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Restricted Planar Location Problems and Applications

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Abstract

Facility location problems in the plane are among the most widely used tools of Mathematical Programming in modeling real-world problems. In many of these problems restrictions have to be considered which correspond to regions in which a placement of new locations is forbidden.

We consider center and median problems where the forbidden set is a union of pairwise disjoint convex sets. As applications we discuss the assembly of printed circuit boards, obnoxious facility location and the location of emergency facilities.

Keywords: Location Theory, Restrictions, Applications

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1 Introduction

One of the most successful models in mathematical programming is the location model. A large body of literature (see, for instance, [Domschke and Drexl, 1984]) is witness to the history of the development of location theory and its various successful applications.

In this paper we deal with planar location problems. We consider M existing locations $Ex_1, \ldots, Ex_M$ where each location is assumed to be a point in the plane with coordinates $Ex_m = (a_{m_1}, a_{m_2})\ \forall m \in \mathcal{M} := \{1, \ldots, M\}$. The set of the existing facilites is often denoted Ex.

We want to find N new locations $\text{New}_1, \ldots, \text{New}_N$, again assumed to be points in the plane with coordinates $\text{New}_n = (x_{n_1}, x_{n_2})\ \forall n \in \mathcal{N} := \{1, \ldots, N\}$. The set of new facilities is denoted NEW. If $N = 1$, we drop the index 1 in NEW as well as in the coordinates, such that $\text{New} = (x_1, x_2)$ is a new facility in a single facility location problem in the plane.

In order to evaluate the quality of a set of new facilites, weights w_{mn} are given for all $m \in M$ and $n \in N$ which may, for instance, correspond to the frequencies of use between existing facility Ex_m and new facility New_n. Correspondingly, v_{lk} are weights for all pairs $l, k \in \mathcal{N}$ of new facilities. Moreover a distance function $d(R, S)$ is defined between any two points $R, S \in \mathbb{R}^2$. Although some of the results of this paper are valid for more general distance functions we will restrict ourselves to the l_p-metric defined by

$$d((r_1, r_2), (s_1, s_2)) = l_p((r_1, r_2), (s_1, s_2)) := (|r_1 - s_1|^p + |r_2 - s_2|^p)^{\frac{1}{p}}$$

We consider the functions f and g defined by

$$f(\text{New}) := f_{single}(\text{New}) + f_{multi}(\text{New})$$

where

$$f_{single}(\text{New}) := \sum_{m \in \mathcal{M}} \sum_{n \in \mathcal{N}} w_{mn} d(Ex_m, \text{New}_n)\ ,$$

$$f_{multi}(\text{New}) := \sum_{m \in \mathcal{M}} \sum_{r \in \mathcal{N}, r > n} v_{nr} d(\text{New}_n, \text{New}_r)\ ,$$

and

$$g(\text{New}) := \max\{g_{single}(\text{New}), g_{multi}(\text{New})\}$$

where

$$g_{single}(\text{New}) := \max\{w_{mn} d(Ex_m, \text{New}_n) : m \in \mathcal{M}, n \in \mathcal{N}\}$$

and

$$g_{multi}(\text{New}) := \max\{v_{nr} d(\text{New}_n, \text{New}_r) : n, r \in \mathcal{N}\}\ .$$

The problem

$$minimize\{f(\text{New}) : \text{New} \in \mathbb{R}^{2N}\}$$

is the **N-facility median problem** (N-MP), whereas

$$minimize\{g(\text{New}) : \text{New} \in \mathbb{R}^{2N}\}$$

is the **N-facility center problem** (N-CP). For N=1 we omit N and use **MP** and **CP**, respectively. If we formulate results which hold for median as well as for center problems we will often use the denotation N-LocP or LocP.

The main issue of this paper are restrictions in N-LocP which often arise when facility location is used as a model in practice. In order to show the importance of this approach we discuss three examples which serve as motivation for the following.

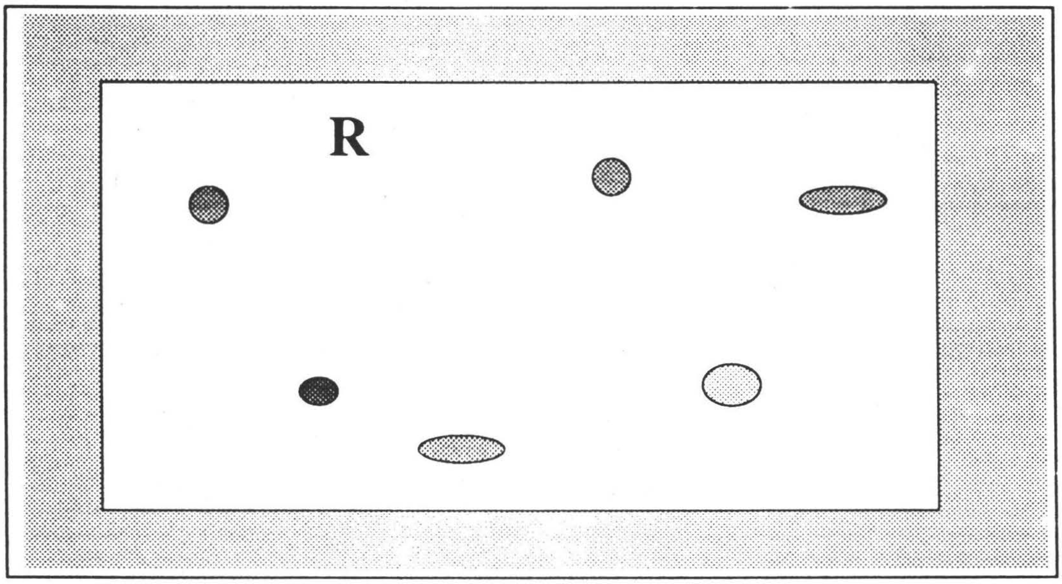

Figure 1.1: PCB with 6 parts and 4 part types. The 4 bins holding the 4 part types can be placed anywhere in the shaded region .

Example 1.1 (Assembly of printed circuit boards). *Mathematical programming methods have been used by various researchers to model the assembly of printed circuit boards (PCB) using robots. We restrict ourselves to discussing the approach for a robot with a single robot arm (see [Francis et al., 1989] and [Foulds and Hamacher, 1990]).*

Consider PCBs represented by a rectangle $R = [0, a] \times [0, b]$ *in which* M *parts* $m \in \mathcal{M} := \{1, \ldots, M\}$ *have to be inserted at fixed insertion points* $P_m = (p_{m_1}, p_{m_2})$, *respectively.*

Each of the parts belongs to one of N *part types, for example transistors, capacitators, etc..*

Different part types are stored in different bins and we denote with F *the feasible region in which the bins can be placed. In many situations* F *will be a non-convex set, for instance, the Euclidean plane without the interior of the rectangle* R, $F := \mathbb{R}^2 \setminus int(R)$.

In order to maximize the throughput of PCB through the assembly line it is necessary to find a location of the N *bins in such a way that the resulting travel distance of the robot is minimized. Therefore one part of the robot assembly problem is the solution of a restricted location problem.*

Example 1.2 (Obnoxious facility planning). *An area within location theory which has attained considerable attention in the last years (see for instance [Erkut and Neuman, 1989]) is the location of undesirable locations (obnoxious facility planning). The model which is mostly used in this context is the maximization of overall distances between existing and new facilities. Another promissing model which, as far as we know, has never been applied is to exclude certain regions $R_1, \ldots, R_L$ from placing the undesirable facility, and deal with the union R of these sets as restricting set of a facility location problem.*

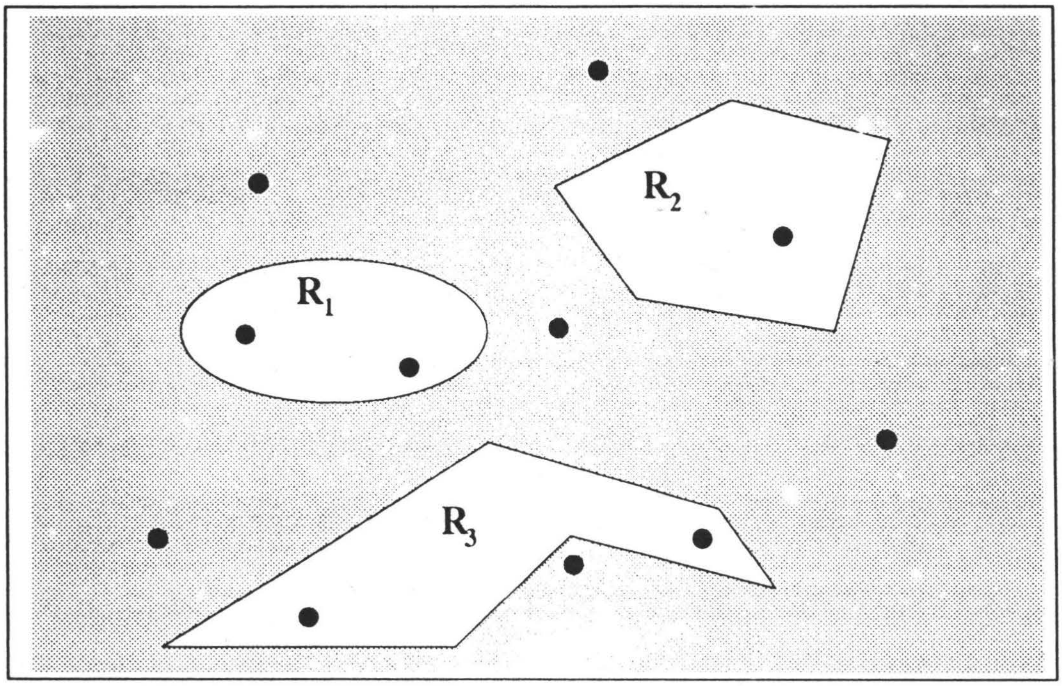

Figure 1.2: Existing facilities (dots) and forbidden regions R_1, R_2 and R_3 for placing new facilities. The shaded region is feasible for placing new facilities.

This is obviously again a restricted facility location problem. From a mathematical point of view it seems to be more complicated than the first example, since the set R in Example 2 which is excluded from placing new facilities is much less structured than in Example 1.

