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Pricing American options in the Heston model: a close look on incorporating correlation

Vorwort

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Berichterstattung heißt hier Dokumentation des Transfers aktueller Ergebnisse aus mathematischer Forschungs- und Entwicklungsarbeit in industrielle Anwendungen und Softwareprodukte – und umgekehrt, denn Probleme der Praxis generieren neue interessante mathematische Fragestellungen.

Prof. Dr. Dieter Prätzel-Wolters
Institutsleiter

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Pricing American options in the Heston model: a close look on incorporating correlation

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Abstract

We introduce a refined tree method to compute option prices using the stochastic volatility model of Heston. In a first step, we model the stock and variance process as two separate trees and with transition probabilities obtained by matching tree moments up to order two against the Heston model ones. The correlation between the driving Brownian motions in the Heston model is then incorporated by the node-wise adjustment of the probabilities. This adjustment, leaving the marginals fixed, optimizes the match between tree and model correlation. In some nodes, we are even able to further match moments of higher order. Numerically this gives convergence orders faster than $1/N$, where N is the number of discretization steps. Accuracy of our method is checked for European option prices against a semi closed-form, and our prices for both European and American options are compared to alternative approaches.

Keywords: Heston model, American options, moment matching, correlation, tree method

1 Introduction

The model of Heston [1993] ranks among the most popular stochastic volatility models. As remarked by Gatheral [2006], amongst others, relaxing the constant volatility assumption of the Black-Scholes model leads to a more flexible framework for explaining empirically observed option prices. The model owes parts of its popularity to the semi closed-form for European call and put option prices, see Heston [1993], as well as Albrecher et al. [2007] for a more stable formulation. Pricing of more complex derivatives like American options is not possible in this semi closed-form framework though, so for this purpose other techniques like Monte Carlo simulations, finite difference methods, and tree-based methods have been proposed. Determining the continuation value for American style options is an issue within the Monte Carlo framework. As pointed out in Hull [2008], finite difference methods like e.g. Crank and Nicolson [1947] and tree methods are closely related. Both of them basically face the same complexity problems and can easily cope with early-exercise features. In this paper we focus on tree-based methods.

As already pointed out in Heston [1993], the correlation between volatility and stock price has a strong impact on skewness effects and is therefore an important parameter.

Adequately incorporating non-zero correlation to a tree method while insisting on proper transition probabilities poses a major problem, see Beliaeva and Nawalkha [2010] or Leisen [2000]. This problem is not only of theoretic interest but has arisen in calibrating the model to real-world data. Several authors have focused on this calibration to market data, obtaining highly negative correlation values, see for example Nandi [1998] for calibrating the model to options on the S&P 500 index. The considered time series is from 21 January 1991 to 10 April 1992 and the observation covers 9,548 call and put prices. The estimated correlation between volatility and the index returns is -0.79 with a standard error of 0.016, indicating a highly negative true value. Another calibration is done in Bakshi et al. [1997] for S&P 500 option price data as well, covering 38,749 call options from June 1988 to May 1991 with an estimated correlation of -0.64, again a pronouncedly negative value.

Literature on tree-based approximation of the Heston model basically lists three approaches to assess this issue. The first one alters the considered processes such that correlation does not affect the Brownian increments but influences the process as a direct parameter, see e.g. Hull and White [1990]. The authors decouple the processes via orthogonalising the Brownian motions. The distinct trees are then treated as independent and marginal probabilities only have to be multiplied to get joint ones. This concept also arises in Beliaeva and Nawalkha [2010], where the authors suggest a transformation of the stock process that consists of three parts. The first part is the standard logarithmic transformation of the stock price. The second part is a product of the correlation, the instantaneous variance and volatility of the variance process. This factor allows for the transformed stock process to be conditionally independent of the variance process. The third component is a deterministic function of time which, suitably defined, smoothes out negative probabilities arising when the volatility process becomes small. According to the authors though, the convergence of the method is negatively affected by realistic correlation values. They compare their numerical results to those obtained in a lattice-based approach by Guan and Xiaoqiang [2000]. The respective results are also taken for our comparisons later.

The second line of research comprises methods where the tree contains enough successors to match the correlation exactly, see e.g. Leisen [2000]. However, proper probabilities with values in $[0, 1]$ can only be guaranteed after simplifications, which result in $-\sqrt{0.75} < \rho < \sqrt{0.75}$, an asymptotic correlation constraint. Moreover, the approximation is done with eight successors so the generated tree is complex and potentially slow.

The third group of authors, e.g. Boyle [1988], Hilliard and Schwartz [1996] or, in the context of interest rates Hull and White [1994], adjusts the joint probabilities to capture non-vanishing covariance. Boyle [1988] considers two correlated state variables, using two combined binomial trees, where a further fifth node arises if both processes remain unchanged at the same time and prob-

abilities are found by matching moments. This parametrization however cannot avoid negative transition probabilities, so an additional stretch parameter has to be introduced. Hilliard and Schwartz [1996] start with independent binomial models for the stock and volatility processes where the joint probabilities are then adjusted for the non-vanishing covariance of the processes. As in the setting of Boyle [1988], the authors choose a scaling factor so that the four transition probabilities come as close as possible to one fourth. They determine the option price as average of the m and $m + 1$ step layers, and state convergence only for zero correlation.

Our approach for pricing American options consists of a tree-based model with a particular emphasis on an adequate incorporation of the correlation in the Heston model.

Like Hilliard and Schwartz [1996], we start with independence, i.e. with two separate trees with product probabilities for the joint values of variance and logarithmic stock price processes. More specifically we use a recombining binomial tree for the variance and a recombining trinomial tree for the logarithmic stock price. Secondly, keeping the marginal distributions fixed, we shift mass to match correlation node-wise. An exact node-wise match of the covariance is not always possible while maintaining proper transition probabilities. We derive an optimal adjustment which produces the closest admissible match, and, where an exact match is possible, even moments of third order are partially matched. This adjustment is done in a static and computationally efficient way. We check our results of European calls against the semi closed-form solution given in Albrecher et al. [2007] and obtain highly accurate prices even for a small number of discretizing steps respectively moderate step sizes. We further compare our results to the prices given in Beliaeva and Nawalkha [2010] and Guan and Xiaoqiang [2000] for speed and accuracy.

The setup of this paper is as follows. Section 2 gives the model formulation in the context of Heston's stochastic volatility model. We state a recursive formula to determine moments for the exact Heston distribution and further illustrate the approximation methods we consider for the variance and logarithmic stock price process. The main part, Section 3, is devoted to the probability adjustment. We provide convergence results, a complexity analysis and derive numerical examples in Section 4. We further compare our method to other approaches. Finally, Section 5 concludes and gives an outlook.

2   Setup and Notation

We consider the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ with $\mathbb{P}$ being the risk neutral probability measure. The risk neutral dynamics of the stock price process S_t and the variance process V_t in the Heston model are given by

$$\mathrm{d}S_t = (r - d)\,\mathrm{d}t + \sqrt{V_t}\,\mathrm{d}W_t^S, \tag{1}$$

$$\mathrm{d}V_t = \kappa\,(\theta - V_t)\,\mathrm{d}t + \eta\sqrt{V_t}\,\mathrm{d}W_t^V, \tag{2}$$

where $S_0 = s_0$ and $V_0 = v_0$ are the initial values and r and d are constant interest rate and dividend yield, respectively. The variance itself is modeled as a stochastic process V_t where κ denotes the speed of mean reversion to the long term variance level θ and η is the volatility of the variance. The two Brownian motions W_t^S and W_t^V are correlated with constant correlation ρ. Further we assume the stability condition $2\kappa\theta > \eta^2$ to be satisfied.

In the sequel, we consider the growth adjusted logarithmic state space transformation of (1),

$$\mathrm{d}X_t = \mathrm{d}\log\left(S_t e^{-(r-d)t}\right) = -\frac{1}{2}V_t\,\mathrm{d}t + \sqrt{V_t}\,\mathrm{d}W_t^S, \tag{3}$$

with $x_0 = \ln(s_0)$. The remaining process is independent of r and d and therefore its approximation is less prone to negative probabilities, see Leisen [2000]. The part $e^{(r-d)t}$ is used later on to adjust the nodes when we determine option values.

For moment matching purposes, the following lemma provides recursions to determine the moments in the Heston model. For simplicity, in moment matching we use linearizations of these moments as given in the subsequent corollary, which is confined to the terms needed later. Proofs for both lemma and corollary are delegated to the appendix.

Lemma 1:
For $i, j \in \mathbb{N}_0$, the moments of the Heston model

$$m^{i,j}(t) := \mathbb{E}(X_t^i V_t^j)$$

are given by the recursion

$$m^{i,j}(t) = x_0^i v_0^j e^{-j\kappa t} + \int_0^t e^{-j\kappa(t-s)} b^{i,j}(s)\,\mathrm{d}s, \tag{4}$$

where

$$b^{i,j}(t) = \frac{i(i-1)}{2} m^{i-2,j+1}(t) - \frac{i}{2} m^{i-1,j+1}(t) + ij\eta\rho m^{i-1,j}(t)$$
$$+ \frac{j\left(2\kappa\theta + \eta^2(j-1)\right)}{2} m^{i,j-1}(t),$$

and with initial condition $m^{0,0}(t) = 1$ and $m^{i,j}(t) = 0$ for negative indices.

Corollary 2:
The first-order Taylor approximations of the moments $\mathbb{E}(X_t)$, $\mathbb{V}ar(X_t)$, $\mathbb{E}(V_t)$, $\mathbb{V}ar(V_t)$, $\mathbb{C}ov(X_t, V_t)$ and $\mathbb{E}(X_t^2, V_t)$ in the Heston model are given as

$$\mathbb{E}(X_t) = x_0 - \frac{1}{2}v_0 t + \mathcal{O}\left(t^2\right),$$

$$\mathbb{V}ar(X_t) = v_0 t + \mathcal{O}\left(t^2\right),$$

$$\mathbb{E}(V_t) = v_0 + \kappa(\theta - v_0) t + \mathcal{O}\left(t^2\right),$$

$$\mathbb{V}ar(V_t) = \eta^2 v_0 t + \mathcal{O}\left(t^2\right),$$

$$\mathbb{C}ov(X_t, V_t) = \eta\rho v_0 t + \mathcal{O}\left(t^2\right),$$

$$\mathbb{E}\left(X_t^2 V_t\right) = v_0^2 t + x_0^2 (v_0 + \kappa\theta t - \kappa v_0 t) - v_0 x_0 (v_0 - 2\eta\rho) t + \mathcal{O}\left(t^2\right)$$

Our tree-based method comes up with a discrete time approximation $(\hat{S}_t, \hat{V}_t)$ of the bivariate process (S_t, V_t), where we suppress the dependence of the approximation on the number of discretizing steps N. In the sequel, we write $\Delta t = T/N$ for the size of the time step.

(S_t, V_t) is a time-homogeneous Markov process. Conditioned on the information at time t the distribution of the process at the next step $t + \Delta t$ depends only on the current information, i.e. on the location of the node (S_t, V_t). Accordingly, we work with conditional probabilities given the event $\{S_t = s, V_t = v\}$, and, when considering the approximation, given the event $\{\hat{S}_t = s, \hat{V}_t = v\}$.

