(t)$ denotes the time-t stock price throughout the whole work. We focus on stocks which pay dividends $D_i > 0$ and assume they are payable at discrete, known times $T_1 < T_2 < \cdots < T_i < \cdots$. Now, the question is, *what happens when a stock pays dividends.* Thus, the following section is revealing.

1.2.1 Ex-Dividend

In general there are different types of dividends, as cash dividends, stock dividends or other dividends. Within this work we only consider cash dividends, which are usually paid yearly or quarterly. Sometimes, there also exist so called special/extra dividend payments which are nonrecurring and used to payout extraordinarily high earnings. As they are rare and strongly depend on the decision of the company's management we omit them in our investigations.
For a shareholder, who wants to receive a dividend, the following dates are important. Note that they succeed as they are listed.

- *Declaration date* (also known as *announcement day*), the day on which the board of directors announces the next dividend payment. This disclosure incorporates the amount and the forthcoming connected dates.

- *Ex-dividend date* (or short *ex-date*), the day on which the stock goes ex-dividend, i.e. the price of the stock jumps down (compare with Assumption 1.1 and its explanations). This day is also important to solve the

question *who receives the dividend*. For more information see the forthcoming date:

- *Record date*, the day on which the shareholder who receives the dividend is determined. Therefore, the shareholder needs to be registered in the company's record. This is the case if he or she owns the stock on its ex-dividend day. For this reason also the trading day before the ex-dividend date has a name: *cum-dividend date*. Thus, this is the last day where someone can buy the stock to receive the dividend.
  The ex-dividend date is set according to the rules of the stock exchange which is typically set two trading days prior to the record date (in the US). The record date does not exist everywhere: in Germany for example the ex-dividend date adopts its function.

- *Payment date*, the date on which the shareholder receives the dividend amount (on his bank account or as a check).

After distinguishing between the different dates we want to point out that we do not pay attention to the record date in this work. Furthermore, for simplicity we suppose that the ex-dividend and payment date coincide. Note that $S(T_i)$ is the ex-dividend price, i.e. the price of the stock after the dividend payment.

If stocks are paying dividends this also has an impact to the corresponding stock index. Thereby, we differentiate between a price index and a total return index:

Remark 1.1 (Price Index vs. Total Return Index) *The total return index takes a reinvestment of the dividend as basis, i.e. on an ex-dividend date the price is not affected by the dividend, whereas the price index only considers the prices of its constituents. Many blue-chip stock indices are available in both versions and one need to be careful of which we are talking about. So, if someone mentions the German stock index DAX he talks about the total return index. In contrast, for example the US Dow Jones, the British FTSE-100, the Japanese Nikkei-225 and the French CAC-40 are price indices.*

1.2.2 Behavior on the Ex-Dividend Date

As we already mentioned, the stock price goes down on the ex-dividend date. Therefore, we assume the following:

Assumption 1.1 *The drop in the stock price at the ex-dividend date is equal to the dividend amount D_i.*

Assumption 1.1 is supported by a number of articles, which examine the behavior of stock prices at or around the ex-dividend date empirically and theoretically. In the no tax framework the drop in the stock price generally coincides with the dividend payment. Conversely, when tax effects are present, there are two main hypothesis: Elton and Gruber (1970) evolve the *tax-clientele* hypotheses, where the stock price drops by a factor α, with

$$\alpha \triangleq \frac{S(T_i-) - S(T_i)}{D_i} = \frac{1-\tau_o}{1-\tau_c}\,. \tag{1.1}$$

$S(T_i-)$ is the price of the stock before it goes ex-dividend, τ_o is the tax rate on dividend payments[1] and τ_c is the *capital gains tax rate*. If dividends are taxed at a higher rate than income, this results in $\alpha < 1$. By contrast, if both are taxed at the same rate, as e.g., Germany, Switzerland and France (see PKF International (2014)), we obtain $\alpha = 1$. As we additionally analyze US data we are also interested in their tax rates: In PKF International (2014) it is written *"For corporations, capital gains are taxed at the same rates applicable to ordinary income"*. Hence, only in some cases for an individual with special tax brackets it can happen that α differs from one. To handle this the forthcoming discussion and Remark 1.2 are helpful.

Alternatively, the *short-term trading* theory of Kalay (1982), Lakonishok and Vermaelen (1986), is based on the hypothesis that, around the ex-dividend day, the shareholder clientele changes. Within the tax framework there is a difference between short-term and long-term capital gains. The latter one are gains from investments which are held longer than one year. The income that someone receives from investments held less than one year concerns to short-term capital gains, which are taxed as ordinary income. Hence, in the short-term trading theory α is determined by the relative importance of short-term traders, i.e. $\alpha = 1$.
On the empirical side, Barone-Adesi and Whaley (1986) use Roll's formula for American call options (see Remark 1.3 for more details) to estimate α and show that it is not significantly different from one.

Remark 1.2

(i) We can extend our methods in the case that α is constant and different from one, via multiplying by this specific α where appropriate.

(ii) With straightforward no-arbitrage arguments it can easily be seen, that Assumption 1.1 is fulfilled if every individual dividend payment is repli-

[1] We use the index o as this tax rate is also called *ordinary tax rate*.

cable. This holds especially in a complete financial market model or with deterministic dividend payments.

Remark 1.3 (Roll's Formula) *Roll (1977) developed a valuation formula for an American call option, where he composed three European call options. Geske (1979) specified this formula and finally, Whaley (1981) corrected it (that is why it is sometimes called Roll-Geske-Whaley formula). They suppose that the decline in the stock price is equal to αD on the ex-dividend day t^*, where we only have one dividend payment in $[t, T]$.*
Then, the value of an American call option is[2]

$$\begin{aligned} C^A(t,T,S,K) =& C(t,T,S,K) + C(t,t^*-\varepsilon,S,S^*) \\ & - C(t,t^*-\varepsilon,C(t,T,S,K),S^*+\alpha D-K)\,, \end{aligned}$$

where $\varepsilon > 0$, $\varepsilon \cong 0$ and S^* the stock price above which the American call option is exercised early, i.e. $C(t^*, T-t^*, S^*, K) = S^* + \alpha D - K$.

1.2.3 Influence of the Dividend Announcement

Not only the dividend itself but also its announcement can impact the stock price. Korn and Rogers (2005) model the stock price and include different announcement settings: Let $p(t,T)$ denote the time-t discount factor for cash flows at time T with $t \le T$. They define the stock price via

$$S(t) \triangleq \mathbb{E}_t\Big[\sum_{i:\,T_i>t} p(t,T_i)D_i\Big]\,,$$

and model the dividend via an exponential Lévy process. Note that in $t = T_i$ this $S(t)$ is the ex-dividend price by definition. One can also define a so called cum-dividend price process $\tilde{S}$, which equals

$$\begin{aligned} \tilde{S}(t) &\triangleq \mathbb{E}_t\Big[\sum_{i:\,T_i\ge t} p(t,T_i)D_i\Big] = S(t) + \mathbb{E}_t[p(t,t)D_i\mathbb{1}_{\{T_i=t\}}] \\ &= S(t) + D_i\,, \end{aligned}$$

in $t = T_i$.[3] This gives already the idea to take an early announcement into account. From the declaration time up to the payment they split the stock price into two components: *The present value of the next, known dividend,*

[2] Note that within this remark we use the notation $C(t,T,S,K)$ for the time-t price of a call option with maturity T, underlying S and strike K. Moreover, we indicate the American call option with an upper index A, i.e. C^A.

[3] Hence, in their approach Assumption 1.1 holds.

which is deterministic and an ex-dividend stock price process. This is an ex-dividend price in the sense of removing the dividend from the stock price.

Furthermore, Korn and Rogers (2005) use their stock models and different announcement settings for option pricing as there is also an impact on the price of a derivative, if it has a dividend paying stock as underlying.

Bar-Yosef and Sarig (1992) also investigate the effect of dividend announcements on stock and option prices. Therefore, let $C(t)$ denote the price of a European call option and $P(t)$ the price of a European put option with underlying S, strike $K > 0$, and maturity T. Then, they use a dividend estimator which is based on

$$D^*(t,T) + (\Delta p - \Delta c) = S(t) + P(t) - C(t) - Kp(t,T)\,, \tag{1.2}$$

where $D^*(t,T)$ is the present value of the outstanding dividends (for more details see Definition 2.1) and Δp (Δc) the price difference between the European and American style put (call). To quantify dividend surprises they compare two Equations of the form (1.2) that are calculated from prices prior and after the dividend announcement, both before the ex-dividend day.

In some countries the dividend announcement is immediate before the ex-dividend day, i.e. the announcement has no or respectively a negligible influence to the stock price. For example in Germany it is common that the ex-dividend date is the trading day after the general business meeting, where they announce the dividend.[4] Contrarily, in the US it is usual to pay a dividend quarterly and often the dividends are declared more than two weeks in advance. Sometimes, it can be that the next dividend is announced before the actual one is paid or even that all dividends for one year are announced at the same time. Hence, in reality different announcement settings can be observed, which can influence the stock price. In this work we first omit a closer analysis of the announcement date in Chapters 2 and 3, whereas in Chapter 4 we also investigate the case of an early announcement.

[4] On 1st of January 2017, this has changed to at least three trading days. But as all analyzed datasets within this work are from before 2017, we still act on the assumption of the next trading day.

Chapter 2

Discrete Dividend Estimation by No-Arbitrage

In this chapter we deduce a method for estimating dividend payments by option data. Therefore, we extend Grün (2014) to stochastic dividend payments and improve the estimation method. The essential parts of this chapter are published in Desmettre, Grün, and Seifried (2017) and is presented here in more details. This chapter is organized as follows: Section 2.1 provides the notations and setting. In Section 2.2 we generalize the put-call parity, which we use in Section 2.3 for estimating dividends. In Section 2.4 we apply our approach to data from German blue-chips and in Section 2.5 to Swiss and French ones. During this chapter we emphasize the differences to Grün (2014).

2.1 Gerneral Framework

We first provide a short repetition of the notations we have introduced so far: $S(t)$ denotes the time-t stock price and $p(t,T)$ the time-t discount factor for cash flows at time T where $t \leq T$. We focus on the next n dividend payments D_i and assume they are payable at discrete, known times $t < T_1 < T_2 < \cdots < T_n \leq T$, where $S(T_i)$ is the ex-dividend price. For further analysis we need the following Definition:

Definition 2.1 *The time-t present value of expected future dividend payments up to time T is denoted by*

$$D(t,T) \triangleq \sum_{i:\, t<T_i\leq T} p(t,T_i)\mathbb{E}_t^{T_i}[D_i]\,, \tag{2.1}$$

where $\mathbb{E}_t^{T_i}[\cdot]$ is the time-t conditional expectation under the T_i-forward measure.

Remark 2.1 *The T_i-forward measure used in* (2.1) *is the unique pricing measure implied by market prices of options with maturity T_i, where $p(t,T_i)$ is*

used as the numéraire. The use of this T_i-forward measure has the computational advantage of no discounting of the final payoff. This is a direct consequence of the change of numéraire approach. The general approach of changing the numéraire was developed by Geman, El Karoui, and Rochet (1995). Therefore, we first need the definition of a ***numéraire pair****: Let $\mathbb{Q}^* \sim \mathbb{P}$ be a probability measure on $\mathcal{F}(T)$ and X a price process of a portfolio. $(\mathbb{Q}^*, X)$ is called a numéraire pair, if $X(t) > 0$ for all $t \in [0,T]$ and the X-discounted price process of every asset S, i.e. $\frac{S(t)}{X(t)}$, is a local $\mathbb{Q}^*$- martingale on $[0,T]$. Geman, El Karoui, and Rochet (1995) showed: If $(\mathbb{Q}^*, X)$ is a numéraire pair then for a contingent claim $\mathcal{C}$ it follows:*

$$\mathcal{C}(t) = X(t)\mathbb{E}_t^{\mathbb{Q}^*}\left[\frac{\mathcal{C}(T)}{X(T)}\right]. \tag{2.2}$$

So if we use $p(t,T_i)$ as numéraire and dividends as contingent claim we get the summands of $D(t,T)$ with Formula (2.2).

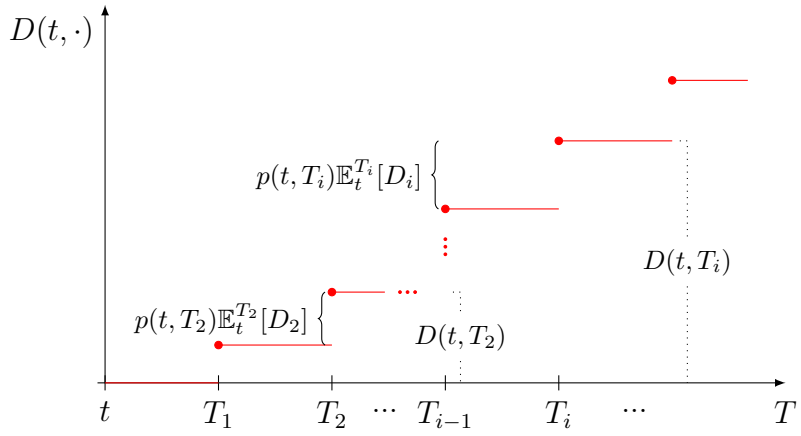


Figure 2.1: Visualization of $D(t,T)$ as a function in T.

Figure 2.1 visualizes Definition 2.1, where it displays $D(t,T)$ as a function in T with fixed t. The red line looks like a step function with stairs/jumps on every dividend payment day T_i of size $p(t,T_i)\mathbb{E}_t^{T_i}[D_i]$. Consequently, we can use the two successive "step values", i.e. $D(t,T_i)$ and $D(t,T_{i-1})$ to estimate the present value of a single dividend payment.

2.2 Put-Call Parity with Discrete Dividends

Now, we also consider prices of European call, $C(t)$ and put options, $P(t)$, with underlying S, strike $K > 0$, and maturity T. The following theorem puts these two quantities into relation with $D(t,T)$. It can be seen as a generalization of the classical put-call parity to dividend paying stocks; see, e.g., Hull (2012).

Remark 2.2 *Within the proof we use the notation F_T for the T-forward price of the stock S, fixed at time t and assume that all needed forwards are traded in the market. If this is not the case, we can change Assumption 1.1 to replicable dividends (compare with Remark 1.2). For more details and the adapted proof of Theorem 2.1 see Appendix A.*

Theorem 2.1 (Put-Call Parity with Dividends) *In the case of a dividend paying stock under the assumption of no-arbitrage the following parity holds:*

$$S(t) - D(t,T) + P(t) = C(t) + Kp(t,T)\,. \tag{2.3}$$

Proof. As in the proof of the traditional put-call parity we use simple no-arbitrage conditions and we argue by contradiction.

1. Suppose that $\mathbf{S(t) - D(t,T) + P(t) < C(t) + Kp(t,T)}$:
In this case we can construct an arbitrage opportunity as follows:

At Time t:

- sell the call $C(t)$ and borrow the amount $Kp(t,T)$ in cash,
- buy the put $P(t)$ and the underlying asset $S(t)$,
- take a short position in the T_i- -forward on S and a long position in the T_i -forward for every $t < T_i \leq T$. Furthermore, borrow the amount $p(t,T_i)[F_{T_i-} - F_{T_i}]$ until T_i for every dividend date.

From Assumption 1.1 it follows that

$$S(T_i-) - S(T_i) = D_i\,,$$

as T_i- is the time directly before the stock goes ex-dividend. Hence, the amount borrowed for a single position in the third part of the strategy is

$$\begin{aligned} p(t,T_i)[F_{T_i-} - F_{T_i}] &= p(t,T_i)\left[\frac{S(t)}{p(t,T_i)} - \frac{S(t) - p(t,T_i)\mathbb{E}_t^{T_i}[D_i]}{p(t,T_i)}\right] \\ &= p(t,T_i)\frac{p(t,T_i)\mathbb{E}_t^{T_i}[D_i]}{p(t,T_i)} \\ &= p(t,T_i)\mathbb{E}_t^{T_i}[D_i]\,, \end{aligned}$$

where the first equation is due to the *cost of carry formula* of the forward price in combination with Assumption 1.1. So the total amount borrowed accumulates to

$$\sum_{i:\, t<T_i\leq T} p(t,T_i)\Big[F_{T_i-} - F_{T_i}\Big] = D(t,T)\,.$$

So the total position reforms to $-C(t) - Kp(t,T) + P(t) + S(t) - D(t,T)$ with time-t cash flow $C(t) + Kp(t,T) - P(t) - S(t) + D(t,T) > 0$.

Time T_i:
The stock pays dividends and we need to repay the credit, which directly settles up with the difference of the forward positions:

$$\begin{aligned} &D_i - \underbrace{[F_{T_i-} - F_{T_i}]}_{\text{repay credit}} + \underbrace{F_{T_i-} - S(T_i-)}_{\text{short position}} - \underbrace{F_{T_i} + S(T_i)}_{\text{long position}} \\ &= D_i - [S(T_i-) - S(T_i)] = 0\,, \end{aligned}$$

where the last equation follows by Assumption 1.1. Hence, the total cash flow is equal to 0.

Time T:
If $T = T_n$ we can again compensate the credit repayment and the dividend payment with the difference in the forward positions. Otherwise, we have that the forward positions are already 0. In both cases we obtain

$$-C(T) - K + S(T) + P(T) = 0\,,$$

where this equation follows from the classical put-call parity. Hence, the strategy constructed above is a riskless gain and thus an arbitrage opportunity. So we must have

$$S(t) - D(t,T) + P(t) \geq C(t) + Kp(t,T)\,. \tag{2.4}$$

2. Suppose that $\mathbf{S(t) - D(t,T) + P(t) > C(t) + Kp(t,T)}$:
By exchanging "sell" and "buy" in the previous step it follows that

$$S(t) - D(t,T) + P(t) \leq C(t) + Kp(t,T)\,. \tag{2.5}$$

Combining (2.4) and (2.5) ensures the result. □

Remark 2.3 *In Grün (2014) we already prove a put-call parity with discrete dividends. The main difference to* (2.3) *is the assumption of a deterministic risk-less rate. In this case it is enough to have a look at the final wealth: the future value of the sum over the incurred dividends is equal to the future value of $D(t,T)$ (with $p(t,T) = \exp(-r(t)(T-t))$).*

2.3 Estimation of Dividends and Discount Factors

Theorem 2.1 can be used to estimate $D(t,T)$ from the stock price $S(t)$, put and call prices $P(t)$ and $C(t)$, and the discount factor $p(t,T)$ via

$$D(t,T) = S(t) + P(t) - C(t) - Kp(t,T)\,.$$

While the former three are available as market data, we need to bootstrap the corresponding discount factor. Alternatively, we could use prices of safe bonds instead. But this accompanies with some issues, e.g. we would refer to another market, differences in liquidity, and also funding costs. Hence, we need a discount curve, which is specific to that market. Therefore, the implicit discount curve in stock option prices is the best choice.
In this section we establish the bootstrap method and explain the further estimation base. First we repeat the methods from Grün (2014) and illustrate the challenges which occur within the calculation. Afterwards, we derive the new approach and emphasize its improvements.

2.3.1 The Box Spread Method

In Grün (2014) we developed the so called *box spread method* for bootstrapping the discount curve. The basic idea is simple: We considered two pairs of put and call options (P_1, C_1) and (P_2, C_2), respectively, with the same underlying, the same maturity, but different strike prices K_1 and K_2, and applied Theorem 2.1 to each pair to obtain

$$\begin{aligned} S(t) - D(t,T) + P_1(t) &= C_1(t) + K_1 p(t,T)\,,\\ S(t) - D(t,T) + P_2(t) &= C_2(t) + K_2 p(t,T)\,. \end{aligned}$$

By subtracting the first equation from the second one and subsequent arranging, we achieved the following representation of the discount curve:

$$\begin{aligned} P_2(t) - P_1(t) &= C_2(t) - C_1(t) + (K_2 - K_1)p(t,T)\\ \Longrightarrow p(t,T) &= \frac{1}{K_2 - K_1}\Big[C_1(t) - C_2(t) + P_2(t) - P_1(t)\Big]\,. \end{aligned} \tag{2.6}$$

Note that the representation (2.6) depends on neither the spot price of the underlying nor the unknown dividends.

Remark 2.4 *A portfolio of a long bull call spread and a long bear put spread with strikes K_1 and K_2 and the same maturity T is called a box spread $BS(t)$,*

$$BS(t) = \underbrace{C_1(t) - C_2(t)}_{\textit{bull call spread}} + \underbrace{P_2(t) - P_1(t)}_{\textit{bear put spread}}\,. \tag{2.7}$$

It is easy to see that

$$BS(t) = (K_2 - K_1)p(t,T)\,, \tag{2.8}$$

compare also Hull (2012). Since (2.7) *and* (2.8) *together is equivalent to* (2.6)*, we refer to the latter as the box spread method. This approach already exists for some time. For instance, Billingsley and Chance (1985) use it for testing the efficiency of the options market under the assumption of deterministic interest rates. Moreover, Ronn and Ronn (1989) explore box spread arbitrage conditions and derive arbitrage bounds for American options under transaction costs.*

Involving the box spread method the resulting dividend estimate reads as follows:

$$D^*(t,T) = S(t) + P(t) - C(t) - Kp^*(t,T)\,. \tag{2.9}$$

From now on, we use a * to indicate prices bootstrapped from market data.

Bar-Yosef and Sarig (1992) previously use a dividend estimator of the form (2.9) to measure the effect of dividend announcements on stock and option prices. For more details see Section 1.2.3. Note that their analysis is based on both American and European options. They focus on one time point (the dividend announcement day) and have a shorter time horizon than the analysis of this thesis, which aims to estimate dividends for up to 2-5 years.

In this setting the analysis was separated into two parts:

1. We developed an algorithm for the calculation of the relevant discount factors based on the box spread (2.6).

2. With the results of Step 1. we computed the implied dividend estimates $D^*(t,T)$ via the put-call parity approach (2.9).

As explained above we needed to select two put-call pairs with different strike prices to compute (2.6). Therefore, we fixed a percentage deviation γ from the *at-the-money* price and selected the resulting values as the strikes used for the box spread. The following Algorithm 2.1 shows the details.

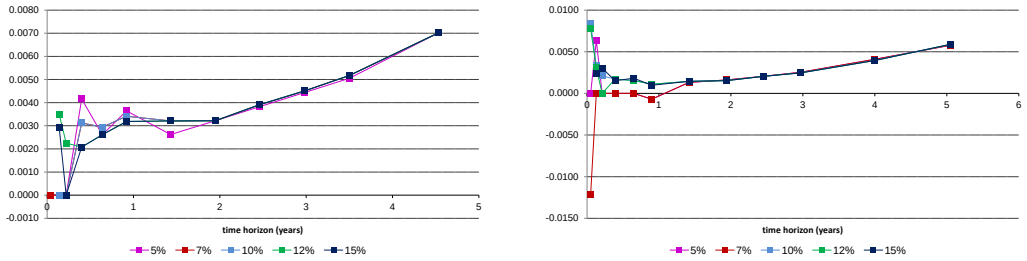
Algorithm 2.1 *Input: Table with prices of put-call pairs and corresponding maturities, strikes and spots, percentage* γ

- *Determine two values* $\tilde{K}_1$ *via rounding* $(1-\gamma)*S(t)$ *and* $\tilde{K}_2$ *via rounding* $(1+\gamma)*S(t)$.

- *For each maturity, choose K_1 as the maximal available strike, in the option market data for the relevant spot date, smaller than or equal to $\tilde{K}_1$ and K_2 as the minimal available strike greater than or equal to $\tilde{K}_2$.*
- *Look up the associated put and call prices and compute*

$$p^*(t,T) = \frac{1}{K_2 - K_1}\Big[C_1(t) - C_2(t) + P_2(t) - P_1(t)\Big].$$

In the further analysis we ran through Algorithm 2.1 with five different percentage deviations 5%, 7%, 10%, 12% and 15%. In addition, we calculated the arithmetic average over the resulting discount factors to use it for the dividend estimation.
Theorem 2.1 holds for one single put-call pair with the same strike and maturity. Hence, one could determine the present value of dividends from this single pair. In practice, this may lead to inaccurate results due to for example misquotes, liquidity issues or rounding errors. Furthermore, that approach does not take the available information of all option pairs into account. In order to deal with that, we estimated the dividends for each pair and then built the average for every maturity.



(a) 2012-08-03 (b) 2014-02-05

Figure 2.2: Zero yield curves for Siemens at two spot dates(box spread method)

Remark 2.5 (Challenges with the Box Spread Approach) *While working with the data, we got to know, that the data was rounded. With this some issues incurred within the discount curves. Thus, we had a closer look at the zero yields.*

- *For maturities smaller than one year, the curves often had a "ragged" structure and differed a lot from each other (compare with Figure 2.2). After that the curves got closer together. As the values of the zero yield curves were calculated via $-\frac{\log(p(t,T))}{\tau}$ (τ is the time to maturity), a rounding of the data is carried more into account for smaller time to maturities.*

- *Furthermore, it was noticeable that the curves with γ equal to 5% or 7% had bigger outliers and differed more from the other curves (see again with Figure 2.2), whereas normally one would suppose, that a smaller γ would be better. This could also be explained by the bigger weighting of the data rounding: in the calculation of the box spread (Equation* (2.6)*) the difference $K_2 - K_1$ was in the denominator, which is smaller for smaller percentage deviation.*

Using only two put-call pairs was also a disadvantage of the box spread, as misquotes can influence the calculation. There is also more information via the other pairs available, but with this approach we failed to take it into account.

2.3.2 Linear Regression

In Remark 2.5 we have seen, that there are some issues with the box spread approach. In order to improve our approach, we now need to deal with in particular data rounding, outliers and also using all available information. Therefore, we develop a regression-based method, which uses all available put-call pairs and simultaneously calculates both the discount curve and the present value of dividend payments.
The idea behind this is again simple: Consider *all* pairs of put and call options (P_j, C_j), with the same underlying S, the same maturity T, and different strike prices K_j where $j = 1, \dots, m$ and apply Theorem 2.1 to each pair to obtain

$$S(t) - D(t,T) + P_j(t) = C_j(t) + K_j p(t,T) + \varepsilon_j \,, \tag{2.10}$$

where we add a potential error term ε_j for the data issues. The ε_j are assumed to be i.i.d. with $\mathbb{E}[\varepsilon_j] = 0$ and finite variance. After rearranging Equation (2.10) and defining Y_j, X_j, a and b as follows:

$$\underbrace{P_j(t) - C_j(t)}_{Y_j} - \Big[\underbrace{p(t,T)K_j}_{aX_j} + \underbrace{(D(t,T) - S(t))}_{b}\Big] = \varepsilon_j \,, \tag{2.11}$$

we can perform the linear regression. The details can be seen in the below Theorem 2.2.

Theorem 2.2 (Dividend Estimation) *The least squares estimator of $D(t,T)$ is given by*

$$D^*(t,T) \triangleq \hat{b} + S(t) \,. \tag{2.12}$$

The important parameter $\hat{b}$ is defined as

$$\hat{b} \triangleq \bar{Y} - \hat{a}\bar{X} \quad \textit{with} \quad \hat{a} \triangleq \frac{\frac{1}{m}\sum_{j=1}^{m}(X_j - \bar{X})(Y_j - \bar{Y})}{\frac{1}{m}\sum_{j=1}^{m}(X_j - \bar{X})^2} \,,$$

where $X_j \triangleq K_j$, $Y_j \triangleq P_j(t) - C_j(t)$ *for* $j = 1, \dots, m$, *and* $\bar{X}$ *and* $\bar{Y}$ *are the respective sample means.*

Proof. We can execute the linear regression via minimizing the sum over the potential error terms ε_j in Equation (2.11):

$$\min_{a,b} \sum_{j=1}^{m} \varepsilon_j^2 \triangleq \min_{a,b} \sum_{j=1}^{m} (Y_j - aX_j - b)^2 \,,$$

where

$$Y_j \triangleq P_j(t) - C_j(t) \,, \qquad a \triangleq p(t,T) \,,$$
$$X_j \triangleq K_j \,, \qquad b \triangleq D(t,T) - S(t) \,.$$

The result follows directly with the ordinary least squares estimators. □

Corollary 2.3 *In the notation of Theorem 2.2, the market-implied discount factor is given by*

$$p^*(t,T) \triangleq \hat{a} \,. \tag{2.13}$$

Note that the different maturities of the option need not coincide with the payment days. Because of the structure of $D(t,T)$ (compare Definition 2.1) it is sufficient to consider time-points with $\tilde{T}_i \geq T_i$. This can also be seen in Figure 2.3, colored in blue.

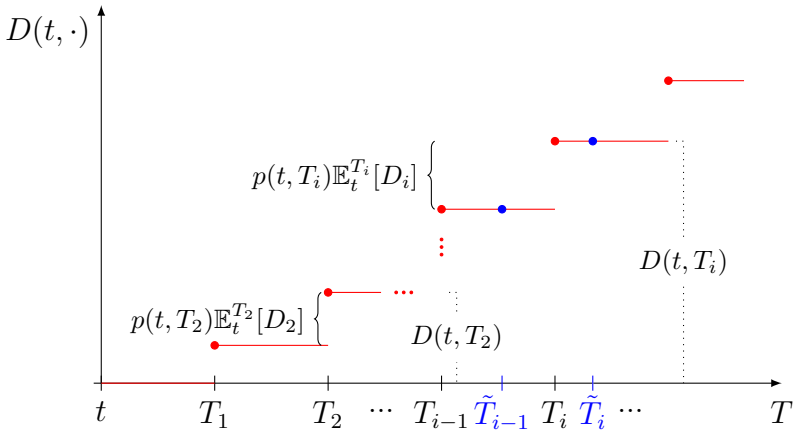


Figure 2.3: Visualization of $D(t,T)$ as a function in T including the options' maturities.

Then, using the estimator from Theorem 2.2, we can directly calculate the time-t net value of the time-T_i dividend payment via

$$D^*(t,\tilde{T}_i) - D^*(t,\tilde{T}_{i-1}) \quad \text{where} \quad T_{i-1} \leq \tilde{T}_{i-1} < T_i \leq \tilde{T}_i < T_{i+1} \,.$$

Moreover, the time-t forward price $\mathbb{E}_t^{T_i}[D_i]$ of an individual dividend payment D_i due at time T_i, denoted by $D_{i,t}^*$, can subsequently be approximated via[5]

$$D_{i,t}^* \triangleq \frac{D^*(t,\tilde{T}_i) - D^*(t,\tilde{T}_{i-1})}{p^*(t,\tilde{T}_{i-1}) + (p^*(t,\tilde{T}_i) - p^*(t,\tilde{T}_{i-1}))\frac{T_i - \tilde{T}_{i-1}}{\tilde{T}_i - \tilde{T}_{i-1}}}\,. \tag{2.14}$$

If $T_i = \tilde{T}_i$, i.e. the time horizon of $D(t,T_i)$ coincides with a maturity $\tilde{T}_i$ for which option market data are available, than Equation (2.14) is exact. Otherwise, the relevant discount factor $p(t,T_i)$ is approximated via linear interpolation.

Remark 2.6 *For estimating dividends one could also use stock futures or forwards compare e.g., Golez (2014) instead of options. Within this approach another challenge arises, as the required discount curves can not be calculated without additional data (such as for example options).*

2.4 Results for DAX Constituents

After having established the theoretical framework, we apply our estimating methodology to dividends of individual stocks of German blue-chips. In particular, we investigate in detail the resulting estimates (2.12) and (2.13) and analyze their applicability in practice. Afterwards, where reliable historical data are available, we benchmark our results against realized dividend payments.

2.4.1 Data Basis

In this section we shortly explain, which data we use for the application of the estimation method: As mentioned before we focus on German blue-chips. Therefore, we request data from stocks, which are constituent in the German stock index DAX[6] and where relevant derivatives; e.g. European options, are available. From the 30 constituents the following 14 meet our restrictions: *Adidas, Allianz, BASF, Bayer, Commerzbank, Daimler, Deutsche Bank, Deutsche Telekom, Infineon Technologies, Merck, Munich Re, RWE, SAP, Siemens*. In Germany dividends are paid once per year. The dividend amounts are set at the general business meetings of the companies and are announced on the same day. About one day later the stock goes ex-dividend, which is

[5] In the case where we have more maturities $\tilde{T}_i$ with $T_i \leq \tilde{T}_i < T_{i+1}$, we chose the first one in the timeline to calculate the $D_{i,t}^*$.

[6] Status of the DAX composition: January, 2015.

the time T_i, we are focusing on. Commerzbank is the only stock which does not pay dividends at all. But still it makes sense to consider the estimation results. In the first part of the analysis, in Sections 2.4.2 and 2.4.3, we illustrate the estimates on five different spot dates 2011-03-29, 2012-08-03, 2013-11-06, 2014-02-05 and 2014-07-08, which we selected at random. We reject to take equidistant spot dates, as then the time between the dividend payment and the estimation would always be the same.
In the second part, in Section 2.4.4, we restrict attention to six of the stocks. We execute our analysis to all data available for every Wednesday between 2011-01-01 and 2013-12-31. Afterwards, we perform aggregate statistics to the results. In total we have a look on 126'367 put-call pairs, the details of the available data can be seen in Table 2.1. For requesting the market data we use Thomson Reuters' Datastream.

Data basis	**BASF**	**Bayer**	**Daimler**	**Merck**	**Munich Re**	**Siemens**
Total Number of Put-Call Pairs	24'406	25'996	26'820	9'451	16'565	23'129
Number of Available Spot Dates	146	146	144	83	110	147
Average Number per Spot Date	167.16	178.05	186.25	113.87	150.59	157.34
Min Strike	28	20	16	48	52	40
Max Strike	120	180	92	180	280	180

Table 2.1: Analyzed data for the aggregate statistics.

2.4.2 Dividends and Discount Curves

As explained in the Subsection 2.3.2 we get (2.12) and (2.13) via linear regression. To visualize the results we put three or four stocks, where the size of the dividends is close together, in one Figure. Figures 2.4, 2.5, 2.6 and 2.7 illustrate the present value of the dividends $D^*(t,T)$ as a function in the maturity T for each spot date t.
As we want to check the applicability in practice we need some properties, which a good estimate should fulfill. We can get these properties by looking at the Definition of $D(t,T)$:

- As $D(t,T)$ is the sum over the present values of the expected dividends $D_i > 0$, an estimate should always be positive.

- Furthermore, a noticeable jump should be seen every time a dividend was paid; e.g. in the case of German stocks once per year.

- Assembling these two points, clearly the estimate should increase in the time T.

Having a closer look at the figures, all functions of the estimates have something in common: the structure looks like a "staircase". Sometimes there is not a "step" but a straight line, especially in Figure 2.4 with $T > 3$ years. This happens as there is only one maturity per year for the put-call pairs available.

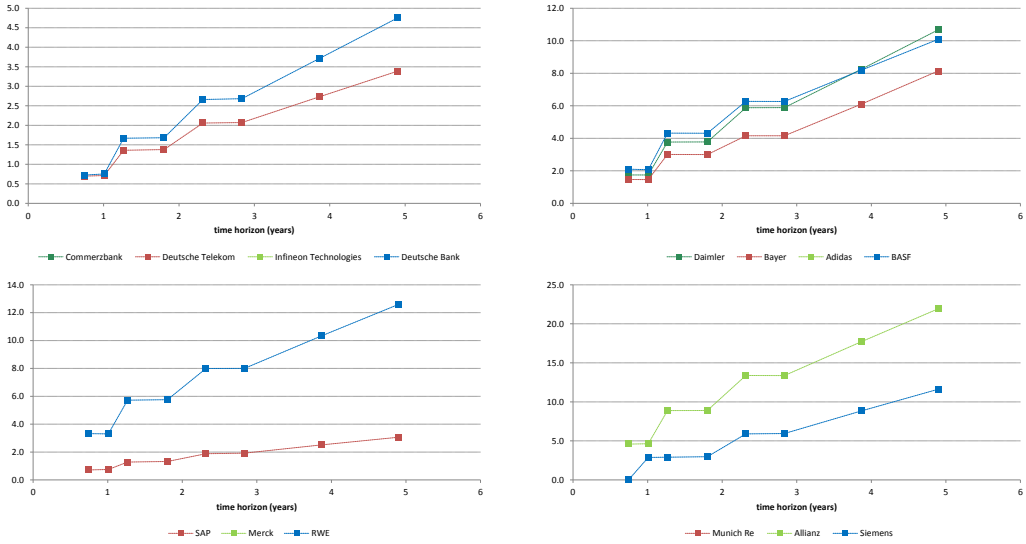


Figure 2.4: Present value of dividends $D^*(t, T)$ (spot date t = 2011-03-29).