Example 1.3 (Location of emergency facilities). *A well-known application of center problems is the placement of an emergency facility. Suppose that our first objective in locating a new emergency facility is to satisfy given response constraints in each existing facility, i.e. the maximal distance between* Ex_m *and* New *should be bounded by a number MD_m, $\forall m \in \mathcal{M}$. If we draw circles with radius MD_m around each Ex_m, we may want to find a location which is feasible with respect to the response time restictions, and optimal with respect to the overall distance traveled by the emergency vehicles. Such a bicriteria facility location problem can be modeled by enforcing the intersection of the circles as region in which the facility has to be placed.*

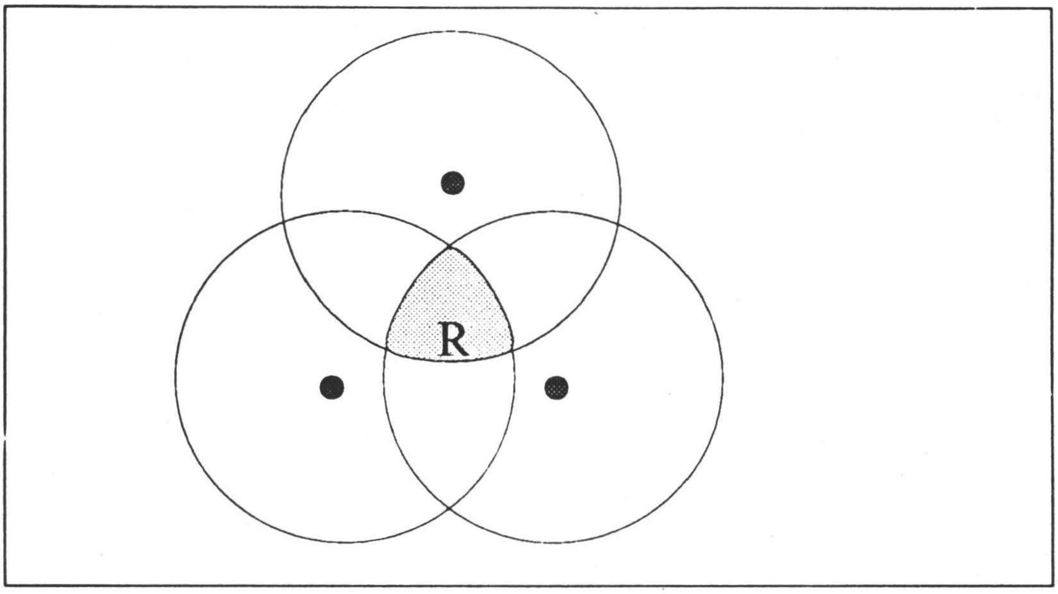

Figure 1.3: Three existing facilities with circles corresponding to maximal allowable respond times and shaded region as feasible region for placing an emergency facility.

This is again a restricted location problem, but it seems to be of a different flavour than the previous two examples. The emphasis is on enforcing a region for placing new facilities instead of forbidding regions.

We will come back to this problem when we have discussed suitable choices of distance functions and algorithms for solving the corresponding restricted loctation problems.

After having motivated the necessity of restrictions we now formally introduce restricted location problems. Let R be a subset of $\mathbb{R}^2$ (the **restricting set**), let $int(R)$ be the interior of R, and let $F := \mathbb{R}^2 \setminus int(R)$ (the **feasible** set).

Then

$$minimize\{f(\text{New}) : \text{New} \in F^N\}$$

is the **N-facility restricted median problem** (N-RMP), and

$$minimize\{g(\text{New}) : \text{New} \in F^N\}$$

is the **N-facility restricted center problem** (N-RCP). If $N = 1$ we often abbreviate 1-RMP and 1-RCP by RMP and RCP. In cases were we discuss results which hold both for center and median problems we also use the denotation N-RLocP or RLocP for the general restricted N-facility location problem and the general restricted 1-facility location problem, respectively. If we want to emphasize the distance function, which is used in the definition of the objective function, we sometimes write N-RLocP-d for the N-facility restricted location problem with respect to distance function d. N-RMP-l_1 is, for instance, an N-facility restricted median problem with respect to the rectilinear distance function.

With Opt^* we denote the family of optimal location sets **New** $=$ ($\mathbf{New}_1, \ldots, \mathbf{New}_N$) of a given (unrestricted location problem, while $Opt(R)^*$ is the family of optimal location sets of N-RLocP with respect to restricting set R.

Obviously, N-LocP is a relaxation of N-RLocP such that we can solve N-RLocP if the intersection of Opt^* with F is non-empty. Any **New** taken from this intersection will be optimal.

Example 1.4. *Suppose we want to solve* RMP-l_1 *with respect to two existing facilities, weights $w_{11} = w_{21} = 1$, and R as shown in Figure 1.4 .*

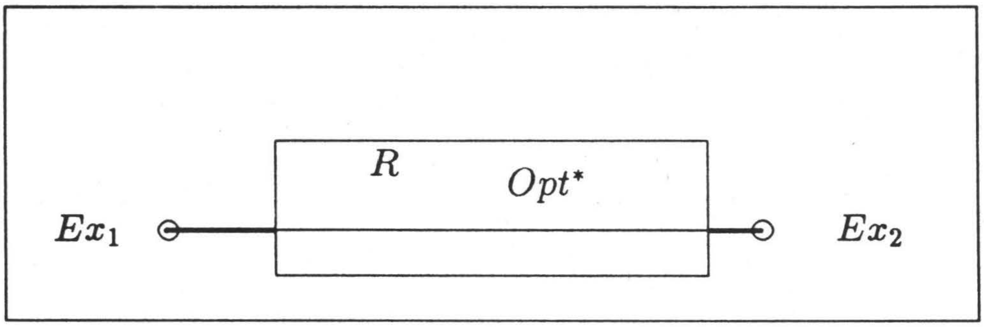

Figure 1.4: An instance of a RMP-l_1 with 2 existing facilities.

The optimal solution set Opt^* of the unrestricted problem MP-l_1 *is given by the line between Ex_1 and Ex_2. Its intersection $Opt^*(R)$ with $F = \mathbb{R}^2 \backslash int(R)$ is indicated in Figure 1.4 by the bold part of this line. Hence* RMP-l_1 *is trivially solvable in this case.*

Example 1.5. *If we extend the previous example by just one more existing facility, then* MP-l_1 *has a unique optimal solution Opt^* which is no longer feasible for* RMP-l_1 *(see Figure 1.5).*

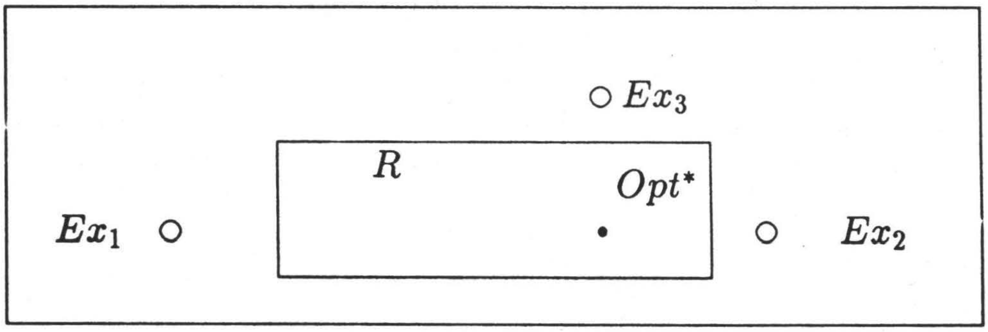

Figure 1.5: An instance of a RMP-l_1 with 3 existing facilities.

In this situation it becomes apparent that we need algorithms for dealing with restricting sets.

The only literature on restricted location problems we are aware of is [Francis and White, 1974] in which a contour line approach is given. (We review this approach in Section 2.3), and [Hamacher and Nickel, 1991] in which a combinatorial algorithm to RMP is described.

A related, but different problem is the location problem with obstacles in which travel barriers are considered (see, for instance [Larson and Sadiq, 1983] and [Batta et al., 1989]).

In the next section we will formulate some general results for N-LocP and N-RLocP. In Section 3 we will concentrate on N-RMP and develop combinatorial algorithms for RMP-l_1, RMP-l_∞, and RMP-l_2^2. These approaches can be extended to 2-RMP, and used for heuristic procedures for N-RMP. In particular we will show how our approach improves current facility location models by taking the space into account which is used by existing and new facilities. In the fourth section we will deal with N-facility center problems. In Section 5 we come back to our motivational problems and show how the practical problems of Examples 1-3 can be tackled by the theory developed previously. Section 6 concludes the paper with a short description of the public domain software which was developed based on the results of this paper.

2 Mathematical Analysis of (Restricted) Location Problems

In this section we will discuss some general properties of location problems. In particular, we address the relation between Opt^* and $Conv\{Ex_1, \ldots, Ex_m\}$, the convex hull of $Ex_1, \ldots, Ex_m$, and show that in the unrestricted case for center and median problems the optimal solution sets are contained in special rectangles which dependent on the coordinates of the existing facilities. For the restricted case we will prove the fundamental property that the optimal location is always contained in the boundary of the restricting set. This will give rise to a generic algorithm which reduces the restricted location problem to a one-variable optimization problem.

2.1 Unrestricted Location Problems

The following theorem is central for our theory and well-known from the literature. (see, for instance [Love et al., 1988] or [Nickel, 1991])

Theorem 2.1. *f and g are convex funtions.*

2.1.1 Location of Optimal Solutions Sets

The motivation for this subsection is as follows: If we find the solution set Opt^* of the unrestricted problem and discover that $Opt^* \cap F \neq \emptyset$, the restricted location problem is solved. Therefore we want to know more about the structure of Opt^*.

We first address the relation of Opt^* and $Conv\{Ex_1, \ldots, Ex_m\}$. [Love et al., 1988, Property 2.3] claim that $Opt^* \subseteq Conv\{Ex_1, \ldots, Ex_m\}$ for median problems. In fact, they provide a proof for $d = l_2$ which can be generalized to $d = l_p$ for even p. For general p this result is wrong as the following example shows. (see also Figure 2.1)

Example 2.1. *$M = 2$, $Ex_1 = (3,5)$, $Ex_2 = (3,4)$, all weights are equal and $d = l_1$. In this example Opt^* is the rectangle $[3,5] \times [3,4] \not\subseteq Conv\{Ex_1, Ex_2\}$.*

We therefore state as our first therorem a weakend version of this result.