2.1 Binomial variance tree approximation

As indicated above, we approximate the variance process V_t in (2) with given initial variance v_0 by a binomial tree $\hat{V}_t$. The process V_t has a square-root form and, as mentioned by Nelson and Ramaswamy [1990], Leisen [2000] and Beliaeva and Nawalkha [2010], constructing a binomial tree for it naïvely would lead to a non-recombining, hence computationally inefficient approximation since the process is heteroscedastic. To retain computational efficiency we follow Nelson and Ramaswamy [1990] and adopt a transformation of the state space

$$Z_t = \frac{2}{\eta}\sqrt{V_t}.$$

By Itô's Lemma we get

$$\mathrm{d}Z_t = \left[\left(\frac{2\kappa\theta}{\eta^2} - \frac{1}{2}\right)\frac{1}{Z_t} - \frac{\kappa}{2}Z_t\right]\mathrm{d}t + \mathrm{d}W_t^V, \qquad z_0 = \frac{2\sqrt{v_0}}{\eta}.$$

Approximating Z_t by a binomial tree leads to equally spaced nodes z as multiples of $\sqrt{\Delta t}$ due to the unit variance. Via the back-transformation $R(z) = \frac{z^2\eta^2}{4}$ we obtain values for the original variance process again.

Since negative values for Z_t and $\hat{Z}_t$ correspond to negative variance values, we only use nodes for $\hat{Z}_t$ with $z \geq 0$, and nodes below zero are set to zero. This setting results in a smallest variance value equal to zero, which however will not

be attained when going through the tree from the root to the leaves. A fine discretization is needed around $z = 0$, because of the unbounded slope of the square root and since small variance values are important for pricing purposes. In contrast to Nelson and Ramaswamy [1990] we do not create negative nodes which would be attained with probability 0.
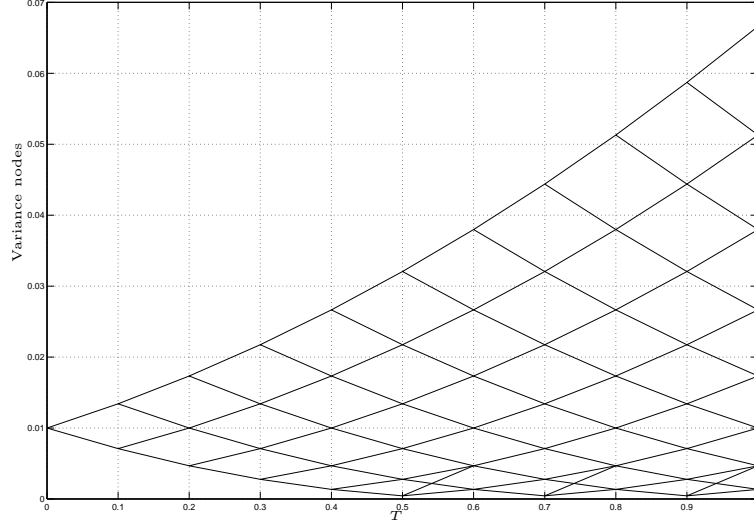
Figure 1: *Realization of the binomial variance tree for $v_0 = \theta = 0.01$, $\kappa = 1$, $\eta = 0.1$ and $T = 1$, where we discretized into $N = 10$ time steps.*

Figure 1 shows a realization of the variance tree for $v_0 = \theta = 0.01$. The other parameters are $\kappa = 1$, $\eta = 0.1$, and we discretized $T = 1$ into $N = 10$ equidistant time steps. Note that for the limit Heston model, the stability condition is satisfied and hence local time in 0 is 0. Due to the mean-reversion of the underlying variance process, small variance values yield a positive drift, truncating the tree from below. In order to determine successor nodes and transition probabilities, Nelson and Ramaswamy [1990] and Beliaeva and Nawalkha [2010] pass over again to the variance process, since Z_t contains a singularity in 0. In addition, to match the drift in the mean reversion setting of (2), multiple jumps have to be introduced. Let $R(z)$ the actual variance node of $\hat{V}_t$ and let further

$$\mu_{R(z)} := R(z) + \kappa(\theta - R(z))\Delta t,$$

be the drift at $R(z)$. We determine possible successors v_1 and v_2 as nodes

enclosing $\mu_{R(z)}$ as

$$v_{1|2} = R\left(z + j_{1|2}\left(z\right)\sqrt{\Delta t}\right),$$

where

$$j_1\left(z\right) := \max_{j \in 2\mathbb{Z}+1}\left\{j \mid R\left(z + j\sqrt{\Delta t}\right) \le \mu_{R(z)}\right\}$$

$$j_2\left(z\right) := \min_{j \in 2\mathbb{Z}+1}\left\{j \mid R\left(z + j\sqrt{\Delta t}\right) > \mu_{R(z)}\right\}.$$

The definition of $j_1(z)$ and $j_2(z)$ ensures that the drift always lies between v_1 and v_2, such that the transition probabilities are always well defined. The fact that we only allow j to be odd restricts the jump to valid nodes at each stage.

For determining transition probabilities, we follow the concept of moment matching in order to enforce weak convergence. Having obtained valid successors positions, we choose the probabilities such that their expectation matches the respective moment of the Heston model. Let

$$\mathbb{P}_{\hat{V}}\left(v_2\right) := \mathbb{P}\left(\hat{V}_{t+\Delta t} = v_2 \mid \hat{V}_t = R\left(z\right)\right)$$

denote the probability of an up jump to v_2. We formulate the first moment condition

$$\mathbb{E}\left(\hat{V}_{t+\Delta t} - \hat{V}_t \mid \hat{V}_t = R\left(z\right)\right) \overset{!}{=} \mu_{R(z)},$$

$$\iff \quad v_1\left(1 - \mathbb{P}_{\hat{V}}(v_2)\right) + v_2\mathbb{P}_{\hat{V}}(v_2) = \kappa\left(\theta - R\left(z\right)\right)\Delta t,$$

resulting in

$$\mathbb{P}_{\hat{V}}(v_2) = \frac{\kappa\left(\theta - R\left(z\right)\right)\Delta t - v_1}{v_2 - v_1},$$

where we used the linearized version of $\mathbb{E}(V)$ as it is given in Corollary 2. Note that we suppress the dependence on both, the current value v and the time step Δt.

2.2 Trinomial stock price tree approximation

The diffusion component of $\mathrm{d}X_t$ in (3) depends on the variance process V_t at time t. Accordingly, the approximation $\hat{X}_t$ of X_t depends on the approximated variance process $\hat{V}_t$, and a trinomial x-grid would in general not recombine. To circumvent this, we introduce a standard variance step $\hat{v}$ that represents the smallest allowed log-stock jump size in our model. We define grid nodes for $\hat{X}_t$ as nodes with distance $\Delta x := \sqrt{\hat{v}\Delta t} > 0$ at each occurring variance level. Beliaeva and Nawalkha [2010] for example set $\hat{v} = v_0$. We come back to the choice of $\hat{v}$ at a later stage. The link to the variance tree is given through the following relation. Conditioned on being at node v of the variance process, the height of a log-stock price jump is given through

$$k\left(v\right) = \left\lceil \frac{\sqrt{v\Delta t + \frac{v^2}{4}\left(\Delta t\right)^2}}{\Delta x} \right\rceil = \left\lceil \sqrt{\frac{v\left(4 + v\Delta t\right)}{4\hat{v}}} \right\rceil, \tag{5}$$

whereas $\lceil \cdot \rceil$ denotes the ceiling function. So $k(v)$ can be considered as the multiplicator of standard jumps in order to reach the instantaneous variance. As shown in Proposition 3 below, this choice of $k(v)$ yields positive probabilities. In particular, the additional term $\frac{v^2}{4}(\Delta t)^2$ guarantees positivity of the middle node probability. For fixed $\hat{v}$ a higher variance value v leads to higher jumps. Restricting $k(v)$ to be an integer results in a recombining efficient tree. The successors of a node x are then given as

$$x_{1|3} := x \mp k(v)\sqrt{\hat{v}\Delta t}, \qquad x_2 := x, \tag{6}$$

where x_1 is the down, x_2 the middle and x_3 the up node.

Next, we determine transition probabilities for the stock price process by matching the tree moments to the ones of the limit distribution. To this end, let

$$\mathbb{P}_{\hat{X}}(x_i) := \mathbb{P}\left(\hat{X}_{t+\Delta t} = x_i \mid \hat{X}_t = x, \hat{V}_t = v\right), \qquad i = 1, 2, 3,$$

be the probabilities for an up, mid or down move of the stock process at variance level v. The transition probabilities are obtained in Proposition 3

Proposition 3:
For Δt small enough and with $k(v)$ given in (5), the setting

$$\mathbb{P}_{\hat{X}}\left(x_{1|3}\right) = \frac{4v + v^2\Delta t \pm 2vk(v)\sqrt{\hat{v}\Delta t}}{8k(v)^2\hat{v}}, \tag{7}$$

$$\mathbb{P}_{\hat{X}}(x_2) = 1 - \mathbb{P}_{\hat{X}}(x_1) - \mathbb{P}_{\hat{X}}(x_3),$$

is a probability with the same first two moments as the limit distribution.

The proof is given in the appendix. In what follows we join both trees and adjust for non-zero correlation.

3 Combining the trees and matching correlation

So far the logarithmic stock price and variance processes were treated separately and we could generate admissible probabilities for the marginal movements. For the joint approximation, we now have to incorporate the correlation between them by defining suitable joint probabilities. Since we want to recur to Donsker's theorem for weak convergence of the joint tree against the joint model distribution, in addition we have to match the covariance between $\hat{X}_t$ and $\hat{V}_t$ against the Heston model one between variance V_t and log-stock process X_t.