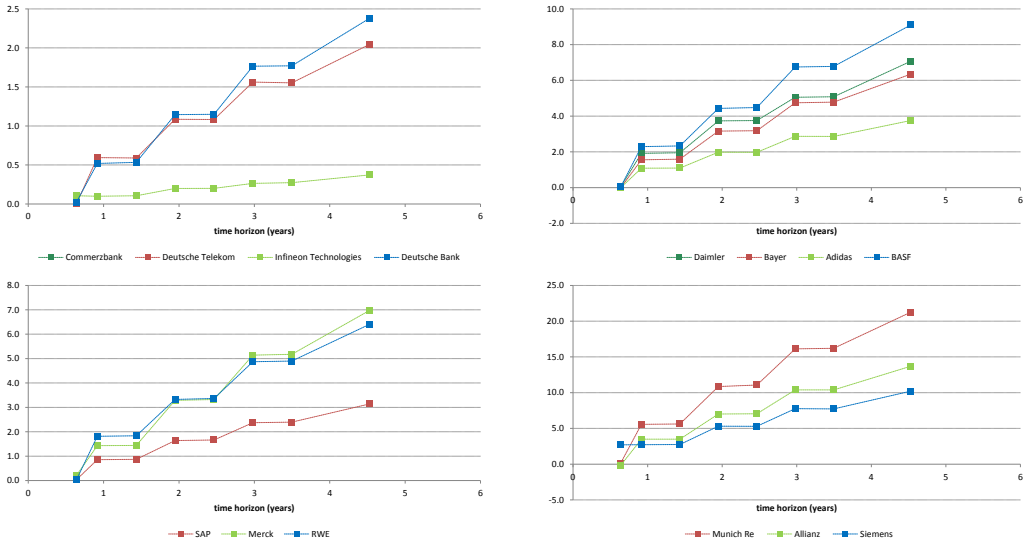


Figure 2.5: Present value of dividends $D^*(t, T)$ (spot date t = 2012-08-03).

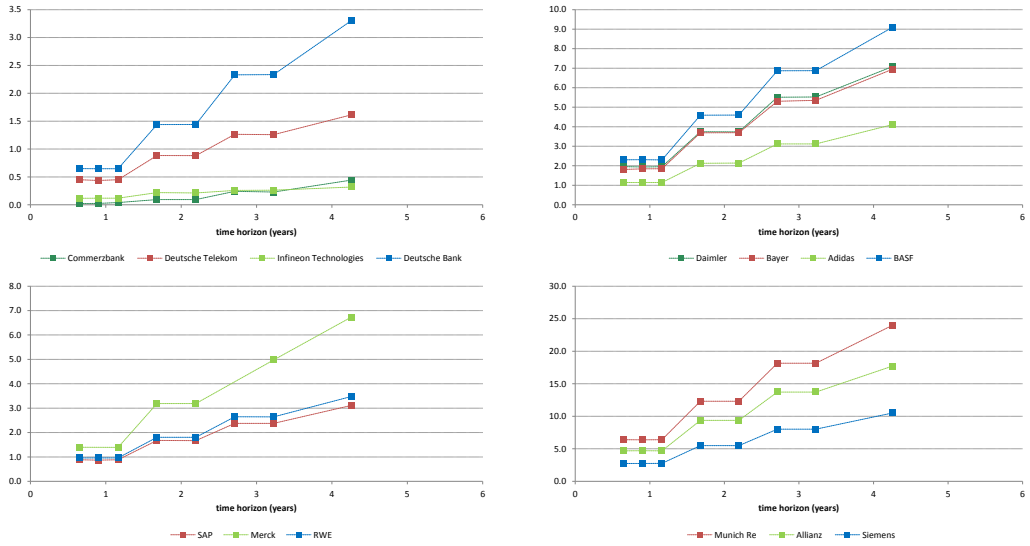


Figure 2.6: Present value of dividends $D^*(t,T)$ (spot date t = 2013-11-06).

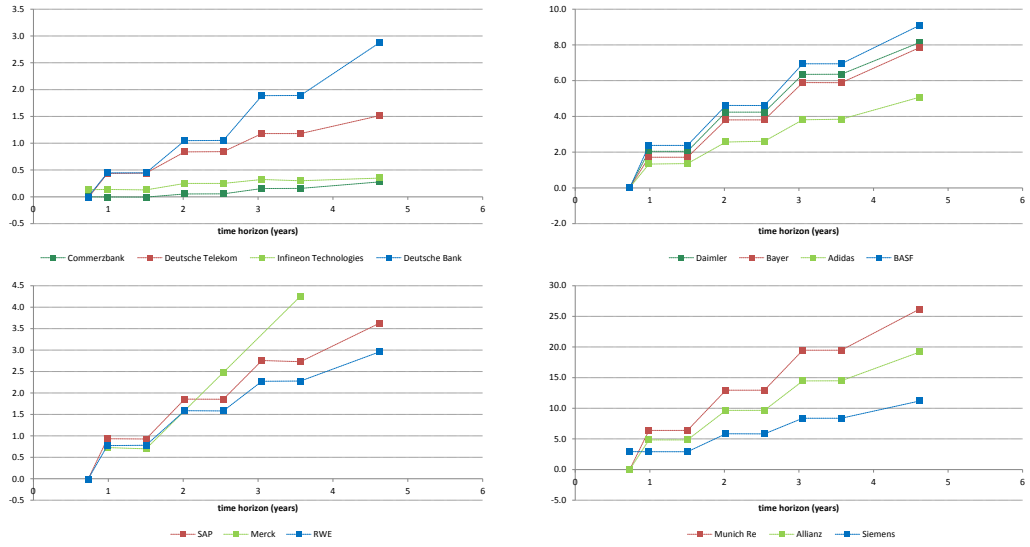


Figure 2.7: Present value of dividends $D^*(t,T)$ (spot date t = 2014-07-08).

Overall, all the properties are apparently fulfilled. Additionally, these figures are close to the visualization of the definition of $D(t,T)$ (compare Figure 2.1). Now, we also have a look to the estimate (2.13) for the discount curve. Therefore, we display the zero yield curves of the estimates $p^*(t,T)$ as a function of the time until maturity ($\tau = T - t$) in Figures 2.8 and 2.9 for two different spot dates.

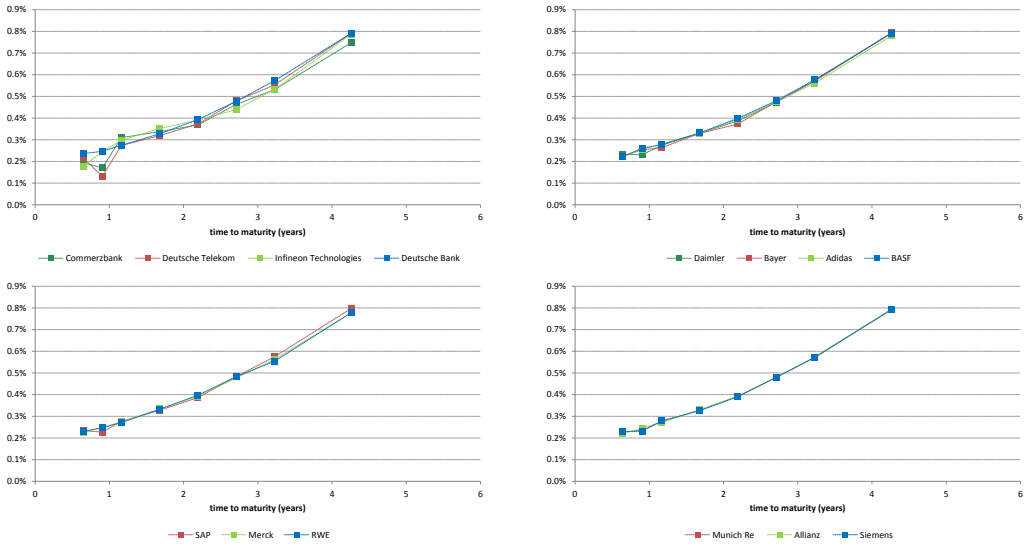


Figure 2.8: Market-implied zero yield curves (spot date 2013-11-06).

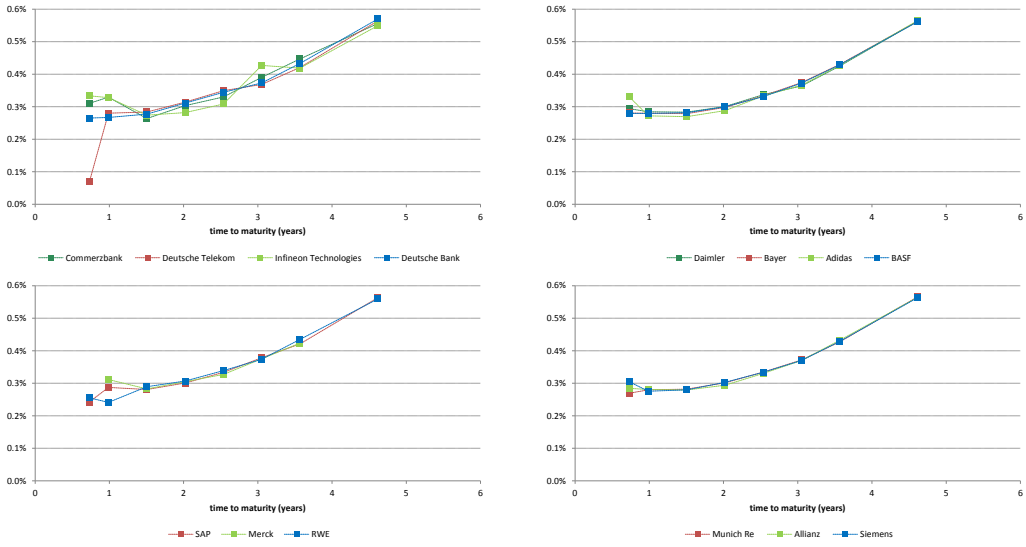


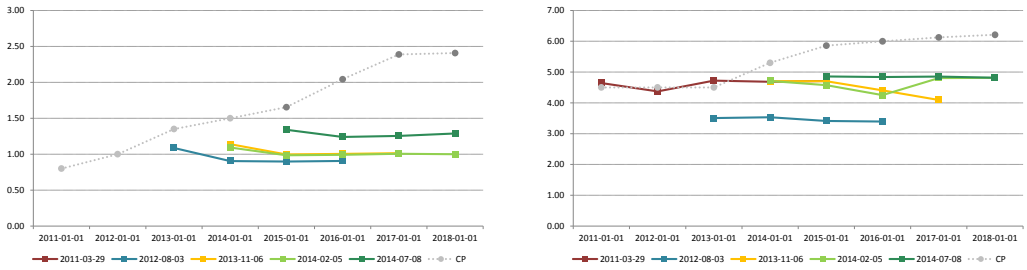
Figure 2.9: Market-implied zero yield curves (spot date 2014-07-08).

As already mentioned in Remark 2.5 there was an issue with data rounding. Unfortunately, here the zero yields of some stocks also differ from the others for small τ. So we decided to take only put-call pairs with $\tau > \frac{1}{2}$ year into account. This does not downgrade our approach as in our example dividends are paid once per year, so in most of the cases we do not need these maturities. Otherwise, there are enough maturities available with $\tau < 2$ years for the

estimation. Essentially for $\tau > 1$ year the zero yields and thereby the discount curves do not differ a lot across the different underlyings. As we have seen, that the estimates are working well in practice one question still remains: *Does using a linear regression based approach make sense from a statistical point of view?* An indicator for this are the R^2 values. It can reach values between 0% and 100%. If it is equal to 0% it implies that there is no linear relation and the line does not fit the data. On the other hand a R^2 of 100% is an indicator for a perfect linear relation and a line which fits the data. In our examples the R^2 always exceed 99.99%. Hence, it supports the usage of the linear regression based method. We come back to this in the Subsection 2.4.4, where we analyze our results for a bigger dataset.

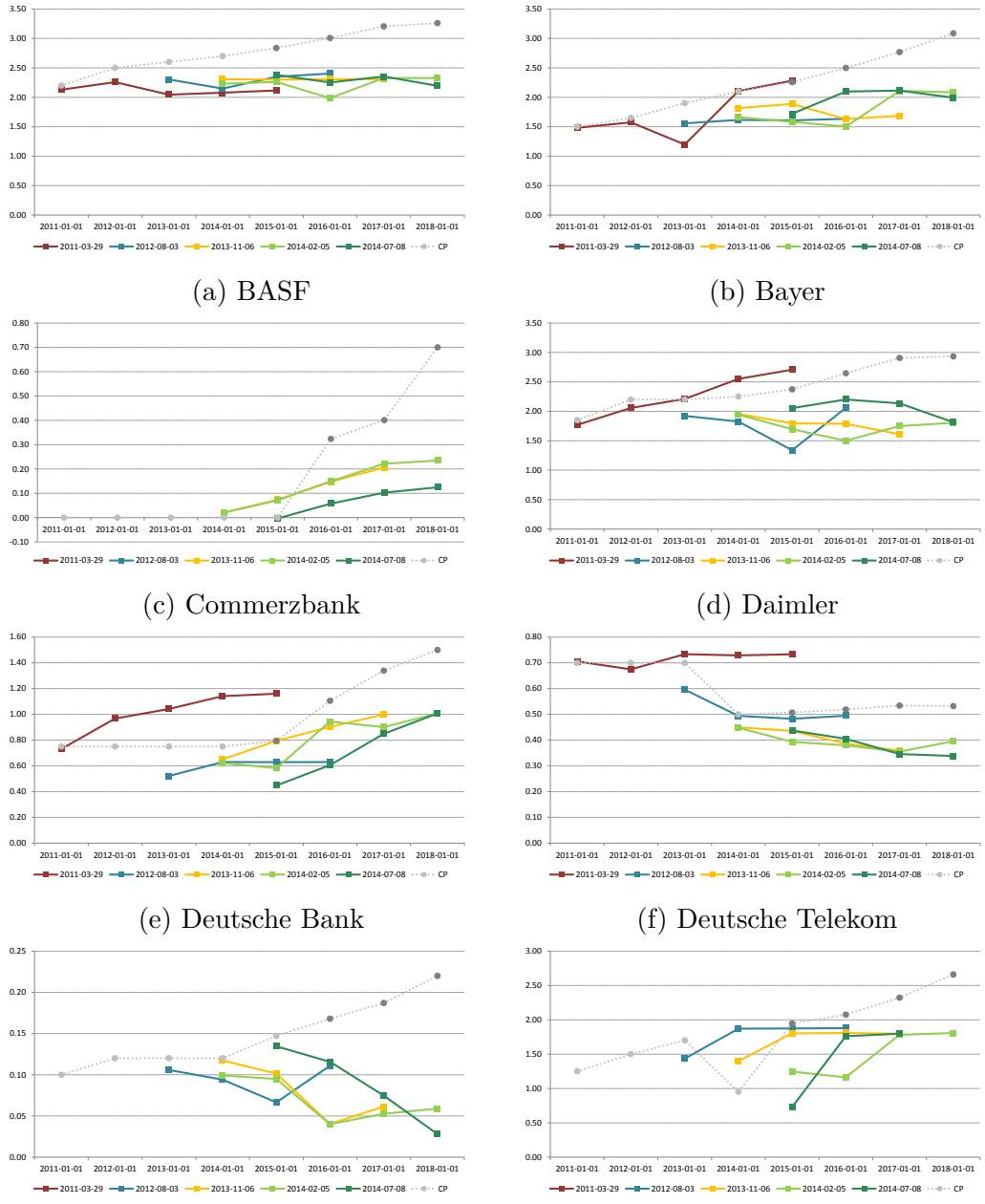
2.4.3 Benchmarking the Results

In this section we benchmark our estimated dividends with the historical incurred values. Therefore, we need to calculate the D^*_{i,t_k} via (2.14), where t_k is the data request date. Note that D^*_{i,t_k} is a *market-implied forward price* for the dividend payment D_i. This is important as we expect a variation from the actual incurred value for that reason. Coincidence is only given when dividends are assumed to be deterministic, hence known in advance.
Figures 2.10, 2.11 and 2.12 display for each stock and different spot dates t_k the resulting estimates via distinctive colors. The incurred values are colored in light gray. Note that there was a stock split of 1:2 for Merck at 2014-06-30 to which the incurred value for 2014 is already adjusted. Also, observe that SAP payed an additional special dividend of 0.35 € in 2012, that was not known on the spot date. Apart from these outliers, our estimates seem to be steady and in line with the actual payed dividends. One might ask, *why the estimates for one payment date vary across the different spot dates.* This is due to the flow of new information that is available in the market.



(a) Adidas (b) Allianz

Figure 2.10: Dividend estimates at different spot dates and benchmark against historical dividends and commercial forecasts.



(a) BASF

(b) Bayer

(c) Commerzbank

(d) Daimler

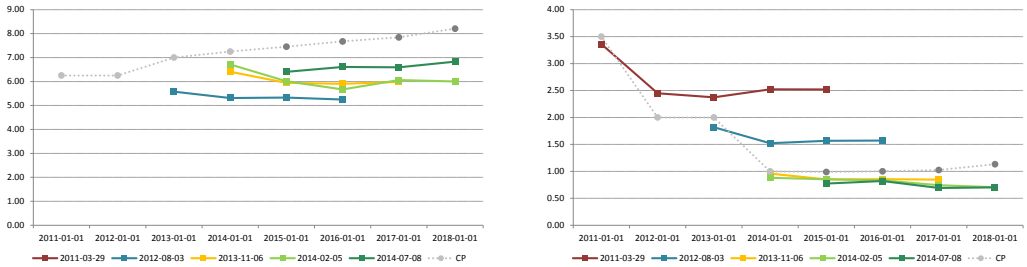
(e) Deutsche Bank

(f) Deutsche Telekom

(g) Infineon Technologies

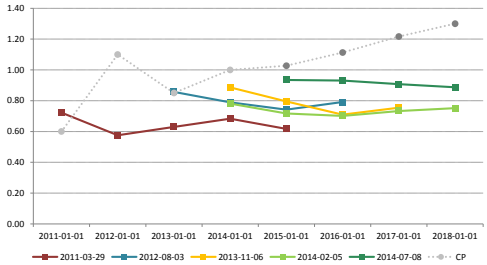
(h) Merck

Figure 2.11: Dividend estimates at different spot dates benchmarked against historical dividends and commercial forecasts.

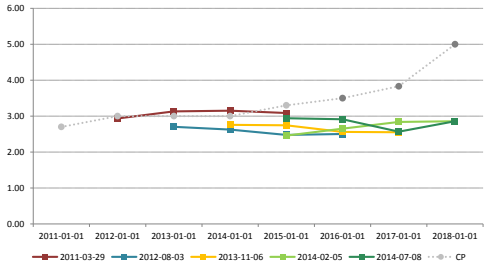


(a) Munich Re

(b) RWE



(c) SAP



(d) Siemens

Figure 2.12: Dividend estimates at different spot dates benchmarked against historical dividends and commercial forecasts.

For example, in half a year a lot can happen: options can expire, new ones are issued and more information about the company's performance are available. In Section 2.4.4 we have a closer look to the differences between the incurred and the estimated values. Therefore, we perform aggregate statistics, where we also take the time between the estimation and the actual payment into account.
In Figures 2.10, 2.11 and 2.12 there are also dark gray points expanding the light gray line, which illustrate a commercial prediction for payment days beyond 2014. Its calculation basis is a firm value model, that uses a projection of the company's equity (more details are not available to us). As the request date of the commercial estimate is the 2014-07-08, it should only be compared with the dark green line, the estimate of the same spot date. The commercial forecast and our estimate are in line for the initial dividend payment. But afterwards the commercial prediction estimates increasing dividends for each stock and it exceeds our estimate, which is mostly stable.

2.4.4 Aggregate Statistics

As already mentioned we now have a closer look to the results of a bigger dataset. Therefore, we restrict our analysis to six stocks: *BASF, Bayer, Daim-*

ler, Merck, Munich Re and Siemens. The details of the data basis can be seen in Section 2.4.1. Table 2.2 shows the results of the performed aggregate statistics.

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation	Estimate for Year	Empirical Volatility
BASF									
1 Year	146	8%	13%	12%	47%	0%	8%	2011	46%
2 Years	145	9%	20%	20%	31%	1%	6%	2012	45%
3 Years	112	5%	17%	17%	24%	8%	4%	2013	10%
4 Years	62	5%	19%	18%	26%	12%	4%	2014	6%
5 Years	11	4%	24%	24%	25%	21%	1%	2015	6%
Total	476	7%	17%	18%	47%	0%	7%		16%
Bayer									
1 Year	146	8%	12%	12%	42%	0%	9%	2011	32%
2 Years	142	10%	23%	23%	43%	0%	8%	2012	25%
3 Years	112	8%	26%	29%	44%	0%	11%	2013	10%
4 Years	61	5%	21%	24%	46%	0%	15%	2014	7%
5 Years	14	1%	3%	3%	5%	2%	1%	2015	9%
Total	475	8%	19%	22%	46%	0%	12%		13%
Daimler									
1 Year	144	9%	13%	13%	34%	0%	7%	2011	24%
2 Years	146	10%	22%	22%	40%	0%	10%	2012	30%
3 Years	106	7%	22%	17%	47%	0%	18%	2013	8%
4 Years	57	3%	12%	10%	29%	0%	9%	2014	6%
5 Years	6	2%	12%	11%	15%	10%	2%	2015	7%
Total	459	8%	18%	15%	47%	0%	13%		12%
Merck									
1 Year	81	14%	20%	22%	40%	1%	11%	2013	28%
2 Years	84	2%	3%	1%	13%	0%	4%	2014	9%
3 Years	50	1%	3%	3%	7%	0%	2%	2015	6%
Total	215	6%	10%	5%	40%	0%	11%		13%
Munich Re									
1 Year	109	8%	12%	12%	24%	0%	6%	2012	25%
2 Years	110	10%	23%	23%	30%	12%	5%	2013	20%
3 Years	76	10%	26%	25%	41%	18%	7%	2014	18%
4 Years	25	5%	20%	19%	35%	18%	3%	2015	18%
Total	320	9%	19%	19%	41%	0%	8%		19%
Siemens									
1 Year	146	7%	9%	9%	40%	0%	6%	2012	60%
2 Years	147	8%	14%	14%	24%	0%	7%	2013	14%
3 Years	100	6%	17%	19%	37%	0%	10%	2014	9%
4 Years	49	3%	11%	7%	26%	3%	6%	2015	9%
Total	442	6%	13%	10%	40%	0%	8%		20%

Table 2.2: Aggregate statistics.

The Table can be separated into two parts, which can be distinguished via the vertical line. The first part focuses on the disparity between the estimate and the actual incurred value, whereas the second one displays the volatility of the evolution in the estimate. As not all values are self-explanatory we now specify the calculation method:

First part of the Table

As we wish to consider the time $\tau_{k,i} \triangleq T_i - t_k$ between the estimation (spot date t_k) and the payment day T_i, we distinguish several *estimation periods* Δ, i.e., $\Delta - 1Y \leq \tau_{k,i} < \Delta$ where Y denotes year(s) and $\Delta = 1Y, ..., 5Y$. Furthermore, we denote the estimate for $\mathbb{E}_{t_k}^{T_i}[D_i]$ at time t_k by D_{i,t_k}^* and focus on the relative difference between the incurred and estimated dividend:

$$\hat{D}_{k,i} \triangleq \left| \frac{D_{i,t_k}^* - D_i}{D_i} \right| ,$$

for all spot and payment dates. Again, it is important that D_{i,t_k}^* is the approximation of the time-t forward price of an individual dividend payment (compare (2.14)). With the *counter* C denoting the corresponding number of request dates that are used for the statistics and N as the total number, we can define the *weighted average* as

$$\frac{1}{C} \cdot \sum_{k=1}^{N} \sum_{i=1}^{n} w_{k,i} \cdot \hat{D}_{k,i} \cdot \mathbb{1}_{\{\Delta - 1Y \leq \tau_{k,i} < \Delta\}} ,$$

where

$$w_{k,i} = 1 - \frac{\tau_{k,i}}{(Year(T_i) - Year(t_k) + 1) \cdot 365} .$$

This average assigns more weight to estimates which are expected to be more accurate, e.g. for $\tau_{k,i}$ small. Furthermore, the data are aggregated in terms of the classical *mean* ($w_{k,i} = 1$), the *median*, and both the *worst* and the *best case*. Additionally, we calculate the *standard deviation*, which is in terms of the classical average.

Second part of the Table

Here we examine the volatility of the estimate itself. Hence, we investigate its evolution as a function of the request date t_k for a fixed payment date T_i (indicated via column *estimate for year*). Thus, the *empirical volatility* s_K is defined via

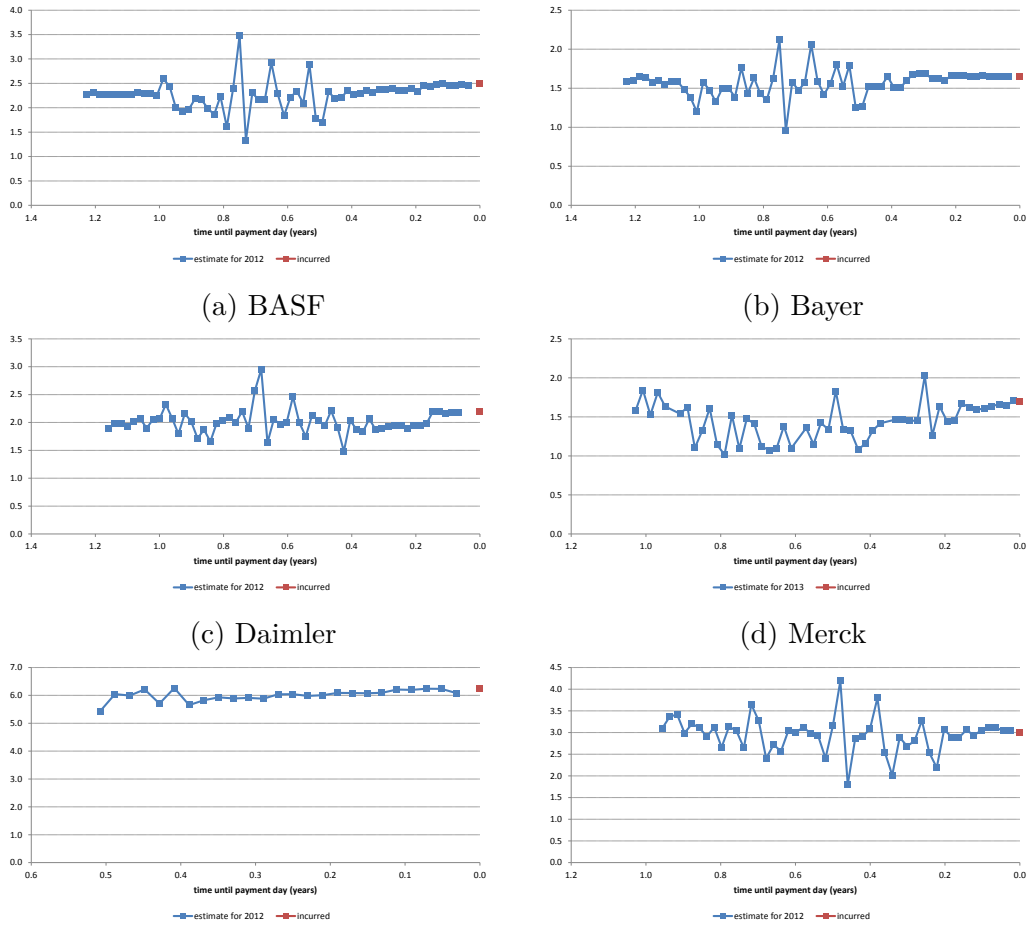
$$s_K^2 = \frac{1}{K} \sum_{k=1}^{K} (D_{i,t_{k+1}}^* - D_{i,t_k}^*)^2 ,$$

where K denotes the number of available estimates corresponding to T_i.

After knowing how the values are determined, we now discuss the results of Table 2.2: As we already mentioned at the beginning of Section 2.4.3 it is important to understand, that D_{i,t_k}^* is the *market-implied forward price*, which we compare with the actual, ex-post values of the dividends. Hence, non-zero

deviations in Table 2.2 should be expected as we deal with stochastic dividends. Even in an ideal setting non-zero deviations would occur.

We observe that classical averages and medians are close together, 90% of the values are less than 25%. The deviations exceeding 25% result from $\tau_{k,i} \geq 3$. Because one quarter of discrepancy together with the worst case values seem to be high, one could make an overhasty conclusion that the estimate does not



(a) BASF (b) Bayer

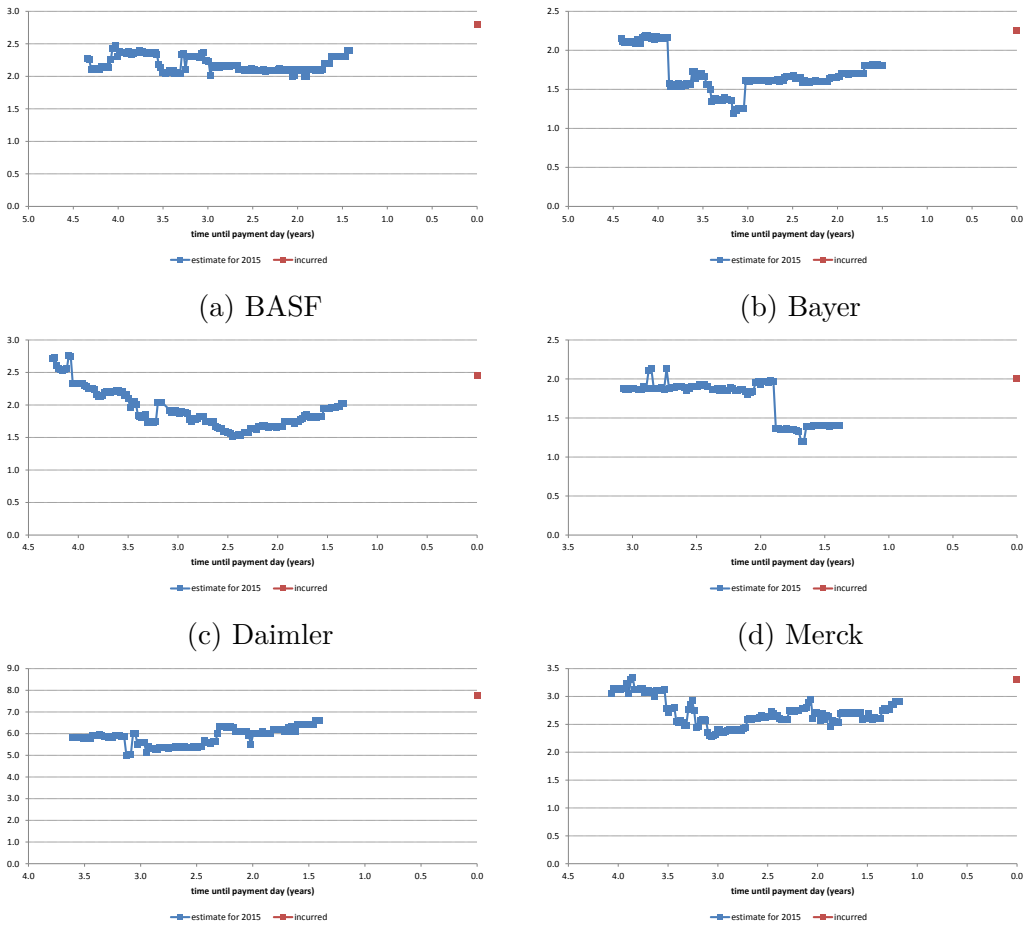
(c) Daimler (d) Merck

(e) Munich Re (f) Siemens

Figure 2.13: Time evolution of the dividend estimates for the payment in year 2012 as a function of the spot dates.[7]

[7] Note that there was no data available for Merck in 2012, so for Merck Figure 2.13 displays the year 2013.

perform well. But the main reason for these values is the time $\tau_{k,i}$ between the estimation and the payment day: For $\tau_{k,i}$ small the estimate performs better as more derivatives and more relevant market information are available. But still the time frame is one year, where especially for the next dividend payment a lot of information can accumulate. This also reflects in the empirical volatility, as the estimate for the former years is more volatile.



(a) BASF (b) Bayer

(c) Daimler (d) Merck

(e) Munich Re (f) Siemens

Figure 2.14: Time evolution of the dividend estimates for the payment in year 2015 as a function of the spot dates.

Figures 2.13 and 2.14 visualize the progress of the estimate, which was used to calculate the empirical volatility. The blue line displays the estimate for a fixed payment day T_i as a function in the spot days t_k, whereas the red point shows the actual incurred dividend amount. In Figure 2.13 it can be seen, that the estimate is volatile but in the end close to the actual incurred value.

In contrast the estimates in Figure 2.14 seem to be less volatile, but a trend towards the incurred value is recognizable. Note that there is white space in between the last blue point and the red point, because the series of spot dates stopped in December 2013. Overall, it results that one should take the $\tau_{k,i}$ into account. Therefore, we calculate the weighted average, where the weights are higher for smaller $\tau_{k,i}$. In our example the weighted average is in all cases smaller or equal to 10% (compare Table 2.2). This together with the best case values, which are typically below 18% for $\tau_{k,i} \leq 3$, now supports the statement from the section before, that the estimates are almost in line with the incurred values. Besides a limitation of our methodology for estimation periods with few available data and/or long prediction intervals is observable.

	BASF	**Bayer**	**Daimler**	**Merck**	**Munich Re**	**Siemens**	**Total**
R^2	99.99998%	99.99998%	99.99991%	99.99999%	99.99999%	99.99993%	99.99996%

Table 2.3: Average values of R^2.

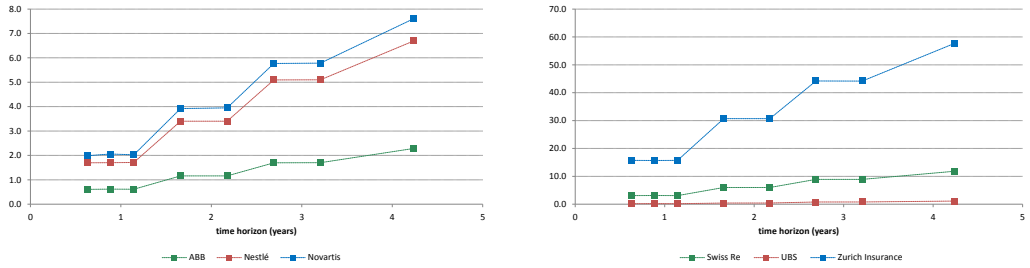
In the last paragraph of Subsection 2.4.2 we discussed the usage of a regression based approach, where we inspect the R^2 values. Table 2.3 shows the average R^2 values for the bigger dataset of the current Subsection 2.4.4. All R^2 values exceed 99.99%, which supports our regression based estimation method. These high R^2 come from the no-arbitrage nature of the method.

2.5 More Results

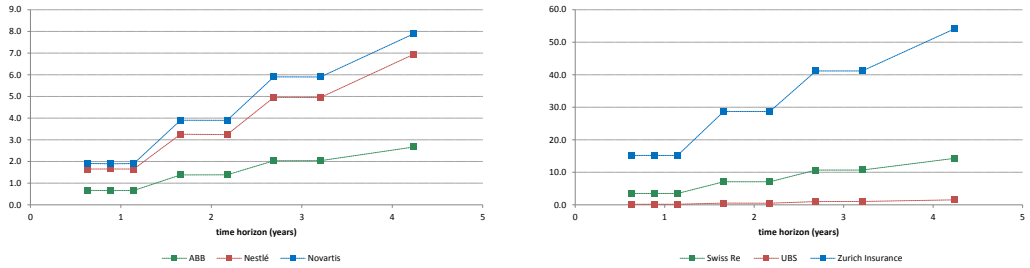
In the previous section we analyzed our estimate with German data. As we want to show, that the estimate is also performing well within other markets we now apply our methodology to Swiss and French blue-chips. We restrict attention to six stocks per country, where suitable derivatives data are available. *ABB, Nestlé, Novartis, Swiss Re, UBS and Zurich Insurance (short Zurich)* are the explored Swiss stocks and *AXA, BNP, Carrefour, Sanofi, Société Générale (abbreviated as SocGén) and Veolia* the French ones. All mentioned stocks pay their dividend once per year.

Remark 2.7 *Note that our method also works with payment periods less than one year. This changes the number of stairs within one year in the figures showing the present value $D^*(t,T)$ (e.g. Figures 2.4 to 2.7). Furthermore, some payments could be aggregated in one step, as for higher maturities there are less options available. So our method is limited with respect to the available maturities.*

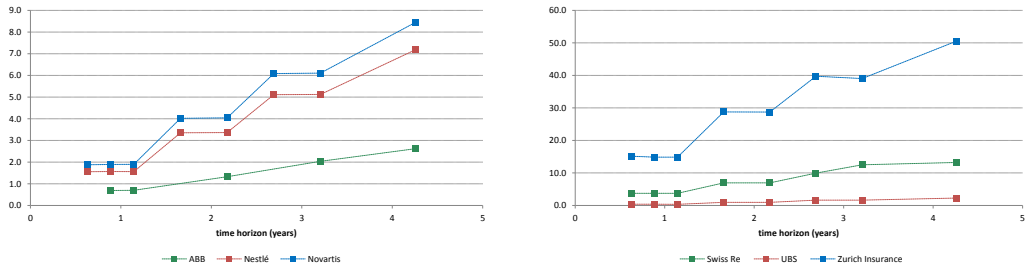
We show the results for the spot dates 2011-11-09 2012-11-14, 2013-11-13, 2014-11-12 and 2015-11-11, e.g. every second Wednesday in November from 2011 to 2015. The market data are again from Thomson Reuters' Datastream.