Theorem 2.2. *1. Let $p \in \mathbb{N} \cup \{\infty\}$ and let Opt^* be the set of optimal locations for MP$-l_p$. Then*

(a) $Opt^* \cap Conv\{Ex_1, \ldots, Ex_M\} \neq \emptyset$, and

(b) $Opt^* \subseteq Conv\{Ex_1, \ldots, Ex_M\}$ if p is even and $p < \infty$.

(c) Let Opt^* be the set of optimal solutions of CP-l_2 and let all weights be equal.
Then $Opt^* \subseteq Conv\{Ex_1, \ldots, Ex_m\}$.

Proof.
ad 1a) We use the hyperbolic approximation of $f(X)$

$$f_\varepsilon(X) := \sum_{m \in \mathcal{M}} w_m((x_1 - a_{m_1})^2 + \varepsilon)^{\frac{p}{2}} + (x_2 - a_{m_2})^2 + \varepsilon)^{\frac{p}{2}})^{\frac{1}{p}}$$

and the corresponding minimization problem, defined as

$$(\mathrm{MP})_\varepsilon := \underset{X \in \mathbb{R}^2}{\text{minimize}}\, f_\varepsilon(X)\,.$$

We will now list some properties of $f_\varepsilon(X)$ which can be found in [Morris and Verdini, 1979].

- $f_\varepsilon(X)$ is strictly convex. i.e. $(\mathrm{MP})_\varepsilon$ has an uniquely defined minimum X_ε^*.
- $X_\varepsilon^* \in int(Conv\{X_1, \ldots, X_M\}$.
- $\max_{X \in \mathbb{R}^2} |f_\varepsilon(X) - f(X)| \leq \delta(\varepsilon) = 2^{\frac{1}{p}} \varepsilon^{\frac{1}{2}} (\sum_{m \in \mathcal{M}} w_m)$

An immediate consequence of these properties is that $f_\varepsilon(X)$ converges uniformly to $f(X)$. Hence there exists a $\rho(\varepsilon)$ such that for an appropriate small ε

$$|X_\varepsilon^* - Opt^*| \leq \rho(\varepsilon)\,,$$

holds, and the $\rho(\varepsilon)$-neighbourhood of X_ε^* is totally contained in $Conv\{Ex_1, \ldots, Ex_M\}$.

ad 1b) Because p is even we can rewrite $f(X)$ as

$$\sum_{m \in \mathcal{M}} w_m((x_1 - a_{m_1})^p + (x_2 - a_{m_2})^p)^{\frac{1}{p}}$$

In order to find candidates for elements of Opt^* we compute the partial derivatives of $f(X)$ in $\mathbb{R}^2 \backslash \{Ex_1, \ldots, Ex_M\}$[1].

$$\frac{\partial f}{\partial x_i} = \sum_{m \in \mathcal{M}} w_m l_p(X, Ex_m)^{\frac{1}{p}-1}(x_i - a_{m_i})^{p-1} \text{ with } i = 1,2$$

$$\nu_j := w_m l_p(X, Ex_m)^{\frac{1}{p}-1} > 0 \text{ for all } m \in \mathcal{M}$$

and we get with

$$\begin{aligned} \frac{\partial f}{\partial x_i} = 0 \quad &\Longleftrightarrow \quad \sum_{m \in \mathcal{M}} \nu_j (x_i - x_{m_i})^{p-1} = 0 \\ &\Longleftrightarrow \quad \sum_{m \in \mathcal{M}} \nu_j (x_i - x_{m_i})^{p-2} x_i = \sum_{m \in \mathcal{M}} \nu_j (x_i - x_{m_i})^{p-2} x_{m_i} \end{aligned}$$

If $\sum_{m \in \mathcal{M}} \nu_j (x_i - a_{m_i})^{p-2} = 0$ for $i = 1$ or 2 all existing facilicties are lying on a horizontal or vertical line, respectively, and therefore in $Conv\{Ex_1, \ldots, Ex_M\}$. Otherwise we can express x_i as

$$\begin{aligned} x_i &= \frac{\sum_{m \in \mathcal{M}} \nu_j (x_i - a_{m_i})^{p-2} x_{m_i}}{\sum_{m \in \mathcal{M}} \nu_j (x_i - a_{m_i})^{p-2}} \\ &\Longrightarrow \quad X \in Conv\{Ex_1, \ldots, Ex_M\} \end{aligned}$$

ad 3) This result is proved by [Nair and Chandrasekaran, 1971].

□

Theorem 2.2 shows that only in special cases $Conv\{Ex_1, \ldots, Ex_m\} \subseteq R$ can be used as criterion to exclude the trivial case that some $X^* \in Opt^*$ is feasible for RLocP. We therefore introduce two rectangles R_{min} and R_∞ which will be used to characterize the optimal locations of any location problem considered in this paper.

$$\begin{aligned} x_{min_1} &:= \min_{i \in \mathcal{M}} a_{i_1} \\ x_{max_1} &:= \max_{i \in \mathcal{M}} a_{i_1} \\ x_{min_2} &:= \min_{i \in \mathcal{M}} a_{i_2} \\ x_{max_2} &:= \max_{i \in \mathcal{M}} a_{i_2} \\ R_{min} &:= [x_{min_1}, x_{max_1}] \times [x_{min_2}, x_{max_2}], \end{aligned}$$

[1] There is no need for examining the existing facilities because they are of course contained in $Conv\{Ex_1, \ldots, Ex_M\}$. Hence the partial derivative exists.

R_{min} is the smallest rectangle with sides parallel to the x- and y-axis containing all existing facilities Ex_m, $m \in \mathcal{M}$.

Correspondingly, R_∞ is the smallest rectangle with 45° and −45° sides. That is, R_∞ is defined by the leftmost and rightmost 45°, and the highest and lowest −45° lines through the points Ex_m, $m \in \mathcal{M}$, respectively. (see Figure 2.1)

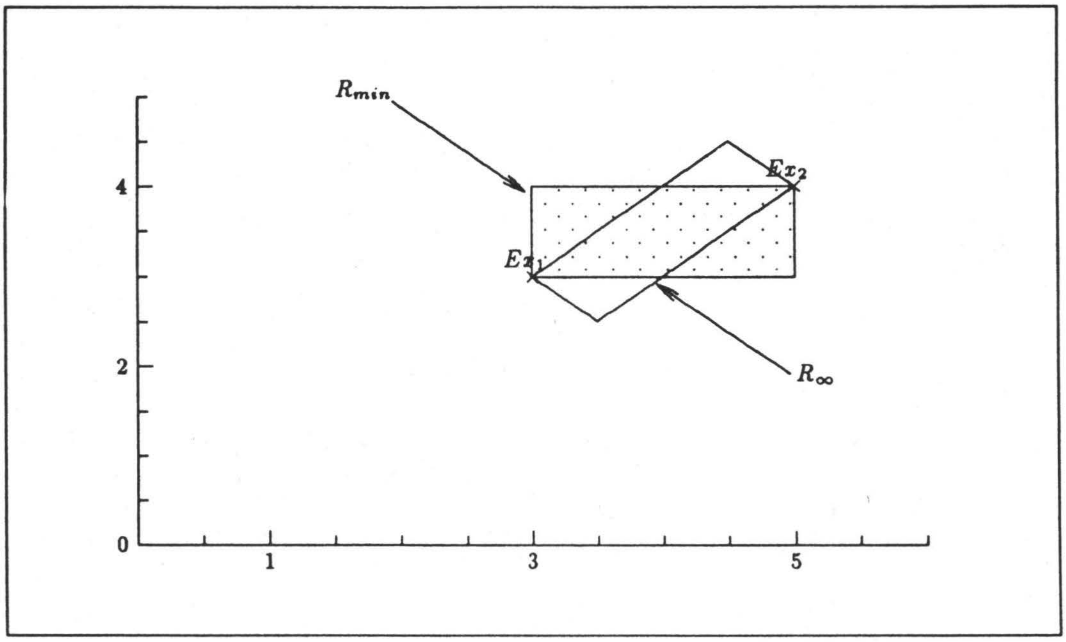

Figure 2.1: R_{min} and R_∞ for two existing facilities.

Using these rectangles we can prove the following results.

Theorem 2.3. *For $p \in \mathbb{N}$ the optimal set of locations for* LocP-l_p *satisfies*

$$Opt^* \subseteq R_{min} \; .$$

Moreover R_{min} is the smallest possible set containing Opt^*.

Proof. Suppose $X = (x_1, x_2) \in Opt^*$ with $X \notin R_{min}$.

Wlog $x_{min_1} \leq x_1 \leq x_{max_1}$ and $x_2 > x_{max_2}$.

Then we get for $X' := (x_1, x_{max_2}) \in R_{min}$.

$$\begin{aligned} l_p(X, Ex_m) &= (|x_1 - a_{m_1}|^p + (x_2 - a_{m_2})^p)^{\frac{1}{p}} \\ &> (|x_1 - a_{m_1}|^p + (x_2 - x_{max_2})^p)^{\frac{1}{p}} \\ &= d(X', Ex_m) \text{ for } m \in \mathcal{M} \end{aligned}$$

All other cases for $X \notin R_{min}$ are treated similarly. Hence X' improves the objective value of X (for center as well as for median problems) contradicting the optimality of X. The fact that R_{min} is the smallest possible set is proved by Example 2.1.

□

In order to extend the result of Theorem 2.3 to $p = \infty$ we consider the transformation $T(X)$ defined by

$$T(X) := \frac{1}{2}(x_1 + x_2, x_2 - x_1)$$

and consequently

$$T^{-1}(X) = (x_1 - x_2, x_1 + x_2)\,.$$

The reader may easily verify the following result which is also used in [Francis and White, 1974]

$$\begin{aligned} l_\infty(X,Y) &= l_1(T(X),T(Y)) &\quad (2.1)\\ l_1(X,Y) &= l_\infty(T^{-1}(X),T^{-1}(Y))\,. &\quad (2.2)\end{aligned}$$

Using this transformation we obtain

Corollary 2.4. *For $p = \infty$ the optimal set of locations for* LocP-l_p *satisfies*

$$Opt^* \subseteq R_\infty$$

and R_∞ is the smallest set satisfiying this property.