We follow Hull and White [1990], starting with the product model of stochastic independence and determine the six joint probabilities as

$$\mathbb{P}_{ij} := \mathbb{P}_{\hat{X}}(x_i) \cdot \mathbb{P}_{\hat{V}}(v_j), \qquad i = 1, 2, 3, \ j = 1, 2,$$

where we recall that our short hand notation omits the dependence on the time discretization Δt and the current node $\hat{V}_t = v$. Note that the probabilities

are invariant for different values of $\hat{X}_t$. These probabilities are then adjusted to match the given correlation structure as well as possible while maintaining proper transition probabilities. Let $\widetilde{\Pi} = \Pi + \theta$, i.e. $\widetilde{\mathbb{P}}_{ij} = \mathbb{P}_{ij} + \theta_{ij}$, for $i = 1, 2, 3$ and $j = 1, 2$, be the changed probability structure, where Π is the product probability structure assuming independence and θ_{ij} the shifted mass to adjust Π, see Table 1.

	v_1	v_2
x_1	$\mathbb{P}_{11} + \theta_{11}$	$\mathbb{P}_{12} + \theta_{12}$
x_2	$\mathbb{P}_{21} + \theta_{21}$	$\mathbb{P}_{22} + \theta_{22}$
x_3	$\mathbb{P}_{31} + \theta_{31}$	$\mathbb{P}_{32} + \theta_{32}$

Table 1: *Additional mass to adjust the independent probabilities.*

The requirement of $\widetilde{\Pi}$ to form a probability translates into corresponding requirements for θ_{ij}. To keep things simple, we leave the marginal, already moment-matched probabilities untouched and only vary θ_{ij} in this invariance region, i.e. for fixed marginals. As a consequence we have zero row sums and column sums for θ_{ij}, i.e.

$$\theta_{i2} = -\theta_{i1}, \qquad i = 1, 2, 3, \tag{8}$$

and

$$\theta_{1j} + \theta_{2j} + \theta_{3j} = 0, \qquad j = 1, 2. \tag{9}$$

As displayed in Table 2, using equations (8) and (9), we may reduce the complexity of our problem to two dimensions with parameters θ_{11} and θ_{21} to choose for an adjustment of the transition probabilities.

	v_1	v_2
x_1	$\mathbb{P}_{11} + \theta_{11}$	$\mathbb{P}_{12} - \theta_{11}$
x_2	$\mathbb{P}_{21} + \theta_{21}$	$\mathbb{P}_{22} - \theta_{21}$
x_3	$\mathbb{P}_{31} - (\theta_{11} + \theta_{21})$	$\mathbb{P}_{32} + (\theta_{11} + \theta_{21})$

Table 2: *Adjustment of probability mass $\widetilde{\Pi}$ while maintaining marginal distributions.*

For a proper transition probability, $\widetilde{\Pi}$ has to satisfy further restrictions on θ_{11} and θ_{21}. For $i = 1, 2$, we have to ensure that $0 \leq \mathbb{P}_{i1} + \theta_{i1} \leq \mathbb{P}_{\hat{X}}(x_i) = \mathbb{P}_{i1} + \mathbb{P}_{i2}$ which reduces to $\theta_{i1} \leq \mathbb{P}_{i2}$ and $\theta_{i1} \geq -\mathbb{P}_{i1}$. Additionally we have to require $0 \leq \mathbb{P}_{31} - (\theta_{11} + \theta_{21}) \leq \mathbb{P}_{\hat{X}}(x_3) = \mathbb{P}_{31} + \mathbb{P}_{32}$ and get $\theta_{11} + \theta_{21} \leq \mathbb{P}_{31}$ and

$\theta_{11} + \theta_{21} \geq -\mathbb{P}_{32}$. In total, we have identified six constraints for the two variables which are summarized in

$$\theta_{i1} \leq \mathbb{P}_{i2}, \qquad\qquad \theta_{i1} \geq -\mathbb{P}_{i1}, \qquad i = 1, 2, \qquad (10)$$

$$\theta_{11} + \theta_{21} \leq \mathbb{P}_{31}, \qquad\qquad \theta_{11} + \theta_{21} \geq -\mathbb{P}_{32}. \qquad (11)$$

Denote by $\mathcal{A}$ the set of all values $(\theta_{11}, \theta_{21})$ such that the constraints given in (10) and (11) hold.
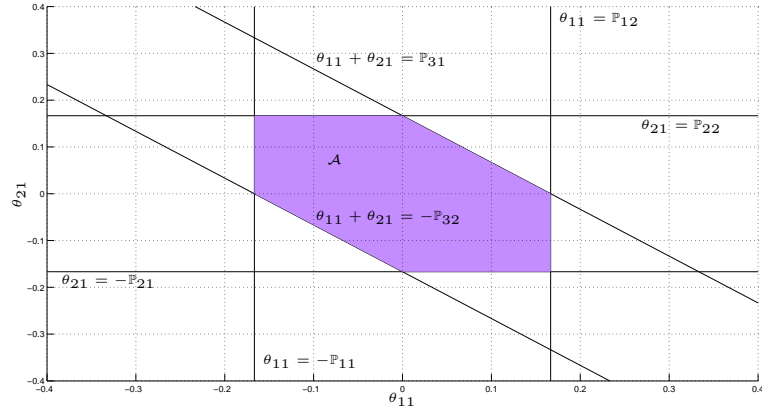
Figure 2: *The linear constraints that bound $\mathcal{A}$ and provide its vertices. All probabilities are equal to $\frac{1}{6}$, thus they define the largest simplex possible.*

For different underlying probabilities the set $\mathcal{A}$ is illustrated in Figure 2 and Figure 3.

Note that the constraints depend on the probabilities for the stock and the variance. Denote by $\mathbb{E}_{\widetilde{\Pi}}(\cdot)$ and $\mathbb{V}\mathrm{ar}_{\widetilde{\Pi}}(\cdot)$ the expectation and variance under the adjusted probabilities. We define the short hand notations $\hat{X} := \hat{X}_{t+\Delta t}$ and $\hat{V} := \hat{V}_{t+\Delta t}$ conditioned on $\hat{V}_t = v$. As a consequence of the untouched marginals, the moments are unaffected by the change of the probabilities. Observe that

$$\mathbb{E}_{\widetilde{\Pi}}\left(\hat{X}\right) = \mathbb{E}\left(\hat{X}\right), \quad \mathbb{E}_{\widetilde{\Pi}}\left(\hat{X}^2\right) = \mathbb{E}\left(\hat{X}^2\right),$$

$$\mathbb{E}_{\widetilde{\Pi}}\left(\hat{V}\right) = \mathbb{E}\left(\hat{V}\right), \quad \mathbb{E}_{\widetilde{\Pi}}\left(\hat{V}^2\right) = \mathbb{E}\left(\hat{V}^2\right),$$

Figure 3: *Linear constraints that bound $\mathcal{A}$. The probability for the stock to stay in the middle node is close to zero. In the limit, the tree reduces to the binomial case.*
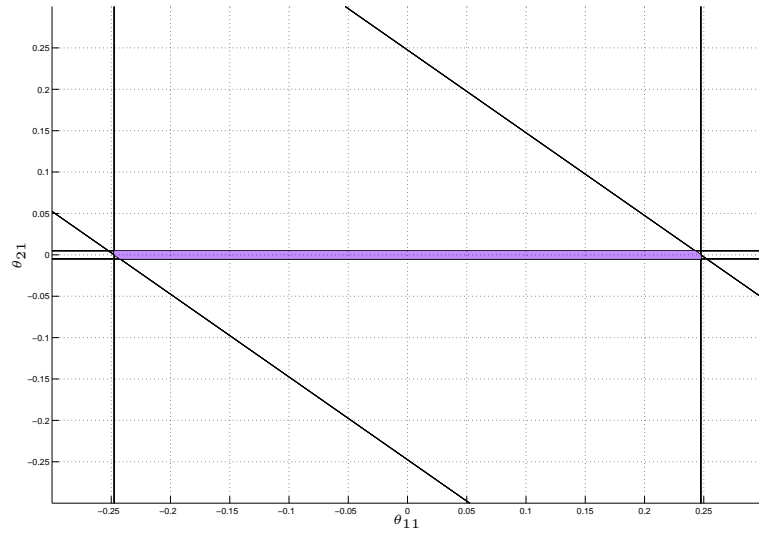
By Corollary 2, the condition for matching the covariance is given by

$$\mathbb{C}\mathrm{ov}(\hat{X}, \hat{V}) \overset{!}{=} \mathbb{C}\mathrm{ov}(X, V) = \eta \rho V \Delta t$$

where by obvious calculations

$$\begin{aligned}
\mathbb{C}\mathrm{ov}_{\widetilde{\Pi}}(\hat{X}, \hat{V}) &= \mathbb{E}_{\widetilde{\Pi}}(\hat{X}\hat{V}) - \mathbb{E}(\hat{X})\mathbb{E}(\hat{V}) \\
&= (v_1 - v_2)\left[\theta_{11}\left(x_1 - x_3\right) + \theta_{21}\left(x_2 - x_3\right)\right].
\end{aligned}$$

To measure the approximation quality of the moment match, we use the squared Euclidean distance

$$\tilde{d}\left(\theta_{11}, \theta_{21}\right) := \left(\mathbb{C}\mathrm{ov}(X, V) - \mathbb{C}\mathrm{ov}_{\widetilde{\Pi}}(\hat{X}, \hat{V})\right)^2, \tag{12}$$

where θ_{11} and θ_{21} have to satisfy the constraints given in (10) and (11), i.e. $(\theta_{11}, \theta_{21})$ lies in $\mathcal{A}$. Our optimal adjustment then minimizes (12), i.e.

$$\min_{(\theta_{11}, \theta_{21}) \in \mathcal{A}} \tilde{d}\left(\theta_{11}, \theta_{21}\right). \tag{13}$$

The objective function $\tilde{d}$ is displayed in Figure 4, and the respective optimization problem is solved in Proposition 4, the proof of which is given in the appendix.
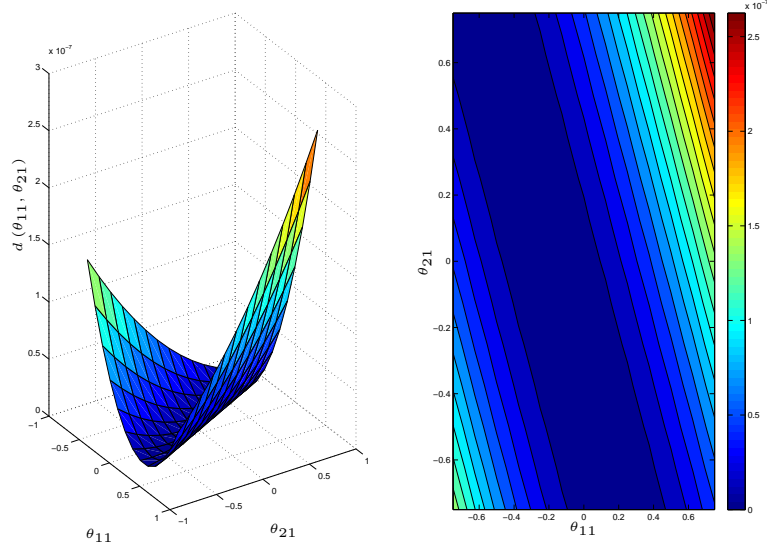

Figure 4: *Objective function $\tilde{d}\left(\theta_{11}, \theta_{21}\right)$ for $\rho = -0.75$. The other parameters are $v_1 = 0.05$, $v_2 = 0.06$, $x_1 = -0.02$, $x_2 = 0$ and $x_3 = -x_1$.*

Proposition 4: *(i) The admissible set $\mathcal{A}$ is a closed, convex, bounded simplex in $\mathbb{R}^2$. Its set of extremal points $\mathcal{E}$ consists of at most 6 vertices.*

(ii) The objective function $\tilde{d}$ given in (12) is continuous and convex.

(iii) The level sets
$$N_d := \{(\theta_{11}, \theta_{21}) \mid \tilde{d}(\theta_{11}, \theta_{21}) = d\},$$

up to the degenerate case $d = 0$, are formed by two parallel linear functions h_1, h_2 given by

$$h_{1|2}: \quad \theta_{21} = \frac{\mathbb{C}ov(X, V) \pm \sqrt{d}}{(x_2 - x_3)(v_1 - v_2)} - \theta_{11}\frac{(x_1 - x_3)}{(x_2 - x_3)}.$$

In the degenerate case $d = 0$, the level set N_0 is given by

$$h_0: \quad \theta_{21} = \frac{\mathbb{C}ov(X, V)}{(x_2 - x_3)(v_1 - v_2)} - \theta_{11}\frac{(x_1 - x_3)}{(x_2 - x_3)}.$$

(iv) The infimum of $\tilde{d}$ on $\mathcal{A}$ is attained. For $L_0 := N_0 \cap \mathcal{A} \neq \emptyset$, L_0 is precisely the set of all minimizers. If $L_0 = \emptyset$, there is a unique minimizer in $\mathcal{E}$.