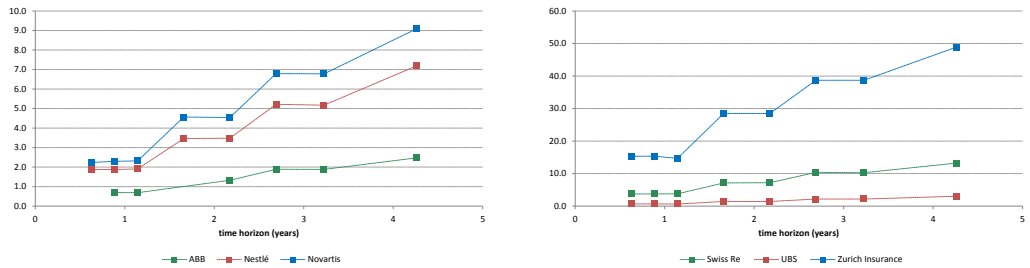
(a) 2012-11-14



(b) 2013-11-13

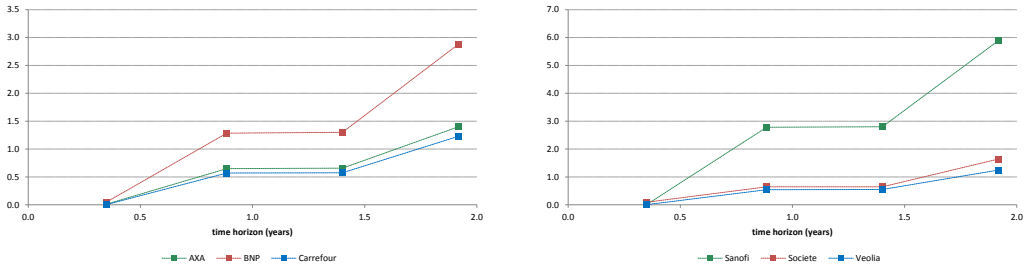


(c) 2014-11-12

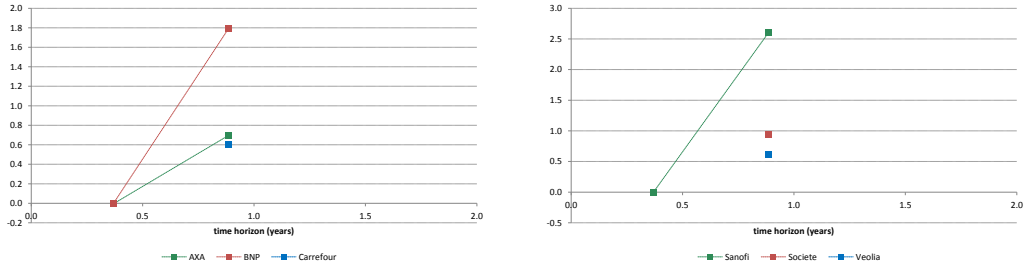


(d) 2015-11-11

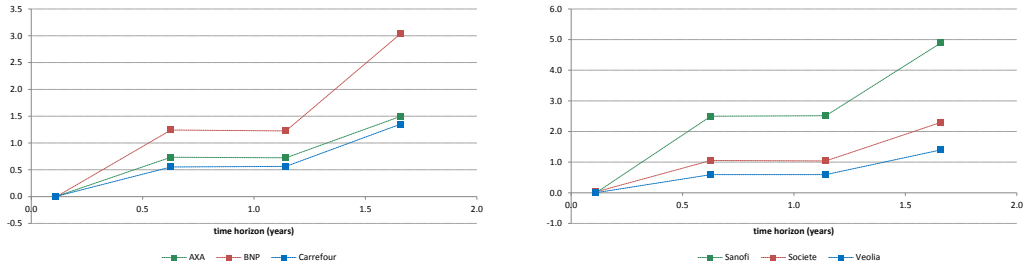
Figure 2.15: Present value of dividends $D^*(t,T)$ (Switzerland).



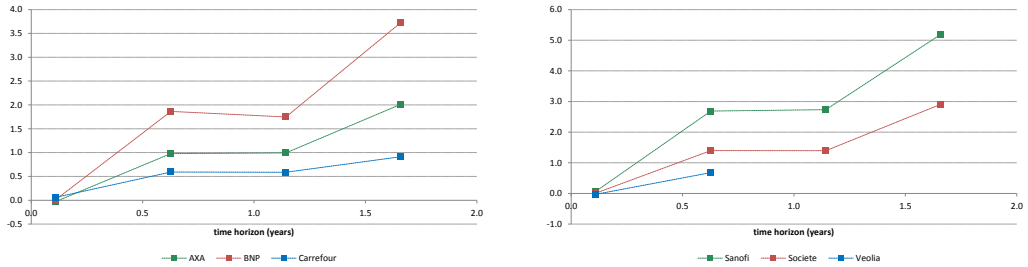
(a) 2012-11-14



(b) 2013-11-13



(c) 2014-11-12



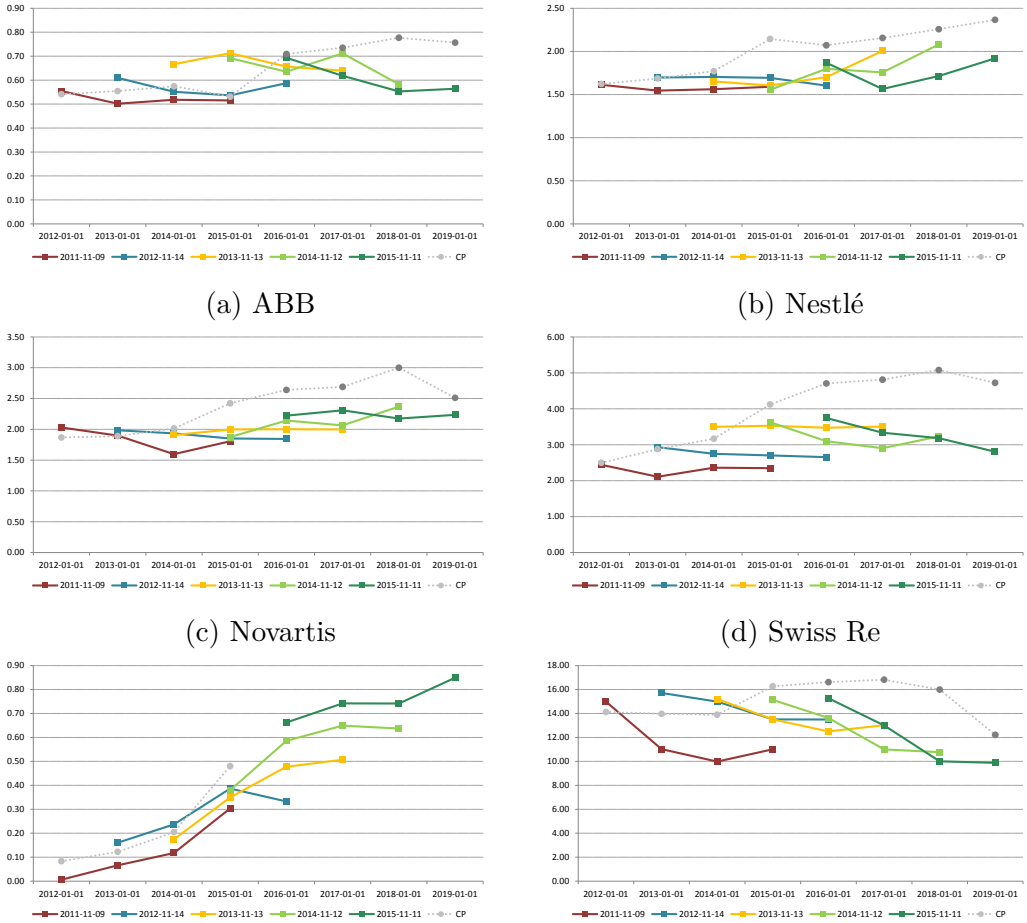
(d) 2015-11-11

Figure 2.16: Present value of dividends $D^*(t,T)$ (France).

Figure 2.15 and Figure 2.16 illustrate the estimate of the present value $D^*(t,T)$ as a function of T for the different spot dates. Observe that the times until maturity of the French derivatives are in general less than two years and for some spot dates even less than one year. Nevertheless, the estimate has the

same “staircase” structure as the German ones and hence fulfills the required properties (compare Subsection 2.4.2).

Additionally, Figure 2.17 and Figure 2.18 illustrate the estimated forward price $\mathbb{E}_t^{T_i}[D_i]$ benchmarked against the historical incurred values augmented with a commercial forecast. Note that the commercial forecast is queried at 2015-11-11, such that it should only be compared with the estimate on that spot date, e.g. the dark green line. For UBS the commercial estimate was not available. These results are in line with the results from the German stocks.

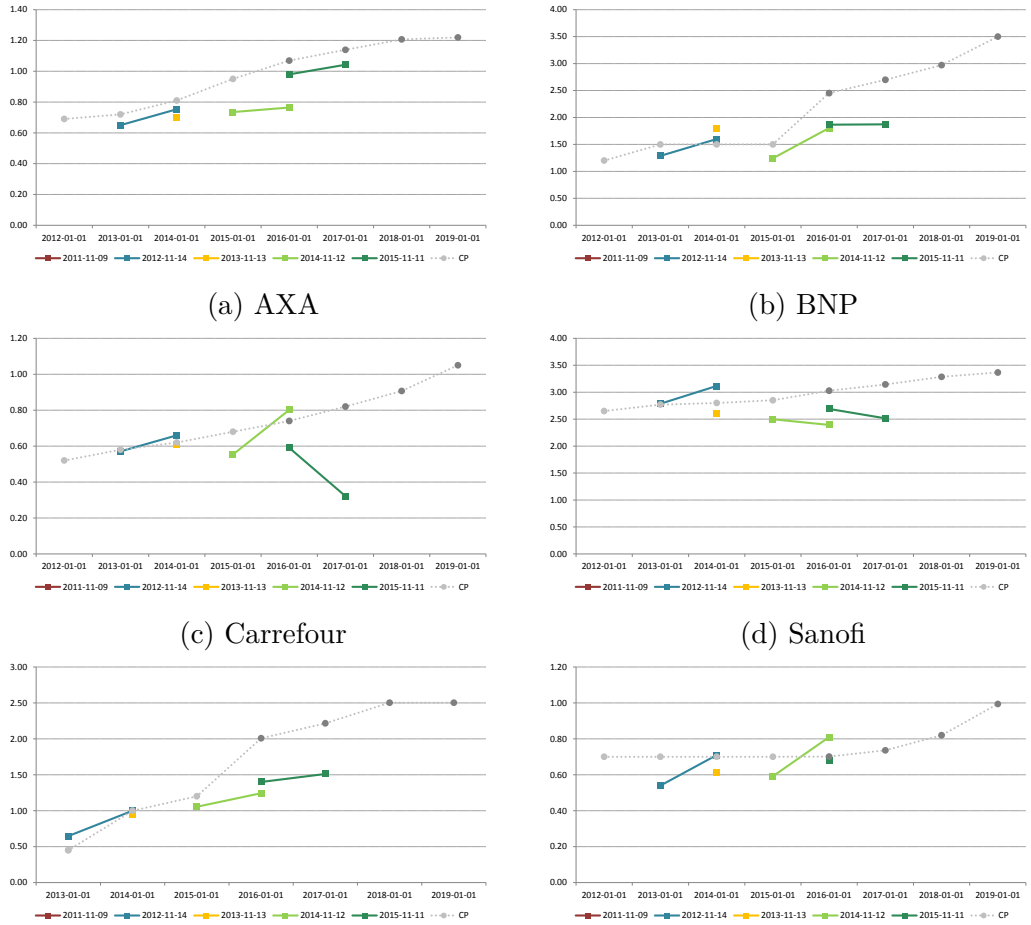


(a) ABB (b) Nestlé

(c) Novartis (d) Swiss Re

(e) UBS (f) Zurich Insurance

Figure 2.17: Dividend estimates at different spot dates benchmarked against historical dividends and commercial forecasts (Switzerland).



(a) AXA

(b) BNP

(c) Carrefour

(d) Sanofi

(e) Société Générale

(f) Veolia

Figure 2.18: Dividend estimates at different spot dates benchmarked against historical dividends and commercial forecasts (France).

Furthermore, we perform the same aggregate statistics as explained in Subsection 2.4.4. The analysis deals with all available data for every Wednesday between 2011-01-01 and 2013-12-31. The details for the Swiss dataset can be seen in Table 2.4 and for the French one in Table 2.5.

Data basis	**ABB**	**Nestlé**	**Novartis**	**Swiss Re**	**UBS**	**Zurich**
total number of put-call pairs	13'401	14'041	14'746	16'209	15'220	14'944
number of available spot dates	110	111	110	107	105	109
average number per spot date	121.83	126.50	134.05	151.49	144.95	137.10
min strike	7.2	36	32	26.53	5.6	100
max strike	40	100	120	140	36	480

Table 2.4: Database (Switzerland).

Data basis	AXA	BNP	Carrefour	Sanofi	SocGén	Veolia
total number of put-call pairs	2'472	2'939	2'511	2'401	3'726	2'453
number of available spot dates	75	65	64	75	75	65
average number per spot date	32.96	45.22	39.23	32.01	49.68	37.74
min strike	6	18	8	40	8	4.8
max strike	28	80	40	160	64	20

Table 2.5: Database (France).

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation	Estimate for Year	Empirical Volatility
ABB									
1 Year	109	8%	10%	10%	25%	0%	6%	2012	3%
2 Years	109	6%	15%	8%	36%	1%	12%	2013	3%
3 Years	77	1%	4%	2%	32%	0%	5%	2014	3%
4 Years	26	2%	8%	9%	18%	0%	5%	2015	3%
Total	321	5%	10%	7%	36%	0%	9%		2%
Nestlé									
1 Year	112	3%	4%	4%	10%	0%	2%	2012	3%
2 Years	110	5%	11%	8%	26%	2%	8%	2013	4%
3 Years	75	6%	18%	20%	26%	0%	5%	2014	4%
4 Years	25	6%	24%	24%	28%	18%	3%	2015	4%
Total	322	4%	11%	8%	28%	0%	9%		5%
Novartis									
1 Year	109	4%	5%	4%	13%	0%	3%	2012	8%
2 Years	110	6%	9%	6%	26%	0%	7%	2013	5%
3 Years	67	7%	21%	21%	34%	1%	7%	2014	7%
4 Years	17	6%	22%	22%	26%	13%	4%	2015	8%
Total	303	5%	11%	8%	34%	0%	9%		8%
Swiss Re									
1 Year	107	6%	8%	8%	20%	1%	4%	2012	10%
2 Years	106	7%	16%	15%	28%	0%	6%	2013	10%
3 Years	71	11%	35%	35%	46%	19%	6%	2014	7%
4 Years	21	11%	44%	45%	46%	43%	1%	2015	7%
Total	305	8%	20%	15%	46%	0%	13%		17%
UBS									
1 Year	104	15%	22%	20%	52%	0%	16%	2013	1%
2 Years	104	9%	20%	25%	53%	0%	14%	2014	2%
3 Years	73	10%	32%	26%	68%	1%	21%	2015	2%
4 Years	22	16%	67%	67%	82%	38%	8%	2016	3%
Total	303	12%	27%	25%	82%	0%	20%		3%
Zurich Insurance									
1 Year	108	8%	10%	10%	22%	2%	4%	2012	36%
2 Years	109	3%	7%	8%	21%	0%	4%	2013	50%
3 Years	73	5%	17%	17%	28%	6%	7%	2014	58%
4 Years	21	8%	28%	26%	32%	20%	4%	2015	36%
Total	311	6%	12%	9%	32%	0%	8%		49%

Table 2.6: Summary statistics (Switzerland).

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation	Estimate for Year	Empirical Volatility
AXA									
1 Year	76	8%	12%	12%	32%	0%	9%	2013	3%
2 Years	31	3%	7%	7%	21%	1%	4%	2014	2%
Total	107	7%	11%	9%	32%	0%	8%		2%
BNP									
1 Year	65	8%	12%	7%	40%	0%	11%	2013	5%
2 Years	33	3%	7%	7%	8%	5%	1%	2014	7%
Total	98	6%	10%	7%	40%	0%	10%		6%
Carrefour									
1 Year	65	6%	9%	9%	26%	0%	6%	2013	4%
2 Years	42	7%	9%	7%	32%	3%	7%	2014	3%
Total	107	7%	9%	8%	32%	0%	6%		3%
Sanofi									
1 Year	75	3%	4%	4%	13%	0%	3%	2013	5%
2 Years	32	5%	10%	11%	11%	6%	1%	2014	5%
Total	107	3%	6%	7%	13%	0%	4%		5%
Société Générale									
1 Year	75	9%	13%	6%	46%	0%	14%	2014	4%
2 Years	34	9%	21%	1%	56%	0%	26%	2015	9%
Total	109	9%	16%	5%	56%	0%	19%		7%
Veolia									
1 Year	68	11%	17%	20%	46%	0%	9%	2013	3%
2 Years	30	0%	1%	1%	3%	0%	1%	2014	9%
Total	98	8%	12%	14%	46%	0%	11%		7%

Table 2.7: Summary statistics (France).

Table 2.7 and 2.6 show the results for the aggregate statistics and Table 2.8 the corresponding R^2 values. The values are in the same range as the analogue amounts of the German result table (compare Table 2.2). One can again observe that the averages and medians are almost close together, where 86% of the values are less than or equal to 25%. Moreover, 86% of the weighted average values are less than or equal to 10%. Once again the R^2 values support the regression based methodology.

Switzerland	ABB	Nestlé	Novartis	Swiss Re	UBS	Zurich	Total
R^2	99.99991%	99.99998%	99.99997%	99.98306%	99.99986%	99.99993%	99.99717%
France	**AXA**	**BNP**	**Carrefour**	**Sanofi**	**SocGén**	**Veolia**	**Total**
R^2	99.99975%	99.99946%	99.99978%	99.99998%	99.80988%	99.99748%	99.96636%

Table 2.8: Average values of R^2 for Switzerland and France.

2.6 Conclusion

In this chapter we developed a regression based, no-arbitrage methodology for estimating dividends, which uses market data of vanilla call and put options. The advantages are:

- All available option data are integrated in the calculation. This is an improvement of the method developed in Grün (2014), as there we use the box spread method which relies on only two put-call pairs.
- The joint estimation of the dividends and the marked-implied discount curve at the same time, is also beneficial. Hence, we can estimate the size of the dividends and how the market evaluates it.
- Furthermore, it is model-free, simple to use and robust.

In practice our method performed well with European blue-chips. The estimates are in line with the actual incurred values with small moderate deviations: More than 90% of the weighted average deviations are less than or equal to 10%. Also, in an ideal setting we would expect deviations from the historical incurred dividends. Moreover, the R^2 values exceed 99.8%, which results from the usage of the put-call parity together with its no-arbitrage arguments and supports the regression based method.
Overall, the method is restricted by the available option data:

- For estimation periods with few data and/or long prediction intervals it can happen that an estimation is not possible.
- The method is limited by the existing maturities especially for payment periods which are smaller than one year.

Nevertheless, the estimate can be helpful as a benchmark for existing dividend estimates, or also as a stand-alone alternative to these solutions.

Chapter 3

Estimation of Outstanding Future Dividend Payments with American Options

In Chapter 2 we developed a methodology for estimating discrete dividend payments based on the put-call parity for European options. Not for every stock these options are available. Brooks (1994) showed that the put-call parity cannot be used for the estimation of dividends when using American type options, due to the fact that the early exercise premium of puts is priced systematically different than early exercise premiums of call options. Nevertheless, we examine, how we can transfer the results from using the put-call parity as before to the case of American options to develop an estimate. The essential parts of this chapter are summarized in Desmettre, Grün, and Korn (2017). This article is conditionally accepted by the journal *Quantitative Finance*.

In Section 3.1 we repeat the notations and give the new setting. With the American counterpart of the put-call parity we can estimate boundaries for the dividends in Section 3.2. Inside this section we give results of German data and visualize problems which occur with US underlyings. To handle these issues we extend our estimation methodology in Section 3.3 and analyze the corresponding results for data from constituents of the Dow Jones in Section 3.4. We make a robustness check with respect to the inserted discount factors in Section 3.5. In this section we furthermore, backtest our method against the one developed in Chapter 2 and against a simple method, which is commonly used by practitioners. Within the whole chapter we visualize the setting and the difference to the one of Chapter 2 for a better understanding.

3.1 General Framework and Put-Call Boundaries

The fundamentals do not differ substantially from the ones of Chapter 2, nevertheless we repeat some important notations and settings and describe the new framework.

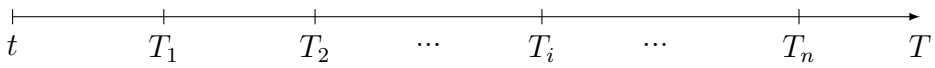


Figure 3.1: Time horizon with dividend payment days T_i.

Again we focus on the next n dividend payments, payable at discrete, known times $T_1 < T_2 < \cdots < T_n \leq T$. Figure 3.1 illustrates the time horizon including the dividend payment days with $t < T_1$. The figure will be extended with further time points and definitions at a later stage. In addition, Table 3.1 repeats the important notations from before.

Notation	**Explanation**
$S(t)$	time-t price of a stock
$p(t,T)$	time-t discount factor for cash flows at time T
$C(t)$	time-t price of a European call option
$P(t)$	time-t price of a European put option
D_i	dividend payment, payable at time T_i
$D(t,T)$	time-t present value of expected future dividend payments up to time T

Table 3.1: Repetition of the notations.

As we now focus on American style options, we denote the price of an American call (put) option with underlying S, strike $K > 0$, and time horizon T with an A as upper index, e.g. $C^A(t)$ ($P^A(t)$).

In this chapter we need an additional assumption, which is relevant in the proof of the forthcoming important Theorem 3.1:

Assumption 3.1 *The time-t discount factor satisfies:* $1 > p(t,T) > 0$ *for* $t < T$.

Remark 3.1 (Negative Interest Rate) *In general, Assumption 3.1 is valid due to no-arbitrage given there is a riskless investment opportunity with non-negative interest rates. But nowadays a negative interest rate can really be observed from time to time, e.g. $p(t,T)$ can be greater than one. Hence, in practice, while applying the theory to data, we need to be careful as the results of this chapter do not need to hold. However, in all our examples Assumption 3.1 is fulfilled.*

For now we have the basic notations and settings to start with the development of an estimate which uses American style options. In Chapter 2 we used the put-call parity as estimation basis. Now, the question is, *how does the analog approach for American options look like?*

If the stock does not pay dividends the American call option and the European call option are the same. Nevertheless, the "put-call parity" for American options differs from the traditional one (see e.g. Cox and Rubinstein (1985)):

$$C^A(t) + Kp(t,T) \leq P^A(t) + S(t) \leq C^A(t) + K\,.$$

So it gives no longer a parity, but arbitrage boundaries. In the setting with the stock paying discrete dividends the American call does not coincide with the European one. The following remark provides more details.

Remark 3.2

(i) In case of a dividend paying stock it can be profitable to early exercise the American call option. Therefore, only a date $t^* < T_i$ immediately before an ex-dividend date can be considered:
If the option holder early exercise the call, he receives the stock in exchange for the strike ($S(t^*) - K$) and then collects the dividend in T_i. Hence, at T_i his portfolio value is equal to

$$S(T_i) - K + D_i = S(t^*) - D_i - K + D_i = S(t^*) - K\,,$$

where the first equality comes from Assumption 1.1. If he instead does not exercise the American call option the stock jumps down by the amount of the dividend (again due to Assumption 1.1) and his portfolio value is equal to $C^A(T_i) = C^A_{S(t^*)-D_i}(T_i)$.[8] So in total we have that

$$C^A(t^*) = \max\left[S(t^*) - K, C^A_{S(t^*)-D_i}(t^*)\right]\,.$$

[8] This notation displays that the underlying is equal to $S(t^*) - D_i$.

Let S^* be the stock price with $S^* - K = C^A_{S^*-D_i}(t^*)$, then for all stock prices $S(t^*) \geq S^*$ it is profitable to early exercise the American call option (compare Cox and Rubinstein (1985)).

(ii) Roll (1977) use that specific S^*, which separates the exercise and non-exercise regions, to develop Roll's Formula for pricing American call options (for more details see Remark 1.3).

(iii) It is not optimal to early exercise the American call option in $t^* < T_i$ if

$$D_i \leq K[1 - e^{-r(T_{i+1}-T_i)}], \tag{3.1}$$

where r is the riskless interest rate (see e.g. Hull (2012)). Furthermore, Relation (3.1) is approximately equal $D_i \leq Kr(T_{i+1} - T_i)$, such that the call is not exercised early when the dividend yield is less than the riskless rate. In the past this was often the case for which reason in a lot of articles it still was assumed that the American and European call option are the same. Now, the riskless interest rate is quite small and the dividend amounts are higher so it is often profitable to early exercise the American call option.

Theorem 3.1 shows the relation between the American call and put in the event of a dividend paying stock, i.e. we have at least one $T_i \in [t, T]$.

Theorem 3.1 (Put-Call Relation/Arbitrage Bounds with Dividends) *Under the assumption of no-arbitrage it holds:*

$$C^A(t) + Kp(t,T) \leq P^A(t) + S(t) \leq C^A(t) + K + D(t,T)\,. \tag{3.2}$$

Proof. We separate the proof into two parts one for each inequality and again argue by contradiction:

1. $\mathbf{C^A(t) + Kp(t,T) \leq P^A(t) + S(t)}$:
Suppose that $C^A(t) + Kp(t,T) > P^A(t) + S(t)$. Now, consider the following trading strategy:

Time t:

- buy the American put option P^A and the stock S,
- sell the American call option C^A and borrow the amount $Kp(t,T)$.

From this strategy we get the position $P^A(t) + S(t) - C^A(t) - Kp(t,T)$ with time-t cash flow greater than zero. If the holder of the call option does not

exercise it before a dividend payment day T_i we receive the dividend D_i (as we hold the stock) and cash it in.

Early exercise of the call option in t^*:
If the call options is exercised early only a time point t^* immediate before a dividend payment day is possible. The option holder gets the stock in return to K and the position changes to:

$$\begin{aligned} &P^A(t^*) + S(t^*) - S(t^*) + K - Kp(t^*, T) + \sum_{i:\, t<T_i<t^*} D_i \frac{1}{p(T_i, T)} \\ &\quad = P^A(t^*) + K[1 - p(t^*, T)] + \sum_{i:\, t<T_i<t^*} D_i \frac{1}{p(T_i, T)} > 0\,, \end{aligned}$$

where this is true due to Assumption 3.1 and $P^A(t^*) > 0$. Also, if the last summand is equal to zeros, i.e. $t^* < T_1$, still the sum is greater than zero.

Time T:
If the call option was not exercised early we have the following position:

$$\begin{aligned} &P^A(T) + S(T) - C^A(T) - K + \sum_{i:\, t<T_i\leq T} D_i \frac{1}{p(T_i, T)} \\ &= P(T) + S(T) - C(T) - K + \sum_{i:\, t<T_i\leq T} D_i \frac{1}{p(T_i, T)} \\ &= \sum_{i:\, t<T_i\leq T} D_i \frac{1}{p(T_i, T)} > 0\,, \end{aligned}$$

where we used the traditional put-call parity for the last equation. As this is a riskless gain, the constructed strategy is an arbitrage opportunity.

2. $\mathbf{P^A(t) + S(t) \leq C^A(t) + K + D(t, T)}$:
Suppose that $P^A(t) + S(t) > C^A(t) + K + D(t, T)$. Then, with the following trading strategy we can construct an arbitrage opportunity:

Time t:

- sell the American put option P^A and the stock S,
- buy the American call option C^A and cash in K,
- take a long position in the T_i- -forward on S and a short position in the T_i -forward for every $t < T_i \leq T$. Furthermore, cash in the amount $p(t, T_i)[F_{T_i-} - F_{T_i}]$ until T_i for every dividend date.

Again from Assumption 1.1 it follows that the total amount cashed in is $\sum_{t<T_i\leq T} p(t,T_i)[F_{T_i-}-F_{T_i}] = D(t,T)$ (compare with the proof of Theorem 2.1). So by this strategy we get the position $-P^A(t) - S(t) + C^A(t) + K + D(t,T)$ with time-t cash flow greater than zero. Thereby we again can monitor the different timepoints:

Time T_i:
The stock distributes dividends which we have to pay to the buyer of the stock. Furthermore, two of the forward contracts need to be settled and we get the payback of the cashed in money:

$$\begin{aligned} &- D_i - \underbrace{F_{T_i-} + S(T_i-)}_{\text{long position}} + \underbrace{F_{T_i} - S(T_i)}_{\text{short position}} + \underbrace{[F_{T_i-} - F_{T_i}]}_{\text{payback}} \\ &= -D_i + [S(T_i-) - S(T_i)] = 0\,, \end{aligned}$$

where the last equation follows by Assumption 1.1. Hence, the total cash flow is equal to 0.

Early exercise in t^*:
If the holder of the put option exercises it before the maturity at time $t < t^* < T$ he gets the amount K in exchange for the stock. We can distinguish between two cases: If $t^* \neq T_i\ \forall i$ then the value of the portfolio is equal:

$$\begin{aligned} &[S(t^*) - K] - S(t^*) + C^A(t^*) + K\frac{1}{p(t,t^*)} + D(t^*,T) \\ &= C^A(t^*) + K\left[\frac{1}{p(t,t^*)} - 1\right] + D(t^*,T) > 0\,, \end{aligned}$$

where the inequality holds because $C^A(t^*) \geq 0$, $\frac{1}{p(t,t^*)} > 1$ by Assumption 3.1 and $D(t^*,T) \geq 0$ by its definition. If $t^* = T_i$ for one i then there is an additional cashflow as explained before. As it is equal to zero the value of the portfolio does not change. So the strategy gives an arbitrage opportunity due to the riskless amount of money we gain.

Time T:
If $T = T_n$ we can again compensate the payback of the amount cashed in and the dividend payment with the difference in the forward positions. Otherwise, we have that the forward positions are already 0. As the put was not exercised early we obtain in both cases

$$-P^A(T) - S(T) + C^A(T) + K\frac{1}{p(t,T)} = K\left[\frac{1}{p(t,T)} - 1\right] > 0\,,$$

via using the traditional put-call parity. This is feasible as at maturity $C^A(T) = C(T)$ and $P^A(T) = P(T)$. Hence, we retrieve a riskless gain and thus the constructed trading strategy is an arbitrage opportunity. □

Summing up the answer to the question about an analogue approach for American options is Theorem 3.1. Now, it needs to be clarified if *this theorem can be used for the estimation of dividends*. The following section deals with this question.

3.2 Estimation of Dividend Boundaries

In this section we want to investigate the applicability of Theorem 3.1 to estimate dividends with American options. Note that we do not have a parity anymore, so it is only possible to derive boundaries with an estimation approach which uses the put-call relation as a basis. Hence, out of inequality (3.2) and the fact that $D(t,T) \geq 0$ by definition, it results:

$$D(t,T) \geq D_l(t,T) \triangleq \max(P^A(t) - C^A(t) + S(t) - K, 0)\,, \tag{3.3}$$

where the subscript l is chosen to indicate a lower boundary. In this case fortunately the discount factor is not needed for the calculation.
As we have a lower bound, we also would like to have an upper bound. A straightforward boundary is the stock price $S(t)$ itself, but this is not a very tight one. Another simple idea to obtain a bound is as follows: We use the relation between a European and an American put option; e.g. $P(t) \leq P^A(t)$ and use the put-call parity (2.3) as follows:

$$\begin{aligned} D(t,T) &\overset{(2.3)}{=} P(t) - C(t) + S(t) - Kp(t,T) \\ &\leq P^A(t) - C(t) + S(t) - Kp(t,T) \\ &\leq P^A(t) \qquad\quad + S(t) - Kp(t,T)\,. \end{aligned}$$

If $P^A(t) < Kp(t,T)$ this bound is tighter so we have

$$D_u(t,T) \triangleq \min\left(S(t), P^A(t) + S(t) - Kp(t,T)\right) \geq D(t,T)\,, \tag{3.4}$$

where the subscript u is chosen to indicate an upper boundary. Again observe that the maturities of the option need not coincide with the payment days. As already explained after Corollary 2.3 it is adequate to examine time points $\tilde{T}_i$ with $T_i \leq \tilde{T}_i < T_{i+1}$ for the estimation of $D(t,T_i)$. Furthermore, note that there are more than one lower and one upper bound, as both bounds are valid for every strike price K. We solve this problem via choosing the maximum of

the lower bounds, denoted by $D_l^*(t, T_i)$ and the minimum of the upper bounds, denoted by $D_u^*(t, T_i)$.[9] Figure 3.2 illustrates this setting, where the light green line exemplifies the definition of $D(t, T)$, the red one the upper bound and the blue one the lower bound.

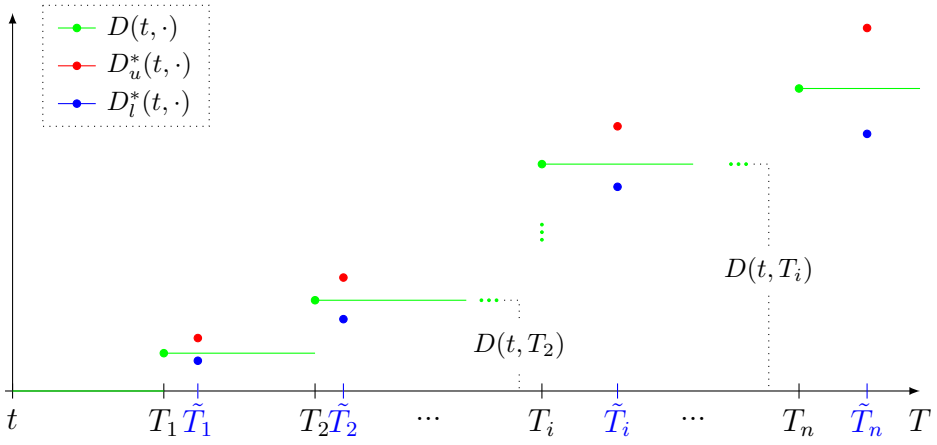


Figure 3.2: Visualization of $D(t, \cdot)$ and the corresponding lower and upper bounds.

In Chapter 2 we also estimated the time-t forward price $\mathbb{E}_t^{T_i}[D_i]$ of an individual dividend payment. In the setting with American options we have boundaries for $D(t, T)$, such that we can only receive boundaries for an individual dividend payment. Remark 3.3 gives the details, but as these boundaries worsen, we focus on estimating the present value of all dividends until a specific date.

Remark 3.3 (Boundaries for an Individual Dividend Payment) *Note that just taking the difference from the two upper bounds of $D(t, T_i)$ and $D(t, T_{i-1})$ and the lower ones respectively, does not work. The following inequalities show the correct lower bound:*

$$\begin{aligned} p(t, T_i)\mathbb{E}_t^{T_i}[D_i] &= D(t, T_i) - D(t, T_{i-1}) \\ &\geq D_l^*(t, T_i) - D(t, T_{i-1}) \geq D_l^*(t, T_i) - D_u^*(t, T_{i-1})\,. \end{aligned}$$

Analogue the upper bound is equal:

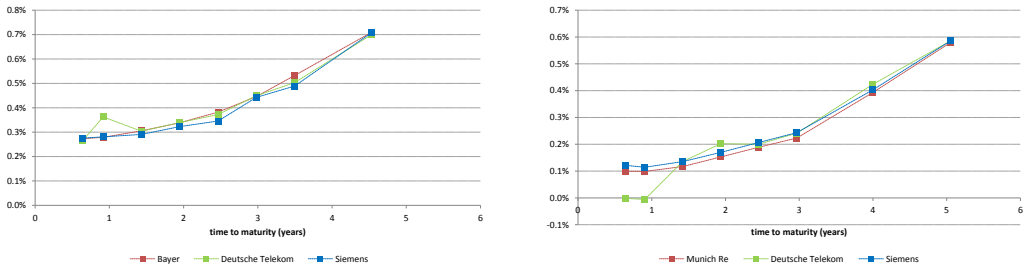
$$\begin{aligned} p(t, T_i)\mathbb{E}_t^{T_i}[D_i] &= D(t, T_i) - D(t, T_{i-1}) \\ &\leq D_u^*(t, T_i) - D(t, T_{i-1}) \leq D_u^*(t, T_i) - D_l^*(t, T_{i-1})\,. \end{aligned}$$

[9] $D_l^*(t, T_i)$ and $D_u^*(t, T_i)$ are not independent from $\tilde{T}_i$, but for the sake of clarity we waive an additional index. In cases where we have more than one $\tilde{T}_l$ with $T_i \leq \tilde{T}_l < T_{i+1}$ and it makes a difference, we explain how to deal with it.

3.2.1 Results for German Underlyings

In this section we visualize the method and results based on the boundaries (3.3) and (3.4). Therefore, we examine three examples with German stocks as underlying at two dates of request 2012-08-03 and 2014-02-05.[10] We select *Bayer, Deutsche Bank* and *Siemens* from the examples in Chapter 2, such that we can compare the resulting values with the European dividend estimates.

In (3.3) we fortunately do not need the discount factor $p(t,T)$, but for the upper bound it is important. For these once we, therefore, use the discount factors, which we estimated within Section 2.4 in these examples. As mentioned in Remark 3.1 we need to be careful with the discount factors. Nevertheless, in our examples we only need to take care of two values, which are slightly negative. Therefore, Figure 3.3 shows the zero yields of the market-implied discount curves which resulted from the estimate in Chapter 2.



(a) 2012-08-03 (b) 2014-02-05

Figure 3.3: Market-implied zero yield curves.