Proof. The result follows from (2.1), (2.2) and Theorem 2.3, since $T(R_\infty) = R_{min}$.

□

In the following we will denote

$$R_{box} := \begin{cases} R_{min} & : \ p \in \mathbb{N} \\ R_\infty & : \ p = \infty \end{cases}$$

to avoid the clumsy distinction depending on p.

We will now discuss the case of N facilities and obtain analogous results to the ones obtained in the previous special case of a single facility.

Theorem 2.5. *For $p \in \mathbb{N} \cup \{\infty\}$ the optimal set of locations for N-*MP*-l_p satisfies*

$$Opt^* \subseteq R^N_{box}\ .$$

Proof. First we ignore the part f_{multi} in f , which expresses the interactions between the new facilities.
We solve N single location problems and get a feasible solution X' for N-MP-l_p.
By Theorem 2.3 we know that $X' \in R^N_{box}$.
The theorem is proved by observing that we only improve the objective value if we stay inside R^N_{box}.

We can use similar arguments as in Theorem 2.3 and Corollary 2.4 using the interpretation of X' and X as points in R_{box}^N and in $\mathbb{R}^{2N} \setminus R_{box}^N$, respectively. (More details can be found in [Nickel, 1991].)

□

In the case of center problems we can only prove a weaker result:

Theorem 2.6. *Let $p \in \mathbb{N} \cup \{\infty\}$ and let $\{\mathbf{New}_1, \ldots, \mathbf{New}_N\}$ be an optimal set of locations for N-CP-l_p. Then*

$$\mathbf{New}_i \in R_{box}^N$$

for at least one $i \in \{1, 2, \ldots, N\}$ and R_{box} is the smallest set with this property. In particular, this means that if R_{box} is a subset of the forbidden region the solution of the N-CP is never a solution of the N-RCP.

Proof. First we ignore the part g_{multi} in g, which expresses the interactions between the new facilities.
We solve N single center problems and get a feasible solution $X' = \{\mathbf{New}_1, \ldots, \mathbf{New}_N\}$ for N-CP-l_p.
By Theorem 2.3 and Corollary 2.4 we know that $X' \in R_{box}^N$.
Since the objective function is convex the theorem is proved by observing that we only improve the objective value if at least one of the new locations stays inside R_{box}^N.

1. If g_{single} defines the maximum of g, then we can not improve the objective function by changing X', and at least the element in X' which defines this maximum must not leave R_{box} to preserve optimality. In this case we may however get alternative optima where all but one of the N new locations lie outside R_{box}.

2. If g_{multi} defines the maximum of g, an improvement of the objective function can only be obtained by moving two of the new facilities towards each other.
Since R_{box} is convex at least the two moved facilities are still contained in R_{box}^N.

The minimality of R_{box} follows from the special case $N = 1$ in Theorem 2.3.

□

Remark. The proof above has an interesting algorithmic application. If we solve the N unrestricted center problems independently (i.e. ignoring g_{multi}) and recognize that g_{single} defines the maximum, we have also solved N-CP.

2.2 Restricted Location Problems

In this section we consider a restricting set $R \subseteq \mathbb{R}^2$ which is connected.

We assume in the following that $Opt^* \subseteq int(R)$, which is, for instance, satisfied if $R_{box} \subseteq int(R)$. Hence we exclude the trivial case that an optimal solution of N-LocP-l_p is feasible for N-RLocP-l_p.

Theorem 2.7. *Let $Opt^*(R)$ be the set of optimal locations for N-RLocP-l_p and let ∂R be the boundary of R.*
Then

$$Opt^*(R) \subseteq (\partial R)^N \ .$$

Proof. Let $Y \in (F \backslash \partial R)^N$, and let $X^* = (\mathbf{New}_1, \ldots, \mathbf{New}_N)$ be any optimal solution of N-LocP-l_p. By our assumption on R Theorem 2.5 implies $X^* \in int(R^N)$, and therefore $X_{\partial R^N} \in \partial R^N \subseteq F^N$ exists such that

$$X_{\partial R^N} \in \{tY + (1-t)X^* : 0 < t < 1\} \ .$$

By Theorem 2.1 f (g) is a convex function, therefore $f(Y) > f(X_{\partial R^N})$ (the same for g), i.e., Y is not an optimal solution of N-RLocP-l_p.

□

Theorem 2.7 can be generalized to restricting sets which are disjoint unions of connected subsets of $\mathbb{R}^2$ (see Theorem 3.8).

An immediate consequence of Theorem 2.7 is the following algorithm for solving RLocP-l_p, provided a parameter description $\partial R = \{\gamma(t) : 0 \leq t \leq 1\}$ is known for ∂R.

<u>*One-Variable Algorithm for* RLocP-l_p</u>

1. Find the set of optimal locations Opt^* of the unrestricted problem LocP-l_p. If some $\mathbf{New} \in Opt^*$ is feasible $\longrightarrow$ **STOP.**
2. Apply a one-variable algorithm to the problem

$$\min_{1 \leq t \leq 1} \{h(\gamma(t))\}$$

For solving Step 2 specialized algorithms can be applied (see, for instance [Miffin, 1991] and references therein).

2.2.1 Location Problems and Level Sets

Although the result of Theorem 2.7 has – to the best of our knowledge – never been proved rigorously, it has been used in the level curve (or contour line) approach for solving restricted problems (see [Francis and White, 1974]). In this section we will introduce level curves and level sets and reformulate (restricted) location problems using these concepts. An immediate consequence of this reformulation is the level curve algorithm for solving restricted location problems. This generic algorithm is used to derive efficient procedures for RMP-l_2^2, RLocP-l_1 and RLocP-l_∞.

For h=f or h=g and $z \in \mathbb{R}_+$ the **level curve** $L(z)$ and the **level set** $L_{\leq}(z)$ (both with respect to z and h) is defined by

$$L(z,h) := \{X \in \mathbb{R}^2 : h(X) = z\}$$

and

$$L_{\leq}(z,h) := \{X \in \mathbb{R}^2 : h(X) \leq z\}$$

respectively. In the following we will omit the reference to h whenever this is possible without causing any confusion.

Using level curves and level sets we can reformulate LocP and RLocP.

Theorem 2.8.

a) *z^* is the optimal objective value of* LocP
$\Leftrightarrow z^* = min\{z : L(z) \neq \emptyset\}$

b) *z^F is the optimal objective value of* RLocP
$\Leftrightarrow z^F = min\{z : L(z) \cap F \neq \emptyset\}$

c) *In* a) *and* b) *$L(z)$ can be replaced by $L_{\leq}(z)$*

d) *X is an optimal solution of* RMP *with $h(X) = z$ if and only if there exists a $z \in \mathbb{R}_+$, such that*

$$L(z) \cap \partial R \neq \emptyset \tag{2.3}$$

and

$$L_{\leq}(z) \subseteq R \tag{2.4}$$

The proof of a)-c) is obvious, whereas d) is proved in [Hamacher and Nickel, 1991].

Based on Theorem 2.8 the following procedure can be used to solve RLocP.

Level Curve Approach for Solving RMP

1. Find level curve $L(z)$ satisfying (2.3) and (2.4)
2. Identify $X \in L \cap \partial R$ with $f(X) = z$
3. Output: $Opt^*(R)$:= set of all X of Step 2

The level curve approach can be implemented by applying a search procedure to values of z until (2.3) and (2.4) are satisfied or any other stopping criterion terminates the procedure. This implementation of the procedure is however quite unsatisfactory, since there is no finite bound on its time complexity for finding the exact solution.

On the other hand, this approach leads in some special cases to efficient procedures for solving restricted location problems, as we will see in the following sections.

3 Restricted Median Problems

In this section we will use our theory stated in the previous section to develop efficient algorithms for the most important cases of restricted median problems.

We will start with one facility problems and distance functions l_1, l_2^2 and l_∞.

3.1 One Facility Problems

3.1.1 Solving RMP-l_2^2

In this section we consider the squared Euclidean distance function:

$$d(\text{New}, Ex_m) = l_2^2(\text{New}, Ex_m) = (x_1 - a_{m_1})^2 + (x_2 - a_{m_2})^2 \ .$$

We first state the following result the proof of which can be found, for instance, in [Francis and White, 1974].

Theorem 3.1. *The optimal solution Opt^* of l_2^2-MP is uniquely defined. The level curves of $f(X)$ are circles with center Opt^*.*

Corollary 2.8 and Theorem 3.1 immediately yield the following result.

Theorem 3.2. *RMP-l_2^2 can be solved by finding the maximal radius r^* of a circle with center Opt^*.*

Notice that the problem of finding r^* in Theorem 3.2 can be solved if we are able to compute the minimal Euclidean distance between Opt^* and ∂R. In particular, if R is a polyhedron the next result follows from Theorem 3.2.

Corollary 3.3. *Let $R := \{X \in \mathbb{R}^2 : a_i X \leq b_i, a_i, b_i \in \mathbb{R}, i = 1, \ldots, I\}$ be a convex polyhedron. For $i = 1, \ldots, I$ let $P_i := (p_{i_1}, p_{i_2})$ be the orthogonal projection point of Opt^* on $\{X : a_i X = b_i\}$. Then*

$$Opt^*(R) := argmin\ \{f(P_i) : i = 1, \ldots, I\}^2$$

is the set of optimal solutions of RMP-l_2^2.

The resulting algorithm for solving RMP-l_2^2 for polyhedra has a time-complexity of $O((I \cdot C) \cdot \log I)$ where C is the complexity for computing the projection of Opt^* onto a line. Hence RMP-l_2^2 can be solved in polynomial time for this special case.

If R has no polyhedral structure, we may approximate R by a polyhedron (see [Gruber, 1983], [Burkard et al., 1991] and [Rote, 1990]) and apply Corollary 3.3 to it. Since the distance between ∂R and the boundary of the approximating polyhedron can be made abritary small we can produce solutions X which get arbitrarily close to the optimal solution of RMP-l_2^2.

[2] With $argmin$ $\{f(P_i) : i = 1, \ldots, I\}$ we denote the set of arguments in the set $\{P_i : i = 1, \ldots, I\}$ in which $\min\{f(P_i) : i = 1, \ldots, I\}$ is attained.