It therefore suffices to determine the objective function only at the finite set of extreme points if $L_0 = \emptyset$. Whenever $L_0 \neq \emptyset$ and contains more than one point, the set of minimizers in $\mathcal{A}$ is a line segment, and there is room for further improvement, as distinct minimizers, despite producing the same minimal value of $\tilde{d}$, may well produce trees giving different option prices. We detail this in Section 3.1.

Remark 5: *(a) A general optimization device to cope with quadratic optimization problems on a simplex is given by the Frank-Wolfe algorithm, which is also known as the convex combination algorithm, compare Frank and Wolfe [1956]. In our context, as mentioned, it is much cheaper though, to evaluate our objective function on the few extremal points of $\mathcal{A}$.*

(b) If $L_0 = \emptyset$ the constraints become active, i.e. the problem given in (13) becomes a linear optimization problem and can be solved with common simplex methods. Still, in our case, it is far cheaper to evaluate our objective function on the few extremal points of $\mathcal{A}$.

(c) Proposition 4 can easily be generalized to general multinomial $\times$ multinomial trees in which to derive optimal moment matches, as the tree moments are linear in the respective cell probabilities. This again reduces to a minimization on the finite set of extremal points of the respective simplex or, in case of exact matches leaves room for higher order matches.

3.1 Matching a higher moment

Whenever the covariance can be matched exactly and L_0 is a line segment, we can choose among all exact matches. This gives us a further degree of freedom,

which can be used to match parts of a higher order cumulant in the Edgeworth expansion with the rationale to asymptotically bring us closer to the exact Heston distribution. To check whether this additional adjustment is possible at all, we determine the frequency of L_0 being a line segment as a function of ρ for a specific set of parameters in Figure 5.

For moderate values of ρ the covariance can be matched exactly in a large amount of cases, in more than half of the cases for $\rho \in [-0.4, 0.4]$. So indeed, the additional adjustment is a relevant improvement and we can try to match a next higher moment at least partly while sticking to the other moments and choices we previously made.
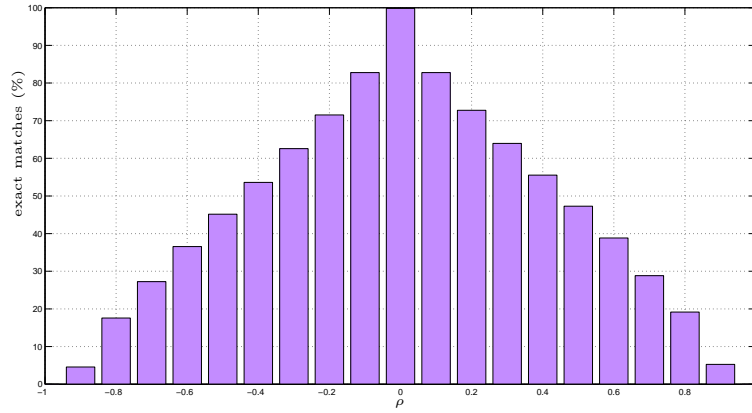
Figure 5: *Number of exact covariance matches in per cent of all variance levels for different correlation levels. A moderate covariance can be matched exactly in more cases than a high covariance. Model and option parameters are $s_0 = K = 100$, $T = 1$, $r = 0.04$ and $d = 0.03$. Variance parameters are $v_0 = \theta = 0.09$, $\eta = 0.2$ and $\kappa = 2$. We use $\hat{v} = 0.01$ and discretize with $N = 500$ steps.*

To realize this additional adjustment, let $I_1 = (\theta_{11}^1, \theta_{21}^1)$ and $I_2 = (\theta_{11}^2, \theta_{21}^2)$ be the two points where the boundary of $\mathcal{A}$ and N_0 intersect. Since the slope of N_0 is negative, we can order I_1 and I_2 such that I_1 is the one with the higher ordinate, the upper left intersection. As is easily seen, each point on the line between I_1 and I_2, which may be parameterized by the convex combinations $I(\lambda) = I_1 + \lambda (I_2 - I_1)$, leaves the marginal distributions of $\hat{X}$ and $\hat{V}$ unchanged and leads to the same exactly matched covariance. Varying λ, we can hence match one extra moment of third order. In fact, in our setup of two processes there are four joint moments of third order, i.e. $\mathbb{E}(\hat{X}^3)$, $\mathbb{E}(\hat{X}^2\hat{V})$, $\mathbb{E}(\hat{X}\hat{V}^2)$, and $\mathbb{E}(\hat{V}^3)$. As the marginals remain fixed, we can either match the mixed moment

$\mathbb{E}(\hat{X}^2 \hat{V})$ or $\mathbb{E}(\hat{X}\hat{V}^2)$. We choose to match $\mathbb{E}(\hat{X}^2 \hat{V})$, since heuristically the option payoff is closer linked to the log-stock price process X. For consistency, we use the linear approximation of $\mathbb{E}(X^2 V)$ which is given in Corollary 2 and want to minimize the squared Euclidean distance between this value and the corresponding moment obtained through the tree. Let further be $\Delta v := (v_1 - v_2)$.

Proposition 6:
The corresponding moment in the tree is given by

$$\begin{aligned}
\mathbb{E}_{\widetilde{\Pi}}\left(\hat{X}^2 \hat{V}\right) = & \mathbb{E}\left(\hat{X}^2\right)\mathbb{E}\left(\hat{V}\right) + \theta_{11}^1\left(x_1^2 - x_3^2\right)\Delta v + \theta_{21}^1\left(x_2^2 - x_3^2\right)\Delta v \\
& + \lambda\Delta v\left[\left(\theta_{11}^2 - \theta_{11}^1\right)\left(x_1^2 - x_3^2\right) + \left(\theta_{21}^2 - \theta_{21}^1\right)\left(x_2^2 - x_3^2\right)\right].
\end{aligned} \tag{14}$$

Restricting λ to be in $[0, 1]$ results in an optimal λ^*, which is given in

Corollary 7:
$\lambda^* = \min(\max(0, \lambda_0), 1)$ *for*

$$\lambda_0 = \frac{\mathbb{E}\left(\hat{X}^2\hat{V}\right) - \mathbb{E}\left(\hat{X}^2\right)\mathbb{E}\left(\hat{V}\right) - \Delta v\left[\theta_{11}^1\left(x_1^2 - x_3^2\right) + \theta_{21}^1\left(x_2^2 - x_3^2\right)\right]}{\Delta v\left[\left(\theta_{11}^2 - \theta_{11}^1\right)\left(x_1^2 - x_3^2\right) + \left(\theta_{21}^2 - \theta_{21}^1\right)\left(x_2^2 - x_3^2\right)\right]}.$$

The proofs of Proposition 6 and Corollary 7 are given in the appendix. Next we state characteristics and numerical results of our pricing algorithm.

4 Numerics

This section discusses various numerical aspects of our approach, including numerical convergence order, computational complexity and accuracy. As a reference we use the semi closed-form prices from Albrecher et al. [2007] and compare our algorithm with the pricing routines given by Guan and Xiaoqiang [2000] (GX) and Beliaeva and Nawalkha [2010] (BN). Our results are computed using Matlab on two 2.0 GHz Intel Centrino processors.

4.1 Pricing of European options

To verify and validate the implemented tree methods we use the corrected semi closed-form European call price formula given by Albrecher et al. [2007]. The price $c(K, T)$ of an European call with strike K and maturity T is given as

$$c(K, T) = \frac{1}{2}\left(s_0 e^{-dT} - K e^{-rT}\right) + \frac{1}{\pi}\int_0^\infty f_1(u) - K e^{-rT} f_2(u) \, \mathrm{d}u$$

with spot s_0, constant interest rate r and constant dividend yield d. The values $f_1(u)$ and $f_2(u)$ are given by

$$f_1(u) = \Re\left(\frac{e^{-iu\ln(K)}\varphi(u - i)}{iu e^{rT}}\right), \qquad f_2(u) = \Re\left(\frac{e^{-iu\ln(K)}\varphi(u)}{iu}\right)$$

where $\Re(\cdot)$ denotes the real part of a complex number. The Heston characteristic function $\varphi(\cdot)$ is given by

$$\varphi(u) = e^{A_1(u) + A_2(u) + A_3(u)}$$

with

$$A_1(u) = iu \left[\ln(s_0) + (r - d) T \right],$$

$$A_2(u) = \frac{\theta\kappa}{\eta^2} \left((\kappa - \rho\eta iu - h(u)) T - 2\ln \left[\frac{1 - g(u) e^{-h(u)T}}{1 - g(u)} \right] \right),$$

$$A_3(u) = \frac{v_0 (\kappa - \rho\eta iu - h(u)) \left(1 - e^{-h(u)T} \right)}{\eta^2 \left(1 - g(u) e^{-h(u)T} \right)}$$

and

$$g(u) = \frac{\kappa - \rho\eta iu - h(u)}{\kappa - \rho\eta iu + h(u)}, \qquad h(u) = \sqrt{(\rho\eta iu - \kappa)^2 + \eta^2 (iu + u^2)}$$

and with i the imaginary unit. The improper integral has to be integrated numerically, but as stated by Kahl and Jäckel [2005], this can be done in a reasonable fashion using adaptive Gauss-Lobatto quadrature.

4.2 Numerical convergence order

To examine the convergence of our model, we compute the logarithmic price differences of the tree prices and the semi closed-form for a European call and plot them against the logarithmic number of discretizing steps $\tilde{N} := \log(N)$. A linear regression shows that the slope parameters are close to -1 or even less for all correlations, so in a respective Edgeworth expansion the leading error term would occur in the skew. In fact, achieving orders beyond 1 means that at least to some extent we have matched higher moments than mean and covariance. The results are shown in Figure 6.

The three panels of the plot illustrate the convergence of our method for $\rho = -0.75$, $\rho = 0$, and $\rho = 0.75$. The linear regressions for the negative and positive correlation values yield slope parameters of approximately -1.1, and -1.2, respectively, indicating an effect of our partial matching of a higher moment. If $\rho = 0$, the approximated covariance $\mathbb{Cov}(\hat{X}, \hat{V})$ is 0 and $\mathcal{A}$ degenerates to the origin, so the additional degree of freedom vanishes, which explains the slope of approximately -1 here.