Figure 3.4 displays the lower bounds (3.3) (in blue) and the upper bounds (3.4) (in red) for Bayer, where the date of request t is the 2014-02-05. Note that the bounds depend on the maturity and additionally on the strike price which reflects in the multiple occurrence of the strike prices at the x-axis. Therefore, we also illustrate the same data in Figure 3.5, a 3D plot with the x-axis displaying the strike price K and the y-axis equal the time to maturity $T-t$. One might think that the 3D plot is sufficient to show the bounds, but we also show Figure 3.4, as there the curves do not overlay each other resulting in a more clear representation.

[10] As we will improve the method in Section 3.3 we now only focus on a small data set with two request dates for the sake of illustration. In the results section (see 3.4) we execute a bigger data set with ca. 155 dates of request.

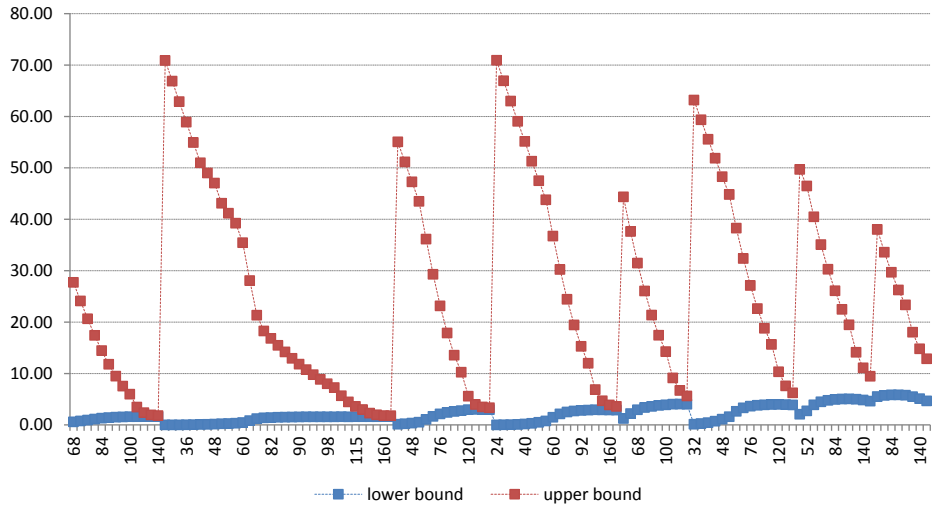


Figure 3.4: Lower and upper bounds for $D(t,T)$ with t equal 2014-02-05 and underlying Bayer dependent on the strike and maturity (reflected in the multiple occurrence of the strike prices at the x-axis).

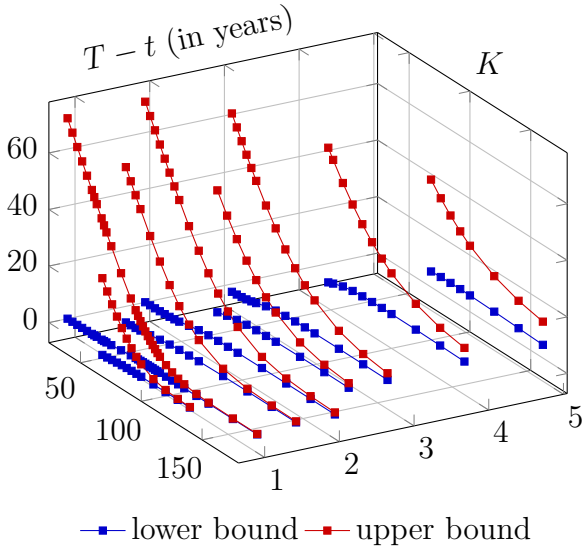
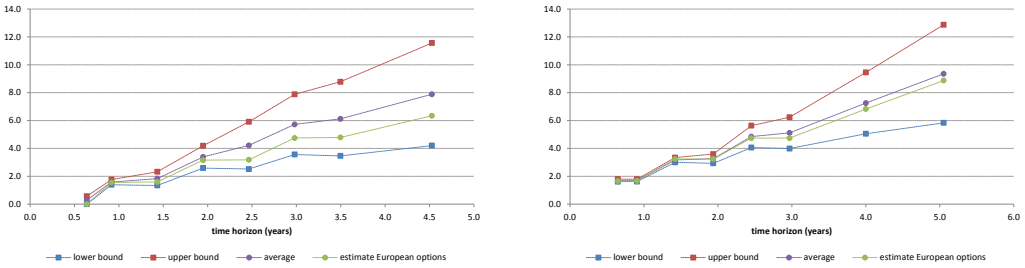


Figure 3.5: 3D Plot of the lower and upper bounds for $D(t,T)$ with t equal 2014-02-05 and underlying Bayer dependent on the strike and the time until maturity.

Furthermore, it is easier to understand the next steps and the resulting figure for the visualization of $D_l^*(t,T_i)$ and $D_u^*(t,T_i)$. For a fixed maturity in both

figures the lower and upper bounds become closer for bigger strike prices. In addition, it is observable, that for bigger maturities less option data are available and hence the lower and upper bounds are not so close anymore.

The next step is to select the maximum lower bound $D_l^*(t, T_i)$ and the minimum upper bound $D_u^*(t, T_i)$ as explained at the beginning of Section 3.2. In the actual example this results in Figure 3.6b. Furthermore, we include the European options estimate from Section 2.4.2 and the average between the two boundaries. For all times until maturity the European option estimate is between the two boundaries which is true due to no-arbitrage. In that example



(a) 2012-08-03 (b) 2014-02-05

Figure 3.6: Lower and upper bound for $D(t, T)$ compared with the European option estimate (underlying Bayer).

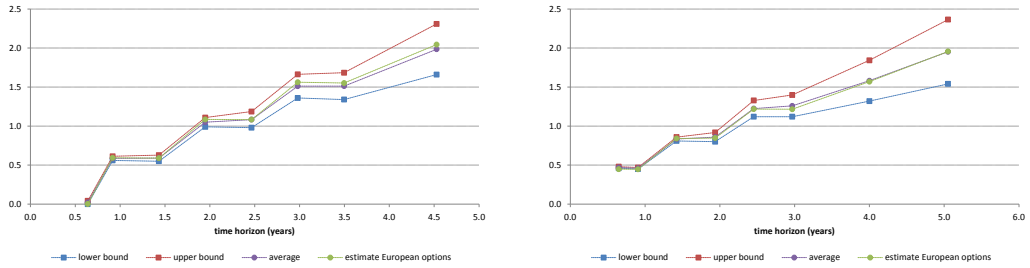
the average between the lower and upper bound is close to the European estimate. For a time to maturity of $T - t$ less than 2 years all values are close together. The space between the two bounds gets bigger depending on the time until maturity. The reason for this is explained in Remark 3.4:

Remark 3.4 *The difference between the upper and lower bound is*

$$\begin{aligned} D_u^*(t, T_i) - D_l^*(t, T_i) &= P^A(t) + S(t) - Kp(t, T_i) - (P^A(t) - C^A(t) + S(t) - K) \\ &= C^A(t) + K(1 - p(t, T_i)), \end{aligned} \tag{3.5}$$

where we suppose for the sake of illustration that the upper and lower bound are reached for the same strike price, $P^A(t) + S(t) - Kp(t, T_i) \leq S(t)$ *and* $P^A(t) - C^A(t) + S(t) - K \geq 0$. *In Figure 3.4 we can see, that for every maturity both* $D_u^*(t, T_i)$ *and* $D_l^*(t, T_i)$ *are reached for big strike prices. Hence, the American call option is out of the money and its market price is low. So Equation* (3.5) *is driven by the second summand: For every maturity these strike prices do not differ a lot from each other, but for a longer time until maturity* $p(t, T_i)$ *gets smaller such that the second summand gets bigger.*

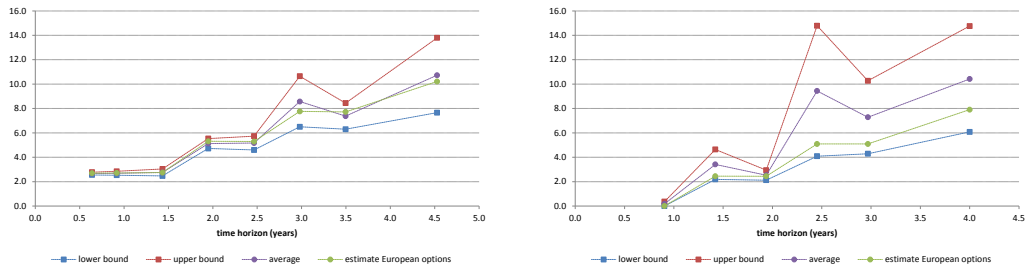
Figure 3.6a shows the resulting bounds for Bayer on the request date 2012-08-03. The same figures for underlying Deutsche Telekom and Siemens are displayed in Figures 3.7 and 3.8, respectively.



(a) 2012-08-03 (b) 2014-02-05

Figure 3.7: Lower and upper bound for $D(t,T)$ compared with the European option estimate (underlying Deutsche Telekom).

All these figures show almost the same behavior of the estimates. In the examples with Siemens as underlying we can observe a huge peak in the red line (upper bound). This is due to the fact that for this fixed maturity the available strike prices are smaller resulting in bigger American put market prices, compared to the ones of the other (close) maturities. Overall, for small $T-t$ this method is applicable in practice. A disadvantage, especially for bigger $T-t$, is the dependence on the available options with $K >> S(t)$.



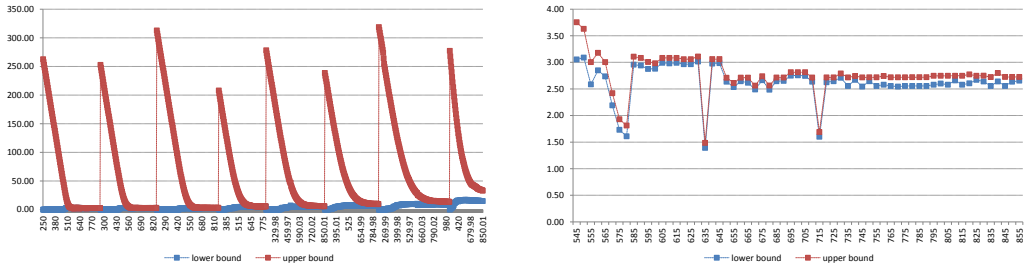
(a) 2012-08-03 (b) 2014-02-05

Figure 3.8: Lower and upper bound for $D(t,T)$ compared with the European option estimate (underlying Siemens).

3.2.2 Problems with US Underlying

In this section we perform the same method with *Apple* as underlying asset.[11] We do not have an algorithm for calculating the discount factor with American options, hence we take data from the USD-LIBOR rates and forward rate agreements (FRA). For discount factors which have a different time to maturity as the ones calculated via LIBOR and FRA, we use linear interpolation.[12]

Note that for the request date 2014-02-05 all bond prices fulfill Assumption 3.1. With this data we can only calculate $p(t,T)$ with $T-t$ smaller or equal than two years. This is enough because for US underlyings the option maturities are only two years in the future. Furthermore, US assets often pay dividends every third month, such that we have lot of payment days within this timeline. Figure 3.9a displays the results for 2014-02-05 as date of request. The structure is the same as the one of the results of the Bayer example. It is noticeable, that there are a lot more data points i.e. there are more American options available for Apple. At first glance the results make a good impression and the



(a) Full picture.[13] (b) Extract for maturity 2014-02-22.

Figure 3.9: Lower and upper bounds for $D(t,T)$ and zoom inside for more details (underlying Apple and date of request 2014-02-05).

spaces between the bounds seem to be very small. Unfortunately, this is a deceptive impression. Therefore, Figure 3.9b shows an extract from Figure 3.9a for maturity 2014-02-22 and strike prices, which are greater than the stock

[11] We also performed the method with other US underlyings with the same problems. We only show the results for Apple as we improve the method in Section 3.3 and then perform the new method with all stocks from the Dow Jones.

[12] We shortly summarize the relation beween the LIBOR/FRA and the discount factor in Subsection 3.4.1.

[13] There was a 7-1 stock split at 2014-06-09, hence we needed to adjust the strike prices for maturities which were after that date. This is the reason for the strike prices with two decimal places.

price (US$ 512.59). So it displays the region where we normally select $D_l^*(t, T_i)$ and $D_u^*(t, T_i)$. In this extract we can see that the boundaries are not monotone, instead they have a stagged structure and some outliers are noticeable. If we now choose $D_l^*(t, T_i)$ and $D_u^*(t, T_i)$, the upper bound is smaller than the lower bound, which violates the no-arbitrage condition.

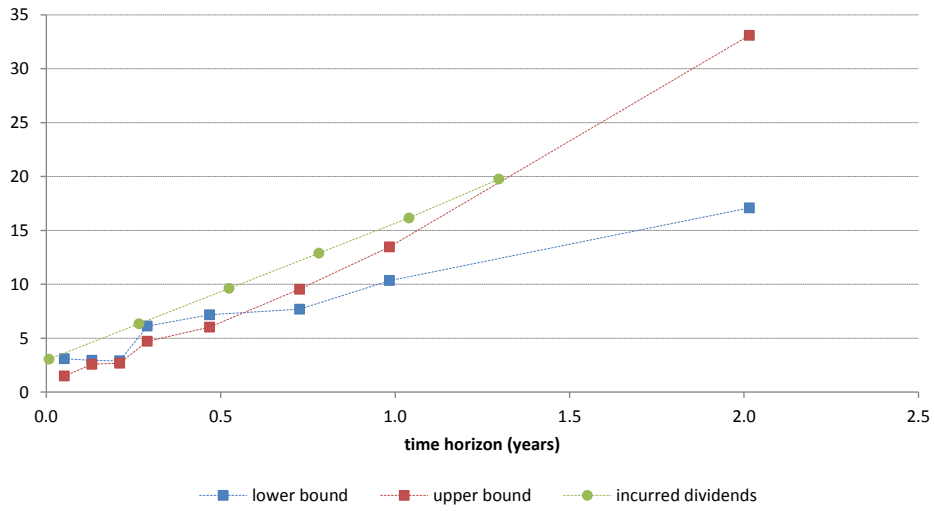
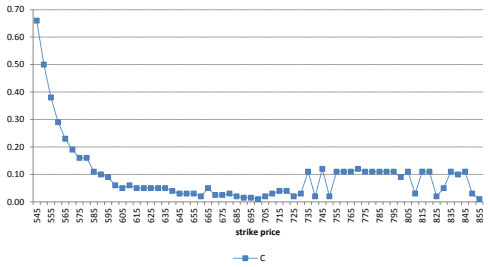
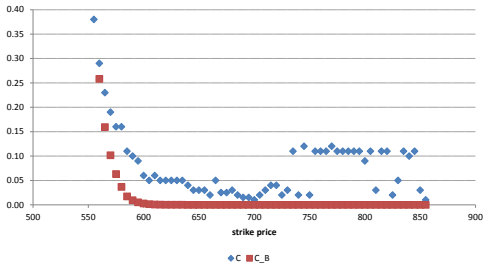


Figure 3.10: $D_l^*(t, \cdot)$ and $D_u^*(t, \cdot)$ compared with the actual incurred $D(t, T)$ (underlying Apple and date of request 2014-02-05).

Figure 3.10 illustrates the resulting values for the whole dataset. Note that the green line now displays the actual incurred $D(t, T)$, as for US underlyings we do not have the European estimate. The round markers of the green line show the starting point of the "stairs" (compare with the green line in Figure 3.2), such that we need to compare these values with the lower and upper bound which come next on the time line. For the first five maturities we have the case, that the upper bound is underneath the lower one, hence a violation of no-arbitrage. Now, the question is *where does that come from?* In order to get to the bottom of this, we have a closer look inside the data-set. Therefore, Figure 3.11a visualizes the price of the American call option as a function of the strike price (again greater than the stock price), where the maturity is equal to 2014-02-22. This figure shows that there is an issue with the market data, because the function of the call option price is not strictly decreasing, especially if it is deep out of the money. Normally, out of the money options are not often traded, hence the price is fault-prone. Furthermore, it can happen, that the put and call prices and/or the stock price do not match. Remark 3.5 explains how we try to deal with this data issues.



(a) Market data of American call option.



(b) Fit to Black-Scholes option price.

Figure 3.11: Illustration of American call option data as a function of big strike prices and fit to a Black-Scholes price (underlying Apple with date of request 2014-02-22).

Remark 3.5 *To handle the flawed data problem we fit the data to Black-Scholes prices. Therefore, we need to estimate the volatility σ and the riskless rate r from the Black-Scholes formula. Within we estimate $S_{j,i}$, which denotes the stock price belonging to the put-call parity with strike K_j and Maturity $\tilde{T}_i$ (in total $N+2$ parameters). In this setting we have an under-determined system of equations, hence we fit put and call options at the same time to have $2N$ data points. Unfortunately, this does not work well: Figure 3.11b shows the result for the American call option in Figure 3.11a, where the red line displays the fitted curve (named C_B). The problem now is, that the price is just set to zero. This is due to fitting puts and calls at the same time: the put option is deep in the money and its price is a lot bigger compared with the call option which is in the 10 cents region. Hence, setting the call option to zero only gives a small summand in the error amount function we are minimizing. So taking the moneyness as a weight inside this minimizing function does also not help to deal with that problem.*

Remark 3.6 summarizes the problems which can occur with the actually developed method. Hence, we will work on this in the next section to improve it.

Remark 3.6 (Problems with the Method) *In order to get close boundaries we need to work with call options which have a high strike price $K >> S(t)$. Hence, these options are not traded and their market prices are often flawed such that they are not in line with no-arbitrage. Additional the upper and lower bound only rely on one put-call pair respectively such that outliers have a big influence to the resulting value.*

3.3 Estimation of Outstanding Dividends

Beforehand, we highlighted some problems which occur with our method, especially for US underlyings. In this section we improve the method such that we do not have only boundaries. Moreover, we focus on market data from at-the-money (ATM) options, such that we avoid taking fault-prone data. Therefore, we first develop a method in Subsection 3.3.1, which is an intuitive progress to enhance the present method. Then, we generalize it and introduce a least squares method in 3.3.2. In the course of the chapter we abridge that method, such that it has a smaller data basis and thus is faster.

3.3.1 An Intuitive Method

Apart from the data issues, explained in Remark 3.6 the approach to calculate lower and upper bounds for the incurred value by (3.3) and (3.4) has two general drawbacks:

- For each available strike price K we have one lower and one upper bound, and it is not always obvious which ones to pick.
- For real world applications we are interested in one specific value as approximation for the incurred outstanding dividends.

The main idea to improve the present method is easy: Motivated by Figures 3.6, 3.7 and 3.8, where the European estimate is close to the average between the upper and lower bound, we suppose that there is a linear combination of a special lower and upper bound which is close to the real value. Now, the question is *how do we get the special lower and upper bound?* The answer is again easy: as it is standard, we base our calculations on ATM options. In the following we explain this method in more detail:
The first step is to calculate the lower bound (3.3) and the upper bound (3.4) with ATM options (the notation is adjusted with an "ATM" as an index) for maturity $\tilde{T}_i$ with $T_i \leq \tilde{T}_i < T_{i+1}$, i. e.

$$D_l^{ATM}(t,\tilde{T}_i) \triangleq \max(P_{ATM}^A(t) - C_{ATM}^A(t) + S(t) - K_{ATM}, 0)\,, \tag{3.6}$$

$$D_u^{ATM}(t,\tilde{T}_i) \triangleq \min\left(S(t), P_{ATM}^A(t) + S(t) - K_{ATM}p(t,\tilde{T}_i)\right). \tag{3.7}$$

As a next step we calculate the estimator as a linear combination of these two boundaries

$$D^*(t,T_i) \triangleq \lambda D_u^{ATM}(t,\tilde{T}_i) + (1-\lambda)D_l^{ATM}(t,\tilde{T}_i)\,.$$

Now, we need to know *where we get λ from.* The short answer is from historical data and also Equations (3.6) and (3.7). Therefore, we first expand our

setting: For every date of request t we need historical dates ($< t$) for the estimation of λ, which are named t_j with $j = 1, 2, ..., J$. This is illustrated in Figure 3.12. It is important that we have enough t_j to get a reliable estimator for λ. In Section 3.4.1 we give more details about the time horizon, the frequency between the dates of request and these historical dates.

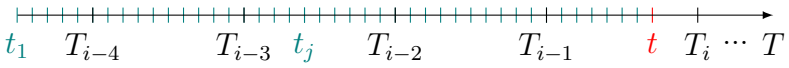


Figure 3.12: Time horizon with historical dates.

Let $\lambda_{m,j}$ be the factor of the linear combination of the lower and upper bound for the dividends from t_j up to time T_m, such that for every t_j we have

$$D(t_j, T_m) = \lambda_{m,j} D_u^{ATM}(t_j, \tilde{T}_m) + (1 - \lambda_{m,j}) D_l^{ATM}(t_j, \tilde{T}_m)\,,$$

which results in an estimate for $\lambda_{m,j}$

$$\lambda^*_{m,j} = \frac{D(t_j, T_m) - D_l^{ATM}(t_j, \tilde{T}_m)}{D_u^{ATM}(t_j, \tilde{T}_m) - D_l^{ATM}(t_j, \tilde{T}_m)}\,.^{14}$$

After calculating $\lambda^*_{m,j}$ for every historical date and every dividend payment date we estimate λ as the arithmetic average

$$\lambda^* = \frac{1}{N} \sum_{j=1}^{J} \sum_{m=1}^{i-1} \lambda^*_{m,j}\,,$$

where N is the number of available[15] $\lambda^*_{m,j}$. Overall, the resulting dividend estimate now reads as follows

$$D^*(t, T_i) \triangleq \lambda^* D_u^{ATM}(t, \tilde{T}_i) + (1 - \lambda^*) D_l^{ATM}(t, \tilde{T}_i)\,.$$

In Remark 3.6 we explained problems which occurred with the method of Section 3.2. In the following we shortly present the improvements of the method in this section.

[14] If there are more $\tilde{T}_l$ with $T_m \leq \tilde{T}_l < T_{m+1}$, which can be used for the calculation of the lower and upper bound, we use the average with respect to l over all $\lambda_{m,j,l}$, where the l in the lower index indicates the usage of $\tilde{T}_l$.

[15] On the request date t_j we can not calculate $\lambda^*_{m,j}$ for all m as some dividend payment dates are before t_j and the lower and upper bounds can only be calculated for a time horizon up to 2 years.

Remark 3.7 (Improvements) *The main advantage of the intuitive method is that we have a single estimator for $D(t, T_i)$ and no boundaries. Hence, it is more useful in practice. It is easy and intuitive to use. Furthermore, we rely on ATM options where the data are very reliable.*

As the name of this subsection already indicates, this is only an intuitive method. Therefore, we generalize it in the following subsection. Some resulting applications of this method are presented in Section 3.4 and Appendix B.

3.3.2 The Δ Method

Again we focus on a linear combination of the bounds calculated with ATM options, i.e. (3.6) and (3.7) and historical dates $t_j < t$ with $j = 1, 2, ..., J$. Additionally, we define the time between estimation and payment day as $\tau_i \triangleq T_i - t$, the time between estimation and the corresponding maturity as $\tilde{\tau}_i \triangleq \tilde{T}_i - t$ and write $\tilde{\tau}_{j,i}$ when we use the historical date t_j.

Using the above notations we now assume that the outstanding dividends from t_j up to time T_m are a linear combination of the lower and upper bound with coefficient $\lambda^{(\Delta)}$, i.e.

$$D(t_j, T_m) = \lambda^{(\Delta)} D_u^{ATM}(t_j, \tilde{T}_m) + (1 - \lambda^{(\Delta)}) D_l^{ATM}(t_j, \tilde{T}_m) + \varepsilon_{j,m}^{(\Delta)}, \qquad (3.8)$$

where Δ is a fixed estimation period with $\tilde{\tau}_{j,m} \in (\Delta - 3M, \Delta]$, where M stands for month. The $\varepsilon_{j,m}^{(\Delta)}$ represent the potential error in the data and assumed to be i.i.d. with finite variance and $\mathbb{E}(\varepsilon_{j,m}^{(\Delta)}) = 0$.

We then use a least squares approach over realized dividends and historical boundaries to obtain an optimal weight of the estimation boundaries. The results of this method are summarized in Theorem 3.2.

Theorem 3.2 (Least Squares Estimation of Discrete Dividends) *For a fixed estimation period Δ with $\Delta - 3M < \tau_i \leq \tilde{\tau}_i \leq \Delta$ a least squares estimate for the outstanding dividends from time t up to T_i is given as the linear combination of upper and lower bounds calculated with the help of ATM options, i.e.*

$$D^*(t, T_i) \triangleq \lambda^{*(\Delta)} D_u^{ATM}(t, \tilde{T}_i) + (1 - \lambda^{*(\Delta)}) D_l^{ATM}(t, \tilde{T}_i), \text{[16]} \qquad (3.9)$$

[16] If there are more $\tilde{T}_l$ with $T_i \leq \tilde{T}_l \leq T_{i+1}$ which can be used for the calculation of the lower and upper bound, we use the closest maturity to the payment day T_i.

where the least squares optimal weight $\lambda^{*(\Delta)}$ *is given by*

$$\lambda^{*(\Delta)} = \frac{\sum_{j=1}^{J}\sum_{m=1}^{i-1}(D(t_j,T_m) - D_l^{ATM}(t_j,\tilde{T}_m))(D_u^{ATM}(t_j,\tilde{T}_m) - D_l^{ATM}(t_j,\tilde{T}_m)) \cdot \mathbb{1}_{\{\tilde{\tau}_{j,m}\in(\Delta-3M,\Delta]\}}}{\sum_{j=1}^{J}\sum_{m=1}^{i-1}(D_u^{ATM}(t_j,\tilde{T}_m) - D_l^{ATM}(t_j,\tilde{T}_m))^2 \cdot \mathbb{1}_{\{\tilde{\tau}_{j,m}\in(\Delta-3M,\Delta]\}}}.$$

Proof. We use Equation (3.8) for every historical date of request $t_j < t$. Then, according to the least squares method we minimize the overall squared error:

$$\sum_{j=1}^{J}\sum_{m=1}^{i-1}(\varepsilon_{j,m}^{(\Delta)})^2 = \sum_{j=1}^{J}\sum_{m=1}^{i-1}\Big[D(t_j,T_m) - D_l^{ATM}(t_j,\tilde{T}_m) - (D_u^{ATM}(t_j,\tilde{T}_m) - D_l^{ATM}(t_j,\tilde{T}_m))\lambda^{(\Delta)} \cdot \mathbb{1}_{\{\tilde{\tau}_{j,m}\in(\Delta-3M,\Delta]\}}\Big]^2,$$

via setting the derivative equal to zero, i.e.

$$2\sum_{j=1}^{J}\sum_{m=1}^{i-1}\Big[\Big(D(t_j,T_m) - D_l^{ATM}(t_j,\tilde{T}_m) - (D_u^{ATM}(t_j,\tilde{T}_m) - D_l^{ATM}(t_j,\tilde{T}_m))\lambda^{(\Delta)}\Big) (-1)(D_u^{ATM}(t_j,\tilde{T}_m) - D_l^{ATM}(t_j,\tilde{T}_m)) \cdot \mathbb{1}_{\{\tilde{\tau}_{j,m}\in(\Delta-3M,\Delta]\}}\Big] = 0\,.$$

This results in the least squares estimator

$$\lambda^{*(\Delta)} = \frac{\sum_{j=1}^{J}\sum_{m=1}^{i-1}(D(t_j,T_m) - D_l^{ATM}(t_j,\tilde{T}_m))(D_u^{ATM}(t_j,\tilde{T}_m) - D_l^{ATM}(t_j,\tilde{T}_m)) \cdot \mathbb{1}_{\{\tilde{\tau}_{j,m}\in(\Delta-3M,\Delta]\}}}{\sum_{j=1}^{J}\sum_{m=1}^{i-1}(D_u^{ATM}(t_j,\tilde{T}_m) - D_l^{ATM}(t_j,\tilde{T}_m))^2 \cdot \mathbb{1}_{\{\tilde{\tau}_{j,m}\in(\Delta-3M,\Delta]\}}}.$$

□

Remark 3.8 (Ex-post Dividends) *For the calculation of* $\lambda^{*(\Delta)}$ *we need the quantities* $D(t_j,T_m)$ *which are not known at the historical date* t_j*. However, only the actual ex-post dividends are available to us. In order to use the ex-post value* $D^{ex\text{-}post}(t_j,T_m)$ *in our calculation we need to be careful, as also in an ideal setting these two values do not coincide. This changes the resulting method from Theorem 3.2 as follows: In Equation* (3.8) *we replace*

$$D(t_j,T_m) = D^{ex\text{-}post}(t_j,T_m) + \delta_{j,m}^{(\Delta)}\,,$$

where $\delta_j^{(\Delta)}$ *is the error resulting by using the ex-post value, which is assumed to be of finite variance with* $\mathbb{E}[\delta_{j,m}^{(\Delta)}] = 0$*. Then, we can perform the same steps as in the proof via minimization of*

$$\sum_{j=1}^{J}\sum_{m=1}^{i-1}(\varepsilon_{j,m}^{\prime(\Delta)})^2 = \sum_{j=1}^{J}\sum_{m=1}^{i-1}(\varepsilon_{j,m}^{(\Delta)} - \delta_{j,m}^{(\Delta)})^2\,.$$

The small deviations, represented in the results Section 3.4 support the approach to use the ex-post dividends.

Remark 3.9 (Bootstrapping of the Discount Curve) *For the calculation of the upper bound* (3.7) *and the present value of the actual, ex-post dividends* $D^{ex\text{-}post}(t_j, T_m)$ *we need different discount factors* $p(t_j, T_m)$*. As these values are not available to us we need to bootstrap them. More details on how we handle this are given in Subsection 3.4.1.*

Observe that both remarks 3.8 and 3.9 are also valid respectively important for the intuitive method. The following Remark 3.10 illustrated the differences between both methods.

Remark 3.10 (Differentiating Between the Intuitive and the Δ Method) λ^* *is estimated via the arithmetic average over all available* $\lambda_{i,j}$ *within the intuitive method. Whereas the* Δ *method restricts the data relating to a fixed estimation period and performs a least squares estimation to get* $\lambda^{*(\Delta)}$*.*

Figure 3.13 visualizes the method developed in Theorem 3.2. On the left hand side before the dashed, red line with color light teal the estimation of $\lambda^{(\Delta)}$ is exemplified for some dates. In order to have a clear figure we only highlight the estimation on date t_j with a bold line. These values lie in the past and are thus known at time t. The right part after the dashed, red line displays the estimation in the future with Equation (3.9) using the $\lambda^{*(\Delta)}$ from the left hand side. Some resulting applications of this method are represented in the next section.

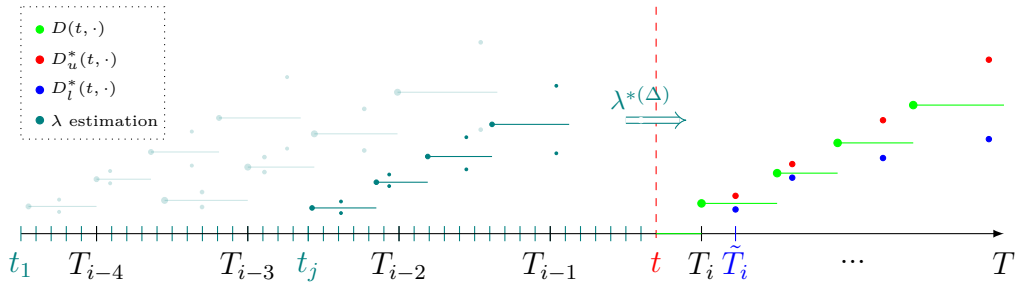


Figure 3.13: Visualization of the estimation of $\lambda^{*(\Delta)}$ and $D(t, T_i)$.

3.4 Results for Dow Jones Constituents

In this section we apply the method from Subsections 3.3.1 and 3.3.2 to data from US underlyings constituent in the Dow Jones and analyze its applicability

in practice. The actual constituents of the Dow Jones (since June 2015) and their corresponding industry and index weights are listed in Table 3.2.

Company Name	**Industry**	**Index Weight**
3M	Conglomerate	5.84
American Express	Consumer finance	2.91
Apple	Consumer electronics	4.72
Boeing	Aerospace and defense	5.38
Caterpillar	Construction and mining equipment	3.16
Chevron	Oil and gas	3.58
Cisco	Computer networking	1.03
Coca-Cola	Beverages	1.51
Du Pont	Chemical industry	2.22
ExxonMobil	Oil and gas	3.11
General Electric	Conglomerate	0.99
Goldman Sachs	Banking, Financial services	7.83
The Home Depot	Home improvement retailer	4.22
IBM	Computers and technology	6.20
Intel	Semiconductors	1.12
Johnson & Johnson	Pharmaceuticals	3.72
JPMorgan Chase	Banking	2.51
McDonald's	Fast food	3.63
Merck	Pharmaceuticals	2.18
Microsoft	Software	1.66
Nike	Apparel	4.18
Pfizer	Pharmaceuticals	1.27
Procter & Gamble	Consumer goods	3.07
Travelers	Insurance	3.76
UnitedHealth Group	Managed health care	4.54
United Technologies	Conglomerate	4.14
Verizon	Telecommunication	1.78
Visa	Consumer banking	2.55
Wal-Mart	Retail	2.77
Walt Disney	Broadcasting and entertainment	4.40

Table 3.2: Constituent Dow Jones Industrial Average.

We again calculate the aggregate statistics as in Chapter 2 (see Section 2.4.4). Therefore, we first give the details of the data basis.

3.4.1 Data Basis

First of all we do not only focus on one time point t but on different equidistant time points $t_k < T$ as dates of request (see Figure 3.14).

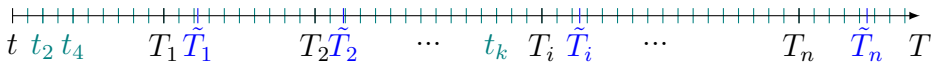


Figure 3.14: Time horizon including the dates of request.

Hence, we need to adapt our notations in Sections 3.3.1 and 3.3.2 with a k which indicates the considered date of request, i.e. $t_j^{(k)}$ for the historical dates and λ_k (respectively $\lambda_k^{(\Delta)}$). In contrast to the situation of Theorem 3.2 we now also face several estimation periods $\Delta = 3M, 6M, ..., 24M$. We then apply our methodology to data from the Dow Jones[17]. Most of these stocks pay dividends quarterly. The dates of request t_k are every Wednesday between 2012-01-01 and 2013-12-31. For the estimation of λ_k^* respectively $\lambda_k^{*(\Delta)}$ we use an interval of one year before these request date, i.e. $t_1^{(k)} = t_k - 1Y$. Hence, the overall data is requested from 2011-01-01 up to 2013-12-31 with a weekly frequency from Thomson Reuters' Datastream. We chose the time interval between $t_1^{(k)}$ and t_k equal to one year, as within this time usually four dividend payments happen such that we have enough datapoints in particular for every Δ.
As already explained in the foregoing section (see Remark 3.9) we need to bootstrap the discount curve. Therefore, we use data from the USD-LIBOR Rate, $L(t,T)$; and the forward rate agreement (FRA), $F(t,T)$ as follows

1. Calculation of $p(t_k, T_i)$ via

$$p(t_k, T_i) = \frac{1}{1 + L(t_k, T_i)\tau_{k,i}},$$

   with $\tau_{k,i} \triangleq T_i - t_k$, for all T_i and t_k where $L(t_k, T_i)$ exists.

2. Use the FRA to get more discount factors

$$p(t_k, T_i) = \underbrace{p(t, t^*)}_{\text{known from 1.}} \frac{1}{1 + F(t^*, T_i)(T_i - t^*)},$$

   where $F(t^*, T_i)$ exists.

[17] Note that we adjusted data, e.g. the strike price and the actual incurred dividend, for some stocks due to a stock split. Additionally, we omit the stock Visa as the data was corrupted corresponding to a stock split in 2015 and hence not usable.

3. In the case where we have a different time to maturity as the ones calculated via LIBOR and FRA, we use linear interpolation.

In the following we show some results for both the intuitive and the Δ method. The before explained data basis is the same for both methods.

3.4.2 Results of Applying the Intuitive Method

Table 3.3 displays the overall aggregate statistics performed with the results concerning the data from all stocks constituent in the Dow Jones. Appendix B contains these statistics on the level of single stocks. In Table 3.3 we again distinguish between the estimation periods Δ, in particular we focus on the following time between estimation and payment day $\tau_{k,i}$, i.e. $\Delta - 3M < \tau_{k,i} \leq \Delta$, with $\Delta = 3M, 6M, 9M, ..., 24M$. In Section 2.4.4 we explained the aggregate statistics in detail, here we repeat the important facts.

Δ	**Counter**	**Weighted Average**	**Average**	**Median**	**Worst Case**	**Best Case**	**Standard Deviation**
3 M	2751	10%	12%	9%	81%	0%	10%
6 M	2774	6%	8%	6%	75%	0%	8%
9 M	1522	5%	10%	7%	80%	0%	9%
12 M	597	4%	11%	9%	55%	0%	10%
15 M	556	6%	12%	10%	61%	0%	9%
18 M	589	6%	14%	13%	63%	0%	9%
21 M	515	4%	17%	16%	65%	0%	11%
24 M	213	5%	20%	19%	71%	1%	13%
Total	9517	7%	11%	8%	81%	0%	10%

Table 3.3: Aggregate statistics for the intuitive method with underlyings constituent in the Dow Jones.