3.1.2 Solving RMP-l_1 and RMP-l_∞

Now we consider the two distance functions

$$d(\mathbf{New}, Ex_m) = l_1(\mathbf{New}, Ex_m) = |x_1 - a_{m_1}| + |x_2 - a_{m_2}|$$

and

$$d(\mathbf{New}, Ex_m) = l_\infty(\mathbf{New}, Ex_m) = max\ \{|x_1 - a_{m_1}|, |x_2 - a_{m_2}|\}\ .$$

First, the structure of the level curves will be examined.

Let $a'_{1_1}, \ldots, a'_{P_1}$ be the different values of the first coordinates of the existing facilities in increasing order, such that

$$a'_{1_1} < a'_{2_1} < \ldots < a'_{P_1}$$

holds. $a'_{1_2}, \ldots, a'_{Q_2}$ are defined analogously with respect to the second coordinates of Ex_M for $m \in \mathcal{M}$.

Additionally we define $a'_{0_1} = a'_{0_2} = -\infty$ and $a'_{P+1_1} = a'_{Q+1_2} = \infty$.

Now we get a decomposition of the $\mathbb{R}^2$ into rectangles

$$< t, s > := \{(x_1, x_2) : a'_{t_1} \leq x_1 \leq a'_{t+1_1}\ , a'_{s_2} \leq x_2 \leq a'_{s+1_2}\}\ ,$$

for $t \in \mathcal{P}_0 := \{0, 1, 2, \ldots, P\}$ and $s \in \mathcal{Q}_0 := \{0, 1, 2, \ldots, Q\}$.

From our definition we know that

$$\bigcup_{\substack{t \in \mathcal{P}_0 \\ s \in \mathcal{Q}_0}} < t, s > = \mathbb{R}^2\ .$$

Now we are ready to state a property of the level curves (for a proof see [Francis and White, 1974]) for RMP-l_1.

Theorem 3.4. *The level curves of $f(X)$ for $d = l_1$ are linear in $< t, s >$ for all $t \in t \in \mathcal{P}_0$, $s \in \mathcal{Q}_0$.*

For the following we will use the denotation

$$\mathcal{H} := \left\{(x_1, x_2) : x_1 = a'_{p_1}, p \in \mathcal{P}_0 \backslash \{0\}\right\} \cup \left\{(x_1, x_2) : x_2 = a'_{q_2}, q \in \mathcal{Q}_0 \backslash \{0\}\right\}$$

for the set of lines, determining the rectangles $< t, s >$.

We call these lines in the following **construction lines** of the given RMP-l_1.

The next theorem shows that it is sufficient to consider only the at most $4 \times M$ many points lying on the intersection of R with the construction lines as candidates for optimal solutions of RMP-l_1.

Theorem 3.5. *If the restricting set R is convex,* RMP-l_1 *has an optimal solution* New* $\in$ *Opt* such that*

1. New* $\in \partial R$ and
2. New* $\in \mathcal{H}$.

Proof. By Theorem 2.7 we can restrict ourselves to $X \in \partial R$.
If $X \in (\partial R \cap <t,s>)$ does not lie on one of the construction lines, the level curve through X will not change its slope S_{ts} in X.
Let L_{ts} be the linear segment of the level curve through X with slope S_{ts}. Since R is convex the following cases are possible.

Case 1 L_{ts} crosses ∂R in X, i.e. there exist $U \in L_{ts} \cap F \setminus R$ and $V \in L_{ts} \cap int(R)$. Therefore $f(X) = f(U)$ and consequently, X cannot be optimal by Theorem 2.7.

Case 2 L_{ts} is a supporting hyperplane of R in X.

1. If $L_{ts} \subseteq R$, then there is also a point $Y \in L_{ts}$ where the level curve changes its slope. Since X has the same objective value as Y we can replace X by Y.
2. If $L_{ts} \not\subseteq R$, then, by the same arguments as in Case 1, X cannot be optimal.

□

Summing up we get the following algorithm:

<u>*Construction Line Algorithm for the* RMP-l_1</u>

1. Compute $\mathcal{H}$.
2. Determine $\{Y_1, \ldots, Y_K\} = \mathcal{H} \cap \partial R$.
3. Let New* $\in argmin\{f(Y_1), \ldots, f(Y_K)\}$ and L the level curve through New*.
4. <u>Output:</u> $Opt^*(R) = L \cap \partial R$.

Notice that the construction line algorithm to solve RMP-l_∞ is an algorithm with polynomial time complexity, provided that R is encoded in such a way that the intersection of lines with ∂R can be computed in polynomial time. In particular if R is a convex polyhedron given by I halfspaces, then $Y_1, \ldots, Y_K$ can be determined in $O(I)$ and since $K \leq 4 \times M$ the overall complexity of the algorithm is $O(I + M \cdot \log M)$.

If $d = l_\infty$, the results of this section hold, mutatis mutandis, using Transformation T (see Section 2). The construction lines are in this case 45° and −45° lines through the points Ex_m, $m \in \mathcal{M}$.

In the special case where R is a rectangle, the construction line algorithm can be further improved.

Theorem 3.6. *Let* **New*** *be an optimal solution of* MP-l_1, $R = [a_1, b_1] \times [a_2, b_2]$ *and let* $X_l = (a_1, x_2^*)$, $X_a = (x_1^*, b_2)$, $X_r = (b_1, x_2^*)$ *and* $X_b = (x_1^*, a_2)$ *be the projection of* **New*** *to the side of* R *which is left, above, right, and below* **New***, *repectively.*
Then any

$$\text{New}^F \in argmin\,\{f(X_l), f(X_a), f(X_r), f(X_b)\}$$

is in $Opt^*(R)$ *of* RMP-l_1.

For a proof see [Hamacher and Nickel, 1991].

Notice that the latter case is of particular importance in the PCB application introduced in Section 1. Since RMP is in this situation embedded in a larger problem, we are also interested in maintaining integrality of data. This question is addressed in the next result.

Corollary 3.7. *If* R *is a rectangle parallel to the* x_1 *and* x_2 *axes and all input data is integer, then the procedure based on the construction line approach will always find an integer optimum solution of* RMP-l_1 *and* RMP-l_∞.

Proof. For $d = l_1$ the result follows immediately from Theorem 3.6. For $d = l_\infty$ the construction lines are 45° and −45° lines and therefore intersect the integral boundary of R in integer points. Hence the result holds for RMP-l_∞ as well.

□

3.1.3 Extensions

The algorithms of the previous sections can easily be modified to accomodate the case where R is the union of pairwise disjoint, convex sets $R_1, \ldots, R_K$ with $K > 1$. Notice that this is a more realistic model in most real-world problems.

In this situation we will first solve the unrestricted problem MP-l_p. Then the following result is an immediate consequence of the results of Section 2.

Theorem 3.8. *Either an optimal solution* New* $\in Opt^*$ *solves* RMP-l_p, *or there exists some* k *such that* $Opt^*(R)$ *is a subset of* ∂R_k.

Proof. If the intersection of Opt^* with the feasible solution set F of RMP-l_p is empty, then the convexity of Opt^* and the assumptions on the sets $R_1, \ldots, R_k$ imply that there exists some k such that Opt^* is a subset of R_k. Hence we can duplicate the proofs of Section 2, where R and ∂R is replaced by R_k and ∂R_k, respectively.

□

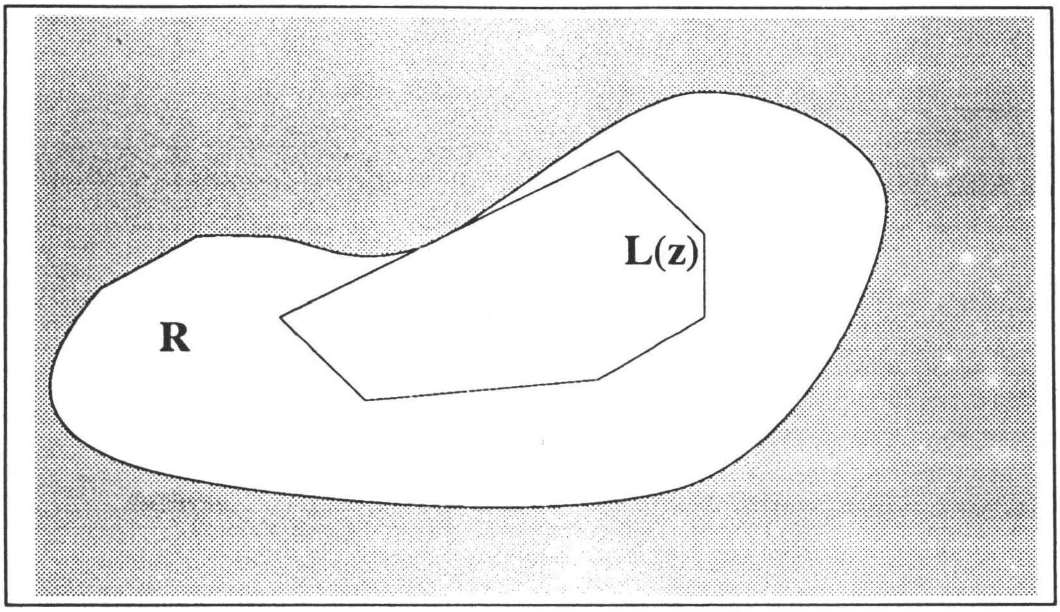

Figure 3.1: A restricting set R with an infinite number of candidate points on ∂R which can lie on the interior of a linear piece of a level curve.

The theory of RMP developed so far can be extended to more special cases of non-convex restricting sets R. If we review the proof of Theorem 3.5 for RMP-l_1 it becomes apparent that we can apply a combinatorial algorithm as in the case of convex sets whenever we have only a finite set of points on ∂R which can lie on the interior of linear pieces of a level curve. For arbitrary non-convex sets this situation is not given as is indicated by Figure 3.1.