4.3 Computational Complexity

Due to differences in hardware architecture and programming language, a fair timing comparison between BN and our method (RSS) is not trivial. Still

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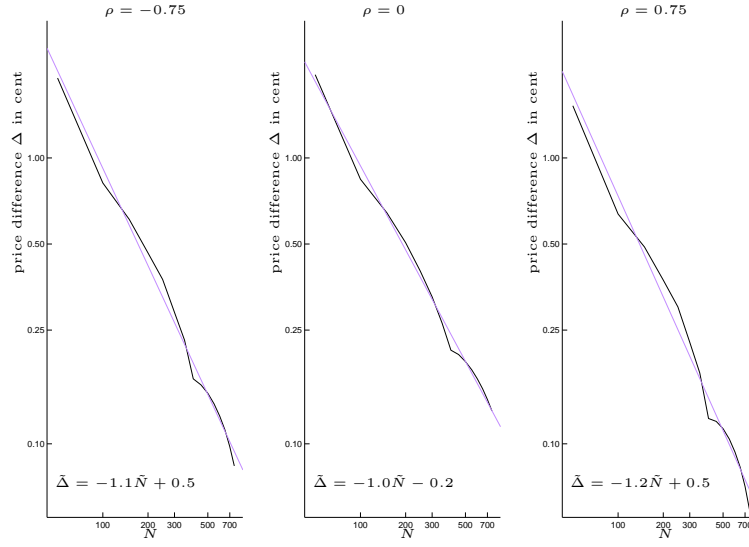
Figure 6: *Convergence of the model for a European call for different correlation levels and increasing number of discretizing steps N. The ordinate shows the absolute price difference Δ of the tree price and the analytic solution in cent. Regression lines for $\tilde{N} := \log(N)$ and $\tilde{\Delta} := \log(\Delta)$. Model parameters are $s_0 = 100$, $r = 0.04$, $d = 0.03$, $v_0 = 0.09$, $\kappa = 2$, $\theta = 0.09$, $\eta = 0.2$. The option matures in $T = 1$ year with strike price $K = 100$. We further use $\hat{v} = 0.01$.*

computational complexity can be compared to some extent. To achieve roughly equal complexity for both approaches, we must choose our standard variance $\hat{v}$ in a way that compensates the fact that we differ from the BN approach in the number of successor nodes. Since BN use trinomial trees for both the variance and the log-stock tree, they end up with $N_V^{BN} \times N_X^{BN} \times 9$ operations per time slice, where N_V^{BN} and N_X^{BN} are the amount of the variance and the log-stock nodes in a particular time step. Denoting by N_V^{RSS} and N_X^{RSS} the respective numbers in our approach, we need $N_V^{RSS} \times N_X^{RSS} \times 6$ operations for the same time slice. If we assume that the state space of the variance process is discretized according to Nelson and Ramaswamy [1990] for both approaches, N_V^{BN} essentially equals N_V^{RSS}. With a restriction to the same range of log-stock prices for both approaches, the only difference arises in the grid widths Δx^{BN} and Δx^{RSS}. Both methods have equal complexity if

$$9 \times \frac{1}{\Delta x^{BN}} \approx 6 \times \frac{1}{\Delta x^{RSS}}.$$

BN replace Δx^{BN} by $\sqrt{v_0 \Delta t}$, so v_0 is used to space the log-stock grid while we use a grid width of $\Delta x^{RSS} = \sqrt{\hat{v} \Delta t}$. This means that for the same order of complexity in both methods, we have to set $\hat{v}$ to $\frac{4}{9} v_0$, which is approximately 44% of the initial variance, hence a conservative comparison is obtained for $\hat{v} = \frac{1}{2} v_0$. Of course this is only a heuristic, since we e.g. have ignored the truncation method in the BN approach. More specifically, they incorporate a bound on the log-stock tree that truncates the tree from above and below suppressing highly improbable nodes. Such a truncation would be possible in our setup as well. As it is not clear a priori in which way it influences the convergence of the method, though, in this paper we decided to only focus on the correlation matching, skipping any truncation.

4.4 Accuracy

In the following we state some results of our method for European and American options and different sets of input parameters. Throughout this part, we determine price deviations through the mean absolute percentage error (MAPE),

$$\text{MAPE} = \frac{|\text{ analytic price} - \text{model price }|}{\text{analytic price}}.$$

Table 3 states European call prices calculated with our model and the pricing algorithms of Beliaeva and Nawalkha [2010] and Guan and Xiaoqiang [2000].

We use the parameters $r = 0.05$, $d = 0$, $\kappa = 3$, $\theta = 0.04$, $\eta = 0.1$ and $\rho = -0.1$ to price the call maturing in 1, 3 and 6 months with strike price $K = 100$. We further use $\hat{v} = 0.02$. We vary the initial spot price from 90 to 110 to investigate the behavior of out-of-the-money and in-the-money calls and set the initial variance to 0.04, 0.09 and 0.16. We use $N = 200$ steps to discretize the lifetime of the option. Note that due to the changing maturity, also the time step Δt varies from $\Delta t = 0.0004$ to $\Delta t = 0.0013$ and $\Delta t = 0.0025$

T	v_0	s_0	analytic	BN	GX	RSS	Δ BN	Δ GX	Δ RSS
1/12	0.04	90	0.0857	0.0857	0.0865	0.0857	0.01%	0.92%	0.00%
		95	0.6508	0.6510	0.6504	0.6507	0.02%	0.07%	0.02%
		100	2.5108	2.5109	2.5088	2.5116	0.01%	0.08%	0.04%
		105	5.9944	5.9943	5.9938	5.9943	0.00%	0.01%	0.00%
		110	10.5241	10.5239	10.5244	10.5238	0.00%	0.00%	0.00%
	0.09	90	0.4335	0.4341	0.4331	0.4336	0.14%	0.09%	0.04%
		95	1.4551	1.4565	1.4623	1.4562	0.10%	0.49%	0.08%
		100	3.5455	3.5468	3.5552	3.5471	0.04%	0.27%	0.04%
		105	6.7897	6.7899	6.7944	6.7901	0.00%	0.07%	0.01%
		110	10.9253	10.9247	10.9245	10.9257	0.01%	0.01%	0.00%
	0.16	90	1.0115	1.0135	1.0123	1.0127	0.20%	0.08%	0.11%
		95	2.3696	2.3723	2.3730	2.3723	0.12%	0.14%	0.11%
		100	4.6017	4.6040	4.6061	4.6050	0.05%	0.10%	0.07%
		105	7.7198	7.7207	7.7231	7.7207	0.01%	0.04%	0.01%
		110	11.5766	11.5759	11.5778	11.5781	0.01%	0.01%	0.01%
						MAPE ($T = 1/12$)	0.05%	0.16%	0.04%
1/4	0.04	90	0.8852	0.8855	0.8859	0.8857	0.03%	0.08%	0.06%
		95	2.2588	2.2592	2.2631	2.2584	0.02%	0.19%	0.02%
		100	4.6105	4.6108	4.6151	4.6118	0.01%	0.10%	0.03%
		105	7.9290	7.9289	7.9310	7.9286	0.00%	0.02%	0.01%
		110	12.0006	12.0002	11.9998	12.0006	0.00%	0.01%	0.00%
	0.09	90	1.9024	1.9060	1.9028	1.9043	0.19%	0.02%	0.10%
		95	3.6147	3.6186	3.6178	3.6171	0.11%	0.09%	0.07%
		100	6.0703	6.0733	6.0719	6.0730	0.05%	0.03%	0.05%
		105	9.2385	9.2400	9.2412	9.2407	0.02%	0.03%	0.02%
		110	13.0088	13.0084	13.0099	13.0104	0.00%	0.01%	0.01%
	0.16	90	3.1355	3.1420	3.1373	3.1387	0.21%	0.06%	0.10%
		95	5.0826	5.0888	5.0855	5.0884	0.12%	0.06%	0.11%
		100	7.6177	7.6224	7.6212	7.6214	0.06%	0.05%	0.05%
		105	10.7087	10.7111	10.7125	10.7108	0.02%	0.04%	0.02%
		110	14.2873	14.2871	14.2906	14.2903	0.00%	0.02%	0.02%
						MAPE ($T = 1/4$)	0.06%	0.05%	0.04%
1/2	0.04	90	2.3272	2.3280	2.3263	2.3290	0.03%	0.04%	0.08%
		95	4.2367	4.2374	4.2374	4.2367	0.02%	0.02%	0.00%
		100	6.8817	6.8819	6.8810	6.8832	0.00%	0.01%	0.02%
		105	10.2068	10.2066	10.2071	10.2077	0.00%	0.00%	0.01%
		110	14.0910	14.0903	14.0896	14.0929	0.00%	0.01%	0.01%
	0.09	90	3.6447	3.6508	3.6462	3.6479	0.17%	0.04%	0.09%
		95	5.7520	5.7576	5.7545	5.7555	0.10%	0.04%	0.06%
		100	8.4366	8.4405	8.4396	8.4403	0.05%	0.04%	0.04%
		105	11.6550	11.6568	11.6580	11.6583	0.02%	0.03%	0.03%
		110	15.3337	15.3332	15.3362	15.3353	0.00%	0.02%	0.01%
	0.16	90	5.1748	5.1836	5.1797	5.1804	0.17%	0.10%	0.11%
		95	7.4382	7.4455	7.4444	7.4443	0.10%	0.08%	0.08%
		100	10.1664	10.1711	10.1737	10.1726	0.05%	0.07%	0.06%
		105	13.3245	13.3260	13.3332	13.3305	0.01%	0.07%	0.05%
		110	16.8632	16.8612	16.8742	16.8676	0.01%	0.07%	0.03%
						MAPE ($T = 1/2$)	0.05%	0.04%	0.05%
						MAPE	0.05%	0.08%	0.04%

Table 3: *European call prices calculated with our model (RSS) and with the pricing routines of Guan and Xiaoqiang [2000] (GX) and Beliaeva and Nawalkha [2010] (BN) for moderate correlation $\rho = -0.1$. The strike price is $K = 100$. The other parameters are $r = 0.05$, $d = 0$, $\kappa = 3$, $\theta = 0.04$ and $\eta = 0.1$. We further use $\hat{v} = 0.02$ and discretize the lifetime of the option into $N = 200$ equidistant time steps. The last 3 columns state the deviation to the analytic solution in per cent. The mean absolute percentage error (MAPE) is given in the last row.*

for $T = 1/12$, $T = 1/4$ and $T = 1/2$, respectively. Columns 5 to 7 contain the model prices and columns 8 to 10 state the relative deviations. In general, the table shows that the approximation is very close to the analytic solution. In this low correlation setting, the MAPEs of BN, GX, and RSS are 0.05%, 0.08% and 0.04%, respectively and the maximal deviation of our model is 0.11%. This is less than the 8 highest deviations of BN (0.21%, 0.20%, 0.19%, 0.17% twice, 0.14% and 0.12% twice) and less than the 5 largest deviations of GX (0.92%, 0.49%, 0.27%, 0.19% and 0.14%). As in case of BN and GX, RSS behaves better for in-the-money than for out-of-the-money options and the approximation quality is decreasing in v_0. The average run time to value the option is 1.887 seconds.

In the next setting, we investigate more realistic correlations as empirically observed in equity markets, see Bakshi et al. [1997] and Nandi [1998]. All parameters except for the correlation remain unchanged, we set $\rho = -0.7$. We compare our model with the BN procedure and employ $N = 50$, $N = 200$ and $N = 500$ steps to discretize the lifetime of the options. Table 4 states the results.