We calculate the relative deviation $\hat{D}_{k,i}$ between the estimate and the ex-post, actually incurred dividends

$$\hat{D}_{k,i} \triangleq \left| (D^*(t_k, T_i) - D^{\text{ex-post}}(t_k, T_i))/D^{\text{ex-post}}(t_k, T_i) \right| ,$$

for all spot and payment dates. Note that also in an ideal setting the relative deviation is not equal to zero due to the stochastic nature of dividends and comparing the estimate with the ex-post incurred values, compare also with

Remark 3.8. The *counter* C denotes the corresponding number of request dates that the statistics are based on. The data are aggregated in terms of alternative averages, including the classical *mean* and *median*, the *worst* and *best case* and the *weighted average*, i.e.

$$\frac{1}{C} \cdot \sum_{k=1}^{N} \sum_{i=1}^{n} w_{k,i} \cdot \hat{D}_{k,i} \cdot \mathbb{1}_{\{\Delta - 3M \leq \tau_{k,i} \leq \Delta\}} \quad \text{with} \quad w_{k,i} = 1 - \frac{\tau_{k,i}}{(Year(T_i) - Year(t_k) + 1) \cdot 365},$$

where N denotes the total number of request dates. This average assigns more weight to estimates where $\tau_{k,i}$ are small. Finally, we display the *standard deviations* of successive dividend estimates.
Observe that the overall aggregate statistics in Table 3.3 are calculated via first taking the deviations $\hat{D}_{k,i}$ for every stock and then performing the calculations as explained before. Moreover, note that we do not weight the deviations with the index weight of its corresponding stock.

At this point we do not interpret the results in Table 3.3 as the Δ method provides a proper mathematical framework, hence it is more relevant. We then compare the results of both methods with each other. Right now Table 3.3 with small deviations from the incurred values underlined by a weighted average of 7%, an average of 11% and a median of 8% indicates that it is worth it to also apply the Δ method.

3.4.3 Results of Applying the Δ Method

Before we have a closer examination of the aggregate statistics, we illustrate the corresponding Figure 3.2 with the market data.

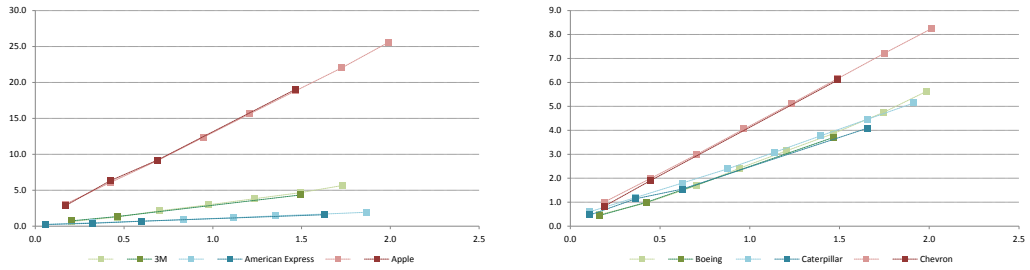


Figure 3.15: Incurred dividends $D^{\text{ex-post}}(t,T)$ (light mark) and their estimate $D^*(t,T)$ (dark mark) as a function of T (t = 2013-06-12).

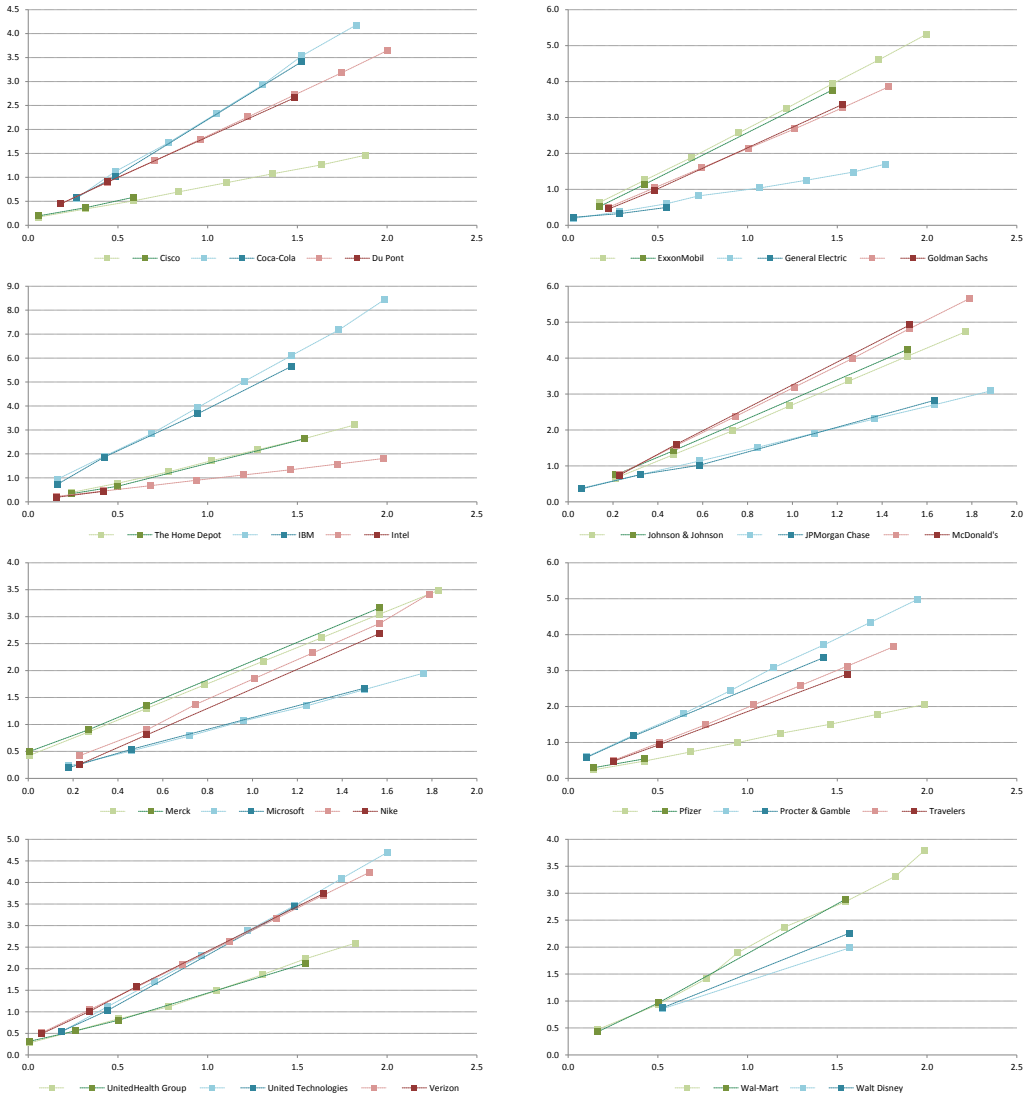


Figure 3.16: Incurred dividends $D^{\text{ex-post}}(t,T)$ (light mark) and their estimate $D^*(t,T)$ (dark mark) as a function of T (t = 2013-06-12).

Therefore, Figures 3.15 and 3.16 visualize exemplary the results of applying the Δ method to stocks constituent in Dow Jones, for one spot date t = 2013-06-12, as a function of T. Thereby, the light colored markers and lines represent the actual incurred, ex-post value of the dividends and the dark ones the corresponding estimate $D^*(t,T)$. The marker always belongs to the next payment date on the time horizon (x-axis). For all stocks, the estimate and the incurred values are close together. Observe that not for every light marker there is also a dark one. This is due to the fact that not for every payment date T_i there are corresponding options with maturity $T_i \leq \tilde{T}_i < T_{i+1}$.

Table 3.4 shows the aggregate statistics of the Δ method. The corresponding tables on the level of the single stocks are in Appendix C. Note that we applied the Δ method out of sample, i.e. it is calibrated and then applied to data points on which the calibration does not rely on.

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	2751	13%	15%	12%	91%	0%	12%
6 M	2774	6%	8%	6%	84%	0%	8%
9 M	1522	4%	8%	6%	73%	0%	7%
12 M	597	2%	7%	5%	39%	0%	7%
15 M	556	3%	6%	4%	34%	0%	5%
18 M	589	2%	6%	5%	35%	0%	5%
21 M	513	2%	6%	4%	29%	0%	6%
24 M	212	1%	6%	5%	39%	0%	6%
Total	9514	7%	9%	7%	91%	0%	10%

Table 3.4: Aggregate statistics for the Δ method.

Evaluation of the results

The values of the average and median in Table 3.4 are between 4% and 15% with an overall value of 9% and 7% respectively. The weighted average is in the range of 1% to 13% with a total value of 7%, thereby it is always smaller than the normal average. The overall worst case is 91% but therefore the best case is equal to 0% for every Δ. The standard deviation is between 5% and 12% with an overall deviation of 10%. It is noticeable that the values for $\Delta = 3M$, including the worst case of 91%, are the highest and by far away from the other ones. The reason for this is, that new information in the market effects the next dividend payment (compare with Figure 2.13 and its explanation), hence the ones with the smallest time between estimation and dividend payment, i.e. $\Delta = 3M$ for a quarterly payment. This characteristic behavior can also be seen on the level of the individual stocks, compare with the tables in Appendix C.
If we now compare these values with the results of the intuitive method in Table 3.3, we can observe that the Δ method outperforms or is at least equal to the intuitive method in more than 88% of the estimations. Only for $\Delta = 3M$ the latter one operates better. For the Δ method this is again due to the flow of information in the market, effecting the first dividend payment. In this case

the intuitive method performs better as the historical weight λ^* is an arithmetic mean which does not distinguish between the estimation periods.

In total the estimate has a good performance, as the total weighted average, average and median are less than 10%. Hence, our estimate is useful in practice and can be used as a benchmark or as stand alone estimate.

3.4.4 Further Prospects of the Intuitive Method

In the cases that there is not enough historical data available or if someone has no access to a data provider, there is a possibility to handle this. Then, one can use the intuitive method with just one historical date of request t_1, such that $\lambda^* = \frac{1}{N}\sum_{i=1}^{n} \lambda^*_{i,1}$. Note that this also has the effect that the method is faster. We also run this method for the data set from before and choose $j = 1$ and $t_1^{(k)} = t_k - 1Y$. We again choose 1 year as time horizon such that there are typically 4 dividend payments in between. Table 3.5 shows the resulting values and Appendix D gives the tables on the level of the single stocks. On the one hand the performance of the weighted average, average and median is of the same size as the one of the intuitive method applied for the full historical data set. On the other hand the worst case is a lot worse. Hence, using this method needs a careful handling.

Δ	**Counter**	**Weighted Average**	**Average**	**Median**	**Worst Case**	**Best Case**	**Standard Deviation**
3 M	2747	12%	14%	11%	107%	0%	13%
6 M	2770	8%	11%	8%	131%	0%	10%
9 M	1515	6%	10%	8%	146%	0%	10%
12 M	584	4%	11%	8%	146%	0%	11%
15 M	554	7%	14%	13%	63%	0%	10%
18 M	586	6%	15%	14%	64%	0%	10%
21 M	512	4%	16%	14%	60%	0%	11%
24 M	206	4%	19%	16%	140%	0%	15%
Total	9474	8%	13%	10%	146%	0%	12%

Table 3.5: Aggregate statistics with historical data from 1 year ago.

Furthermore, it can happen, that there is no data available one year ago (this is noticeable in the reduced counter of 9474 instead of 9514). Then, one needs to choose another day.

3.5 Robustness Check and Backtests

After having seen, that the Δ method performs well in practice, we also make some tests if the method is robust concerning its input values. There, the only input which needs to be checked is the discount factor. Additionally, we will compare the Δ method with the one developed in Chapter 2 as well as to the so called simple method. When writing method in the following we refer to the Δ method if not stated otherwise.

3.5.1 Robustness Check

Δ	**Counter**	**Weighted Average**	**Average**	**Median**	**Worst Case**	**Best Case**	**Standard Deviation**
3 M	2751	13%	15%	12%	91%	0%	12%
6 M	2774	6%	8%	6%	84%	0%	8%
9 M	1522	4%	8%	6%	**72%**	0%	7%
12 M	597	2%	7%	5%	**40%**	0%	7%
15 M	556	3%	6%	4%	34%	0%	5%
18 M	589	2%	6%	5%	35%	0%	**6%**
21 M	513	2%	**7%**	**5%**	**30%**	0%	6%
24 M	212	1%	**7%**	**6%**	**40%**	0%	6%
Total	9514	7%	**10%**	7%	91%	0%	10%

Table 3.6: Aggregate statistics with a discount factor equal to 1.

Δ	**Counter**	**Weighted Average**	**Average**	**Median**	**Worst Case**	**Best Case**	**Standard Deviation**
3 M	2751	13%	15%	12%	91%	0%	12%
6 M	2774	6%	8%	6%	**83%**	0%	8%
9 M	1522	4%	**7%**	6%	**74%**	0%	7%
12 M	597	2%	**6%**	**4%**	**40%**	0%	7%
15 M	556	3%	**5%**	4%	34%	0%	5%
18 M	589	2%	6%	**4%**	**34%**	0%	5%
21 M	513	**1%**	6%	4%	29%	0%	**5%**
24 M	212	1%	**7%**	5%	**43%**	0%	6%
Total	9514	7%	9%	**6%**	91%	0%	10%

Table 3.7: Aggregate statistics with a discount factor based on an interest rate of 2%.

Within this section we check the robustness of our method with respect to the discount factor. Therefore, we also perform our method with a discount factor equal to 1 and with a discount factor using a constant interest rate of 2%. Table 3.6 shows the analogue aggregate statistics as in Table 3.4 for the discount factor equal to 1 and Table 3.7 the one with an interest rate of 2%. The results are close to the ones in Table 3.4. There are only some variations of 1% and one of 4%, which are both highlighted in bold font. Hence, our method is robust against the input of different discount factors.

3.5.2 Backtesting against the European Method

Now, we compare this method with the one developed in Chapter 2, which we refer to as European method. This method is based on the put-call parity for options of European type and provides a no-arbitrage estimate. In contrast the method developed in this chapter uses options of American type and a least squares estimate based on put-call bounds. We now refer to it as American method. The stocks focused on in Chapter 2 pay dividends once per year, whereas here nearly all payment periods are 3 month. The examined European options have maturities up to 5 years in the future including up to 5 payment dates, whereas the American ones here have maturities of up to 2 years, but include up to 8 payment dates.

	Counter		**Weighted Average**		**Average**		**Median**		**Worst Case**		**Best Case**		**Standard Deviation**	
Δ	**E**	**A**	**E**	**A**	**E**	**A**	**E**	**A**	**E**	**A**	**E**	**A**	**E**	**A**
3 M	86	120	5%	7%	5%	9%	5%	7%	17%	36%	0%	0%	4%	7%
6 M	90	75	8%	7%	10%	9%	11%	8%	27%	66%	2%	0%	6%	10%
9 M	110	67	12%	9%	17%	13%	16%	12%	37%	46%	1%	0%	6%	8%
12 M	99	54	11%	7%	19%	12%	19%	10%	32%	26%	3%	1%	6%	6%
15 M	91	53	6%	4%	13%	9%	13%	6%	26%	50%	1%	0%	6%	12%
18 M	89	31	8%	7%	16%	19%	16%	12%	28%	57%	8%	0%	4%	17%
21 M	109	13	9%	4%	19%	8%	18%	7%	29%	29%	8%	2%	4%	7%
24 M	103	7	8%	2%	20%	6%	19%	6%	32%	10%	1%	1%	7%	3%
> 24 M	407	0	6%		19%		18%		39%		1%		6%	
Total	964	420	9%	7%	17%	11%	16%	8%	39%	66%	0%	0%	7%	10%

Table 3.8: Backtesting the results with the European option method.

We choose five stocks from Germany which were also examined in Section 2.4, and for which options of European *and* American type are available: *BASF, Bayer, Daimler, Merck* and *Munich Re* with the same request dates as

explained in the data basis (see Subsection 3.4.1). Table 3.8 displays the resulting overall aggregate statistics and in Appendix E there are all aggregate statistics on the level of a single stock. For every statistic we separate the column into two columns next to each other containing the two methods, where the American method is abbreviated with A and the European method with E, such that we can directly compare them.
We can observe that the American method performs better related to the weighted average, the average as well as the median for almost all Δ. Note that the American method is more volatile and the worst case is in most of the cases worse, compensating with a best case which is always better or at least the same. We can see that for these stocks the European method can estimate dividends with a time until the payment, which is greater than two years. This is an advantage if we are interested in a long time horizon $T - t$, but it strongly depends on the available maturities of the options.

Assessment of Both Methods

As European options are only available for a small set of stocks the American method has a more general range of application compared to the European method. In addition, although we use historical data for the estimation of λ, this method is about 10 times faster, both for getting the data and calculating the estimates. This is due to the fact that it only considers ATM options. The advantage of the method using European options is the simultaneous estimation of the dividends and the discount factor, such that no additional data as LIBOR rates and FRAs are necessary. Moreover, it does not need historical data. In total both methods have their advantages, their applicability always depends on the available data, in particular the option types or maturities.

3.5.3 Backtesting against the Simple Method

In order to compare the American method with the well-known simple method, that uses the last incurred dividend as estimate for all upcoming ones, we display the results of the American method (short A) from Table 3.4 together with the resulting overall aggregate statistics of the simple method (S) in Table 3.9. The simple method performs significantly better for $\Delta = 3M$, due to the already explained problems of the American method for small estimation periods which carries over to a better total weighted average and median. However, for longer estimation periods (i.e. from $\Delta > 9M$ on), the American method outperforms the simple one by roughly a factor of 2. Note that this pattern can also be observed on the level of single stocks as detailed in

Appendix F. Thus, if we are interested in a medium-term forecast of dividends, the American method is superior to the simple one.

	Counter		Weighted Average		Average		Median		Worst Case		Best Case		Standard Deviation	
Δ	**S**	**A**	**S**	**A**	**S**	**A**	**S**	**A**	**S**[18]	**A**	**S**	**A**	**S**	**A**
3 M	2845	2751	**3%**	13%	**4%**	15%	**0%**	12%	100%	91%	0%	0%	**9%**	12%
6 M	2926	2774	**4%**	6%	**5%**	8%	**0%**	6%	100%	84%	0%	0%	11%	8%
9 M	2865	1522	4%	4%	**7%**	8%	**4%**	6%	100%	73%	0%	0%	12%	7%
12 M	2975	597	4%	2%	8%	7%	6%	5%	100%	39%	0%	0%	12%	7%
15 M	2828	556	5%	3%	10%	6%	7%	4%	100%	34%	0%	0%	12%	5%
18 M	2971	589	4%	2%	11%	6%	8%	5%	100%	35%	0%	0%	12%	5%
21 M	2924	513	4%	2%	12%	6%	9%	4%	100%	29%	0%	0%	13%	6%
24 M	2887	212	4%	1%	13%	6%	11%	5%	100%	39%	0%	0%	13%	6%
Total	23221	9514	**4%**	7%	9%	9%	**6%**	7%	100%	91%	0%	0%	12%	10%

Table 3.9: Backtesting the results with the simple method.

Furthermore, the worst-case of 100% which is caused by a stock which suddenly changes from a non-dividend paying to a dividend-paying stock (i.e. Apple), indicates that our estimate is much better facing a changing dividend policy. For a detailed comparison of the values we refer the reader to Table C.3 and Table F.3 in the appendix.

3.6 Conclusion

After having evolved a method which estimates dividends with the put-call parity for European options in Chapter 2 we have transfered the theory to American options and developed a practicable method for the estimation of the outstanding dividends via using ATM options. The new method is based on a linear combination of a lower and upper bound for the dividends and a least squares method calculating an estimate for the weight using historical data. We have made a detailed analysis as well as a statistical assessment of the estimate. In the following we summarize our findings:

- In a case study of Dow Jones Industrial Average constituents with 9514 estimations, our method performed well with an overall average deviation from the ex-post value less than 10%.

[18] Note that the worst case equal to 100% for the simple method is caused by Apple which did not pay dividends before August, 2012, resulting in an estimate equal to zero.

- Furthermore, a robustness check using different discount factors as input, reveals only slight variations of about 1% of the resulting aggregate statistics.

- Additionally, we backtest the method with the one developed in Chapter 2 using European type options. In terms of the aggregate statistic, the American method performed better for the total weighted average, average and median. The usage of both methods always depends on the available data especially on the available maturities.

- The backtest with taking the last incurred dividend as estimate also shows a good performance of the method, in particular for estimation periods larger than $9M$.

The small deviation from ex-post values and the robustness check together with the two backtests underline furthermore, that the least squares estimate developed in Theorem 3.2 is the appropriate approach for the estimation of dividends based on American options. Overall, the evolved method in this Chapter which is based on American ATM options is practicable, robust and performs well in practice. Furthermore, it improves the method from Chapter 2.

Chapter 4

Modeling Discrete Dividends and Portfolio Optimization Problems

In the two foregoing chapters we have developed a general estimation framework for the present value of outstanding future dividends. Now, we want to focus on further aspects of discrete dividends, as including them in the stock model and solving portfolio optimization problems.
In Section 4.1 we give the general notation of the chapter and repeat the classical terminal wealth portfolio optimization problem. Afterwards, we consider well-known models which include discrete dividend payments and solve the corresponding terminal wealth problem in Section 4.2. As a next step we extend the stock model in Section 4.3 with an early announcement and generalize the terminal wealth problem to that effect. In Section 4.4, we additionally solve an optimal consumption problem with the restriction that we can only consume dividends.

4.1 Portfolio Optimization in a Nutshell

For a detailed introduction into portfolio optimization we refer to Korn and Korn (2001). Here, we repeat the classical portfolio optimization problem to maximize the terminal wealth within the standard financial market model with one stock and one bond.

Let $(\Omega, \mathfrak{A}, \mathbb{P})$ be a probability space, $[0,T]$ the time horizon and $W(t)$ a Brownian motion with $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ the Brownian filtration generated by W and satisfying the usual conditions where $\mathcal{F}_0$ is $\mathbb{P}$-trivial and $\mathcal{F}_T = \mathfrak{A}$.
Our primary purpose in this chapter is to deal with a dividend paying stock within portfolio problems. Hence, as in the foregoing chapters we refer to S as a dividend paying stock. In order to have a proper basis for our calculations and to outline the differences, we also consider the case of a non-dividend-paying stock which is denoted by $\tilde{S}$.

Standard Financial Market Model

We consider a standard financial frictionless market, which is complete, with a bond B and a stock $\tilde{S}(t)$, where the prices are given by

$$\begin{aligned} B(t) &= B(0)e^{rt}\,, \\ \tilde{S}(t) &= \tilde{S}(0)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma W(t)}\,, \end{aligned}$$

with constant interest rate $r > 0$, constant trend parameter $\mu \in \mathbb{R}$ and constant volatility $\sigma > 0$.

Remark 4.1 *With the notation of the discount factor from the chapters before we get* $p(t,T) = \frac{B(t)}{B(T)} = e^{-r(T-t)}$.

Utility Function

For modeling the preferences of an investor we additionally need a utility function $U(x)$. It should reflect that the investor prefers more to less, is risk averse and at some point he will be saturated. Hence, $U : (0,\infty) \to \mathbb{R}$ is strictly concave, continuous differentiable and satisfies

$$U'(0) \triangleq \lim_{x\downarrow 0} U'(x) = \infty \quad \text{and} \quad U'(\infty) \triangleq \lim_{x\to\infty} U'(x) = 0\,.$$

Now, let us formulate the terminal wealth optimization problem, where the investor wants to maximize the *value function* $J(x_0)$ (the expected utility of the terminal wealth) for a given *initial capital* $x_0 > 0$ via a strategy which is admissible. After the problem definition we explain the notations in more detail.

Problem 1 (Optimization of the Terminal Wealth)

$$\max_{\varphi\in\mathcal{A}_1(x_0)} J(x_0) = \max_{\varphi\in\mathcal{A}_1(x_0)} \mathbb{E}\Big[U(X^\varphi(T))\Big]\,,$$

with admissible set

$$\mathcal{A}_1(x_0) = \Big\{\varphi \in \mathcal{A}(x_0) \mid \mathbb{E}\Big[U(X^\varphi(T))^-\Big] < \infty\Big\}\,.$$

where $\varphi(t) = (\varphi_0(t), \varphi_1(t))$ is a progressively measurable *trading strategy*, i.e. it reflects the number of bonds, respectively stocks at time t and $X^\varphi(t)$ is the corresponding *wealth process*. Moreover, the set $\mathcal{A}(x_0)$ ensures a non-negative

wealth and that the investor does not need to invest additional money in $(t, T]$, i.e.

$$\mathcal{A}(x_0) = \Big\{\varphi \mid \varphi \text{ is self financing, } X^{\varphi}(0) = x_0 \text{ and } X^{\varphi}(t) \geq 0 \text{ for all } t \in [0,T]\Big\}.$$

Within the foregoing formulation and notations we could attach the notation of the trading strategy as well as the wealth process with an $\sim$ to indicate that the corresponding stock is $\tilde{S}$, i.e. does not pay dividends. We omitted the $\sim$ as the formulation and the notations are the same for the setting with dividends. When we write *generalized terminal wealth problem* in the following we refer to Problem 1 with a dividend paying stock.

In the forthcoming subsections we first show the solution of Problem 1 and then calculate the trading strategy for explicit utility functions.

4.1.1 Solution

There are two ways to solve the portfolio optimization problem without dividends, the martingale method and the stochastic control approach. Again we refer to Korn and Korn (2001) for more details. To show the solution of Problem 1 we use the martingale method.

First, note that the wealth process has the form

$$\tilde{X}^{\tilde{\varphi}}(t) = \tilde{\varphi}_0(t)B(t) + \tilde{\varphi}_1(t)\tilde{S}(t)\,,$$

with dynamics

$$\begin{aligned} d\tilde{X}^{\tilde{\varphi}}(t) &= \tilde{\varphi}_0(t)dB(t) \quad + \tilde{\varphi}_1(t)d\tilde{S}(t) \\ &= \tilde{\varphi}_0(t)rB(t)dt + \tilde{\varphi}_1(t)\tilde{S}(t)[\mu dt + \sigma dW(t)]\,. \end{aligned}$$

In some cases one might also be interested in the *portfolio process* $\tilde{\pi}(t)$ (respectively $\pi(t)$) instead of the trading strategy. It denotes the proportion invested in the stock, i.e.

$$\tilde{\pi}(t) \triangleq \frac{\tilde{\varphi}_1(t)\tilde{S}(t)}{\tilde{X}^{\tilde{\varphi}}(t)}\,, \qquad \text{respectively} \qquad \pi(t) \triangleq \frac{\varphi_1(t)S(t)}{X^{\varphi}(t)}\,.$$

With this we can rewrite the dynamics to

$$d\tilde{X}^{\tilde{\varphi}}(t) = \tilde{X}^{\tilde{\varphi}}(t)\Big[\Big[(1-\tilde{\pi}(t))r + \tilde{\pi}(t)\mu\Big]dt + \tilde{\pi}(t)\sigma dW(t)\Big]\,. \tag{4.1}$$

We are going to use Equation (4.1) in Section 4.2.2 to show the connection respectively difference between the setting with and without dividends. Before showing the solution we give some important definitions. Let θ be the market price of risk and $H(t)$ the state price deflator, which are defined as

$$\theta \triangleq \frac{\mu - r}{\sigma}\,,$$
$$H(t) \triangleq p(0,t)e^{-\frac{1}{2}\theta^2 t - \theta W(t)}\,.$$

For $y > 0$ we can define

$$I(y) \triangleq (U')^{-1}(y)\,,$$

the inverse marginal utility and

$$\mathcal{X}(y) \triangleq \mathbb{E}\Big[H(T)I(yH(T))\Big]\,.$$

The following theorem gives the solution of Problem 1 in the standard financial market model with stock $\tilde{S}$.

Theorem 4.1 (Optimal Terminal Wealth) *Let $\mathcal{X}(y) < \infty$ for all $y > 0$ and $y^* \triangleq \mathcal{X}^{-1}(x_0)$, then*

$$\tilde{Y}^* = I(y^*H(T))\,,$$

is the optimal terminal wealth and an optimal $\tilde{\varphi}^*$ respectively $\tilde{\pi}^*$ exists.[19]

Proof. This is a well-known theorem, which is proved in the basic literature in the field of portfolio optimization (for example see once more Korn and Korn (2001)). Note that $\tilde{\varphi}^*$ respectively $\tilde{\pi}^*$ exist as the market is complete. □

4.1.2 Example: Explicit Calculations

In order to give explicit calculations for the terminal wealth and the concerning portfolio process we consider the logarithmic utility function, i.e. $U(x) = \log(x)$. Then, we have

$$I(y) = (U')^{-1}(y) = \frac{1}{y}$$
$$\Longrightarrow \mathcal{X}(y) = \mathbb{E}\Big[H(T)\frac{1}{yH(T)}\Big] = \frac{1}{y}$$
$$\Longrightarrow y^* = \mathcal{X}^{-1}(x_0) = \frac{1}{x_0}\,.$$

[19] Note that in this chapter we mainly use a $*$ to indicate optimality instead of prices bootstrapped from market data as in the chapters before. Only if a T has an upper index $*$ it has a different meaning. This $*$ indicates the announcement date.

From these values we can directly calculate the optimal terminal wealth:

$$\tilde{Y}^* = I(y^* H(T)) = I\Big(\frac{1}{x_0} H(T)\Big) = \frac{x_0}{H(T)}\,.$$

The well-known optimal portfolio process, which exists due to Theorem 4.1 is equal to

$$\tilde{\pi}^*(t) = \frac{\mu - r}{\sigma^2}\,. \tag{4.2}$$

Remark 4.2 *The logarithmic utility function can be seen as a special case of the power utility* $U(x) = \frac{x^\gamma}{\gamma}$ *with* $\gamma = 0$*. For the power utility the more general, well-known results are*

$$\tilde{Y}^* = \frac{x_0}{H(T)}\,,$$

with

$$\tilde{\pi}^*(t) = \frac{1}{1-\gamma}\frac{\mu - r}{\sigma^2}\,.$$

This optimal strategy, which holds the fraction invested in the stock constant is named *Merton-strategy.*

4.2 First Step to Include Discrete Dividends

Now, we return to the setting where the stock is paying discrete dividends $D_i > 0$, payable at known times $0 < T_1 < T_2 < \cdots < T_n \leq T$. The question is *how do we model the price of a dividend paying stock within the standard market model?* There are three well-known models which are used by practitioners. In the following we recapitulate these models and explain how to use these models within the terminal wealth problem.

4.2.1 Three Different Models

Model 1 *Let us assume that in* $t = 0$ *the dividends up to time* T *are already known. Then the stock price separates into*

$$S(t) = S_D(t) + S_E(t)\,,$$

where S_D *is the deterministic component, the present value of the next, yet known dividends, i.e.*

$$S_D(t) = \sum_{i:t<T_i} p(t, T_i) D_i\,,$$

and S_E the ex-dividend stock price

$$S_E(t) = \tilde{S}(t) \quad with \quad S_E(0) = S(0) - \sum_{i=0}^{n} p(0, T_i) D_i \,.$$

This is an ex-dividend price in the sense of removing the present value of the (known) dividends from the stock price.

Figure 4.1 illustrates an example for the stock price components S_D in green and S_E in blue and the price of the stock itself in red for Model 1, where the horizontal axis displays the time horizon in years. We started with an initial stock price value of 70. The dividend amount equals 7, which we choose to be high such that the jumps in the stock price are really recognizable. They are payable at times 1,2,3 and 4.

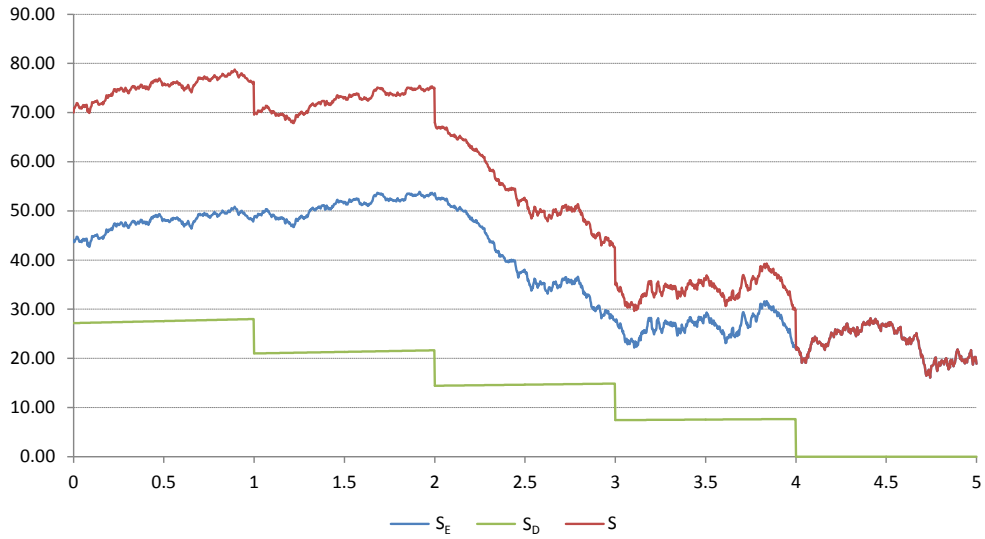


Figure 4.1: Visualization of the stock price and its components in Model 1.

Model 2 *Let us assume that there is no early announcement of the dividends. Then the stock price can be separate into*

$$S(t) = S_C(t) - S_D(t) \,,$$

where now S_D is the sum over all paid dividends, i.e.

$$S_D(t) = \sum_{i:T_i \leq t} D_i \frac{1}{p(T_i, t)} \,,$$

and S_C is the cum-dividend process, which follows a standard geometric Brownian motion, i.e.

$$S_C(t) = \tilde{S}(t) \quad \textit{with} \quad S_C(0) = S(0)\,.$$

Figure 4.2 shows an example for the stock Model 2, where we used the same setting as for the visualization of Model 1. In addition, note that we use the same underlying Brownian motion. It is observable, that the stock price in Model 1 is less volatile and has a trend to decrease, whereas the stock price in Model 2 tends to increase overall. These properties are due to the structure of the particular model.

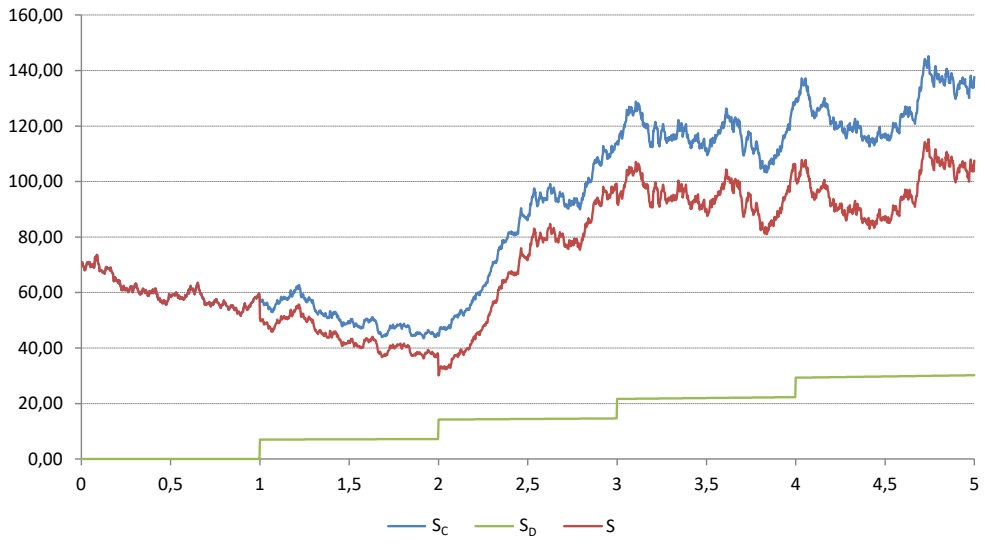


Figure 4.2: Visualization of the stock price and its components in Model 2.

Model 3 *The stock price follows a geometric Brownian motion in between two payment days and jumps down by D_i in T_i, i.e.*

$$dS(t) = d\tilde{S}(t) - \sum_{i=1}^{n} D_i \mathbb{1}_{\{t=T_i\}}\,.$$

The stock price example for Model 3 is visualized in Figure 4.3, again with the same setting and dividend height. The jump in the stock price is clearly observable for the payment time 1 and 2, but not for time 3 and 4.

Figure 4.3: Visualization of the stock price in Model 3.

For more details concerning these three models we refer to Frishling (2002), Bos and Vandermark (2002) and Bos, Gairat, and Shepeleva (2003), who examine them within the option pricing framework.

As we now know, how to include dividends in the stock model of the standard financial market, we can also include them into the terminal wealth portfolio optimization Problem 1. We proceed in the same way as in Section 4.1, after deriving the solution we apply the theory to the example of the logarithmic and power utility function.