But in the following two special cases we can find a combinatorial algorithm:

<u>Convex sets with bumps:</u>
If R_1 and R_2 are two convex sets in $\mathbb{R}^2$ such that $R_1 \cap R_2 \neq \emptyset$, and neither $R_1 \subseteq R_2$ nor $R_2 \subseteq R_1$, and $R := R_1 \cup R_2$ is non-convex. (see Figure 3.2). We call R a **bumpy set.**

Because of the convexity of R_1 and R_2 there exist exactly two points r_1 and r_2 such that

$$r_1, r_2 \in \partial R_1 \cap \partial R_2 \cap \partial R \ .$$

We call r_1 and r_2 the **roots of the bumpy sets.**

Any level curve touching $\partial R = \partial(R_1 \cup R_2)$ from within has linear pieces which have

a) both endpoints in the interior of R, or

b) one or both endpoints on ∂R, or

c) pass through one of the roots of $R_1 \cup R_2$. (see Figure 3.2)

Hence the construction line algorithm will work if we extend the set of $\mathcal{H} \cap \partial R$ by the set of root nodes.

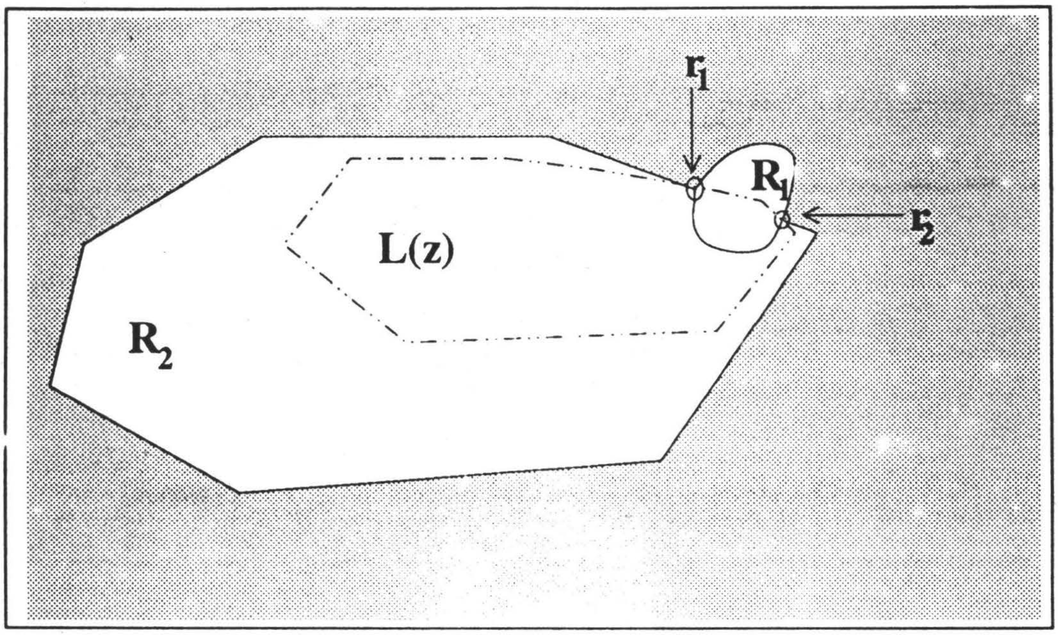

Figure 3.2: A bumpy set with root nodes r_1 and r_2 and the optimal level curve passing through r_1 and r_2.

Obviously this extension also works if a given convex set R' has more than one bump, i.e. bumps $R_1, \ldots, R_K$, where $R_i \cap R_j = \emptyset$ for $i \neq j$. Then we have to find the set $\{r_1, \ldots, r_{2K}\}$ of root points and compare the objective values of the $O(M + K)$ many points

$$(\mathcal{H} \cap \partial R) \cup \{r_1, \ldots, r_{2K}\} \ .$$

Notice that the bumpy set approach can be used to solve N independent 1-facility problems in which the space of the new facilities is taken into consideration.

Complement of polyhedra:

If R is the complement of a polyhedron we get $\partial R = \partial \overline{R}$, where $\overline{R} := \mathbb{R}^2 \backslash R$. Then the linear pieces of any level curve touching ∂R from within have

a) both endpoints in the interior of R, or

b) one or both endpoints on ∂R, or

c) pass through one of the corner points of $\overline{R}$. (see Figure 3.3)

As in the case of bumpy sets the construction line algorithm can be used by extending the set $(\mathcal{H} \cap \partial R)$ by the set of all corner points of $\overline{R}$. We will use this approach to solve the problem of locating emergency facilities (see Section 5).

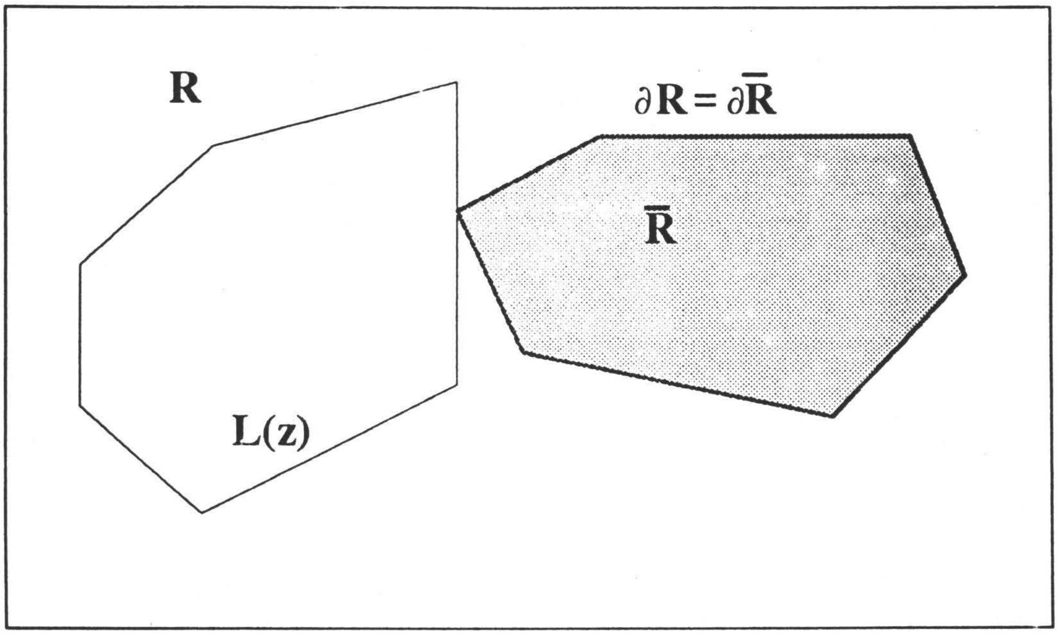

Figure 3.3: A restricting set R which is the complement of a convex polyhedron $\overline{R}$ and a level curve touching ∂R from within.

3.2 Multi Facility Problems

Although no complexity results are known yet the problem of locating more than one new facility seems to be much more difficult than the case of a single facility. The only efficient algorithm we are aware of is for 2-RMP-l_1 and rectangles as restricting sets (see [Nickel, 1991]).

We will extend a linear programming approach of the unrestricted case to fit our needs and give some interesting heuristics.

3.2.1 Solving N-RMP-l_1 and N-RMP-l_∞

As denotation we use in the following

$$d(\text{New}_n, Ex_m) = l_1(\text{New}_n, Ex_m) = |x_{n_1} - a_{m_1}| + |x_{n_2} - a_{m_2}|$$

and

$$d(\text{New}_n, \text{New}_r) = l_1(\text{New}_n, \text{New}_r) = |x_{n_1} - x_{r_1}| + |x_{n_2} - x_{r_2}| \; .$$

We restrict ourselves to the case where R is a convex polyhedron (i.e. R can be described as $AX \leq b$, where A is a $(2 \times L)$-matrix, $X \in \mathbb{R}^2$ and $b \in \mathbb{R}^L$) in order to apply a linear programming based approach. Only the l_1-case is treated explicitly but with the Transformation T of Section 2 we get also a solution method for the l_∞-case.

[Francis and White, 1974] gave the following linear programming formulation which solves N-MP-l_1 and to which we refer in the following as N-MP-l_1-(LP):

$$\text{minimize} \sum_{n\in\mathcal{N}}\sum_{m\in\mathcal{M}} w_{nm}(d^+_{nm1}+d^-_{nm1}) + \sum_{n=1}^{N-1}\sum_{r=n+1}^{r=N} v_{nr}(e^+_{nr1}+e^-_{nr1}) +$$
$$\sum_{n\in\mathcal{N}}\sum_{m\in\mathcal{M}} w_{nm}(d^+_{nm2}+d^-_{nm2}) + \sum_{n=1}^{N-1}\sum_{r=n+1}^{r=N} v_{nr}(e^+_{nr2}+e^-_{nr2}) \quad (3.1)$$

under

$$x_{n_k} - d^+_{nmk} + d^-_{nmk} = a_{m_k} \quad {\scriptstyle n\in\mathcal{N}, m\in\mathcal{M}, k=1,2} \quad (3.2)$$
$$x_{r_k} - x_{n_k} - e^+_{nrk} + e^-_{nrk} = 0 \quad {\scriptstyle n=1,\ldots,N-1, r=n+1,\ldots,N, k=1,2} \quad (3.3)$$
$$(3.4)$$

and

$$d^+_{nmk}, d^-_{nmk}, e^+_{nrk}, e^-_{nrk} \geq 0\ .$$

Since $d^+_{nmk} + d^-_{nmk} = |x_{n_k} - a_{m_k}|$ and $v^+_{nrk} + v^-_{nrk} = |x_{n_k} - x_{r_k}|$, (3.1) coincides with the original objective function.

Let Ω be the set of all N-tupels over $\{1,2,\ldots,L\}$. We denote with A_i the i-th row of the matrix A and with b_i the i-th component of the vector b.

Now we get for every $\mathcal{O} \in \Omega$ the following N conditions for solutions of N-RMP-l_1:

$$A_{\mathcal{O}(n)}X^n \geq b_{\mathcal{O}(n)} \quad {\scriptstyle n\in\mathcal{N}}\ .^3 \quad (3.5)$$

The following algorithm is straightforward.

Algorithm for N-RMP-l_1

1. Let N-RMP-l_1-$(LP)_{\mathcal{O}}$ be defined as the N-MP-l_1-(LP) with the additional N conditions from (3.5).
2. $Opt^*(R_{\mathcal{O}})$:= Optima solution of l_1-(RMP-N)-$(LP)_{\mathcal{O}}$ for all $\mathcal{O} \in \Omega$.
3. Output: $Opt^*(R) := argmin\{f(Opt^*(R_{\mathcal{O}})) : \mathcal{O} \in \Omega\}$.