The last row gives the MAPEs for the corresponding discretizing steps. In nearly all subgroups, the MAPE of our model is smaller than the MAPE of BN. In absolute terms, for $N = 50$, $N = 200$ and $N = 500$, the BN model produces an average deviation in cents of 3.71, 0.98 and 0.35, respectively. The approximation quality of BN strongly depends on the initial variance and pricing errors increase in v_0. The absolute deviations in cent of our model are 1.14, 0.26 and 0.13 for $N = 50$, $N = 200$ and $N = 500$, respectively, which is a remarkable improvement. In consequence of $\hat{v}$, our approach is more stable than the BN method about changes in the initial variance and so obtains smaller pricing errors. While this effect vanishes for $\Delta t \to 0$ for realistic time steps, our approach yields a considerably higher accuracy than the BN method, as can be seen for $N = 50$ and an option maturing in $1/2$ year, resulting in $\Delta t = 0.01$. In this case, for $T = 1/2$, the MAPEs of the BN approach are 1.11%, 0.23% and 0.08% for $N = 50$, $N = 200$ and $N = 500$, respectively. Our model performs better by factors of 3.83, 3.29 and 2.67 for $N = 50$, $N = 200$ and $N = 500$, respectively. This is evidence that our method can be recommended in high correlation settings, even for a small number of discretizing steps.

Next we turn to pricing American put options. For comparison with the BN method we use $r = 0.05$, $d = 0$, $\kappa = 3$, $\theta = 0.04$, $\eta = 0.1$ and $K = 100$. As stated in Beliaeva and Nawalkha [2010], we can improve the pricing accuracy by a control variate technique. First we determine the analytical European put option price p via the put-call parity $p(K, T) = Ke^{-rT} + c(K, T) - s_0$ for non-dividend paying stocks, where we use the corresponding call price c, which is given in Section 4.1. Then the European put is used as a control variate to determine the price of the American put option

$$P = \hat{P} + (p - \hat{p}),$$

where $\hat{P}$ and $\hat{p}$ are the respective American and European put option prices obtained from the tree. The resulting price obtained through the control variate technique by Beliaeva and Nawalkha [2010] (BN + CV) is assumed to be exact

T	v_0	s_0	analytic		Δ BN			Δ RSS	
				$N = 50$	$N = 200$	$N = 500$	$N = 50$	$N = 200$	$N = 500$
1/12	0.04	90	0.0691	0.54%	0.54%	0.25%	0.06%	0.60%	0.50%
		95	0.6232	0.18%	0.04%	0.02%	0.40%	0.01%	0.01%
		100	2.5129	0.06%	0.02%	0.01%	0.18%	0.02%	0.01%
		105	6.0211	0.01%	0.00%	0.00%	0.04%	0.01%	0.00%
		110	10.5423	0.01%	0.00%	0.00%	0.00%	0.00%	0.00%
	0.09	90	0.4063	1.28%	0.15%	0.10%	0.43%	0.13%	0.12%
		95	1.4313	0.61%	0.00%	0.02%	0.29%	0.10%	0.05%
		100	3.5460	0.19%	0.03%	0.01%	0.13%	0.04%	0.01%
		105	6.8125	0.01%	0.03%	0.01%	0.02%	0.00%	0.00%
		110	10.9525	0.02%	0.01%	0.01%	0.00%	0.00%	0.00%
	0.16	90	0.9826	2.61%	0.07%	0.03%	0.25%	0.12%	0.06%
		95	2.3493	1.08%	0.07%	0.00%	0.42%	0.12%	0.04%
		100	4.6010	0.32%	0.06%	0.02%	0.23%	0.07%	0.02%
		105	7.7380	0.03%	0.04%	0.02%	0.02%	0.01%	0.01%
		110	11.6040	0.03%	0.03%	0.02%	0.03%	0.01%	0.00%
	MAPE ($T = 1/12$)			0.47%	0.07%	0.03%	0.17%	0.08%	0.06%
1/4	0.04	90	0.8120	0.76%	0.09%	0.04%	0.65%	0.23%	0.20%
		95	2.2114	0.27%	0.04%	0.01%	0.26%	0.04%	0.02%
		100	4.6192	0.04%	0.04%	0.02%	0.15%	0.01%	0.05%
		105	7.9832	0.03%	0.02%	0.01%	0.08%	0.02%	0.01%
		110	12.0682	0.02%	0.00%	0.00%	0.01%	0.01%	0.01%
	0.09	90	1.8316	1.58%	0.02%	0.03%	0.49%	0.15%	0.08%
		95	3.5738	0.81%	0.08%	0.01%	0.29%	0.09%	0.04%
		100	6.0732	0.48%	0.08%	0.03%	0.19%	0.05%	0.02%
		105	9.2810	0.27%	0.08%	0.03%	0.08%	0.02%	0.01%
		110	13.0747	0.14%	0.06%	0.03%	0.04%	0.01%	0.00%
	0.16	90	3.0709	1.15%	0.30%	0.07%	1.00%	0.10%	0.04%
		95	5.0457	0.40%	0.26%	0.07%	0.74%	0.12%	0.01%
		100	7.6157	0.06%	0.22%	0.07%	0.42%	0.05%	0.03%
		105	10.7399	0.18%	0.18%	0.07%	0.00%	0.02%	0.02%
		110	14.3433	0.26%	0.15%	0.06%	0.04%	0.02%	0.01%
	MAPE ($T = 1/4$)			0.43%	0.11%	0.04%	0.30%	0.06%	0.04%
1/2	0.04	90	2.2262	1.30%	0.10%	0.01%	0.08%	0.13%	0.01%
		95	4.1889	0.68%	0.09%	0.03%	0.17%	0.04%	0.06%
		100	6.9002	0.38%	0.05%	0.02%	0.07%	0.02%	0.04%
		105	10.2797	0.26%	0.03%	0.02%	0.07%	0.02%	0.03%
		110	14.1930	0.22%	0.01%	0.01%	0.00%	0.01%	0.01%
	0.09	90	3.5497	1.82%	0.16%	0.02%	0.57%	0.14%	0.05%
		95	5.7053	1.19%	0.16%	0.04%	0.32%	0.09%	0.04%
		100	8.4453	0.72%	0.16%	0.05%	0.19%	0.06%	0.02%
		105	11.7133	0.50%	0.14%	0.05%	0.10%	0.03%	0.01%
		110	15.4272	0.40%	0.13%	0.05%	0.05%	0.01%	0.00%
	0.16	90	5.0861	2.77%	0.65%	0.20%	1.53%	0.21%	0.05%
		95	7.3913	2.39%	0.54%	0.18%	0.78%	0.13%	0.04%
		100	10.1655	2.05%	0.46%	0.17%	0.34%	0.07%	0.03%
		105	13.3668	0.48%	0.40%	0.15%	0.06%	0.04%	0.02%
		110	16.9406	1.53%	0.35%	0.14%	0.08%	0.01%	0.01%
	MAPE ($T = 1/2$)			1.11%	0.23%	0.08%	0.29%	0.07%	0.03%
	MAPE			0.67%	0.14%	0.05%	0.25%	0.07%	0.04%

Table 4: *Comparison of European call prices determined with BN and our method for correlation $\rho = -0.7$ and discretizing steps $N = 50$, $N = 200$ and $N = 500$. We further use $r = 0.05$, $d = 0$, $\kappa = 3$, $\theta = 0.04$, $\eta = 0.1$, $K = 100$ and $\hat{v} = 0.02$. Columns 5 to 10 show the deviation to the analytic price in per cent.*

T	v_0	ρ	s_0	European Put p	BN + CV $N = 200$	Δ BN $N = 50$	$N = 200$	Δ RSS $N = 50$	$N = 200$
1/12	0.04	-0.1	90	9.6699	10.0000	0.00%	0.00%	0.00%	0.00%
			100	2.0950	2.1254	0.01%	0.01%	0.21%	0.05%
			110	0.1083	0.1091	0.37%	0.18%	0.15%	0.25%
		-0.7	90	9.6533	9.9997	0.00%	0.00%	0.00%	0.00%
			100	2.0971	2.1267	0.08%	0.02%	0.20%	0.04%
			110	0.1265	0.1274	0.63%	0.08%	0.26%	0.26%
	0.16	-0.1	90	10.5957	10.7100	0.09%	0.02%	0.03%	0.01%
			100	4.1859	4.2158	0.28%	0.07%	0.26%	0.08%
			110	1.1608	1.1667	0.00%	0.01%	0.42%	0.13%
		-0.7	90	10.5668	10.6804	0.09%	0.03%	0.05%	0.03%
			100	4.1852	4.2140	0.10%	0.04%	0.29%	0.10%
			110	1.1882	1.1939	1.99%	0.09%	0.40%	0.15%
				MAPE ($T = 1/12$)		0.30%	0.05%	0.19%	0.09%
1/4	0.04	-0.1	90	9.6430	10.1706	0.03%	0.00%	0.00%	0.01%
			100	3.3683	3.4747	0.03%	0.01%	0.22%	0.04%
			110	0.7584	0.7736	0.18%	0.04%	0.29%	0.03%
		-0.7	90	9.5698	10.1206	0.04%	0.01%	0.00%	0.03%
			100	3.3770	3.4807	0.01%	0.04%	0.18%	0.00%
			110	0.8259	0.8416	0.55%	0.01%	0.14%	0.09%
	0.16	-0.1	90	11.8933	12.1819	0.06%	0.08%	0.10%	0.04%
			100	6.3755	6.4964	0.10%	0.13%	0.42%	0.07%
			110	3.0451	3.0914	0.13%	0.12%	0.69%	0.10%
		-0.7	90	11.8287	12.1122	0.41%	0.14%	0.35%	0.09%
			100	6.3735	6.4899	0.07%	0.20%	0.54%	0.11%
			110	3.1011	3.1456	0.87%	0.33%	0.25%	0.14%
				MAPE ($T = 1/4$)		0.21%	0.09%	0.26%	0.06%
1/2	0.04	-0.1	90	9.8582	10.6478	0.04%	0.01%	0.03%	0.02%
			100	4.4126	4.6473	0.06%	0.01%	0.02%	0.05%
			110	1.6220	1.6832	0.17%	0.02%	0.04%	0.12%
		-0.7	90	9.7572	10.5637	0.06%	0.00%	0.03%	0.03%
			100	4.4312	4.6636	0.08%	0.03%	0.11%	0.03%
			110	1.7240	1.7874	0.43%	0.01%	0.15%	0.08%
	0.16	-0.1	90	12.7057	13.3142	0.15%	0.14%	0.20%	0.06%
			100	7.6974	8.0083	0.19%	0.19%	0.40%	0.09%
			110	4.3942	4.5454	0.51%	0.20%	0.57%	0.12%
		-0.7	90	12.6171	13.2172	0.92%	0.29%	0.76%	0.19%
			100	7.6965	7.9998	0.71%	0.41%	0.60%	0.18%
			110	4.4716	4.6201	0.50%	0.59%	0.20%	0.11%
				MAPE ($T = 1/2$)		0.32%	0.16%	0.26%	0.09%
				MAPE		0.28%	0.10%	0.24%	0.08%

Table 5: *American put prices determined with the BN and RSS tree approach. We use $r = 0.05$, $d = 0$, $\kappa = 3$, $\theta = 0.04$, $\eta = 0.1$ and $K = 100$ to price the option. Column 5 states the analytic price of the European put. Column 6 states the 'true' price and the deviations rely to it. We further use $\hat{v} = 0.02$.*

and is used as the reference price. We compare both the BN tree-method prices and ours for $N = 50$ and $N = 200$ discretizing steps and use $\hat{v} = 0.02$, see Table 5.