4.2.2 Derivation of the Solution

Let us first explain how we deal with Model 1. Therefore, we have a closer look at the dynamics of the dividend and ex-dividend component of S:

$$dS_D(t) = rS_D(t)dt - \sum_{i=1}^{n} D_i \mathbb{1}_{\{t=T_i\}}\,,$$
$$dS_E(t) = S_E(t)\Big(\mu dt + \sigma dW(t)\Big)\,.$$

At a later stage we explain, what happens with the dividends, but for the moment let us assume that t is not a payment date. Then, observe that S_D behaves as the bond and S_E obviously as the stock in the standard financial market model. Hence, the main idea to solve the problem is to adjust the

proportion invested in the bond. Overall, the dynamics of S between two payment dates are given by

$$\begin{aligned} dS(t) &= dS_D(t) + dS_E(t) = S_D(t)rdt + S_E(t)[\mu dt + \sigma dW(t)] \\ &= [S_D(t)r + S_E(t)\mu]dt + S_E(t)\sigma dW(t)\,. \end{aligned} \tag{4.3}$$

With this we can also calculate the dynamics of the wealth process as follows

$$\begin{aligned} dX^{\varphi}(t) &= \varphi_0(t)dB(t) + \varphi_1(t)dS(t) \\ &= \varphi_0(t)B(t)rdt + \varphi_1(t)\Big[S_D(t)r + S_E(t)\mu\Big]dt \\ &\quad + \varphi_1(t)S_E(t)\sigma dW(t)\,. \end{aligned} \tag{4.4}$$

As a next step we separate the portfolio process into two parts $\pi(t) = \pi_D(t) + \pi_E(t)$, where π_D is the proportion invested in S_D and respectively π_E invested in S_E, i.e.

$$\pi_D(t) = \frac{\varphi_1(t)S_D(t)}{X^{\varphi}(t)}\,, \qquad\qquad \pi_E(t) = \frac{\varphi_1(t)S_E(t)}{X^{\varphi}(t)}\,.$$

Together with the proportion invested in the bond $\pi_0(t) = \frac{\varphi_0(t)B(t)}{X^{\varphi}(t)} = 1 - \pi(t)$, we insert these two values into Equation (4.4):

$$\begin{aligned} dX^{\varphi}(t) &= X^{\varphi}(t)\pi_0(t)rdt + X^{\varphi}(t)\Big[\pi_D(t)r + \pi_E(t)\mu\Big]dt \\ &\quad + X^{\varphi}(t)\pi_E(t)\sigma dW(t) \\ &= X^{\varphi}(t)\Bigg[\Big[\Big(1 - (\pi_D(t) + \pi_E(t))\Big)r + \pi_D(t)r + \pi_E(t)\mu\Big]dt \\ &\qquad\qquad + \pi_E(t)\sigma dW(t)\Bigg] \\ &= X^{\varphi}(t)\Big[\Big[(1 - \pi_E(t))r + \pi_E(t)\mu\Big]dt + \pi_E(t)\sigma dW(t)\Big]\,. \end{aligned} \tag{4.5}$$

These dynamics look familiar, as they are analogue to the dynamics of the wealth process (4.1) with $\pi_E(t)$ instead of $\pi(t)$. Due to this similarity we can easily solve the generalized terminal wealth problem with the standard procedures, where we get π_E^* as a solution from which we can calculate π^* and π_0^*.

The question which remains is *what happens with the paid dividends*. Let us assume that our investor can but does not have to consume the dividends. As

we want to maximize the terminal wealth, it cannot be optimal to consume the dividends as this reduces the wealth:

$$X^{\varphi}(T_i) = \varphi_0(T_i)B(T_i) + \varphi_1(T_i)[S(T_i-) - D_i] = X^{\varphi}(T_i-) - \varphi_1(T_i)D_i\,,$$

by using Assumption 1.1 which is valid for Model 1. Thus, we need to reinvest the dividends as in Korn and Rogers (2005). Consequently, the number of stocks needs to be adjusted via adding $\frac{\varphi_1(T_i-)D_i}{S(T_i)}$ in T_i.

Remark 4.3 *In the case where the investor must consume dividends we can avoid this consumption via selling all stocks directly before the dividend is payed. After the payment we buy back the relevant amount of stocks, which are more than before (as the stock has fallen by the dividend amount).*

Hence, we can calculate the terminal wealth in the same way as within the standard financial market (compare with Theorem 4.1), i.e.

$$Y^* = I(y^*H(T))\,. \tag{4.6}$$

Remark 4.4 (How to Deal with Model 2 and Model 3)

- *Model 2: We can follow the basic ideas developed for Model 1 and rearrange the money invested in the stock and bond.*
- *Model 3: As the stock price is following a geometric Brownian motion in between two payment dates, we can use the standard method explained in Section 4.1 with reinvesting the dividends (see also Korn and Rogers (2005)).*

4.2.3 Example: Calculation of the Portfolio Process

Let the stock price be modeled via Model 1. Again, we consider the logarithmic utility function for explicit computations. As we have seen before we can solve the generalized terminal wealth problem via slight adjustments.

From the dynamics of the wealth process (4.5) and Equation (4.2) we have the Merton-strategy as optimal proportion invested in S_E, i.e.

$$\pi_E^*(t) = \frac{\mu - r}{\sigma^2}\,.$$

The definition of π_E implies that $\varphi_1^*(t) = \frac{\mu-r}{\sigma^2}\cdot\frac{X^{\varphi}(t)}{S_E(t)}$, which we need to determine $\pi_D^*(t)$ as follows

$$\pi_D^*(t) = \frac{\varphi_1^*(t)S_D(t)}{X^{\varphi}(t)} = \frac{\frac{\mu-r}{\sigma^2}\cdot\frac{X^{\varphi}(t)}{S_E(t)}\cdot S_D(t)}{X^{\varphi}(t)} = \frac{\mu - r}{\sigma^2}\cdot\frac{S_D(t)}{S_E(t)}\,.$$

Accordingly, we can obtain $\pi^*(t)$ and $\pi_0^*(t)$ via

$$\begin{aligned}\pi^*(t) &= \pi_E^*(t) + \pi_D^*(t) = \frac{\mu - r}{\sigma^2} + \frac{\mu - r}{\sigma^2} \cdot \frac{S_D(t)}{S_E(t)} \\ &= \frac{\mu - r}{\sigma^2}\left[1 + \frac{S_D(t)}{S_E(t)}\right] = \frac{\mu - r}{\sigma^2} \cdot \frac{S(t)}{S_E(t)}\,, \end{aligned} \tag{4.7}$$

and

$$\pi_0^*(t) = 1 - \pi^*(t) = 1 - \frac{\mu - r}{\sigma^2} \cdot \frac{S(t)}{S_E(t)}\,,$$

where we need to ensure that if $t = T_i$ we adjust the strategy as explained in respectively before Remark 4.3. Note that in contrast to the classical problem without dividends π^* as well as π_0^* are not constant. They now depend on the relation from the stock and its ex-dividend part. Furthermore, observe that $\frac{S(t)}{S_E(t)} > 1$, i.e. in comparison to Equation (4.2) we need to invest more into the stock.

The corresponding terminal wealth can again be calculated via (4.6)

$$Y^* = I\Big(\frac{1}{x_0}H(T)\Big) = \frac{x_0}{H(T)}\,. \tag{4.8}$$

Remark 4.5 *Corresponding to Remark 4.2 and the foregoing calculations, the solution for the generalized terminal wealth problem with the power utility function is*

$$Y^* = \frac{x_0}{H(T)}\,,$$

with

$$\pi^*(t) = \frac{1}{1-\gamma}\frac{\mu - r}{\sigma^2} \cdot \frac{S(t)}{S_E(t)}\,.$$

4.3 Stock Model: Early Announcement of Single Dividends

Model 2 and 3 do not include an early announcement of the dividends, i.e. the announcement time and payment time coincide. Furthermore, within Model 2 the stock price can go negative if the Brownian path moves a long way down. In contrast, within Model 1 all dividends up to time T are already declared in $t = 0$.

Remark 4.6 (Include the Dividend Estimate) *If we do not know all dividends in $t = 0$ we could calculate S_D as the sum of already known dividends and the estimates of the further ones, where we can use the methods from Chapters 2 and 3.*

In reality, as we have seen in the introduction (Section 1.2.3), dividends are announced at least one day in advance and of course not every dividend is known in $t = 0$. Therefore, let $T_i^* < T_i$ denote the also known announcement time. Usually, one dividend is payed before the next one is declared, nevertheless in some cases the subsequent payment can be already declared before the actual one is paid or respectively several payments can be announced on the same date. Figure 4.4 visualizes an example, where the announcement days are colored in blue.

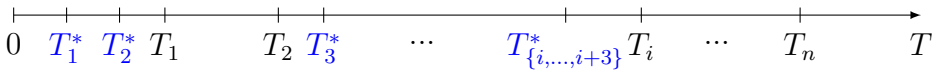


Figure 4.4: Example of a time horizon with dividend payment days T_i and announcement days T_i^*.[20]

Hence, our *aim* is to find a stock model, which

- includes discrete dividend payments,
- an early announcement is possible but not all dividends should be known at the same time,
- and ideally we would like to use the stock process without dividends $\tilde{S}$.

So the *main idea* is that with an early announcement one part of the stock, the dividend gets a certain payment and the wealth need to be shifted towards the bond. This means we will stay close to Model 1, i.e.

$$S(t) = S_D(t) + S_E(t)\,,$$

where S_D is the deterministic component, the present value of the next, yet known dividends and S_E the ex-dividend stock price.

[20] For the sake of clarity, the notation $T^*_{\{i,\ldots,i+3\}}$ stands for a simultaneous announcement of the dividend D_i and the three subsequent ones, i.e. $T_i^* = \ldots = T_{i+3}^*$.

Remark 4.7 *The first thing, which comes someone into mind is to model S_D and S_E as in Model 1 with only taking the already announced dividends into account, i.e.*

$$S(t) = \underbrace{\sum_{i:T_i^* \le t < T_i} p(t,T_i)D_i}_{S_D(t)} + \underbrace{\Big(S(0) - \sum_{i:T_i^* \le t} p(0,T_i)D_i\Big)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma W(t)}}_{S_E(t)} .$$

The problem with this model is that the stock price will also jump on every announcement date T_i^*.

4.3.1 Derivation of Two New Models

In order to define S_D and S_E we first need to make the assumption that the present value of the dividend is equal to a proportion of the stock price just before the declaration, i.e.

$$p(T_i^*, T_i)D_i = \alpha_i S(T_i^*-) ,$$

with $0 < \alpha_i < 1$ known. Now, the idea is to avoid jumps in the stock price at the announcement date and include that another dividend can be announced before the actual one is paid. Therefore, we have a closer look to the stock price at different time points:

We have

$$S(t) = \tilde{S}(t) ,$$

for $t < T_1^*$ as no dividend is announced. If then the first dividend is declared, i.e. $t = T_1^*$ the price of the stock changes to

$$S(t) = \underbrace{\alpha_1 S(T_1^*-)\frac{1}{p(T_1^*,t)}}_{S_D} + \underbrace{(1-\alpha_1)S(T_1^*-)}_{S_E} .$$

With the definition of D_1, $S(t) = \tilde{S}(t)$ for $t < T_1^*$ and the continuity of $\tilde{S}$ we have

$$S(t) = D_1 p(t,T_1) + (1-\alpha_1)\tilde{S}(t) ,$$

for $T_1^* \le t < T_1$. Note that there is no jump at the announcement date and Assumption 1.1 is fulfilled. Next, let the second dividend be announced before

the first one is paid, i.e. $t = T_2^* < T_1$:

$$\begin{aligned} S(t) = &\quad \alpha_2 S(T_2^*-)\frac{1}{p(T_2^*,t)} &&+ (1-\alpha_2)S(T_2^*-) \\ = &\quad D_2 p(t,T_2) &&+ (1-\alpha_2)S(T_2^*-) \\ = &\quad D_2 p(t,T_2) &&+ (1-\alpha_2)[D_1 p(t,T_1) + (1-\alpha_1)\tilde{S}(t)] \\ = &\; D_2 p(t,T_2) + (1-\alpha_2)D_1 p(t,T_1) + (1-\alpha_2)(1-\alpha_1)\tilde{S}(t)\,. \end{aligned}$$

We can do the same calculation for $T_3^* \le t < T_1$ to get

$$\begin{aligned} S(t) = &\; D_3 p(t,T_3) + (1-\alpha_3)D_2 p(t,T_2) + (1-\alpha_3)(1-\alpha_2)D_1 p(t,T_1) \\ &+ (1-\alpha_3)(1-\alpha_2)(1-\alpha_1)\tilde{S}(t)\,. \end{aligned}$$

Overall, we can make a general formulation of the stock price model.

Model 4 *Under the assumption that $p(T_i^*, T_i)D_i = \alpha_i S(T_i^*-)$ the price of the dividend paying stock can be written as*

$$S(t) = \underbrace{\sum_{i:T_i^* \le t < T_i} \Big(\prod_{j>i:T_j^* \le t < T_i} (1-\alpha_j) \Big) D_i p(t,T_i)}_{S_D(t)} + \underbrace{\Big(\prod_{i:T_i^* \le t} (1-\alpha_i) \Big) \tilde{S}(t)}_{S_E(t)}\,.$$

We illustrate an example for the stock price corresponding to Model 4 in red and its components S_D (in green) and S_E (in blue) in Figure 4.5. We again used the same underlying Brownian motion and choose $\alpha_i = 0.15$, such that the dividends are clearly observable in the figure. This time we change the setting concerning the early announced. Hence, the process S_D makes a step up at the dividend announcement and a step down on the payment date. The line in between is slightly increasing because of the interest rate being greater than zero. In this example the third dividend is announced before the second one is paid. The resulting "peak" in S_D for $t \in [1.8, 2]$ is reflected and observable in S_E and S.

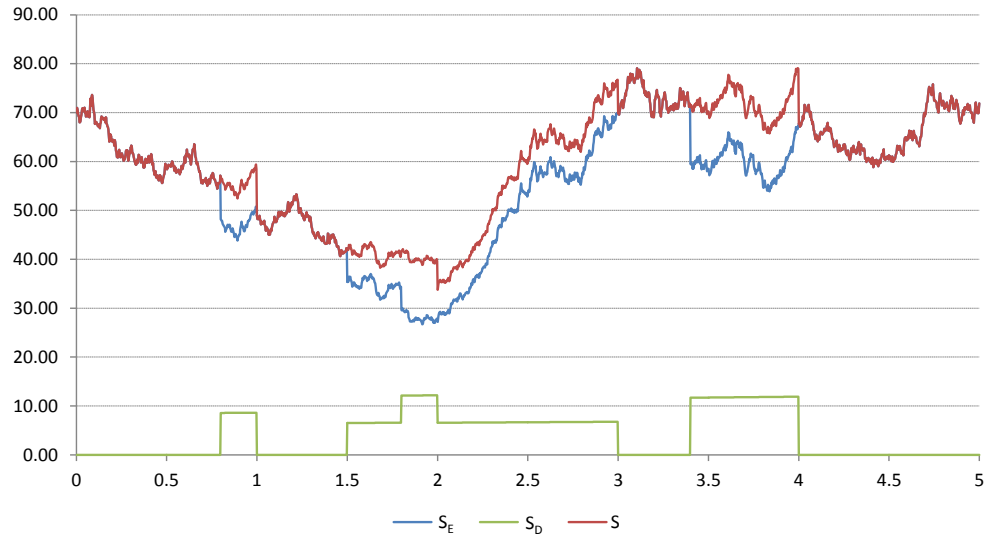


Figure 4.5: Visualization of the stock price and its components in Model 4.

If we have a closer look at the jump in the stock price on the ex-dividend date, we have

$$S(T_i-) - S(T_i) = \prod_{j>i:T_j^* \le t < T_i} (1-\alpha_j) D_i \,,$$

i.e. Assumption 1.1, that the drop in the stock price is equal to the dividend amount is only fulfilled when every dividend declaration is after the previous payment date. As usually a dividend is payed before the next one is announced, Model 4 can be useful in practice. Nevertheless, we slightly change the model to avoid the factors before D_i respectively to ensure that the model fulfills Assumption 1.1 for every announcement setting. Therefore, we need to assume that the dividend amount is a proportion of S_E instead of S, i.e.

$$p(T_i^*, T_i) D_i = \alpha_i S_E(T_i^*-) \,.$$

This is a realistic assumption as only the already payed dividends affect the actual dividend (as they are affected by S_E). In the other model the before announced but not yet payed dividends also affect the amount of the actual dividend. With the current assumption, we can do the same calculations as before to get the following model:

Model 5 *Under the assumption that* $p(T_i^*, T_i)D_i = \alpha_i S_E(T_i^*-)$ *the price of the dividend paying stock can be written as*

$$S(t) = \underbrace{\sum_{i:T_i^* \le t < T_i} D_i p(t, T_i)}_{S_D(t)} + \underbrace{\Big(\prod_{i:T_i^* \le t} (1-\alpha_i)\Big)\tilde{S}(t)}_{S_E(t)} . \tag{4.9}$$

Now, Assumption 1.1 is fulfilled, as

$$\begin{aligned} S(T_j) &= \sum_{i:T_i^* \le T_j < T_i} D_i p(T_j, T_i) + \Big(\prod_{i:T_i^* \le T_j} (1-\alpha_i)\Big)\tilde{S}(T_j) \\ &= \sum_{i:T_i^* \le T_j - < T_i} D_i p(T_j-, T_i) - D_j p(T_j-, T_j) + \Big(\prod_{i:T_i^* \le T_j -} (1-\alpha_i)\Big)\tilde{S}(T_j-) \\ &= S(T_j-) - D_j . \end{aligned}$$

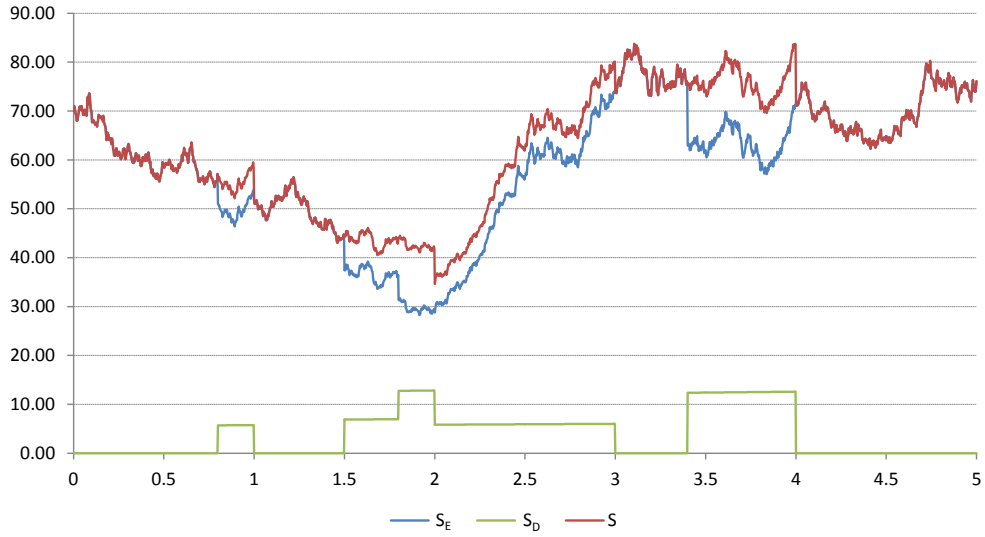


Figure 4.6: Visualization of the stock price and its components in Model 5.

For the same example settings, as we used to visualize Model 4, we display the stock price following Model 5 in Figure 4.6. At first sight this figure seems to be identical to Figure 4.5. Therefore, Figure 4.7 shows both stock price, where the light green line displays S corresponding to Model 4 and the dark green one S concerning to Model 5. We can directly see, that the prices first were the same and after the second dividend payment they differ from each other. This is mainly caused by the different α_i factors.

Figure 4.7: Comparison of the stock price of Model 4 and Model 5.

4.3.2 Optimization Problem 1 using Model 5

As we have seen in the section before, Model 5 is a good model to include early announcement of the dividends. Now, we want to solve Problem 1, where the stock price follows Model 5. Therefore, we calculate the dynamics of S, which is

$$\begin{aligned} dS(t) &= \sum_{i:T_i^* \le t < T_i} r D_i p(t,T_i) - \sum_{i=1}^{n} D_i \mathbb{1}_{\{t=T_i\}} + \Big(\prod_{i:T_i^* \le t} (1-\alpha_i) \Big) d\tilde{S}(t) \\ &= r S_D(t) dt + \underbrace{\Big(\prod_{i:T_i^* \le t} (1-\alpha_i) \Big) \tilde{S}(t)}_{S_E(t)} \Big[\mu dt + \sigma dW(t) \Big] - \sum_{i=1}^{n} D_i \mathbb{1}_{\{t=T_i\}} \\ &= \Big[S_D(t) r + S_E(t) \mu \Big] dt + S_E(t) \sigma dW(t) - \sum_{i=1}^{n} D_i \mathbb{1}_{\{t=T_i\}} . \end{aligned}$$

Note that this dynamics are the same as the dynamics of the stock within Model 1 (compare Equation (4.3)). So we can use one-to-one every step in Subsection 4.2.2 to solve Problem 1. Furthermore, we can also follow the example in Section 4.2.3. The only thing one needs to have in mind is that the definition of S_E and S_D differs. This procedure is valid as the payment days T_i, announcement days T_i^* and the factor α_i are known.

Remark 4.8 (Transformation of the Problem) *There is an alternative way to solve Problem 1. First, express the stock price $\tilde{S}$ without dividends in terms of the stock S and the bond via Equation* (4.9)

$$\begin{aligned}\tilde{S}(t) &= \left(\prod_{i:T_i^*\le t}(1-\alpha_i)^{-1}\right)\left[S(t) - \sum_{i:T_i^*\le t<T_i} D_i p(t,T_i)\right] \\ &= \left(\prod_{i:T_i^*\le t}(1-\alpha_i)^{-1}\right)S(t) - \left(\prod_{i:T_i^*\le t}(1-\alpha_i)^{-1}\sum_{i:T_i^*\le t<T_i}\frac{D_i}{B(T_i)}\right)B(t)\,.\end{aligned}$$

This means we have found a self financing trading strategy to recover $\tilde{S}(t)$. So with this we can transform the generalized optimal terminal wealth problem, where we reinvest dividends, into one without dividends via using $\tilde{S}$ and B. Let $\tilde{\varphi}^* = (\tilde{\varphi}_0^*, \tilde{\varphi}_1^*)$ be the concerning optimal trading strategy. From this we can calculate the amount invested in S and B in the original problem in between two payment dates as follows

$$\begin{aligned}\varphi_0^*(t) &= \tilde{\varphi}_0^*(t) - \tilde{\varphi}_1^*(t)\left(\prod_{i:T_i^*\le t}(1-\alpha_i)^{-1}\sum_{i:T_i^*\le t<T_i}\frac{D_i}{B(T_i)}\right) \\ \varphi_1^*(t) &= \tilde{\varphi}_1^*(t)\left(\prod_{i:T_i^*\le t}(1-\alpha_i)^{-1}\right).\end{aligned}$$

At a payment date we need to reinvest the dividends as explained before and in Remark 4.3. For a more explicit calculation we use the logarithmic utility function. Then, we know that $\tilde{\pi}^*(t) = \frac{\mu-r}{\sigma^2}$, respectively $\tilde{\varphi}_1^*(t) = \frac{\frac{\mu-r}{\sigma^2}X(t)}{\tilde{S}(t)}$ (compare (4.2)*). Overall, we get*

$$\begin{aligned}\pi^*(t) &= \frac{\tilde{\varphi}_1^*(t)\prod_{i:T_i^*\le t}(1-\alpha_i)^{-1}S(t)}{X(t)} \\ &= \frac{\frac{\mu-r}{\sigma^2}X(t)}{\tilde{S}(t)}\frac{S(t)}{\prod_{i:T_i^*\le t}(1-\alpha_i)X(t)} \\ &= \frac{\mu-r}{\sigma^2}\frac{S(t)}{S_E(t)}\,,\end{aligned}$$

which is the same result as with the procedure, we used before.

Remark 4.9 (Modeling the Stock Price via a Dividend Process) *Korn and Rogers (2005) showed that the solution to the portfolio optimization problem and the concerning strategy does not differ from the classical one with their*

dividend paying stock model. Within this investigation they assumed that there is no early announcement. So the question is, what happens when an early announcement is included? Therefore, one should note that the transformation of the problem as we explained in Remark 4.8 also works, if we do not define a model for $\tilde{S}$. Instead, we could model the stock price S as defined in Korn and Rogers (2005), i.e. S equals the present value of all future dividends, where they model the dividend process itself (including an early announcement). Then, we can solve Problem 1 with the transformation and hence, the trading strategy differs from the "plain" Merton-strategy.

After solving the generalized terminal wealth problem we now pay attention to another optimization problem.

4.4 Optimizing the Dividend Consumption

The *optimal consumption* or respectively the *optimal terminal wealth and consumption* are also standard optimization problems. When including dividends, they can both be handled in the analogue way as explained in the foregoing sections. In the actual low interest period, many investors only focus on the dividends as consumption. This is reflected in the quote "*Dividenden sind die bessere Miete*" (in English dividends are the better rent) by the German book author Christian W. Röhl. Hence, he compares the stock with a property, through which he receives a rent, the dividend. In order to take this setting we adapt the optimal consumption problem by the restriction that only dividends can be consumed. Hence, the consumption is equal to $\varphi(T_i)D_i$ in T_i and zero for $t \neq T_i$ for all $i = 1, \ldots, n$, i.e. the consumption process is discrete. The corresponding portfolio problem is

Problem 2 (Optimal Dividend Consumption Problem)

$$\max_{\varphi \in \mathcal{A}(x_0)} J(x_0) = \max_{\varphi \in \mathcal{A}(x_0)} \mathbb{E}\Big[\sum_{i:0<T_i \leq T} U(\varphi(T_i)D_i) \Big],$$

where $\mathcal{A}(x_0)$ and $U(x)$ are defined as explained in Section 4.1.

4.4.1 Derivation of the Solution

For the moment let us assume, that the stock pays only one dividend in $T_1 = T$ with an announcement in T^*. We now investigate a strategy, which is denoted by φ^* and show that it is optimal (this is the reason why we already put a $*$ in the notation):

Step 1: Let $\bar{T} \triangleq T-$ and allow the investor to trade directly before the dividend payment. Then, solve the *generalized optimal terminal wealth problem* with end time $\bar{T}$, i.e.

$$\max_{\psi \in \mathcal{A}_1(x_0)} \mathbb{E}\Big[U(X^{\psi}(\bar{T}))\Big] .$$

Let Y^* as usual denote the optimal terminal wealth, i.e. $Y^* = X^{\psi^*}(\bar{T})$ with ψ^* the corresponding optimal trading strategy and set $\varphi_0^*(t) = \psi_0^*(t)$ respectively $\varphi_1^*(t) = \psi_1^*(t)$ for all $t \in [0, \bar{T})$.
Figure 4.8 shows the strategy together with important time points. Step 2 is specified afterwards.

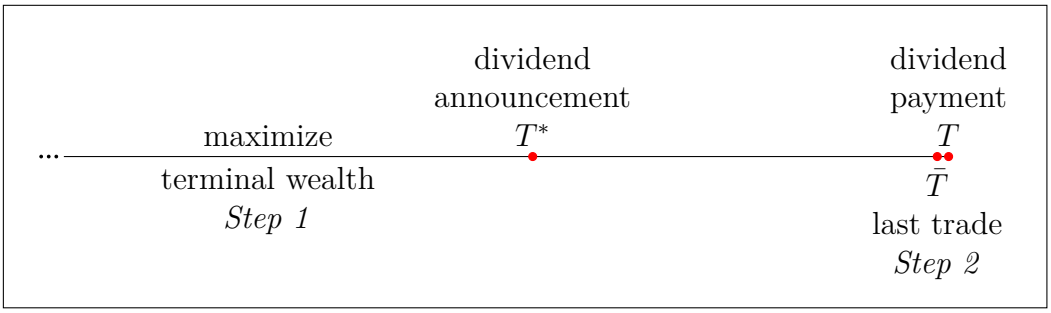


Figure 4.8: Visualization of the strategy.

Step 2: Pay the whole wealth Y^* out via the dividend payment in T. This works by setting $\varphi_1^*(\bar{T}) = \frac{Y^*}{D_1}$, which is possible as we know D_1 at $\bar{T}$. Hence, we need to borrow money via the bond at time $\bar{T}$:

$$X^{\varphi^*}(\bar{T}) = \underbrace{\frac{Y^*}{D_1} S(\bar{T})}_{\text{stock-part}} + \underbrace{\left[Y^* - \frac{Y^*}{D_1} S(\bar{T})\right]}_{\text{bond-part}} .$$

Note that we calculated the amount invested in the bond respectively the bond-part via

$$\begin{aligned} \varphi_0^*(\bar{T}) &= \frac{\pi_0(\bar{T}) X^{\varphi^*}(\bar{T})}{B(\bar{T})} = \frac{\big(1 - \pi(\bar{T})\big) X^{\varphi^*}(\bar{T})}{B(\bar{T})} = \frac{\big(1 - \frac{\varphi_1^*(\bar{T}) S(\bar{T})}{X^{\varphi^*}(\bar{T})}\big) X^{\varphi^*}(\bar{T})}{B(\bar{T})} \\ &= \frac{X^{\varphi^*}(\bar{T}) - \varphi_1^*(\bar{T}) S(\bar{T})}{B(\bar{T})} = \frac{Y^* - \frac{Y^*}{D_1} S(\bar{T})}{B(\bar{T})} . \end{aligned}$$

In T the stock pays the dividend which results in a consumption equal to $\frac{Y^*}{D_1} \cdot D_1 = Y^*$. For the terminal wealth, using Assumption 1.1 it holds

$$\begin{aligned} X^{\varphi^*}(T) &= \frac{Y^*}{D_1} S(T) + \Big[Y^* - \frac{Y^*}{D_1} S(\bar{T}) \Big] \frac{1}{p(\bar{T}, T)} \\ &= \frac{Y^*}{D_1} S(T) + Y^* - \frac{Y^*}{D_1} (S(T) + D_1) \\ &= \frac{Y^*}{D_1} S(T) + Y^* - \frac{Y^*}{D_1} S(T) - Y^* = 0 \,, \end{aligned}$$

i.e. together with $\psi^* \in \mathcal{A}(x_0)$ we have $\varphi^* \in \mathcal{A}(x_0)$ and we can payback the credit via selling the shares of the stock ($\varphi_1^*(T) = 0$).

After specifying the strategy, the question is: *Is φ^* optimal for Problem 2?* Suppose it is not optimal, i.e. there exists a $\hat{\varphi}$ with

$$\max_{\varphi \in \mathcal{A}(x_0)} \mathbb{E}\Big[U(\varphi_1(T) D_1) \Big] = U(\hat{\varphi}_1(T) D_1) > U(\varphi_1^*(T) D_1) \,.$$

As we have seen in step 2, the maximal payout is determined by the wealth immediately before the dividend payment, i.e. we have that $X^{\hat{\varphi}}(\bar{T}) > Y^*$. Thus, this is a contradiction to the choice of Y^*. In total we have

$$\max_{\varphi \in \mathcal{A}(x_0)} \mathbb{E}\Big[U(\varphi_1(T) D_1) \Big] = \max_{\psi \in \mathcal{A}_1(x_0)} \mathbb{E}\Big[U(X^{\psi}(\bar{T})) \Big] \,,$$

with an optimal strategy φ^*, as defined before.

Remark 4.10 *If the dividend payment day is $T_1 < T$ with announcement in T_1^*, we can proceed in the same way with the difference that the terminal optimization problem relates to the end time $\bar{T}_1 \triangleq T_1-$. Afterwards, the wealth process is equal to zero.*

Let us define $\bar{T}_i \triangleq T_i-$ in general, where the investor again is allowed to trade directly before the dividend payments. Now, we change the setting to two dividend payments in $0 < T_1 < T_2 \leq T$ with announcement in T_1^*, respectively T_2^*.

Theorem 4.2 (Optimal Consumption with two Dividend Payments) *In the case of two dividend payments in $[0, T]$, the solution of the optimal consumption problem is*

$$\begin{aligned} \max_{\varphi \in \mathcal{A}(x_0)} \mathbb{E}\Big[U(\varphi_1(T_1) D_1) + U(\varphi_1(T_2) D_2) \Big] \\ = \max_{\psi \in \mathcal{A}_1(x_0)} \mathbb{E}\Big[U(X^{\psi}(\bar{T}_2)) \Big] = U(Y^*) \,, \end{aligned}$$

with reinvesting the dividend payment in T_1 and optimal trading strategy φ^*, which is given by

$$\varphi_0^*(t) = \begin{cases} \psi_0^*(t) & t < \bar{T}_2\,, \\ \frac{Y^* - \frac{Y^*}{D_2} S(\bar{T})}{B(\bar{T})} & t = \bar{T}_2\,, \end{cases} \tag{4.10}$$

$$\varphi_1^*(t) = \begin{cases} \psi_1^*(t) & t < \bar{T}_2\,, \\ \frac{Y^*}{D_2} & t = \bar{T}_2\,, \end{cases} \tag{4.11}$$

where $\psi^* = (\psi_0^*, \psi_1^*)$ is the optimal strategy concerning to Y^*.

Proof. We follow the same strategy as for one dividend payment, i.e. optimizing the terminal wealth with end time $\bar{T}_2$, where we reinvest D_1 and consume Y^* in T_2. This gives us the strategy φ^* as defined in Equations (4.10) and (4.11).
Suppose this strategy is not optimal. With the same arguments as before it is not possible to consume an amount with higher utility in T_2. Hence, it is optimal to consume $\hat{Y}_1 > 0$ in T_1 and $\hat{Y}_2$, in T_2 with

$$\max_{\varphi \in \mathcal{A}(x_0)} \mathbb{E}\Big[U(\varphi_1(T_1)D_1) + U(\varphi_1(T_2)D_2)\Big] = U(\hat{Y}_1) + U(\hat{Y}_2)\,.$$

Now, consider the following strategy: Invest $\hat{Y}_1$ in the bond and consume it in T_2 instead of T_1. Then, it follows that

$$U(\hat{Y}_2) + U(\hat{Y}_1 \frac{1}{p(T_1, \bar{T}_2)}) > U(\hat{Y}_2) + U(\hat{Y}_1)$$

from the properties of U and as $r > 0$. This is a contradiction to the optimality of $\hat{Y}_1$ and $\hat{Y}_2$. Overall, we have

$$\max_{\varphi \in \mathcal{A}(x_0)} \mathbb{E}\Big[U(\varphi(T_1)D_1) + U(\varphi(T_2)D_2)\Big] = \max_{\psi \in \mathcal{B}(x_0)} \mathbb{E}\Big[U(X^{\psi}(\bar{T}_2))\Big]\,,$$

with the optimal φ^*, where we reinvest the first dividend. □

For more dividend payments we have exactly the same structure as in the foregoing theorem and can use the same arguments as in the associated proof. That means we can reinvest the dividends and maximize the wealth until the last payment date, where we consume everything. Thus, we have the following theorem and remark.

Theorem 4.3 (Optimal Dividend Consumption) *For a stock with n dividend payments in* $[0,T]$*, we can solve Problem 2 via:*

$$\max_{\varphi\in\mathcal{A}(x_0)} \mathbb{E}\Big[\sum_{i:0<T_i\leq T} U(\varphi(T_i)D_i)\Big] = \max_{\psi\in\mathcal{A}_1(x_0)} \mathbb{E}\Big[U(X^{\psi}(\bar{T}_n))\Big] = U(Y^*)\,,$$

with reinvesting all dividends D_i *for* $i<n$ *and optimal strategy*

$$\varphi_0^*(t) = \begin{cases} \psi_0^*(t) & t<\bar{T}_n\,, \\ \frac{Y^*-\frac{Y^*}{D_n}S(\bar{T})}{B(\bar{T})} & t=\bar{T}_n\,, \end{cases} \tag{4.12}$$

$$\varphi_1^*(t) = \begin{cases} \psi_1^*(t) & t<\bar{T}_n\,, \\ \frac{Y^*}{D_n} & t=\bar{T}_n\,, \end{cases} \tag{4.13}$$

where $\psi^*=(\psi_0^*,\psi_1^*)$ *is the optimal strategy concerning to* Y^**.*

Remark 4.11 (More Stocks) *The optimal consumption problem with several stocks can also be reduced to a generalized optimal terminal wealth problem by sorting all payment dates and choosing the latest one as end time.*

4.4.2 Example: Calculation of the Strategy

Let us now solve Problem 2 with Theorem 4.3 for the logarithmic utility function.

We can use Subsection 4.2.3 respectively Equation (4.8) to achieve the total consumption, i.e

$$Y^* = \frac{x_0}{H(\bar{T})}\,.$$

The trading strategy can easily be calculated with the help of Equation (4.13) and Equation (4.7)

$$\varphi_1^*(t) = \begin{cases} \frac{\mu-r}{\sigma^2}\cdot\frac{X^{\varphi^*}(t)}{S_E(t)} & t<\bar{T}_n\,, \\ \frac{x_0}{H(\bar{T})D_n} & t=\bar{T}_n\,. \end{cases}$$

4.5 Conclusion

After deriving several methods for the estimation of the outstanding dividend payments in the two foregoing chapters, we focus on modeling the price of a dividend paying stock in this chapter. We found a model (Model 5), which has the following advantages:

- The model includes discrete dividends, no dividend rate processes.
- Multiple, early announcements of dividend payments are possible within this model.
- Assumption 1.1 is valid.
- The model is easy to use as it is based on a deterministic part and the price of a non-dividend paying stock.