As one easily recognizes the time complexity of this algorithm is not polynomial and so it is neccessary to add a subsection with two useful heuristics.

3.2.2 Two heuristics for the N-RMP

The first heuristic consists of a sequential solution of N single-facility problems. Whenever one of the new facilities is placed we add the space R_n occupied by this facility to the restricting set R. In each step of the algorithm $R \cup R_1 \cup \ldots \cup R_N$ is a bumpy set such that we can apply the algorithm of Section 3.1.3 to solve the single facility problems. Moreover we treat all of the new facilities

[3] Here $\mathcal{O}(n)$ means the n-th component of the N-tupel $\mathcal{O}$.

which have already been placed as existing facilites for the next single facility problem. Thus in each of these problems the approximation of the objective function $f(\mathbf{New})$ is improved. In the last step the objective functions of N-RMP and of the single facility problem coincide.

Heuristic for N-RMP considering space constraints for the new facilities

1. $n := 1$
2. Find the optimal location $\mathbf{New}_n$ for the single facility n with respect to the restricting set R and existing facilities $Ex_1, \dots, Ex_M$, $\mathbf{New}_1, \dots, \mathbf{New}_{n-1}$.
3. Extend R by a bump corresponding to the space occupied by the new facility n, i.e. define $R := R \cup R_n$.
4. $n := n + 1$
5. If $n \leq N$, then goto Step 2; otherwise $\longrightarrow$ STOP.

Further improvements of the objective function $f(\mathbf{New})$ can be obtained by reiterating Steps 1 through 5 where all $\mathbf{New}_i$ for $i \neq n$ are treated as existing facilities. These iterations can be repeated until no improvment of $f(\mathbf{New})$ is obtained during an iteration. Notice that this heurstic can also be applied to the unrestricted N-facility problem.

For the second heuristic we assume that $Opt^* \subseteq int(R)$ such that $Opt^*(R) \subseteq (\partial R)^N$ by Theorem 2.7.

Heuristic for N-RMP with search on the boundary

1. $n := 1$
2. Solve the RMP for new facility n.
3. $n := n + 1$
4. If $n \leq N$, then goto Step 2; otherwise continue.
5. Choose two new facilites and move them closer together until no more improvements can be achieved.
6. Treat all pairs of new facilites in this way.
7. Repeat the Steps 5 and 6 with n-subsets ($n = 3, \dots, N$) if needed.

If our assumption is not satisfied and some of the new facilities are in $int(R)$ we restrict ourselves in Step 5 and 7 to pairs and n-subsets, respectively, of new facilities on ∂R.

The two heuristics can also be combined such that the first one is a preprocessing heuristic and the second one (without Steps 1 – 4) is a postprocessing routine.

4 Restricted Center Problems

As we have shown in Section 2.3, level sets and lines of LocP play a crucial role in developing algorithms for solving restricted location problems. Therefore we will first show how a careful analysis of these sets leads to efficient algorithms for solving RCP-l_1 and RCP-l_∞.

In the case of single facility unrestricted center problems with respect to rectilinear distance function we can write the objective function as

$$g(\text{New}) = \max\{w_m(|x_1 - x_{m_1}| + |x_2 - x_{m_2}|) : m \in \mathcal{M}\}$$

where New $= (x_1, x_2)$ is the location of the new facility. For any $z > 0$ we can characterize the level set $L_\leq(z)$ as follows (see [Francis and White, 1974]).

$$\begin{aligned}
& X = (x_1, x_2) \in L_\leq(z) \\
& \iff g(X) \leq z \\
& \iff x_1 + x_2 \leq A_m(z) := \frac{z}{w_m} + x_{m_1} + x_{m_2} \\
& \qquad \text{and} \\
& \qquad x_1 + x_2 \geq B_m(z) := -\frac{z}{w_m} + x_{m_1} + x_{m_2} \\
& \qquad \text{and} \\
& \qquad -x_1 + x_2 \leq C_m(z) := \frac{z}{w_m} - x_{m_1} + x_{m_2} \\
& \qquad \text{and} \\
& \qquad -x_1 + x_2 \geq D_m(z) := -\frac{z}{w_m} - x_{m_1} + x_{m_2} \\
& \iff B(z) \leq x_1 + x_2 \leq A(z) \\
& \qquad \text{and} \\
& \qquad D(z) \leq -x_2 + x_2 \leq C(z)\,, \qquad (4.1)
\end{aligned}$$

where the functions A(z), B(z), C(z), and D(z) are defined by

$$\begin{aligned}
A(z) &:= \min_{m \in \mathcal{M}} A_m(z) \\
B(z) &:= \max_{m \in \mathcal{M}} B_m(z) \\
C(z) &:= \min_{m \in \mathcal{M}} C_m(z) \\
D(z) &:= \max_{m \in \mathcal{M}} D_m(z)
\end{aligned}$$

Using this characterization we can prove the following result.

Theorem 4.1. *1. Either $L_{\leq}(z) = \emptyset$ or $L_{\leq}(z)$ is a – possibly degenerate – rectangle with corner points*

$$\begin{aligned} P_1(z) &:= \frac{1}{2}(A(z) - C(z), A(z) + C(z)) \\ P_2(z) &:= \frac{1}{2}(A(z) - D(z), A(z) + D(z)) \\ P_3(z) &:= \frac{1}{2}(B(z) - D(z), B(z) + D(z)) \\ P_4(z) &:= \frac{1}{2}(B(z) - C(z), B(z) + C(z)) \,. \end{aligned}$$

2. Let $z_{A,B}$ and $z_{C,D}$ be positive real numbers such that $A(z_{A,B}) = B(z_{A,B})$ and $C(z_{C,D}) = D(z_{C,D})$. Then

$$z^* := \max\{z_{A,B}, z_{C,D}\}$$

is the optimal ovjective value z^* of CP-l_1.
Moreover the optimal solution set of CP is

$$Opt^* = L_{\leq}(z^*) = L(z^*) \,.$$

Proof.
ad 1) By condition (4.1) we know that $L_{\leq}(z)$ is either empty or a rectangle with pairwise parallel sides given by

$$\begin{aligned} x_1 + x_2 &= A(z) \\ x_1 + x_2 &= B(z) \\ -x_1 + x_2 &= C(z) \\ -x_1 + x_2 &= D(z) \end{aligned}$$

ad 2) The funcions $A(z)$ and $B(z)$ have the following properties.

- $A(0) < B(0)$ (where we exclude the trivial case of a single existing facility)
- $A(z)$ is strictly increasing and $B(z)$ is strictly decreasing
- $A(z)$ and $B(z)$ are continous, piecewise linear functions

Therefore $z_{AB} > 0$ with $A(z_{AB}) = B(z_{AB})$ is uniquely defined, and is the smallest z such that $A(z) \leq x_1 + x_2 \leq B(z)$ for some $X = (x_1, x_2)$.

Correspondingly, these three properties are also satisfied by the functions $C(z)$ and $D(z)$, i.e. $z_{CD} > 0$ with $A(z_{CD}) = B(z_{CD})$ is uniquely defined, and is the smallest z with $C(z) < x_1 + x_2 < D(z)$ for some $X = (x_1, x_2)$.

Therefore $z^* = \max\{z_{AB}, z_{CD}\}$ is the optimal objective value of CP-l_1 by Theorem 2.8.

Notice that by the definition of z^* the rectangle $L_{\leq}(z^*)$ is always degenerate such that $L_{\leq}(z^*) = L(z^*)$ as claimed in Theorem 4.1.

□

In order to compute z_{AB} we compute for all $i \neq j$ the number $z_{A_iB_j}$ with $A_i(z_{A_iB_j}) = B_j(z_{A_iB_j})$, provided the two lines corresponding to the functions $A_i(z)$ and $B_j(z)$ intersect. This can be done in $O(1)$ time complexity. Then, z_{AB} is the largest of these numbers. The same approach is used to compute z_{CD} and we get the following $O(M^2 \cdot \log M^2)$ algorithm for solving CP-l_1.

Algorithm for CP-l_1

1. Compute for all $i, j \in \mathcal{M}$ with $i \neq j$ and $w_i \neq 0 \neq w_j$:

$$z_{A_iB_j} := \frac{w_i w_j}{w_i + w_j}((x_{j_1} + x_{j_2}) - (x_{i_1} + x_{i_2}))$$

and

$$z_{C_iD_j} := \frac{w_i w_j}{w_i + w_j}((-x_{j_1} + x_{j_2}) - (-x_{i_1} + x_{i_2})).$$

2. Compute

$$\begin{aligned} z_{AB} &= \max\{z_{A_iB_j} : i \neq j\} \\ z_{CD} &= \max\{z_{C_iD_j} : i \neq j\}\ , \text{ and} \\ z^* &:= \max\{z_{AB}, z_{CD}\} \end{aligned}$$

3. Output: $Opt^* = L_{\leq}(z^*) = L(z^*)$

Notice that we can speed up the algorithm by restricting ourselves in Steps 1 and 2 to $i < j$ (since $z_{A_iB_j} = -z_{A_jB_i}$ and $z_{C_iD_j} = -z_{C_jD_i}$).

Using Theorem 4.1 we can efficiently test whether any of the optimal solutions of the unrestricted center problem is feasible for the restricted one. We assume therefore in the following that $Opt^* \subseteq int(R)$.

The result of Theorem 4.1 together with the characterization of optimal solutions sets of restricted location problems (Theorem 2.8) implies the following result which will lead to an efficient algorithm for solving restricted 1-facility center problems with respect to rectilinear distance functions.

Corollary 4.2. *Let R be a restricting set which is convex.*

z^*_R is the optimal objective value of RCP-l_1 *if and only if*

1. $\{P_1(z^*_R), \ldots, P_4(z^*_R)\} \subseteq R$ *and*
2. $P_i(z^*_R) \in \partial R$ *for at least one* $i \in \{1, 2, 3, 4\}$.

Moreover $Opt^*(R)$ is the intersection of ∂R with the rectangle $L_{\leq}(z^*_R)$.

Proof. Immediate consequence of Theorem 4.1 and Theorem 2.8.