Comparing the results for $N = 50$, the maximal deviation of BN is 1.99% with a MAPE of 0.28% while our model yields a maximal deviation of 0.76% and a MAPE of 0.24%. The worst approximation of our model is better then the 3 worst approximations of BN (1.99%, 0.92%, 0.87%). For $N = 200$, BN yields a maximal deviation of 0.59% with a MAPE of 0.10%. The maximal deviation of RSS is 0.26% with a MAPE of 0.08% and our worst approximation is better then the 4 worst of BN (0.59%, 0.41%, 0.33%, 0.29%).

Table 6 states the effect of the standard variance step $\hat{v}$. Intuitively, a smaller spacing suggests a finer grid and a closer price. The results confirm this

intuition.

We determine European call prices for $N = 50$ and $N = 200$ discretizing steps and employ $\hat{v} = 0.02$ and $\hat{v} = v_0$. Once again, we use $r = 0.05$, $d = 0$, $\kappa = 3$, $\theta = 0.04$, $\eta = 0.1$ and $\rho = -0.7$. The strike is set to $K = 100$. We achieve a MAPE of 0.25% for $N = 50$ and 0.07% for $N = 200$ for $\hat{v} = 0.02$. The MAPEs for $\hat{v} = v_0$ are 0.47% for $N = 50$ and 0.25% for $N = 200$. So the results for $\hat{v} = 0.02$ are closer to the analytic price in all subtables, where we grouped the parameters according to the initial variance v_0.

5 Conclusion and Outlook

We have introduced a refined tree method for pricing derivatives in the stochastic volatility model of Heston with a particular focus on incorporating the correlation between the log-stock price and the variance process. To this end, we adjust the transition probabilities of the corresponding product model, leaving the marginals invariant. Among all admissible adjustments we choose the one giving the closest moment match up to second order, where "closest" is determined in a mathematically rigorous way. In the nodes where an exact moment match up to order two is possible, we use the remaining degree of freedom to match a mixed cumulant of higher order of the Heston distribution with the rationale to achieve a lower distributional approximation error. We determine recursions for the joint cumulants of all orders for the Heston limit distribution in Lemma 1. The calculated option values can be determined fast and, more importantly, are highly accurate.

Overall, our results compare favorably to Beliaeva and Nawalkha [2010] although we use a smaller number of successor nodes, i.e. a sparser discretization of the state space. Our numerical evaluations indicate a convergence order of at least $\mathcal{O}(1/N)$. There is evidence that the power of the convergence order can even be increased for non-vanishing correlation due to the matching of higher order cumulants. In particular our approach is recommendable to cover a broad range of realistic correlation values.

In future work, without affecting our results so far, a truncation scheme bounding the number of grid nodes per time slice could help to further reduce computational complexity. Even more so, as indicated in Remark 5(c), our matching technique easily extends to more general schemes of successor nodes, with the perspective for strategies matching even higher order moments.

Acknowledgement

v_0	T	s_0	analytic	Δ RSS ($\hat{v} = 0.02$)		Δ RSS ($\hat{v} = v_0$)	
				$N = 50$	$N = 200$	$N = 50$	$N = 200$
0.04	1/12	90	0.0691	0.06%	0.60%	3.38%	2.81%
		95	0.6232	0.40%	0.01%	1.17%	0.94%
		100	2.5129	0.18%	0.02%	0.15%	0.04%
		105	6.0211	0.04%	0.01%	0.08%	0.08%
		110	10.5423	0.00%	0.00%	0.05%	0.04%
	1/4	90	0.8120	0.65%	0.23%	2.19%	1.53%
		95	2.2114	0.26%	0.04%	0.72%	0.58%
		100	4.6192	0.15%	0.01%	0.03%	0.07%
		105	7.9832	0.08%	0.02%	0.10%	0.08%
		110	12.0682	0.01%	0.01%	0.11%	0.10%
	1/2	90	2.2262	0.03%	0.12%	0.82%	0.87%
		95	4.1889	0.20%	0.04%	0.49%	0.38%
		100	6.9002	0.08%	0.02%	0.07%	0.07%
		105	10.2797	0.07%	0.02%	0.08%	0.07%
		110	14.1930	0.00%	0.01%	0.18%	0.12%
		MAPE ($v_0 = 0.04$)		0.15%	0.08%	0.64%	0.52%
0.09	1/12	90	0.4063	0.43%	0.13%	0.80%	0.59%
		95	1.4313	0.29%	0.10%	0.44%	0.27%
		100	3.5460	0.13%	0.04%	0.25%	0.09%
		105	6.8125	0.02%	0.00%	0.04%	0.00%
		110	10.9525	0.00%	0.00%	0.04%	0.02%
	1/4	90	1.8316	0.49%	0.15%	0.89%	0.31%
		95	3.5738	0.29%	0.09%	0.39%	0.19%
		100	6.0732	0.19%	0.05%	0.31%	0.07%
		105	9.2810	0.08%	0.02%	0.05%	0.02%
		110	13.0747	0.04%	0.01%	0.01%	0.01%
	1/2	90	3.5497	0.51%	0.11%	0.83%	0.21%
		95	5.7053	0.29%	0.07%	0.30%	0.11%
		100	8.4453	0.18%	0.05%	0.18%	0.07%
		105	11.7133	0.10%	0.03%	0.15%	0.01%
		110	15.4272	0.05%	0.01%	0.06%	0.01%
		MAPE ($v_0 = 0.09$)		0.21%	0.06%	0.32%	0.13%
0.16	1/12	90	0.9826	0.25%	0.12%	0.58%	0.26%
		95	2.3493	0.42%	0.12%	0.32%	0.14%
		100	4.6010	0.23%	0.07%	0.25%	0.09%
		105	7.7380	0.02%	0.01%	0.08%	0.03%
		110	11.6040	0.03%	0.01%	0.02%	0.00%
	1/4	90	3.0709	1.00%	0.10%	1.36%	0.22%
		95	5.0457	0.74%	0.12%	0.80%	0.12%
		100	7.6157	0.42%	0.05%	0.41%	0.05%
		105	10.7399	0.00%	0.02%	0.08%	0.05%
		110	14.3433	0.04%	0.02%	0.02%	0.03%
	1/2	90	5.0861	1.53%	0.20%	1.12%	0.14%
		95	7.3913	0.78%	0.12%	0.89%	0.15%
		100	10.1655	0.34%	0.07%	0.42%	0.07%
		105	13.3668	0.07%	0.04%	0.03%	0.04%
		110	16.9406	0.08%	0.01%	0.21%	0.02%
		MAPE ($v_0 = 0.16$)		0.40%	0.07%	0.44%	0.09%
		MAPE		0.25%	0.07%	0.47%	0.25%

Table 6: *Effect of $\hat{v}$ on the prices of European calls. Column 5 and 6 state the absolute percentage difference of the model price to the analytic solution for $\hat{v} = 0.02$. In column 7 and 8 we obtained the absolute percentage difference of the tree method to the closed-form solution for $\hat{v} = v_0$. We further set $r = 0.05$, $d = 0$, $\kappa = 3$, $\theta = 0.04$, $\eta = 0.1$, $\rho = -0.7$ and set $K = 100$.*

A Proofs

Proof of Lemma 1:

Let $i, j \in \mathbb{N}_0$ and let

$$m^{i,j}(t) = \mathbb{E}\left(X_t^i V_t^j\right)$$

under the assumption that x_0 and v_0 are the initial values. We follow the example 4.13 of Björk [2004] and employ the Itô formula

$$\mathrm{d}\left(X_t^i V_t^j\right) = i X_t^{i-1} V_t^j \,\mathrm{d}X_t + j X_t^i V_t^{j-1} \,\mathrm{d}V_t + \frac{i}{2}(i-1) X_t^{i-2} V_t^j (\mathrm{d}X_t)^2 \tag{15}$$
$$+ \frac{j}{2}(j-1) X_t^i V_t^{j-2} (\mathrm{d}V_t)^2 + ij X_t^{i-1} V_t^{j-1} (\mathrm{d}X_t \,\mathrm{d}V_t),$$

where $\mathrm{d}X_t$, $(\mathrm{d}X_t)^2$, $\mathrm{d}V_t$, $(\mathrm{d}V_t)^2$ and $(\mathrm{d}X_t \,\mathrm{d}V_t)$ are known as

$$\mathrm{d}X_t = -\frac{1}{2} V_t \,\mathrm{d}t + \sqrt{V_t} \,\mathrm{d}W_t^S, \qquad (\mathrm{d}X_t)^2 = V_t \,\mathrm{d}t,$$
$$\mathrm{d}V_t = \kappa(\theta - V_t)\,\mathrm{d}t + \eta\sqrt{V_t}\,\mathrm{d}W_t^V, \quad (\mathrm{d}V_t)^2 = \eta^2 V_t \,\mathrm{d}t \tag{16}$$
$$(\mathrm{d}X_t \,\mathrm{d}V_t) = \eta V_t \rho \,\mathrm{d}t$$

where we have used that $\mathrm{d}W_t^S \,\mathrm{d}W_t^V = \rho\,\mathrm{d}t$. Plugging (16) into (15) the integral form of the resulting equation reads as

$$X_t^i V_t^j = x_0^i v_0^j + \int_0^t \frac{i(i-1)}{2} X_u^{i-2} V_u^{j+1} - \frac{i}{2} X_u^{i-1} V_u^{j+1} + ij\eta\rho X_u^{i-1} V_u^j \,\mathrm{d}u$$
$$+ \int_0^t \frac{j\left(2\kappa\theta + \eta^2(j-1)\right)}{2} X_u^i V_u^{j-1} - j\kappa X_u^i V_u^j \,\mathrm{d}u$$
$$+ \int_0^t i X_u^{i-1} V_u^{j+1/2} \,\mathrm{d}W_u^S + \int_0^t j\eta X_u^i V_u^{j-1/2} \,\mathrm{d}W_u^V,$$

where the Brownian integrals vanish when taking expectation.