Furthermore, we have generalized the optimal terminal wealth problem via including discrete dividends. We detect that the trading strategy, respectively the portfolio process differs from the classical one. For example, the optimal portfolio process in the classical problem equals the Merton strategy, i.e. $\frac{\mu-r}{\sigma^2}$ for the logarithmic utility, whereas the generalized strategy is $\frac{\mu-r}{\sigma^2}\frac{S(t)}{S_E(t)}$, where we now invest more in the stock and it is not a constant anymore.
Additionally, we solved a special consumption problem, where the investor is only allowed to consume dividends. We found out that the optimal strategy is to follow the optimal one of the generalized terminal wealth problem with reinvesting every dividend payment for $T_i < T_n$ and consume the whole wealth in T_n.

Appendix A

Replicable Dividends and an Adapted Proof of the Put-Call Parity with Dividends

As explained in Remark 2.2 we change Assumption 1.1 to

Assumption A.1 *Every dividend payment D_i is replicable by market instruments.*

Assumption A.1 is satisfied by supposing that dividends are subject to the same fundamental risk factors as the company's stock price. In particular it holds if the market is complete.

Proof. As in the proof of Theorem 2.1 we use simple no-arbitrage conditions where we change the third part of the trading strategy.

1. Suppose that $\mathbf{S(t) - D(t,T) + P(t) < C(t) + Kp(t,T)}$:
In this case we can construct an arbitrage opportunity as follows:

At Time t:

- sell the call $C(t)$ and borrow the amount $Kp(t,T)$ in cash;
- buy the put $P(t)$ and the underlying asset $S(t)$;
- replicate each dividend cash flow D_i using the replication strategy φ_i, such that $D_i = X^{\varphi_i}(T_i)$, where X^{φ_i} is the wealth process implied by the strategy φ_i. Sell the portfolio of these cash flows and cash in the present value $\sum_{i:\, t<T_i\leq T} X^{\varphi_i}(t) = \sum_{i:\, t<T_i\leq T} p(t,T_i)\mathbb{E}_t^{T_i}[X^{\varphi_i}(T_i)] = D(t,T)$.

This strategy gives the position $-C(t) - Kp(t,T) + S(t) - D(t,T) + P(t)$ with time-t cash flow $C(t) + Kp(t,T) - S(t) + D(t,T) - P(t) > 0$.

Time $\mathbf{T_i}$:
The stock pays dividends, which can directly be used to settle up the payments of the buyer of the dividend-payment-portfolio. Therefore observe that at time T_i it holds $p(T_i, T_i)\mathbb{E}_{T_i}^{T_i}[D_i] = D_i$ as D_i is the actual paid dividend and hence there is a cash flow of $D_i - p(T_i, T_i)\mathbb{E}_{T_i}^{T_i}[X^{\varphi_i}(T_i)] = 0$.

Time T:
If $T \neq T_n$ then $D(T, T) = 0$ and the value of the portfolio is equal $-C(T) - K + S(T) + P(T)$. Otherwise we have $D(T, T) = X^{\varphi_n}(T_n)$ and we can use the same argument as before. Thus the value of the portfolio is given by $-C(T) - K + S(T) + D_i - D(T, T) + P(T) = -C(T) - K + S(T) + P(T)$. In both cases we obtain

$$-C(T) - K + S(T) + P(T) = 0,$$

where the last equation follows from the classical put-call parity. Hence the strategy constructed above is a riskless gain and thus an arbitrage opportunity. So we must have

$$S(t) - D(t, T) + P(t) \geq C(t) + Kp(t, T). \tag{A.1}$$

2. Suppose that $\mathbf{S(t) - D(t, T) + P(t) > C(t) + Kp(t, T)}$:
By exchanging "sell" and "buy" in the previous step it follows that

$$S(t) - D(t, T) + P(t) \leq C(t) + Kp(t, T). \tag{A.2}$$

Combining (A.1) and (A.2) ensures the result. □

Appendix B

Tables of the Aggregate Statistics for the Intuitive Method

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	95	12%	13%	10%	45%	1%	10%
6 M	102	4%	6%	5%	29%	0%	5%
9 M	28	4%	9%	7%	17%	1%	5%
12 M	25	3%	8%	7%	32%	1%	7%
15 M	24	7%	16%	16%	24%	6%	7%
18 M	26	6%	17%	18%	27%	8%	6%
21 M	23	4%	18%	18%	27%	12%	3%
24 M	5	3%	21%	23%	26%	10%	6%
Total	328	6%	11%	9%	45%	0%	8%

Table B.1: 3M

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	94	9%	10%	8%	37%	0%	8%
6 M	103	9%	12%	11%	37%	0%	9%
9 M	65	9%	13%	12%	34%	1%	9%
12 M	26	8%	13%	12%	24%	2%	7%
15 M	20	11%	17%	14%	33%	0%	10%
18 M	26	11%	20%	16%	39%	6%	10%
21 M	26	8%	17%	18%	28%	3%	5%
24 M	14	6%	15%	15%	26%	4%	6%
Total	374	9%	13%	13%	39%	0%	9%

Table B.2: American Express

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	31	6%	7%	5%	18%	0%	5%
6 M	30	2%	3%	3%	10%	0%	3%
9 M	21	3%	4%	4%	8%	0%	2%
12 M	6	4%	8%	7%	15%	5%	4%
15 M	12	3%	7%	8%	11%	2%	3%
18 M	13	4%	13%	12%	17%	9%	3%
21 M	2	3%	11%	11%	11%	11%	0%
24 M	0						
Total	115	4%	6%	5%	18%	0%	5%

Table B.3: Apple

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	94	14%	15%	13%	46%	0%	11%
6 M	103	9%	12%	11%	63%	0%	9%
9 M	69	6%	10%	9%	40%	0%	7%
12 M	25	4%	9%	4%	33%	1%	10%
15 M	24	7%	16%	17%	28%	3%	8%
18 M	26	6%	16%	15%	28%	7%	6%
21 M	21	4%	17%	17%	28%	10%	4%
24 M	4	2%	17%	18%	25%	9%	9%
Total	366	8%	13%	11%	63%	0%	9%

Table B.4: Boeing

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	101	13%	15%	13%	50%	0%	11%
6 M	97	7%	10%	10%	39%	0%	7%
9 M	55	7%	14%	13%	39%	1%	8%
12 M	26	7%	21%	20%	54%	12%	9%
15 M	19	15%	25%	22%	61%	12%	11%
18 M	26	14%	25%	27%	34%	16%	6%
21 M	24	13%	28%	26%	62%	17%	11%
24 M	26	10%	26%	25%	34%	18%	4%
Total	374	10%	17%	16%	62%	0%	11%

Table B.5: Caterpillar

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	97	7%	8%	7%	24%	0%	6%
6 M	100	2%	3%	3%	12%	0%	2%
9 M	51	2%	4%	3%	10%	0%	2%
12 M	24	1%	3%	2%	9%	0%	2%
15 M	25	2%	4%	4%	8%	1%	2%
18 M	26	2%	6%	5%	14%	0%	3%
21 M	24	2%	9%	8%	23%	1%	5%
24 M	8	1%	11%	10%	18%	6%	5%
Total	355	3%	5%	4%	24%	0%	4%

Table B.6: Chevron

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	96	20%	24%	24%	68%	0%	16%
6 M	93	13%	19%	16%	75%	0%	14%
9 M	81	11%	20%	16%	57%	2%	14%
12 M	22	11%	27%	29%	54%	1%	15%
15 M	8	9%	15%	15%	24%	6%	5%
18 M	4	11%	21%	11%	60%	1%	27%
21 M	6	10%	21%	5%	57%	1%	27%
24 M	12	18%	45%	53%	66%	8%	18%
Total	322	15%	22%	17%	75%	0%	16%

Table B.7: Cisco

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	86	6%	7%	5%	29%	0%	6%
6 M	85	4%	5%	5%	13%	0%	3%
9 M	18	2%	6%	5%	13%	0%	3%
12 M	20	3%	10%	10%	20%	0%	4%
15 M	18	6%	13%	11%	23%	4%	6%
18 M	14	4%	13%	12%	23%	5%	6%
21 M	16	3%	16%	16%	20%	7%	3%
24 M	6	2%	16%	15%	23%	14%	3%
Total	263	4%	8%	7%	29%	0%	6%

Table B.8: Coca-Cola

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	93	7%	8%	7%	25%	0%	5%
6 M	99	3%	5%	4%	20%	0%	4%
9 M	19	2%	5%	6%	12%	0%	3%
12 M	12	2%	6%	6%	10%	3%	2%
15 M	15	3%	6%	6%	15%	1%	4%
18 M	21	3%	9%	8%	13%	6%	2%
21 M	12	2%	11%	11%	17%	4%	4%
24 M	1						
Total	272	4%	7%	6%	25%	0%	5%

Table B.9: Du Pont

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	101	8%	10%	9%	33%	0%	7%
6 M	96	4%	7%	5%	26%	0%	5%
9 M	25	5%	12%	11%	21%	3%	4%
12 M	24	2%	6%	6%	19%	0%	5%
15 M	25	3%	7%	7%	12%	0%	3%
18 M	26	4%	12%	12%	18%	5%	3%
21 M	22	4%	19%	16%	33%	9%	7%
24 M	4	2%	17%	14%	28%	12%	7%
Total	323	5%	9%	8%	33%	0%	7%

Table B.10: ExxonMobil

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	102	14%	15%	13%	47%	0%	12%
6 M	85	8%	12%	11%	29%	0%	8%
9 M	99	6%	12%	13%	29%	1%	7%
12 M	23	5%	12%	13%	28%	0%	7%
15 M	9	5%	12%	12%	16%	1%	4%
18 M	8	4%	14%	14%	19%	7%	5%
21 M	6	3%	16%	15%	27%	10%	7%
24 M	4	2%	14%	16%	18%	5%	6%
Total	336	9%	13%	12%	47%	0%	9%

Table B.11: General Electric

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	95	14%	16%	12%	66%	1%	14%
6 M	103	10%	15%	13%	56%	0%	11%
9 M	36	10%	22%	17%	53%	3%	15%
12 M	24	6%	25%	25%	55%	1%	16%
15 M	25	8%	18%	16%	36%	7%	7%
18 M	26	7%	21%	21%	42%	4%	10%
21 M	25	6%	31%	28%	65%	7%	16%
24 M	11	4%	30%	28%	71%	6%	21%
Total	345	10%	19%	16%	71%	0%	14%

Table B.12: Goldman Sachs

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	94	10%	11%	10%	47%	0%	9%
6 M	93	7%	10%	8%	35%	0%	8%
9 M	23	5%	13%	10%	49%	0%	11%
12 M	17	5%	22%	28%	35%	1%	12%
15 M	26	11%	24%	23%	34%	18%	4%
18 M	24	8%	25%	26%	30%	22%	3%
21 M	24	5%	25%	26%	28%	18%	3%
24 M	7	3%	26%	27%	33%	18%	6%
Total	308	8%	15%	13%	49%	0%	10%

Table B.13: The Home Depot

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	95	8%	9%	8%	63%	0%	8%
6 M	98	7%	10%	8%	30%	0%	8%
9 M	36	8%	16%	16%	41%	1%	9%
12 M	37	9%	18%	17%	37%	0%	9%
15 M	24	11%	24%	20%	38%	14%	9%
18 M	25	8%	25%	23%	43%	17%	5%
21 M	13	6%	26%	24%	37%	20%	6%
24 M	4	4%	26%	26%	29%	22%	4%
Total	332	8%	14%	13%	63%	0%	10%

Table B.14: IBM

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	98	7%	8%	6%	34%	0%	7%
6 M	97	5%	7%	6%	25%	0%	5%
9 M	42	4%	7%	7%	16%	0%	4%
12 M	10	3%	8%	8%	14%	0%	4%
15 M	7	5%	11%	11%	16%	3%	5%
18 M	8	3%	10%	10%	14%	6%	3%
21 M	10	3%	13%	13%	21%	5%	5%
24 M	2	1%	5%	5%	6%	5%	0%
Total	274	5%	8%	7%	34%	0%	5%

Table B.15: Intel

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	103	7%	8%	7%	28%	0%	6%
6 M	95	4%	6%	5%	21%	0%	5%
9 M	46	5%	9%	10%	19%	0%	5%
12 M	23	2%	7%	7%	13%	4%	2%
15 M	26	3%	7%	7%	12%	1%	2%
18 M	26	3%	10%	10%	22%	1%	5%
21 M	23	3%	14%	15%	25%	5%	6%
24 M	10	2%	17%	15%	25%	11%	4%
Total	352	5%	8%	8%	28%	0%	6%

Table B.16: Johnson & Johnson

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	95	7%	8%	7%	28%	0%	6%
6 M	102	6%	8%	7%	33%	0%	6%
9 M	54	5%	8%	8%	22%	0%	5%
12 M	6	7%	11%	12%	14%	7%	3%
15 M	16	9%	15%	16%	22%	8%	4%
18 M	21	11%	20%	20%	27%	14%	3%
21 M	14	9%	19%	17%	28%	13%	5%
24 M	7	7%	18%	18%	20%	14%	2%
Total	315	7%	10%	8%	33%	0%	7%

Table B.17: JPMorgan Chase

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	100	6%	7%	6%	27%	0%	5%
6 M	97	4%	5%	5%	15%	0%	4%
9 M	66	3%	6%	6%	14%	0%	3%
12 M	25	1%	5%	5%	11%	0%	3%
15 M	24	3%	6%	6%	14%	0%	3%
18 M	26	3%	8%	9%	14%	2%	4%
21 M	21	2%	8%	6%	18%	3%	4%
24 M	8	1%	8%	7%	14%	5%	3%
Total	367	4%	6%	6%	27%	0%	4%

Table B.18: McDonald's

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	100	8%	9%	7%	31%	0%	7%
6 M	96	4%	5%	4%	24%	0%	5%
9 M	48	3%	5%	4%	19%	0%	5%
12 M	1						
15 M	8	3%	6%	6%	10%	2%	3%
18 M	6	2%	6%	6%	7%	3%	1%
21 M	8	2%	8%	7%	11%	5%	2%
24 M	2	1%	10%	10%	11%	8%	2%
Total	269	5%	7%	5%	31%	0%	6%

Table B.19: Merck

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	94	9%	10%	10%	29%	0%	7%
6 M	100	5%	8%	7%	20%	0%	5%
9 M	46	3%	6%	6%	13%	0%	4%
12 M	14	3%	10%	12%	23%	1%	7%
15 M	9	6%	12%	12%	21%	2%	7%
18 M	12	4%	11%	9%	20%	4%	5%
21 M	10	4%	21%	22%	31%	8%	9%
24 M	5	3%	23%	25%	28%	13%	6%
Total	290	6%	9%	8%	31%	0%	7%

Table B.20: Microsoft

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	103	13%	14%	11%	80%	0%	13%
6 M	95	7%	10%	9%	30%	0%	6%
9 M	44	6%	12%	11%	29%	0%	6%
12 M	24	2%	11%	10%	24%	4%	5%
15 M	22	9%	20%	21%	32%	3%	8%
18 M	26	9%	27%	27%	33%	20%	4%
21 M	24	6%	27%	29%	39%	9%	7%
24 M	16	3%	27%	28%	37%	13%	8%
Total	354	8%	15%	13%	80%	0%	11%

Table B.21: Nike

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	94	9%	10%	8%	74%	0%	12%
6 M	103	4%	6%	5%	43%	0%	7%
9 M	68	3%	6%	6%	17%	0%	4%
12 M	9	1%	4%	4%	9%	0%	4%
15 M	7	2%	4%	5%	6%	2%	1%
18 M	5	3%	10%	10%	14%	5%	4%
21 M	11	2%	11%	12%	17%	3%	5%
24 M	1						
Total	298	5%	8%	6%	74%	0%	8%

Table B.22: Pfizer

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	84	21%	24%	23%	81%	0%	15%
6 M	75	11%	15%	11%	68%	1%	13%
9 M	50	5%	8%	6%	28%	0%	7%
12 M	33	6%	12%	11%	23%	4%	6%
15 M	28	4%	8%	7%	20%	2%	5%
18 M	26	2%	6%	5%	14%	0%	4%
21 M	16	1%	3%	2%	8%	0%	3%
24 M	0						
Total	312	10%	14%	10%	81%	0%	12%

Table B.23: Procter & Gamble

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	100	10%	11%	8%	59%	0%	11%
6 M	96	5%	8%	5%	44%	0%	8%
9 M	49	5%	9%	9%	39%	0%	7%
12 M	23	2%	8%	8%	13%	3%	3%
15 M	26	5%	11%	10%	38%	3%	8%
18 M	25	5%	16%	14%	47%	3%	10%
21 M	25	3%	15%	16%	27%	1%	7%
24 M	14	2%	15%	14%	34%	4%	9%
Total	358	6%	11%	9%	59%	0%	9%

Table B.24: Travelers

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	96	12%	14%	11%	53%	0%	10%
6 M	100	9%	14%	11%	57%	0%	12%
9 M	75	8%	15%	16%	40%	1%	9%
12 M	25	4%	18%	16%	30%	10%	6%
15 M	24	6%	14%	16%	25%	0%	9%
18 M	26	6%	18%	16%	63%	6%	11%
21 M	24	7%	33%	30%	61%	16%	11%
24 M	12	4%	33%	33%	45%	20%	7%
Total	382	9%	16%	15%	63%	0%	11%

Table B.25: UnitedHealth Group

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	95	11%	13%	11%	63%	1%	10%
6 M	103	5%	7%	6%	43%	0%	6%
9 M	72	3%	5%	4%	46%	0%	6%
12 M	25	1%	4%	4%	10%	0%	3%
15 M	24	1%	2%	2%	6%	0%	2%
18 M	26	1%	3%	2%	9%	0%	2%
21 M	20	1%	5%	4%	15%	0%	4%
24 M	5	1%	4%	5%	7%	1%	3%
Total	370	5%	7%	5%	63%	0%	8%

Table B.26: United Technologies

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	98	9%	10%	9%	40%	0%	7%
6 M	100	2%	3%	3%	9%	0%	2%
9 M	95	3%	4%	3%	80%	0%	8%
12 M	21	2%	4%	3%	8%	0%	2%
15 M	9	2%	4%	5%	6%	1%	2%
18 M	13	3%	5%	5%	9%	1%	2%
21 M	15	2%	4%	3%	9%	0%	2%
24 M	7	2%	5%	6%	10%	2%	3%
Total	358	4%	5%	4%	80%	0%	6%

Table B.27: Verizon

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	97	9%	9%	7%	36%	0%	7%
6 M	101	4%	6%	5%	19%	0%	4%
9 M	52	4%	7%	6%	22%	0%	5%
12 M	25	2%	7%	7%	17%	0%	4%
15 M	24	3%	5%	5%	15%	0%	4%
18 M	26	2%	7%	7%	14%	0%	5%
21 M	24	2%	10%	8%	22%	2%	6%
24 M	10	1%	11%	10%	22%	6%	5%
Total	359	5%	7%	6%	36%	0%	6%

Table B.28: Wal-Mart

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	26	17%	19%	20%	37%	5%	9%
6 M	24	14%	21%	23%	36%	4%	10%
9 M	21	8%	19%	19%	44%	1%	12%
12 M	13	2%	13%	11%	25%	2%	8%
15 M	21	4%	10%	9%	29%	0%	7%
18 M	21	4%	12%	11%	33%	1%	8%
21 M	15	4%	20%	18%	34%	1%	11%
24 M	7	1%	5%	4%	10%	1%	4%
Total	148	8%	16%	14%	44%	0%	10%

Table B.29: Walt Disney

Note that if there is only one estimate for a specific Δ, i.e. the corresponding Counter is equal to one, it is not possible or rather it does not make sense to calculate the values in the table.

Appendix C

Tables of the Aggregate Statistics for the Δ Method

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	95	16%	18%	10%	86%	0%	23%
6 M	102	4%	6%	5%	26%	0%	5%
9 M	28	4%	9%	8%	18%	0%	5%
12 M	25	2%	5%	3%	25%	0%	6%
15 M	24	3%	7%	6%	19%	0%	5%
18 M	26	3%	9%	8%	21%	1%	6%
21 M	23	2%	8%	7%	21%	0%	6%
24 M	5	2%	12%	13%	17%	3%	6%
Total	328	7%	10%	6%	86%	0%	14%

Table C.1: 3M

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	94	9%	10%	8%	35%	0%	8%
6 M	103	7%	9%	8%	35%	0%	7%
9 M	65	5%	8%	7%	25%	0%	6%
12 M	26	2%	4%	3%	14%	0%	4%
15 M	20	3%	4%	4%	10%	0%	3%
18 M	26	4%	7%	4%	25%	0%	7%
21 M	26	2%	5%	4%	15%	1%	4%
24 M	14	2%	6%	5%	21%	1%	6%
Total	374	6%	8%	6%	35%	0%	7%

Table C.2: American Express

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	31	9%	10%	9%	22%	0%	6%
6 M	30	3%	5%	3%	15%	0%	4%
9 M	21	2%	3%	2%	7%	0%	2%
12 M	6	2%	3%	2%	9%	0%	4%
15 M	12	1%	2%	1%	11%	0%	3%
18 M	13	2%	4%	4%	9%	0%	3%
21 M	2	0%	0%	0%	1%	0%	0%
24 M	0						
Total	115	4%	5%	4%	22%	0%	5%

Table C.3: Apple

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	94	10%	12%	10%	48%	0%	9%
6 M	103	8%	10%	8%	39%	0%	8%
9 M	69	6%	9%	7%	44%	0%	7%
12 M	25	5%	11%	6%	35%	1%	10%
15 M	24	5%	11%	9%	31%	0%	9%
18 M	26	3%	9%	4%	26%	0%	10%
21 M	21	2%	10%	10%	24%	1%	6%
24 M	4	2%	12%	12%	22%	1%	10%
Total	366	7%	10%	8%	48%	0%	8%

Table C.4: Boeing

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	101	24%	27%	22%	91%	0%	20%
6 M	97	7%	9%	7%	32%	0%	7%
9 M	55	5%	10%	9%	32%	0%	7%
12 M	26	3%	9%	8%	39%	0%	8%
15 M	19	4%	8%	8%	13%	1%	4%
18 M	26	6%	10%	8%	22%	1%	7%
21 M	24	3%	7%	5%	23%	1%	6%
24 M	26	4%	9%	11%	22%	0%	6%
Total	374	10%	14%	11%	91%	0%	14%

Table C.5: Caterpillar

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	97	16%	18%	18%	38%	0%	10%
6 M	100	3%	4%	3%	12%	0%	3%
9 M	51	2%	4%	4%	10%	0%	3%
12 M	24	1%	4%	3%	8%	0%	3%
15 M	25	1%	3%	3%	6%	0%	2%
18 M	26	1%	3%	2%	13%	0%	3%
21 M	24	1%	2%	2%	10%	0%	2%
24 M	8	0%	3%	3%	5%	0%	2%
Total	355	6%	8%	4%	38%	0%	9%

Table C.6: Chevron

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	96	19%	23%	16%	88%	0%	22%
6 M	93	13%	18%	14%	84%	1%	15%
9 M	81	8%	15%	11%	44%	1%	12%
12 M	22	6%	15%	18%	30%	0%	10%
15 M	8	2%	3%	2%	11%	1%	3%
18 M	4	3%	6%	4%	16%	0%	7%
21 M	6	7%	15%	17%	18%	9%	4%
24 M	12	2%	5%	3%	24%	0%	8%
Total	322	12%	17%	13%	88%	0%	16%

Table C.7: Cisco

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	86	11%	13%	11%	33%	0%	10%
6 M	85	4%	5%	4%	14%	0%	4%
9 M	18	2%	5%	5%	14%	0%	4%
12 M	20	1%	4%	3%	11%	1%	3%
15 M	18	2%	5%	3%	14%	1%	4%
18 M	14	2%	5%	4%	12%	0%	4%
21 M	16	1%	4%	4%	9%	0%	3%
24 M	6	0%	4%	3%	10%	1%	3%
Total	263	5%	8%	5%	33%	0%	7%

Table C.8: Coca-Cola

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	93	13%	15%	14%	45%	0%	11%
6 M	99	3%	5%	5%	18%	0%	4%
9 M	19	2%	5%	5%	12%	1%	3%
12 M	12	2%	4%	5%	11%	0%	3%
15 M	15	1%	2%	2%	8%	0%	2%
18 M	21	1%	3%	3%	8%	0%	2%
21 M	12	1%	5%	3%	11%	1%	3%
24 M	1						
Total	272	6%	8%	5%	45%	0%	8%

Table C.9: Du Pont

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	101	14%	16%	15%	55%	0%	11%
6 M	96	4%	6%	5%	18%	0%	5%
9 M	25	4%	10%	9%	21%	1%	5%
12 M	24	1%	4%	3%	13%	0%	3%
15 M	25	2%	5%	5%	9%	1%	2%
18 M	26	2%	5%	5%	14%	0%	4%
21 M	22	1%	7%	8%	15%	0%	4%
24 M	4	1%	5%	5%	8%	4%	2%
Total	323	7%	9%	7%	55%	0%	9%

Table C.10: ExxonMobil

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	102	21%	24%	22%	71%	0%	17%
6 M	85	10%	13%	11%	37%	0%	10%
9 M	99	5%	10%	8%	31%	0%	7%
12 M	23	3%	8%	10%	23%	0%	5%
15 M	9	2%	3%	2%	10%	0%	3%
18 M	8	1%	4%	2%	12%	1%	4%
21 M	6	2%	9%	8%	16%	3%	4%
24 M	4	1%	4%	3%	11%	0%	4%
Total	336	10%	14%	10%	71%	0%	13%

Table C.11: General Electric

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	95	15%	17%	13%	57%	0%	14%
6 M	103	10%	15%	12%	68%	0%	12%
9 M	36	7%	15%	10%	45%	0%	13%
12 M	24	3%	13%	13%	38%	1%	10%
15 M	25	3%	6%	6%	14%	1%	4%
18 M	26	3%	8%	7%	24%	1%	6%
21 M	25	2%	10%	7%	29%	0%	10%
24 M	11	2%	15%	11%	39%	1%	12%
Total	345	9%	14%	11%	68%	0%	12%

Table C.12: Goldman Sachs

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	94	10%	12%	8%	48%	0%	10%
6 M	93	7%	10%	8%	42%	0%	8%
9 M	23	4%	10%	8%	49%	0%	11%
12 M	17	3%	13%	12%	26%	0%	7%
15 M	26	3%	6%	4%	26%	0%	7%
18 M	24	2%	5%	6%	11%	1%	3%
21 M	24	1%	3%	3%	8%	0%	2%
24 M	7	1%	5%	6%	8%	0%	3%
Total	308	6%	9%	7%	49%	0%	8%

Table C.13: The Home Depot

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	95	13%	15%	14%	53%	0%	11%
6 M	98	7%	10%	9%	27%	0%	7%
9 M	36	6%	12%	10%	30%	1%	8%
12 M	37	5%	10%	8%	27%	1%	7%
15 M	24	6%	12%	10%	27%	2%	8%
18 M	25	4%	11%	12%	21%	2%	5%
21 M	13	2%	7%	6%	16%	2%	4%
24 M	4	1%	10%	9%	17%	5%	5%
Total	332	8%	12%	10%	53%	0%	9%

Table C.14: IBM

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	98	12%	14%	14%	34%	0%	8%
6 M	97	5%	7%	6%	36%	0%	6%
9 M	42	4%	7%	6%	15%	0%	4%
12 M	10	2%	6%	5%	14%	1%	5%
15 M	7	4%	8%	6%	15%	2%	5%
18 M	8	1%	4%	4%	8%	0%	2%
21 M	10	1%	7%	7%	17%	0%	6%
24 M	2	1%	4%	4%	4%	4%	0%
Total	274	7%	9%	8%	36%	0%	7%

Table C.15: Intel

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	103	9%	10%	7%	44%	0%	9%
6 M	95	5%	7%	6%	19%	0%	4%
9 M	46	4%	7%	7%	17%	0%	4%
12 M	23	1%	3%	2%	7%	0%	2%
15 M	26	2%	3%	3%	11%	0%	2%
18 M	26	2%	6%	6%	12%	0%	3%
21 M	23	1%	5%	5%	15%	1%	3%
24 M	10	1%	4%	2%	12%	1%	4%
Total	352	5%	7%	5%	44%	0%	6%

Table C.16: Johnson & Johnson

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	95	9%	11%	10%	31%	0%	8%
6 M	102	6%	8%	7%	29%	0%	5%
9 M	54	5%	7%	6%	22%	1%	5%
12 M	6	4%	6%	7%	11%	1%	4%
15 M	16	2%	3%	2%	7%	0%	2%
18 M	21	2%	4%	3%	10%	0%	3%
21 M	14	2%	4%	4%	7%	1%	2%
24 M	7	1%	3%	4%	5%	0%	2%
Total	315	6%	8%	6%	31%	0%	6%

Table C.17: JPMorgan Chase

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	100	6%	7%	6%	28%	0%	6%
6 M	97	4%	6%	6%	24%	0%	4%
9 M	66	2%	4%	4%	11%	0%	3%
12 M	25	1%	4%	3%	10%	1%	2%
15 M	24	2%	3%	2%	11%	0%	3%
18 M	26	2%	4%	3%	10%	0%	3%
21 M	21	0%	2%	2%	5%	0%	1%
24 M	8	0%	3%	2%	10%	0%	3%
Total	367	3%	5%	4%	28%	0%	4%

Table C.18: McDonald's

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	100	11%	13%	13%	41%	0%	8%
6 M	96	3%	4%	3%	20%	0%	4%
9 M	48	2%	4%	4%	16%	0%	4%
12 M	1						
15 M	8	1%	2%	2%	5%	1%	1%
18 M	6	0%	1%	1%	3%	0%	1%
21 M	6	0%	2%	2%	4%	0%	2%
24 M	1						
Total	266	6%	7%	5%	41%	0%	7%

Table C.19: Merck

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	94	13%	15%	14%	38%	0%	10%
6 M	100	6%	8%	7%	27%	0%	6%
9 M	46	4%	7%	7%	15%	0%	3%
12 M	14	3%	8%	8%	19%	0%	6%
15 M	9	4%	9%	7%	18%	2%	6%
18 M	12	1%	4%	3%	11%	0%	4%
21 M	10	2%	10%	9%	21%	0%	8%
24 M	5	2%	12%	12%	18%	5%	5%
Total	290	7%	10%	8%	38%	0%	8%

Table C.20: Microsoft

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	103	12%	13%	10%	54%	0%	11%
6 M	95	6%	8%	7%	26%	0%	6%
9 M	44	4%	8%	7%	21%	0%	5%
12 M	24	1%	4%	3%	16%	0%	4%
15 M	22	2%	5%	6%	10%	1%	3%
18 M	26	3%	9%	9%	15%	0%	4%
21 M	24	1%	7%	5%	26%	1%	7%
24 M	16	1%	6%	5%	18%	1%	5%
Total	354	6%	9%	7%	54%	0%	8%

Table C.21: Nike

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	94	13%	15%	14%	64%	0%	10%
6 M	103	5%	7%	5%	37%	0%	7%
9 M	68	3%	5%	5%	14%	0%	3%
12 M	9	2%	4%	4%	10%	1%	3%
15 M	7	1%	2%	1%	4%	1%	1%
18 M	5	1%	3%	2%	6%	1%	2%
21 M	11	1%	3%	3%	7%	0%	2%
24 M	1						
Total	298	7%	9%	6%	64%	0%	8%

Table C.22: Pfizer

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	84	16%	18%	17%	68%	0%	12%
6 M	75	10%	14%	12%	58%	0%	9%
9 M	50	6%	11%	11%	25%	1%	4%
12 M	33	4%	8%	4%	23%	0%	8%
15 M	28	3%	6%	4%	22%	0%	6%
18 M	26	2%	6%	5%	25%	1%	5%
21 M	16	1%	6%	6%	10%	1%	3%
24 M	0						
Total	312	9%	12%	10%	68%	0%	10%

Table C.23: Procter & Gamble

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	100	12%	14%	12%	67%	0%	11%
6 M	96	5%	8%	6%	34%	0%	7%
9 M	49	4%	7%	6%	37%	0%	7%
12 M	23	1%	4%	3%	18%	0%	4%
15 M	26	3%	7%	4%	26%	0%	8%
18 M	25	2%	6%	3%	35%	0%	8%
21 M	25	2%	7%	6%	23%	0%	6%
24 M	14	1%	7%	5%	17%	2%	5%
Total	358	6%	9%	6%	67%	0%	9%

Table C.24: Travelers

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	96	11%	13%	11%	45%	0%	10%
6 M	100	9%	13%	10%	45%	0%	10%
9 M	75	6%	11%	10%	31%	1%	7%
12 M	25	2%	8%	7%	15%	1%	3%
15 M	24	3%	6%	5%	16%	0%	4%
18 M	26	1%	5%	3%	23%	0%	6%
21 M	24	2%	11%	12%	26%	1%	6%
24 M	12	1%	8%	7%	18%	1%	5%
Total	382	7%	11%	9%	45%	0%	9%

Table C.25: UnitedHealth Group

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	95	13%	15%	14%	66%	0%	10%
6 M	103	5%	8%	6%	50%	1%	6%
9 M	72	3%	6%	5%	45%	0%	6%
12 M	25	1%	4%	3%	12%	0%	3%
15 M	24	2%	4%	3%	7%	1%	2%
18 M	26	2%	6%	7%	12%	1%	3%
21 M	20	1%	4%	4%	19%	0%	4%
24 M	5	1%	4%	2%	13%	1%	5%
Total	370	6%	8%	6%	66%	0%	8%

Table C.26: United Technologies

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	98	11%	12%	12%	28%	1%	5%
6 M	100	4%	5%	4%	16%	0%	3%
9 M	95	2%	3%	2%	73%	0%	7%
12 M	21	1%	2%	1%	7%	0%	2%
15 M	9	1%	2%	1%	4%	0%	1%
18 M	13	1%	2%	1%	5%	0%	1%
21 M	15	1%	2%	2%	4%	0%	1%
24 M	7	1%	2%	2%	5%	1%	2%
Total	358	5%	6%	4%	73%	0%	6%

Table C.27: Verizon

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	97	9%	9%	8%	26%	0%	7%
6 M	101	5%	6%	5%	19%	0%	5%
9 M	52	4%	7%	6%	25%	0%	6%
12 M	25	1%	5%	6%	13%	0%	3%
15 M	24	2%	4%	3%	12%	1%	3%
18 M	26	1%	3%	3%	8%	0%	2%
21 M	24	1%	4%	3%	12%	0%	3%
24 M	10	1%	4%	5%	9%	0%	3%
Total	359	5%	6%	5%	26%	0%	5%

Table C.28: Wal-Mart

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	26	6%	7%	5%	24%	0%	6%
6 M	24	8%	12%	8%	34%	2%	9%
9 M	21	3%	7%	4%	29%	0%	7%
12 M	13	2%	8%	3%	27%	0%	9%
15 M	21	4%	9%	6%	34%	2%	8%
18 M	21	3%	9%	9%	18%	1%	4%
21 M	15	3%	14%	13%	25%	5%	5%
24 M	7	1%	4%	2%	11%	0%	4%
Total	148	4%	9%	8%	34%	0%	7%

Table C.29: Walt Disney

Appendix D

Tables of the Aggregate Statistics for 1Y Method

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	95	11%	13%	11%	45%	0%	9%
6 M	102	6%	9%	9%	33%	0%	6%
9 M	28	3%	8%	8%	18%	0%	5%
12 M	25	3%	8%	4%	34%	1%	8%
15 M	24	8%	18%	19%	24%	6%	5%
18 M	26	7%	20%	22%	35%	1%	8%
21 M	23	4%	17%	17%	29%	10%	4%
24 M	5	3%	22%	25%	25%	13%	5%
Total	328	7%	12%	11%	45%	0%	8%

Table D.1: 3M

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	94	14%	15%	13%	99%	0%	15%
6 M	103	11%	15%	13%	131%	0%	16%
9 M	65	10%	15%	11%	146%	0%	20%
12 M	26	8%	14%	5%	146%	0%	29%
15 M	20	12%	19%	18%	37%	4%	10%
18 M	26	12%	21%	21%	47%	4%	11%
21 M	26	8%	17%	18%	30%	1%	8%
24 M	14	8%	21%	12%	140%	4%	35%
Total	374	11%	16%	13%	146%	0%	18%

Table D.2: American Express

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	31	8%	9%	8%	36%	1%	7%
6 M	30	3%	5%	3%	17%	0%	4%
9 M	21	3%	4%	3%	13%	0%	3%
12 M	6	3%	6%	5%	11%	3%	3%
15 M	12	3%	7%	8%	14%	1%	4%
18 M	13	4%	12%	12%	16%	8%	2%
21 M	2	3%	11%	11%	12%	10%	1%
24 M	0						
Total	115	5%	7%	6%	36%	0%	6%

Table D.3: Apple

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	94	13%	15%	14%	46%	0%	11%
6 M	103	9%	12%	12%	76%	0%	9%
9 M	69	7%	11%	8%	30%	0%	7%
12 M	25	4%	10%	7%	27%	0%	8%
15 M	24	7%	17%	16%	28%	7%	5%
18 M	26	6%	18%	17%	39%	2%	10%
21 M	21	4%	17%	17%	27%	7%	6%
24 M	4	2%	16%	18%	30%	0%	14%
Total	366	9%	14%	13%	76%	0%	9%