□

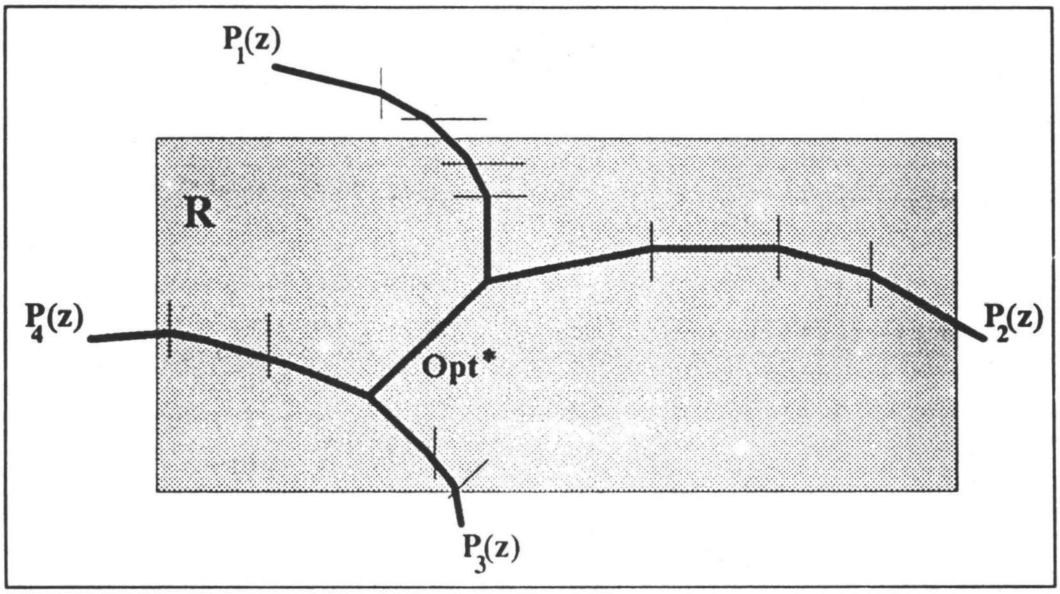

Figure 4.1: Computation of z_R^* by considering the four intersections of piecewise linear functions $P_i(z)$ with ∂R.

Since $P_i(z)$ is for each $i \in \{1,2,3,4\}$ a piecewise linear function, the computation of z_R^* can be reduced to checking $P_i(z)$ for the $O(M^2)$ many breakpoints of $P_i(z)$. Let $\{\zeta_1, \ldots, \zeta_L\}$ be the set of breakpoints of $A(z)$, $B(z)$, $C(z)$, and D(z) sorted by increasing value. By Theorem 4.1 all breakpoints of the functions $P_i(z)$ are contained in $\{\zeta_1, \ldots, \zeta_L\}$. If $\zeta_l \in \{\zeta_1, \ldots, \zeta_L\}$ is the breakpoint with smallest index such that $P_i(\zeta_l) \notin int(R)$, for some $i \in \{1,2,3,4\}$ then $z_R^* \in [\zeta_{l-1}, \zeta_l]$ can be found by computing the intersections of the lines given by the *linear* functions $P_j(z)$, $j = 1,2,3,4$, on the intervall $[\zeta_l, \zeta_{l+1}]$ with ∂R. If this intersection is attained in z_i, then $z_R^* = \max\{z_1, z_2, z_3, z_4\}$. (see Figure 4.1)

Since the breakpoints of $A(z)$ are points at which functions $A_i(z)$ intersect functions $A_j(z)$, and analogously for $B(z)$, $C(z)$, and $D(z)$ the following algorithm will solve RCP-l_1.

Algorithm for RCP-l_1

1. Compute the breakpoints with positive values

$$|z_{A_iA_j}| = |-z_{B_iB_j}| = |((x_{j_1} + x_{j_2}) - (x_{i_1} + x_{i_2}))|$$

and

$$|z_{C_iC_j}| = |-z_{D_iD_j}| = |((-x_{j_1} + x_{j_2}) - (-x_{i_1} + x_{i_2}))|$$

for functions A and B, and C and D, respectively. Let $\{\zeta_1, \ldots, \zeta_L\}$ be the set of these numbers with

$$\zeta_0 := z* < \zeta_1 < \ldots < \zeta_L < \zeta_{L+1} := \infty$$

(where z^* is the optimal objective value of the unrestricted problem).

2. Find the smallest index l such that $P_i(z_l) \in int(R)$, $\forall i = 1,2,3,4$, and $P_i(\zeta_{l+1}) \notin R$, for at least one $i \in 1,2,3,4$.

3. Compute the real numbers $z_i \in [\zeta_l, \zeta_{l+1}]$ such that $P_i(z_i) \in \partial R$.

4. Output: the optimal objective value $z_R^* = \max\{z_1, z_2, z_3, z_4\}$ and the optimal solution set $Opt^*(R) = L(z) \cap \partial R$.

The number L of breakpoints to consider is $O(M^2)$. Sorting of the breakpoints requires $O(M^2 \cdot \log(M^2))$ time. The algorithm performs $O(M^2)$ checks whether $P_i(\zeta_l) \in R$. If the answer to this check is for the first time negative (Step 2), then 4 computations of the intersection of a line with ∂R are needed in Step 3. In the case where R is given as a polyhedron with K facetes this results in a $O(M^2 \cdot \log(M^2) + K)$ algorithm for solving RCP-l_1.

Using transformation T introduced in Section 2.1 we can solve RCP-l_∞ by applying the algorithm for RCP-l_1 to $T(R)$ and the transformed existing facilities $T(Ex_i)$.

The only solution method for N-CP-l_1 is the transformation into a linear program (see [Francis and White, 1974]).

5 Applications of the Theory to PCB Assembly, Undesirable Facilities and Emergency Facilities

In this section we will come back to the examples introduced in Section 1.

5.1 PCB assembly

In the assembly of printed circuit boards the existing facilities correspond to the fixed insertion points $P_m = (p_{m_1}, p_{m_2})$, i.e., we set $a_{m_1} = p_{m_1}$ and $a_{m_2} = p_{m_2}$, $\forall m \in (\mathcal{M})$. Each of the parts belongs to one of N part types $n \in \mathcal{N} := \{1, \ldots, N\}$. If $t(m)$ is the type of part m, we denote $\mathcal{M}_n := \{m \in \mathcal{M} : t(m) = n\}$. The new facilities correspond to the (unknown) locations $X_n = (x_{n_1}, x_{n_2})$ of the bin holding the parts of type n, $\forall n \in \mathcal{N}$.

Suppose the insertion sequence of the parts is given (an optimal sequence with respect to given locations of the bins can be found by applying a traveling salesman algorithm, see [Francis et al., 1989] and [Foulds and Hamacher, 1990]). Without loss of generality we assume that this sequence is (1,2,...,M). For given locations X_n of the bins, this sequence defines in a unique way a **robot tour**

$$tour = (X_{t(1)}, P_1, X_{t(2)}, P_2, \ldots, X_{t(M)}, P_M, X_{t(1)})$$

given by the insertion points and the locations of bins which have to be visited to pick up the corresponding part. Then the length of a robot tour can be written as

$$l(tour) = \sum_{n \in \mathcal{N}} \sum_{m \in \mathcal{M}_n} (d(X_{t(m)}, P_m] + d(P_{m-1}, X_{t(m)}))\ ,$$

where we define $P_0 := P_M$. Obviously, this is the objective function of an N-facility median problem without interaction between the new facilities. Hence we solve N independent facility problems with weights

$$w_{m,n} := \begin{cases} 2 & : \quad m \in \mathcal{M}_n \text{ and } (m+1) \in \mathcal{M}_n \\ 1 & : \quad m \in \mathcal{M}_n \text{ and } (m+1) \notin \mathcal{M}_n \\ 0 & : \quad m \notin \mathcal{M}_n \text{ and } (m+1) \notin \mathcal{M}_n \end{cases},$$

where we define $M + 1 := 0$.

If we solve the N independent location problems MP with respect to these data we conclude from Theorem 2.3 and Corollary 2.4 that any solution of the unrestricted location problems will find locations for the bins which are on the PCB – obviously a solution which is not feasible in the practical context. We therefore introduce a rectangle $[0, a] \times [0, b]$ with $a, b > 0$ containing the PCB including a security distance on the border of the PCB (see Figure 1.1). Depending on the production environement which defines which of the sides of the rectangle are allowed to place the bins, the restricting set may be

- $R = [0, a] \times [0, b]$, if all four sides are feasible, or
- $R = [0, a] \times [0, \infty]$, if the upper side is not feasible, or
- $R = [0, \infty] \times [0, b]$, if the right-hand side is not feasible, or
- $R = [0, a] \times [-\infty, b]$, if the lower side is not feasible, or
- $R = [-\infty, a] \times [0, b]$, if the left-hand side is not feasible.

Correspondingly, one may exclude any combination of sides of the rectangle by a suitable defintion of R.

For all possible choices of R, we can solve the problem of finding the location of the bins by solving N independent RMP with the algorithms of Section 3. If the robot arm has two independent motors such that it can move simultanously in x and y direction we choose the Chebyshev distance l_∞. If on the other hand only one motor is available (implying the robot arm moves sequentially in x and y direction) the rectilinear distance function l_1 is appropiate.

5.2 Obnoxious facility planning

In order to deal with the case of a restricting set

$$R = R_1 \cup R_2 \cup \ldots \cup R_K ,$$

which is the union of connected, pairwise disjoint sets of $\mathbb{R}^2$ we use the result of Theorem 3.8.

Consequently, the obnoxious facility planning problem introduced in Section 1 can be solved with the same complexity as RLocP with respect to a single connected restricting set.

5.3 Location of emergency facilities

In this example which was introduced in Section 1 the restricting set R is the complement of a convex set $\overline{R}$. From Section 3.1.3 we know that we can deal with this problem if $\overline{R}$ is a convex polyhedron. Depending on the type of distance function we may have to approximate $\overline{R}$ by a suitable polyhedron to apply the algorithm introduced in Section 3.1.3. (The approximation can, for instance, be done by the Sandwich approach of [Burkard et al., 1991].)

6 Computer Implementation

The ideas of this paper have been implemented in a software package **RLP** (see [Nickel and Hamacher, 1992]). It runs in the MS-Windows environment on PCs with MS-DOS or OS/2 2.x. It is available through ftp under uranus.mathematik.uni-kl.de.

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