After moving the expectation under the integral sign in the $\mathrm{d}u$-integral and using that $m^{i,j}(t) = \mathbb{E}(X_t^i V_t^j)$, we obtain the equation

$$m^{i,j}(t) = x_0^i v_0^j + \int_0^t b^{i,j}(u) - j\kappa m^{i,j}(u) \,\mathrm{d}u,$$

where

$$b^{i,j}(t) = \frac{i(i-1)}{2} m^{i-2,j+1}(t) - \frac{i}{2} m^{i-1,j+1}(t) + ij\eta\rho m^{i-1,j}(t) \tag{17}$$
$$+ \frac{j\left(2\kappa\theta + \eta^2(j-1)\right)}{2} m^{i,j-1}(t).$$

Differentiating, we obtain the first-order non-homogeneous linear ODE

$$\mathrm{d}m^{i,j}(t) = -j\kappa m^{i,j}(t) + b^{i,j}(t),$$

with initial condition $m^{i,j}(0) = x_0^i v_0^j$. This ODE has the unique solution

$$m^{i,j}(t) = x_0^i v_0^j e^{-j\kappa t} + \int_0^t e^{-j\kappa(t-s)} b^{i,j}(s)\, ds. \tag{18}$$

Term $m^{i,j}$ in (18) is well-defined, since the recursions for term $b^{i,j}$ in (17) can be resolved for every combination $i, j \in \mathbb{N}_0$. To see this, note that term-wise, the sum of the superscripts is decreased passing from left-hand side to right-hand side of (17) in all but one term. In this term the sum remains unchanged, but one index is decreased, compare Figure 7. Hence circular recursions are excluded, and every term can be tracked to one with at least one negative index, for which the term becomes 0.
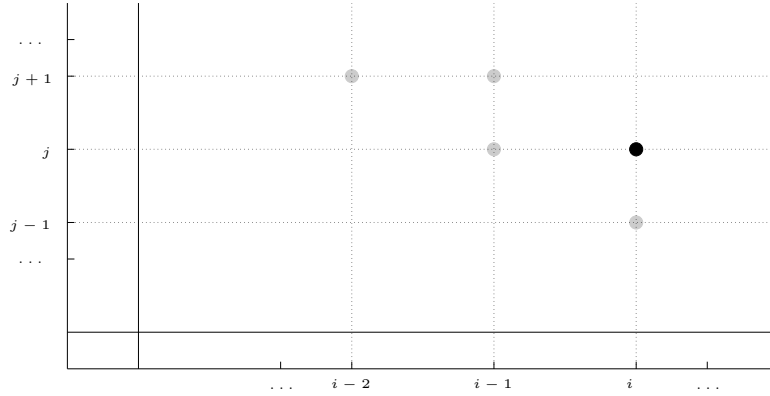
Figure 7: *Recursive structure of the moment generating procedure. Desired moment $m^{i,j}$ (black) depends on moments (gray) that can be determined previously.*

□

Proof of Corollary 2:

Differentiating $m^{i,j}(t)$ from (4) in Lemma 1 with respect to t, we obtain the first-order Taylor expansion

$$m^{i,j}(t) = x_0^i v_0^j + \left(-j\kappa x_0^i v_0^j + b^{i,j}(0)\right) t + \mathcal{O}\left(t^2\right).$$

The respective terms in the corollary are obtained through (17) and ignoring terms of order $\mathcal{O}\left(t^2\right)$. □

Proof of Proposition 3:

To ensure a total probability of 1, we set

$$\mathbb{P}_{\hat{X}}\left(x_2\right) = 1 - \mathbb{P}_{\hat{X}}\left(x_1\right) - \mathbb{P}_{\hat{X}}\left(x_3\right). \tag{19}$$

Matching expectation requires

$$-\frac{v}{2}\Delta t \overset{!}{=} \sum_{i=1}^{3}\left(x_i - x\right)\mathbb{P}_{\hat{X}}\left(x_i\right) \tag{20}$$

and for matching variance we demand

$$v\Delta t \overset{!}{=} \sum_{i=1}^{3}\left(x_i - x\right)^2\mathbb{P}_{\hat{X}}\left(x_i\right) - \left(\sum_{i=1}^{3}\left(x_i - x\right)\mathbb{P}_{\hat{X}}\left(x_i\right)\right)^2, \tag{21}$$

while we obtained the linearized moments by Corollary 2. The equations (19), (20) and (21) reduce to (7), where x can be considered as a level shift, so we can set $x = 0 = x_2$. The term $k(v)$ is independent of x and we obtain $x_1 = -x_3$. We still have to check whether $\mathbb{P}_{\hat{X}}\left(\cdot\right)$ is in fact a probability, i.e. is non-negative. As a consequence of (5), $4v + v^2\Delta t \leq 4k(v)^2\hat{v}$. Therefore $\mathbb{P}_{\hat{X}}(x_1) + \mathbb{P}_{\hat{X}}(x_3) \leq 1$ and $\mathbb{P}_{\hat{X}}(x_2) \geq 0$.

While $\mathbb{P}_{\hat{X}}(x_1) \geq 0$ always, $\mathbb{P}_{\hat{X}}(x_3) \geq 0$ if

$$k\left(v\right) \leq \frac{4 + v\Delta t}{2\sqrt{\hat{v}\Delta t}}, \tag{22}$$

which is satisfied asymptotically for Δt tending to zero since then the right-hand side of (22) tends to infinity. So, setting $k(v)$ to the next larger integer guarantees positivity of the probabilities. This corresponds to our choice of $k(v)$. $\qquad\square$

Proof of Proposition 4:

(i) Each constraint defines a subspace in $\mathbb{R}^2$ that is closed and convex, since the boundaries are given through constraints satisfied with equality. In particular, the boundary belongs to $\mathcal{A}$, too, and hence $\mathcal{A}$ as the intersection of these closed convex sets is closed and convex. It is non-empty as $(\theta_{11}, \theta_{21}) = (0,0) \in \mathcal{A}$. It is bounded, since the constraints imply $0 \leq \theta_{i1} \leq 1$, $i = 1, 2$. As its boundary is formed by the linear restrictions, it is a simplex. As it is bounded, each constraint induces two vertices, and as in each vertex of $\mathcal{A}$ two line segments are joined, we have at most 6 extremal points.

(ii) As the square of an affine function in $(\theta_{11}, \theta_{21})$, $\tilde{d}$ is obviously convex and continuous.

(iii) Let $d \geq 0$. Then

$$\tilde{d}\left(\theta_{11}, \theta_{21}\right) \overset{!}{=} d$$

$$\iff \left[\mathbb{C}\text{ov}(X, V) - (v_1 - v_2)\left[\theta_{11}\left(x_1 - x_3\right) + \theta_{21}\left(x_2 - x_3\right)\right]\right]^2 = d \quad (23)$$

$$\iff \theta_{21} = \frac{\mathbb{C}\text{ov}(X, V) \pm \sqrt{d}}{\left(x_2 - x_3\right)\left(v_1 - v_2\right)} - \theta_{11} \frac{\left(x_1 - x_3\right)}{\left(x_2 - x_3\right)}.$$

(iv) By (i), $\mathcal{A}$ is compact and by (ii), $\tilde{d}$ given in (12) is continuous, so the infimum is attained.

Let $L_0 \neq \emptyset$. Then L_0 is either a singleton and contains the unique minimizer, or it is a line segment, as $\mathcal{A}$ is bounded by (i) and N_0 is the straight line h_0. Each point in L_0 exactly matches the linearized covariance of the Heston model and gives a proper probability.

Assume $L_0 = \emptyset$ and let $\check{d} > 0$ be the minimal value of $\tilde{d}$ on $\mathcal{A}$. Then the simplex $\mathcal{A}$ cannot be enclosed by the level set $N_{\check{d}}$ formed by the two straight lines h_1 and h_2, as otherwise $\check{d}$ could not be minimal.

We next shift from minimizing $\tilde{d}$ to minimizing distances and define

$$\text{dist}(A, B) = \inf\{\|x - y\| \mid x \in A, y \in B\}$$

as the distance of two sets $A, B \subset \mathbb{R}^2$ for $\|\cdot\|$ the Euclidean distance in $\mathbb{R}^2$.

Hence $\text{dist}(\mathcal{A}, h_1) \neq \text{dist}(\mathcal{A}, h_2)$ and we can w.l.o.g. assume that

$$\text{dist}(\mathcal{A}, h_1) < \text{dist}(\mathcal{A}, h_2).$$

Therefore, any minimizer of $\tilde{d}$ in $\mathcal{A}$ achieves $\text{dist}(\mathcal{A}, h_1) = 0$.

In the sequel we show that this minimizer must lie on the boundary $\partial\mathcal{A}$. Let therefore $z \in \mathcal{A}$ and $\pi(z)$ be its orthogonal projection onto h_1. Then on the line segment $[z; \pi(z)]$, the function $y \mapsto \text{dist}(\{y\}, h_1)$ decreases linearly when moving from z to $\pi(z)$. So at the intersection point z_b of $[z; \pi(z)]$ and the boundary $\partial\mathcal{A}$, $\text{dist}(\{z_b\}, h_1) \leq \text{dist}(\{z\}, h_1)$.

In fact, the minimizer is even an extremal point. Note that for two arbitrary line segments g and h and $y \in g$, the function $y \mapsto \text{dist}(\{y\}, h)$ is piecewise linear with at most one kink in the intersection point of g and h, if both intersect. Hence, for compact g, the function $y \mapsto \text{dist}(\{y\}, h)$ attains its minimum in its extremal points or in the intersection of g and h, if it exists. Now z_b itself can be uniquely written as a convex combination $z_b = (1 - \mu_b)z_0 + \mu_b z_1$ of two extremal points $z_0, z_1 \in \mathcal{E}$ for some $\mu_b \in [0, 1]$. Since the line segment $[z_0; z_1]$ does not intersect h_0, the function $y \mapsto \text{dist}(\{y\}, h_0)$ takes its minimal value in z_0 or z_1, hence there is always a minimizer in $\mathcal{E}$.

To show uniqueness, note that we so far have worked with an arbitrary successor node position x_2 in (6). However, x_2 may be treated as a constant shift, so it has no impact on the choice of the probabilities $\tilde{\Pi}$ on

our grid. Hence we are free to set $x_2 = 0$ here, leading to $x_1 = -x_3$ for the other two successor nodes. If both z_0 and z_1 are minimizers, the line segment $[z_0; z_1]$ must be parallel to h_0. This is not the case since the absolute value of the slope of h_1 is 2 and the one of $[z_0; z_1]$ lies in $\{0, 1, \infty\}$, compare (23), (10) and (11). Hence $\tilde{d}(z_0) \neq \tilde{d}(z_1)$ and the minimizer is unique.

$\square$

Proof of Proposition 6:

$$\mathbb{E}_{\widetilde{\Pi}}\left(\hat{X}^2\hat{V}\right) = v_1\left(x_1^2\widetilde{\mathbb{P}}_{11} + x_2^2\widetilde{\mathbb{P}}_{21} + x_3^2\widetilde{\mathbb{P}}_{31}\right) + v_2\left(x_1^2\widetilde{\mathbb{P}}_{12} + x_2^2\widetilde{\mathbb{P}}_{22} + x_3^2\widetilde{\mathbb{P}}_{32}\right) \quad (24)$$

and since $\widetilde{\mathbb{P}}_{ij} = \mathbb{P}_{ij} + \theta_{ij}$, for $i = 1, 2, 3$ and $j = 1, 2$ and

$$\begin{aligned}
\mathbb{E}\left(\hat{X}^2\right)\mathbb{E}\left(\hat{V}\right) &= v_1\left(x_1^2\mathbb{P}_{11} + x_2^2\mathbb{P}_{21} + x_3^2\mathbb{P}_{31}\right) \\
&\quad + v_2\left(x_1^2\mathbb{P}_{12} + x_2^2\mathbb{P}_{22} + x_3^2\mathbb{P}_{32}\right),
\end{aligned}$$

(24) reduces to (14). $\square$

Proof of Corollary 7:

For the unrestricted case with $\lambda \in \mathbb{R}$, we obtain λ_0 for solving (14) for λ. Due to convexity, in the restricted case $\lambda \in [0, 1]$, we hence obtain the optimal λ^* as $\lambda^* = \min(\max(0, \lambda_0), 1)$, as truncated to $[0, 1]$. $\square$

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