Table D.4: Boeing

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	100	14%	16%	12%	58%	0%	13%
6 M	96	7%	11%	10%	42%	0%	8%
9 M	55	8%	15%	15%	43%	1%	8%
12 M	26	7%	21%	20%	57%	7%	10%
15 M	19	16%	26%	22%	63%	7%	13%
18 M	25	15%	26%	29%	36%	14%	8%
21 M	24	13%	28%	26%	60%	11%	12%
24 M	26	10%	26%	27%	37%	16%	6%
Total	371	11%	18%	16%	63%	0%	12%

Table D.5: Caterpillar

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	97	8%	9%	8%	27%	0%	6%
6 M	100	4%	5%	5%	21%	0%	4%
9 M	51	3%	6%	5%	23%	1%	4%
12 M	24	2%	5%	5%	13%	1%	3%
15 M	25	3%	7%	7%	13%	3%	3%
18 M	26	3%	8%	7%	20%	1%	5%
21 M	24	1%	6%	7%	19%	0%	5%
24 M	8	1%	9%	9%	13%	6%	3%
Total	355	4%	7%	6%	27%	0%	5%

Table D.6: Chevron

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	96	27%	32%	26%	105%	0%	24%
6 M	93	16%	23%	20%	88%	2%	18%
9 M	81	12%	21%	20%	60%	1%	14%
12 M	22	11%	24%	22%	48%	7%	12%
15 M	8	13%	21%	22%	29%	6%	7%
18 M	4	11%	20%	12%	57%	0%	26%
21 M	6	11%	23%	10%	55%	5%	24%
24 M	12	15%	37%	36%	64%	11%	14%
Total	322	18%	26%	23%	105%	0%	19%

Table D.7: Cisco

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	84	10%	11%	9%	40%	0%	9%
6 M	84	6%	9%	8%	22%	0%	6%
9 M	18	2%	5%	3%	11%	0%	3%
12 M	19	2%	7%	7%	14%	0%	4%
15 M	18	6%	14%	17%	26%	1%	9%
18 M	14	4%	13%	11%	29%	1%	9%
21 M	16	3%	13%	13%	21%	5%	4%
24 M	6	2%	13%	13%	21%	8%	5%
Total	259	6%	10%	9%	40%	0%	7%

Table D.8: Coca-Cola

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	93	6%	7%	6%	40%	0%	6%
6 M	98	4%	6%	5%	20%	0%	5%
9 M	19	2%	4%	3%	12%	1%	4%
12 M	12	2%	5%	5%	9%	1%	3%
15 M	15	3%	7%	5%	16%	2%	5%
18 M	20	4%	11%	12%	17%	3%	4%
21 M	12	3%	12%	11%	19%	7%	4%
24 M	1						
Total	270	5%	7%	6%	40%	0%	5%

Table D.9: Du Pont

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	101	9%	10%	8%	42%	0%	9%
6 M	96	6%	8%	7%	41%	0%	7%
9 M	25	3%	8%	8%	21%	2%	5%
12 M	24	3%	8%	9%	18%	0%	5%
15 M	25	5%	10%	11%	21%	2%	4%
18 M	26	4%	12%	14%	23%	3%	5%
21 M	22	4%	17%	17%	30%	5%	8%
24 M	4	2%	15%	12%	26%	7%	8%
Total	323	6%	10%	9%	42%	0%	8%

Table D.10: ExxonMobil

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	102	16%	18%	16%	56%	1%	14%
6 M	85	9%	12%	11%	34%	0%	9%
9 M	99	7%	13%	12%	38%	0%	8%
12 M	23	6%	13%	8%	36%	1%	11%
15 M	9	8%	16%	15%	32%	6%	7%
18 M	8	3%	10%	8%	21%	3%	7%
21 M	6	2%	12%	11%	28%	1%	11%
24 M	4	1%	7%	8%	10%	0%	5%
Total	336	10%	14%	12%	56%	0%	11%

Table D.11: General Electric

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	95	19%	22%	15%	107%	0%	20%
6 M	103	12%	17%	14%	59%	1%	13%
9 M	36	8%	17%	18%	48%	0%	12%
12 M	24	5%	21%	18%	56%	1%	16%
15 M	25	9%	20%	24%	40%	1%	11%
18 M	26	7%	22%	19%	46%	2%	11%
21 M	25	6%	27%	24%	60%	5%	14%
24 M	11	3%	26%	20%	59%	0%	17%
Total	345	12%	20%	17%	107%	0%	15%

Table D.12: Goldman Sachs

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	94	14%	16%	14%	92%	0%	14%
6 M	93	11%	15%	14%	38%	0%	10%
9 M	23	5%	13%	8%	55%	2%	13%
12 M	17	5%	21%	27%	34%	2%	12%
15 M	26	11%	25%	27%	41%	4%	8%
18 M	24	9%	28%	27%	41%	14%	8%
21 M	24	5%	24%	23%	46%	10%	7%
24 M	7	3%	25%	26%	35%	18%	6%
Total	308	10%	18%	18%	92%	0%	12%

Table D.13: The Home Depot

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	95	11%	12%	9%	52%	0%	11%
6 M	98	9%	12%	10%	48%	0%	9%
9 M	36	8%	14%	11%	40%	1%	11%
12 M	37	9%	18%	18%	37%	1%	10%
15 M	24	12%	27%	29%	39%	3%	9%
18 M	25	9%	26%	26%	38%	17%	7%
21 M	13	5%	24%	22%	43%	11%	9%
24 M	4	3%	21%	21%	28%	15%	6%
Total	332	9%	16%	13%	52%	0%	11%

Table D.14: IBM

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	98	12%	13%	12%	44%	0%	9%
6 M	97	8%	11%	11%	35%	0%	8%
9 M	42	6%	12%	11%	28%	1%	7%
12 M	10	4%	13%	15%	19%	2%	6%
15 M	7	6%	14%	14%	21%	6%	5%
18 M	8	3%	8%	9%	15%	0%	5%
21 M	10	2%	12%	12%	26%	2%	8%
24 M	2	1%	7%	7%	11%	3%	6%
Total	274	8%	12%	11%	44%	0%	8%

Table D.15: Intel

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	103	9%	11%	8%	45%	0%	8%
6 M	95	5%	7%	7%	20%	0%	5%
9 M	46	5%	8%	6%	22%	0%	6%
12 M	23	2%	6%	5%	18%	0%	4%
15 M	26	4%	9%	9%	18%	1%	5%
18 M	26	3%	10%	9%	22%	0%	7%
21 M	23	3%	12%	12%	23%	4%	6%
24 M	10	2%	13%	12%	17%	9%	3%
Total	352	6%	9%	8%	45%	0%	7%

Table D.16: Johnson & Johnson

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	95	11%	13%	9%	50%	0%	12%
6 M	102	8%	12%	9%	57%	0%	11%
9 M	54	6%	9%	8%	30%	1%	7%
12 M	6	7%	12%	12%	21%	3%	7%
15 M	16	10%	16%	17%	22%	4%	6%
18 M	21	10%	19%	19%	27%	8%	5%
21 M	14	8%	16%	16%	31%	9%	6%
24 M	7	6%	14%	16%	27%	2%	8%
Total	315	9%	12%	10%	57%	0%	10%

Table D.17: JPMorgan Chase

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	100	7%	8%	8%	24%	0%	5%
6 M	97	4%	6%	5%	21%	0%	4%
9 M	66	3%	5%	5%	17%	0%	4%
12 M	25	2%	5%	5%	10%	1%	3%
15 M	24	3%	8%	7%	15%	0%	4%
18 M	26	3%	8%	7%	16%	1%	4%
21 M	21	2%	7%	6%	18%	0%	5%
24 M	8	1%	6%	6%	11%	2%	3%
Total	367	4%	7%	6%	24%	0%	5%

Table D.18: McDonald's

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	100	13%	15%	7%	96%	0%	20%
6 M	96	8%	12%	6%	67%	0%	17%
9 M	48	7%	15%	6%	61%	0%	19%
12 M	1						
15 M	8	3%	6%	6%	10%	2%	3%
18 M	6	2%	5%	5%	8%	1%	3%
21 M	8	1%	6%	6%	10%	1%	3%
24 M	2	1%	10%	10%	13%	7%	4%
Total	269	9%	13%	6%	96%	0%	18%

Table D.19: Merck

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	94	11%	12%	10%	44%	0%	10%
6 M	100	7%	10%	8%	32%	0%	8%
9 M	46	4%	7%	7%	18%	0%	4%
12 M	14	3%	10%	9%	23%	1%	6%
15 M	9	7%	14%	16%	18%	5%	5%
18 M	12	4%	13%	12%	20%	5%	4%
21 M	10	4%	20%	20%	34%	3%	9%
24 M	5	3%	24%	25%	31%	14%	6%
Total	290	7%	11%	9%	44%	0%	9%

Table D.20: Microsoft

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	103	16%	19%	15%	104%	0%	17%
6 M	95	8%	11%	10%	31%	0%	8%
9 M	44	6%	11%	9%	28%	0%	7%
12 M	24	2%	10%	9%	24%	1%	5%
15 M	22	8%	18%	22%	27%	1%	8%
18 M	26	9%	26%	26%	40%	17%	6%
21 M	24	5%	22%	22%	42%	2%	9%
24 M	16	3%	25%	24%	38%	8%	9%
Total	354	9%	16%	14%	104%	0%	12%

Table D.21: Nike

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	94	11%	12%	9%	70%	0%	12%
6 M	103	5%	8%	6%	40%	0%	8%
9 M	68	4%	6%	6%	20%	0%	4%
12 M	9	2%	4%	3%	10%	0%	3%
15 M	7	2%	4%	5%	7%	1%	2%
18 M	5	2%	8%	8%	12%	3%	4%
21 M	11	2%	10%	9%	21%	2%	6%
24 M	1						
Total	298	6%	9%	6%	70%	0%	9%

Table D.22: Pfizer

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	84	18%	20%	20%	78%	0%	14%
6 M	75	11%	15%	11%	69%	1%	13%
9 M	50	5%	9%	8%	24%	0%	6%
12 M	33	3%	5%	5%	13%	0%	4%
15 M	28	5%	10%	7%	30%	1%	8%
18 M	26	3%	9%	9%	24%	0%	6%
21 M	16	1%	5%	4%	12%	0%	3%
24 M	0						
Total	312	9%	13%	10%	78%	0%	12%

Table D.23: Procter & Gamble

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	100	14%	16%	12%	67%	0%	14%
6 M	96	8%	11%	9%	40%	0%	10%
9 M	49	6%	11%	8%	42%	0%	11%
12 M	23	2%	9%	7%	30%	0%	7%
15 M	26	7%	15%	18%	38%	0%	10%
18 M	25	5%	15%	14%	40%	0%	10%
21 M	25	3%	13%	10%	41%	0%	11%
24 M	14	2%	14%	13%	32%	1%	11%
Total	358	8%	13%	11%	67%	0%	11%

Table D.24: Travelers

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	96	12%	14%	10%	61%	0%	12%
6 M	100	11%	16%	13%	58%	0%	12%
9 M	75	8%	15%	13%	45%	1%	10%
12 M	25	4%	17%	15%	32%	5%	8%
15 M	24	8%	18%	16%	37%	2%	12%
18 M	26	8%	23%	22%	64%	5%	12%
21 M	24	6%	28%	27%	53%	16%	8%
24 M	12	4%	29%	30%	43%	15%	9%
Total	382	9%	17%	15%	64%	0%	12%

Table D.25: UnitedHealth Group

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	95	10%	12%	10%	55%	0%	9%
6 M	103	6%	9%	6%	45%	0%	8%
9 M	72	5%	8%	6%	40%	0%	7%
12 M	25	2%	7%	7%	20%	2%	5%
15 M	24	2%	5%	5%	10%	0%	3%
18 M	26	2%	6%	5%	17%	0%	4%
21 M	20	1%	5%	4%	17%	1%	4%
24 M	5	1%	5%	4%	8%	3%	2%
Total	370	6%	9%	7%	55%	0%	8%

Table D.26: United Technologies

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	98	11%	12%	11%	39%	0%	8%
6 M	100	3%	4%	4%	19%	0%	3%
9 M	95	3%	4%	3%	78%	0%	8%
12 M	21	2%	3%	2%	12%	0%	3%
15 M	9	2%	4%	3%	7%	0%	2%
18 M	13	2%	4%	3%	7%	0%	2%
21 M	15	1%	3%	3%	7%	0%	2%
24 M	7	2%	5%	5%	8%	3%	2%
Total	358	5%	6%	4%	78%	0%	7%

Table D.27: Verizon

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	97	10%	11%	9%	31%	0%	8%
6 M	101	6%	9%	8%	28%	0%	6%
9 M	52	5%	9%	8%	22%	0%	6%
12 M	25	2%	9%	9%	26%	1%	5%
15 M	24	3%	8%	7%	19%	0%	6%
18 M	26	3%	8%	9%	15%	0%	5%
21 M	24	3%	12%	12%	23%	1%	7%
24 M	10	1%	11%	11%	14%	2%	4%
Total	359	6%	10%	8%	31%	0%	7%

Table D.28: Wal-Mart

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	25	15%	17%	14%	42%	2%	11%
6 M	23	12%	20%	22%	29%	5%	7%
9 M	14	9%	21%	17%	53%	8%	13%
12 M	1						
15 M	19	5%	11%	11%	32%	1%	9%
18 M	20	3%	9%	7%	31%	2%	7%
21 M	12	4%	19%	18%	35%	4%	9%
24 M	0						
Total	114	9%	16%	14%	53%	1%	10%

Table D.29: Walt Disney

Appendix E

Tables of the Backtest against the European Method

	Counter		Weighted Average		Average		Median		Worst Case		Best Case		Standard Deviation	
Δ	**E**	**A**	**E**	**A**	**E**	**A**	**E**	**A**	**E**	**A**	**E**	**A**	**E**	**A**
3 M	23	24	4%	6%	5%	7%	4%	7%	10%	15%	0%	0%	3%	4%
6 M	23	20	6%	6%	8%	7%	6%	6%	15%	16%	2%	0%	4%	5%
9 M	26	17	10%	11%	14%	16%	15%	16%	22%	29%	1%	7%	6%	6%
12 M	24	16	10%	6%	18%	11%	21%	9%	26%	26%	11%	5%	6%	5%
15 M	25	10	6%	11%	13%	24%	14%	13%	26%	50%	6%	4%	5%	19%
18 M	21	8	6%	15%	15%	42%	14%	51%	17%	57%	10%	5%	2%	20%
21 M	26	0	8%		17%		16%		24%		8%		4%	
24 M	26	2	7%	3%	19%	7%	20%	7%	24%	8%	15%	6%	3%	1%
> 24 M	192		5%		17%		17%		23%		8%		3%	
Total	315	97	7%	8%	15%	14%	16%	9%	26%	57%	0%	0%	5%	13%

Table E.1: BASF

	Counter		Weighted Average		Average		Median		Worst Case		Best Case		Standard Deviation	
Δ	**E**	**A**	**E**	**A**	**E**	**A**	**E**	**A**	**E**	**A**	**E**	**A**	**E**	**A**
3 M	23	24	4%	5%	5%	6%	6%	6%	14%	11%	0%	1%	5%	3%
6 M	24	15	6%	7%	8%	10%	6%	9%	14%	29%	2%	1%	5%	7%
9 M	25	12	11%	9%	16%	13%	15%	13%	29%	18%	2%	10%	5%	2%
12 M	25	9	12%	11%	20%	19%	21%	20%	32%	21%	8%	14%	4%	2%
15 M	24	9	7%	3%	15%	6%	15%	1%	26%	24%	10%	0%	4%	9%
18 M	19	1	6%		14%		14%		19%		9%		3%	
21 M	25	4	8%	2%	17%	4%	17%	4%	23%	7%	11%	2%	4%	2%
24 M	25	1	9%		24%		23%		31%		18%		3%	
> 24 M	95		6%		23%		22%		39%		14%		5%	
Total	228	75	9%	6%	18%	9%	19%	9%	39%	29%	0%	0%	8%	6%

Table E.2: Bayer

	Counter		Weighted Average		Average		Median		Worst Case		Best Case		Standard Deviation	
Δ	**E**	**A**	**E**	**A**	**E**	**A**	**E**	**A**	**E**	**A**	**E**	**A**	**E**	**A**
3 M	23	25	6%	11%	7%	13%	6%	12%	17%	28%	0%	1%	5%	7%
6 M	21	13	10%	9%	13%	11%	13%	8%	23%	32%	4%	1%	4%	10%
9 M	26	16	11%	8%	16%	12%	17%	8%	21%	46%	12%	0%	3%	13%
12 M	26	7	10%	6%	18%	10%	18%	9%	26%	17%	11%	7%	6%	4%
15 M	21	11	6%	2%	13%	6%	14%	5%	26%	20%	4%	1%	6%	5%
18 M	25	7	10%	7%	20%	14%	20%	12%	28%	19%	15%	10%	3%	4%
21 M	24	2	11%	4%	23%	8%	23%	8%	29%	10%	16%	7%	4%	2%
24 M	26	1	9%		23%		23%		32%		14%		8%	
> 24 M	80		7%		22%		22%		33%		6%		8%	
Total	220	82	11%	8%	18%	11%	18%	9%	33%	46%	0%	0%	8%	9%

Table E.3: Daimler

	Counter		Weighted Average		Average		Median		Worst Case		Best Case		Standard Deviation	
Δ	**E**	**A**	**E**	**A**	**E**	**A**	**E**	**A**	**E**	**A**	**E**	**A**	**E**	**A**
3 M	4	21	2%	7%	3%	8%	3%	8%	4%	35%	1%	0%	1%	7%
6 M	5	12	20%	11%	26%	14%	26%	6%	27%	66%	26%	1%	0%	19%
9 M	13	10	20%	10%	28%	14%	26%	16%	37%	26%	26%	0%	4%	9%
12 M	12	9	12%	9%	20%	15%	28%	16%	29%	26%	3%	1%	12%	10%
15 M	8	11	1%	2%	3%	5%	3%	1%	4%	16%	1%	0%	1%	6%
18 M	6	8	10%	3%	18%	7%	18%	7%	18%	13%	18%	0%	0%	4%
21 M	13	4	9%	5%	18%	11%	18%	5%	23%	29%	15%	4%	3%	12%
24 M	13	0	3%		9%		14%		15%		1%		7%	
> 24 M	6		1%		2%		2%		2%		1%		0%	
Total	74	75	10%	7%	16%	10%	18%	7%	37%	66%	1%	0%	11%	11%

Table E.4: Merck

	Counter		Weighted Average		Average		Median		Worst Case		Best Case		Standard Deviation	
Δ	**E**	**A**	**E**	**A**	**E**	**A**	**E**	**A**	**E**	**A**	**E**	**A**	**E**	**A**
3 M	13	26	5%	7%	5%	8%	5%	6%	9%	36%	3%	1%	2%	9%
6 M	17	15	8%	5%	11%	7%	10%	6%	16%	24%	6%	0%	3%	7%
9 M	20	12	11%	5%	15%	8%	15%	7%	21%	16%	12%	2%	3%	5%
12 M	12	13	10%	5%	16%	9%	17%	8%	17%	17%	14%	2%	1%	4%
15 M	13	12	6%	3%	12%	7%	10%	7%	24%	16%	9%	1%	5%	4%
18 M	18	7	7%	5%	15%	12%	16%	13%	21%	13%	8%	11%	4%	1%
21 M	21	3	10%	4%	20%	9%	20%	9%	24%	10%	16%	8%	2%	1%
24 M	13	3	7%	4%	18%	8%	18%	9%	19%	10%	17%	6%	1%	2%
> 24M	34		6%		19%		20%		26%		12%		5%	
Total	127	91	10%	5%	15%	8%	16%	7%	26%	36%	3%	0%	5%	6%

Table E.5: Munich Re

Appendix F

Tables for the Backtest against the Simple Method

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	95	2%	3%	0%	26%	0%	6%
6 M	102	4%	5%	0%	26%	0%	7%
9 M	95	4%	7%	5%	26%	0%	7%
12 M	103	5%	9%	7%	26%	0%	7%
15 M	95	5%	11%	12%	29%	2%	7%
18 M	103	6%	13%	13%	30%	3%	8%
21 M	103	6%	16%	16%	32%	4%	8%
24 M	95	6%	18%	19%	32%	4%	7%
Total	791	5%	10%	8%	32%	0%	9%

Table F.1: 3M

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	94	3%	3%	0%	13%	0%	5%
6 M	103	2%	4%	0%	13%	0%	5%
9 M	94	3%	5%	4%	13%	0%	5%
12 M	103	4%	6%	6%	13%	0%	4%
15 M	95	4%	8%	7%	15%	0%	4%
18 M	102	4%	9%	8%	17%	3%	4%
21 M	103	4%	11%	10%	18%	5%	4%
24 M	94	4%	12%	11%	18%	6%	4%
Total	788	3%	7%	7%	18%	0%	5%

Table F.2: American Express

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	101	13%	15%	0%	100%	0%	33%
6 M	97	18%	28%	7%	100%	0%	43%
9 M	101	18%	35%	9%	100%	0%	44%
12 M	103	18%	36%	10%	100%	0%	43%
15 M	95	17%	37%	11%	100%	2%	43%
18 M	103	13%	37%	12%	100%	3%	43%
21 M	96	12%	39%	13%	100%	4%	42%
24 M	102	12%	38%	14%	100%	4%	41%
Total	798	15%	33%	10%	100%	0%	42%

Table F.3: Apple

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	94	3%	3%	0%	34%	0%	8%
6 M	103	5%	7%	0%	34%	0%	10%
9 M	96	6%	10%	6%	34%	0%	10%
12 M	103	7%	13%	9%	34%	0%	10%
15 M	95	7%	15%	17%	37%	2%	10%
18 M	103	8%	18%	20%	39%	4%	10%
21 M	103	9%	21%	22%	40%	5%	10%
24 M	95	9%	24%	26%	41%	5%	10%
Total	792	7%	14%	9%	41%	0%	12%

Table F.4: Boeing

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	101	3%	3%	0%	13%	0%	6%
6 M	97	3%	5%	6%	13%	0%	5%
9 M	101	3%	6%	7%	13%	0%	5%
12 M	103	4%	8%	9%	13%	0%	4%
15 M	91	5%	10%	9%	16%	4%	4%
18 M	103	5%	11%	11%	18%	6%	4%
21 M	96	5%	13%	14%	19%	7%	4%
24 M	101	5%	14%	15%	20%	8%	3%
Total	793	4%	9%	9%	20%	0%	6%

Table F.5: Caterpillar

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	97	2%	2%	0%	10%	0%	4%
6 M	100	2%	4%	0%	10%	0%	4%
9 M	98	2%	5%	4%	10%	0%	4%
12 M	101	3%	6%	5%	10%	0%	3%
15 M	97	3%	7%	6%	12%	2%	3%
18 M	103	3%	8%	7%	13%	3%	3%
21 M	96	3%	8%	8%	14%	3%	4%
24 M	102	3%	9%	9%	15%	4%	4%
Total	794	3%	6%	5%	15%	0%	4%

Table F.6: Chevron

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	96	11%	11%	0%	43%	0%	15%
6 M	101	11%	16%	18%	43%	0%	14%
9 M	98	11%	20%	18%	47%	0%	17%
12 M	99	11%	22%	18%	48%	0%	17%
15 M	97	12%	25%	18%	51%	3%	18%
18 M	99	11%	26%	20%	54%	4%	19%
21 M	101	10%	28%	21%	56%	6%	19%
24 M	97	11%	29%	22%	57%	6%	19%
Total	788	11%	22%	18%	57%	0%	18%

Table F.7: Cisco

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	96	2%	2%	0%	9%	0%	4%
6 M	101	2%	3%	0%	9%	0%	4%
9 M	97	2%	4%	4%	9%	0%	3%
12 M	103	3%	6%	6%	9%	2%	2%
15 M	96	3%	6%	6%	11%	2%	2%
18 M	103	3%	7%	6%	12%	3%	3%
21 M	98	3%	8%	8%	12%	4%	3%
24 M	99	3%	9%	9%	13%	5%	2%
Total	793	3%	6%	6%	13%	0%	4%

Table F.8: Coca-Cola

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	96	1%	1%	0%	5%	0%	2%
6 M	102	1%	2%	0%	5%	0%	2%
9 M	96	1%	2%	2%	5%	0%	2%
12 M	102	1%	2%	2%	5%	0%	2%
15 M	98	1%	3%	3%	6%	0%	2%
18 M	100	1%	3%	3%	6%	1%	2%
21 M	103	1%	4%	3%	6%	1%	2%
24 M	95	1%	4%	4%	7%	1%	2%
Total	792	1%	3%	3%	7%	0%	2%

Table F.9: Du Pont[21]

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	101	3%	3%	0%	18%	0%	6%
6 M	96	3%	5%	0%	18%	0%	6%
9 M	102	3%	6%	5%	18%	0%	6%
12 M	99	4%	7%	7%	18%	0%	5%
15 M	99	4%	8%	6%	19%	2%	5%
18 M	101	4%	9%	7%	20%	4%	5%
21 M	97	3%	10%	8%	21%	5%	5%
24 M	102	4%	11%	9%	22%	5%	5%
Total	797	3%	8%	7%	22%	0%	6%

Table F.10: ExxonMobil

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	103	4%	3%	0%	14%	0%	5%
6 M	96	3%	4%	0%	14%	0%	5%
9 M	103	4%	6%	6%	14%	0%	5%
12 M	103	4%	8%	8%	14%	0%	3%
15 M	94	4%	9%	9%	15%	4%	3%
18 M	104	4%	10%	10%	15%	6%	3%
21 M	95	4%	11%	11%	16%	6%	3%
24 M	103	4%	13%	12%	17%	6%	3%
Total	801	4%	8%	9%	17%	0%	5%

Table F.11: General Electric

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	95	5%	5%	0%	24%	0%	8%
6 M	103	5%	7%	5%	24%	0%	8%
9 M	95	5%	9%	6%	26%	0%	8%
12 M	102	5%	10%	7%	27%	0%	8%
15 M	97	5%	11%	7%	28%	2%	8%
18 M	102	4%	12%	9%	28%	3%	8%
21 M	99	4%	13%	10%	29%	5%	8%
24 M	99	4%	14%	12%	30%	5%	8%
Total	792	5%	10%	8%	30%	0%	8%

Table F.12: Goldman Sachs

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	94	3%	3%	0%	26%	0%	9%
6 M	102	5%	7%	0%	26%	0%	10%
9 M	94	6%	10%	10%	26%	0%	9%
12 M	102	6%	13%	13%	26%	0%	8%
15 M	103	7%	15%	14%	29%	5%	7%
18 M	95	7%	18%	17%	30%	8%	7%
21 M	103	7%	20%	19%	32%	9%	7%
24 M	96	7%	22%	22%	33%	10%	6%
Total	789	6%	14%	14%	33%	0%	10%

Table F.13: The Home Depot

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	95	2%	3%	0%	12%	0%	5%
6 M	102	3%	4%	0%	12%	0%	5%
9 M	96	3%	6%	5%	12%	0%	4%
12 M	103	4%	7%	7%	12%	0%	3%
15 M	95	4%	9%	9%	14%	3%	3%
18 M	103	4%	10%	10%	15%	4%	3%
21 M	99	4%	12%	12%	16%	6%	4%
24 M	99	4%	13%	14%	18%	6%	3%
Total	792	4%	8%	9%	18%	0%	5%

Table F.14: IBM

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	98	1%	1%	0%	7%	0%	2%
6 M	100	1%	1%	0%	7%	0%	2%
9 M	99	1%	2%	0%	7%	0%	2%
12 M	99	1%	2%	0%	7%	0%	3%
15 M	100	1%	2%	0%	7%	0%	3%
18 M	98	1%	2%	0%	7%	0%	3%
21 M	103	1%	2%	1%	7%	0%	3%
24 M	96	1%	3%	2%	7%	0%	3%
Total	793	1%	2%	0%	7%	0%	3%

Table F.15: Intel

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	103	2%	2%	0%	8%	0%	3%
6 M	95	2%	3%	0%	8%	0%	3%
9 M	103	2%	3%	3%	8%	0%	3%
12 M	97	2%	4%	4%	8%	0%	2%
15 M	101	2%	5%	5%	9%	1%	2%
18 M	103	2%	6%	5%	9%	2%	2%
21 M	95	2%	7%	6%	10%	3%	2%
24 M	103	2%	7%	7%	10%	3%	2%
Total	800	2%	5%	5%	10%	0%	3%

Table F.16: Johnson & Johnson

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	95	4%	5%	0%	21%	0%	8%
6 M	102	4%	6%	0%	21%	0%	8%
9 M	96	4%	8%	2%	21%	0%	8%
12 M	103	5%	9%	8%	21%	0%	8%
15 M	97	5%	11%	12%	22%	0%	7%
18 M	101	5%	12%	14%	22%	2%	7%
21 M	100	5%	14%	14%	23%	3%	7%
24 M	98	6%	15%	17%	24%	3%	7%
Total	792	5%	10%	10%	24%	0%	8%

Table F.17: JPMorgan Chase

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	100	2%	2%	0%	9%	0%	3%
6 M	97	2%	3%	3%	9%	0%	3%
9 M	101	2%	3%	3%	9%	0%	3%
12 M	103	2%	4%	4%	9%	0%	3%
15 M	95	2%	5%	5%	10%	1%	3%
18 M	103	2%	6%	6%	11%	2%	3%
21 M	97	2%	6%	6%	11%	2%	3%
24 M	101	2%	7%	7%	11%	3%	3%
Total	797	2%	5%	5%	11%	0%	3%

Table F.18: McDonald's

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	100	1%	1%	0%	2%	0%	1%
6 M	96	1%	1%	1%	2%	0%	1%
9 M	102	1%	1%	1%	2%	0%	1%
12 M	103	1%	1%	2%	2%	0%	1%
15 M	96	1%	2%	2%	3%	1%	1%
18 M	103	1%	2%	2%	3%	1%	1%
21 M	95	1%	2%	2%	3%	1%	1%
24 M	103	1%	3%	3%	3%	1%	1%
Total	798	1%	2%	2%	3%	0%	1%

Table F.19: Merck

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	94	3%	4%	0%	18%	0%	7%
6 M	103	4%	6%	0%	18%	0%	7%
9 M	94	5%	8%	9%	18%	0%	6%
12 M	102	5%	10%	10%	18%	0%	5%
15 M	103	5%	12%	11%	20%	3%	5%
18 M	94	6%	13%	13%	21%	4%	5%
21 M	102	6%	15%	15%	21%	5%	5%
24 M	95	6%	17%	17%	22%	6%	4%
Total	787	5%	11%	11%	22%	0%	7%

Table F.20: Microsoft

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	103	3%	3%	0%	14%	0%	6%
6 M	95	4%	5%	7%	14%	0%	6%
9 M	103	5%	7%	9%	14%	0%	5%
12 M	102	5%	9%	10%	14%	3%	4%
15 M	95	5%	10%	10%	17%	3%	4%
18 M	103	5%	12%	10%	18%	5%	4%
21 M	95	5%	14%	15%	19%	7%	4%
24 M	103	5%	15%	16%	20%	8%	3%
Total	799	5%	10%	10%	20%	0%	6%

Table F.21: Nike

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	94	2%	2%	0%	9%	0%	4%
6 M	103	2%	3%	0%	9%	0%	3%
9 M	95	3%	4%	4%	9%	0%	3%
12 M	103	2%	5%	4%	9%	0%	2%
15 M	94	3%	6%	5%	11%	2%	2%
18 M	103	3%	7%	6%	12%	3%	3%
21 M	102	3%	8%	8%	12%	4%	3%
24 M	95	3%	9%	8%	13%	5%	2%
Total	789	3%	5%	5%	13%	0%	4%

Table F.22: Pfizer

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	94	1%	2%	0%	7%	0%	3%
6 M	103	2%	3%	0%	7%	0%	3%
9 M	95	2%	3%	3%	7%	0%	3%
12 M	102	2%	4%	4%	7%	0%	2%
15 M	94	2%	5%	5%	8%	2%	2%
18 M	103	2%	6%	5%	9%	3%	2%
21 M	101	2%	6%	6%	9%	3%	2%
24 M	95	2%	7%	7%	10%	4%	2%
Total	787	2%	4%	4%	10%	0%	3%

Table F.23: Procter & Gamble

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	100	2%	2%	0%	11%	0%	4%
6 M	96	2%	4%	0%	11%	0%	4%
9 M	102	2%	5%	3%	11%	0%	4%
12 M	98	3%	6%	5%	11%	0%	3%
15 M	100	3%	7%	6%	12%	2%	3%
18 M	101	3%	8%	6%	13%	3%	3%
21 M	98	3%	9%	8%	14%	4%	3%
24 M	102	3%	10%	9%	15%	5%	3%
Total	797	3%	6%	6%	15%	0%	4%

Table F.24: Travelers

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	96	5%	6%	0%	24%	0%	11%
6 M	100	6%	9%	0%	24%	0%	10%
9 M	98	6%	13%	14%	24%	0%	9%
12 M	103	7%	16%	17%	24%	0%	7%
15 M	96	9%	19%	18%	29%	7%	6%
18 M	103	8%	22%	20%	32%	11%	7%
21 M	101	8%	25%	26%	34%	14%	7%
24 M	97	8%	27%	29%	35%	15%	6%
Total	794	7%	17%	18%	35%	0%	11%

Table F.25: UnitedHealth Group

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	95	2%	2%	0%	10%	0%	4%
6 M	103	3%	4%	0%	10%	0%	4%
9 M	95	3%	5%	5%	10%	0%	4%
12 M	103	3%	6%	6%	10%	0%	4%
15 M	95	3%	6%	7%	10%	0%	3%
18 M	103	3%	7%	8%	12%	2%	3%
21 M	103	3%	8%	8%	13%	3%	3%
24 M	95	3%	9%	9%	14%	3%	3%
Total	792	3%	6%	6%	14%	0%	4%

Table F.26: United Technologies

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	98	1%	1%	0%	3%	0%	1%
6 M	100	1%	1%	0%	3%	0%	1%
9 M	98	1%	1%	1%	3%	0%	1%
12 M	100	1%	2%	2%	3%	0%	1%
15 M	100	1%	2%	2%	4%	1%	1%
18 M	98	1%	3%	2%	4%	1%	1%
21 M	103	1%	3%	3%	4%	1%	1%
24 M	95	1%	3%	3%	5%	2%	1%
Total	792	1%	2%	2%	5%	0%	1%

Table F.27: Verizon

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	97	3%	3%	0%	15%	0%	6%
6 M	101	3%	4%	1%	15%	0%	6%
9 M	97	3%	5%	1%	15%	0%	5%
12 M	101	3%	7%	8%	15%	0%	5%
15 M	97	4%	8%	10%	16%	1%	6%
18 M	102	4%	9%	10%	16%	1%	6%
21 M	102	3%	9%	11%	16%	1%	6%
24 M	97	3%	9%	12%	16%	1%	6%
Total	794	3%	7%	6%	16%	0%	6%

Table F.28: Wal-Mart

Δ	Counter	Weighted Average	Average	Median	Worst Case	Best Case	Standard Deviation
3 M	26	14%	16%	16%	20%	13%	4%
6 M	24	11%	16%	16%	20%	13%	4%
9 M	26	7%	16%	16%	20%	13%	4%
12 M	24	3%	16%	13%	20%	13%	4%
15 M	24	11%	24%	25%	25%	13%	4%
18 M	26	9%	25%	25%	25%	25%	0%
21 M	33	6%	22%	25%	25%	5%	5%
24 M	38	4%	22%	25%	25%	16%	4%
Total	221	8%	20%	20%	25%	5%	5%

Table F.29: Walt Disney

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Scientific Career

09/2000 - 03/2009	Secondary School: Zeugnis der Allgemeinen Hochschulreife (general qualification for university entrance) at *St. Matthias Gymnasium Gerolstein*
10/2009 - 03/2013	Bachelor of Science (B. Sc.) in Mathematics with minor Economics at *University of Kaiserslautern*
04/2013 - 09/2014	Master of Science (M. Sc.) with focus in Financial and Insurance Mathematics and minor Economics at *University of Kaiserslautern*
10/2014 - 09/2017	PhD Student of Prof. Dr. Ralf Korn at *Fraunhofer Institute for Industrial Mathematics* and *University of Kaiserslautern*

Wissenschaftlicher Werdegang

09/2000 - 03/2009	Gymnasium: Zeugnis der Allgemeinen Hochschulreife am *St. Matthias Gymnasium Gerolstein*
10/2009 - 03/2013	Bachelor of Science (B. Sc.) in Mathematik mit Nebenfach Wirtschaftswissenschaften an der *Technischen Universität Kaiserslautern*
04/2013 - 09/2014	Master of Science (M. Sc.) mit Schwerpunkt Finanz- und Versichungsmathematik mit Nebenfach Wirtschaftswissenschaften an der *Technischen Universität Kaiserslautern*
10/2014 - 09/2017	Doktorandin von Prof. Dr. Ralf Korn am *Fraunhofer Institut für Techno- und Wirtschaftsmathematik* sowie an der *Technischen Universität Kaiserslautern*