Diagram expansions in classical stochastic field theory.
I. Regularisations, stochastic calculus and causal Wick's theorem

L. I. Plimak, M. Fleischhauer† and M. J. Collett

Dept. of Physics, University of Auckland, Private Bag 92019, Auckland, New Zealand
†Sektion Physik, Ludwig-Maximilians Universität München, D-80333 München, Germany

Abstract. We aim to establish a link between path-integral formulations of quantum and classical field theories via diagram expansions. This link should result in an independent constructive characterisation of the measure in Feynman path integrals in terms of a stochastic differential equation (SDE) and also in the possibility of applying methods of quantum field theory to classical stochastic problems. As a first step we derive in the present paper a formal solution to an arbitrary c-number SDE in a form which coincides with that of Wick's theorem for interacting bosonic quantum fields. We show that the choice of stochastic calculus in the SDE may be regarded as a result of regularisation, which in turn removes ultraviolet divergences from the corresponding diagram series.

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1. Introduction

The ultimate aim of the results reported in this and following papers is to place Feynman path-integral techniques [0, 0], as used in quantum field theory (QFT), on a well-defined mathematical basis. More precisely, we wish to find a constructive characterisation of the measure on these path integrals; this should be independent of Hilbert space or other q-number techniques. We shall show that there exist simple rules that map an interacting bosonic field to a related (pseudo-) stochastic differential equation (SDE), such that the Feynman paths for the q-number field are the c-number solutions to this SDE. The measure on the Feynman paths is then just the stochastic measure on the solutions to the corresponding SDE. (For stochastic calculus and SDEs see [0, 0] and references therein.)

We find that any SDE is formally solved by a diagram series of a certain structure, which we call a *causal diagram series*. This result may be interesting in itself since it allows one to extend methods specific to diagram approaches (e.g. the Dyson equations) to SDEs. However, its major implications follow from the fact that, using methods developed in [0], all three types of diagram series (i.e., Feynman, Keldysh and Matsubara series) for bosons may be recast in the form of causal diagram series. As a result, a link will be established between boson QFT problems and those in stochastic classical field theory (CFT), expressing Feynman path integrals as classical stochastic averages corresponding to certain SDEs. This results in the possibility of characterising path integrals constructively, in well-defined mathematical terms, and indeed in a way independent of perturbation theory. (Some SDEs we find for Keldysh series have also been derived using phase-space methods in the quantum stochastics approach to quantum optics [0].)

Perhaps the most important result of our investigation is that we encounter a link between regularisations in the diagrams and the stochastic calculus in SDEs. We show that the choice of stochastic calculus in an SDE may be approached as a problem of multiplication of generalised functions [0] (emerging e.g. due to the singular nature of the Wiener process [0]). One solution to this problem is to regularise the equation, i.e. to replace it by a mathematically defined one, specifying a limiting procedure to remove the regularisation. Designing a regularisation procedure is equivalent to specifying a stochastic calculus; e.g. Stratonovich calculus is found by regularising (smoothing) the white noise [0]. In this paper we consider an alternative regularisation procedure, which we term a *causal regularisation*, and show it to be equivalent to introducing Ito calculus.

It turns out that regularising an SDE one modifies the respective causal diagram series in such a way that ultraviolet divergences in diagrams cancel. Moreover, strictly speaking, causal diagrams cannot exhibit ultraviolet divergences at all because their derivation holds only for regularised SDEs. However, if found as a formal transformation of a diagram series in QFT [0], a causal diagram series exhibits ultraviolet divergences the same way as its preimage. This means that perturbation theory cannot completely specify the respective SDE: it specifies it only as a symbolic equation, up to the freedom in stochastic calculus. This mathematical uncertainty manifests itself in ultraviolet divergences (which thus can be ascribed to the measure over the path integrals being not defined properly). Then, if a *renormalisation* procedure is designed to remove divergences, its side effect is to cancel the said uncertainty. Indeed, renormalisation always includes a regularisation of diagrams, and *ipso facto* implies a regularisation in the respective SDE. Hence renormalisation may be regarded as a limiting procedure specifying the stochastic calculus in the SDE for the Feynman paths. For example, it will be shown [0] that the well-known Pauli-Villars regularisation [0] yields a causal regularisation in the respective SDE which in turn specifies it as an Ito SDE (cf also [0]). This link between renormalisation and stochastic calculus, which appears to be of fundamental importance, suggests more extensive mathematical investigation, which may lead to better understanding of the formal grounds of quantum field theory.

In this paper we consider as a preliminary the case of stochastic classical fields obeying stochastic differential equations. We start from defining a causal regularisation of an SDE and show for a simple example that it is indeed equivalent to introducing Ito calculus. For regularised SDEs (and only for them) we then find the central formal result of this paper: a general relation which we call the *causal Wick's theorem*. The reason for this name is that it in essence coincides with Hori's form [0] of Wick's theorem for bosons in QFT [0, 0]. In the following paper [0] we use the causal Wick's theorem to derive a solution to an SDE in a form of a diagram series (in very similar fashion to the way that Wick's theorem proper allows one to derive, e.g., the Feynman diagram series [0, 0, 0]). We shall consider major structural properties of this series and also classify diagram structures related to particular problems in CFT. In turn, this will allow us to approach the *converse problem* of recovering an explicit SDE from a given causal diagram series. This problem plays a key role in deriving SDEs related to problems in QFT, which will be the subject of the papers to follow [0]. (An SDE related to Matsubara series is derived in [0].)

2. Stochastic calculus and regularisations in SDEs.

We consider a c-number field $\psi(r,t)$, which satisfies a generic equation with a source $s(r,t)$,

$$\mathcal{L}\psi(r,t) = s(r,t), \tag{1}$$

where $\mathcal{L}$ is a differential operator. In general, the source s is random and depends on the field ψ; that is, (1) is a stochastic differential equation for the field $\psi(r,t)$. To be specific, we assumed that ψ is a classical field in one-dimensional space; to consider other situations, e.g., a multi-mode field, or a field in three-dimensional space, one should replace in the relations below $\int dr$ by $\sum_r$ (i.e., r is then a mode index) or $\int d^3\mathbf{r}$ respectively. For a single-mode field, the variable r and the summation should simply be dropped. We shall often resort to this last case for simplicity of examples.

Formally solving (1) turns it into an integral equation, (assuming $s(r,t) \to 0$ as $t \to -\infty$)

$$\psi(r,t) = \int_{-\infty}^{\infty} dt' \int dr'\, G(r,r',t-t')\, s(r',t') + \psi_0(r,t), \tag{2}$$

where $G(r,r',t-t')$ is the retarded Green's function of equation (1),

$$\mathcal{L}G(r,r',t-t') = \delta(r-r')\delta(t-t'), \quad G(r,r',t-t') = 0, t < t', \tag{3}$$

(so that the integration in (2) is in fact from minus infinity to t), and ψ_0 is the *in*-field, i.e. $\psi = \psi_0$ before the source is on. The *in*-field obeys the free version of equation (1), $\mathcal{L}\psi_0(r,t) = 0$. Note that the existence of a retarded Green's function is not guaranteed for an arbitrary $\mathcal{L}$; the conditions (3) single out equations that may have physical meaning.

The retarded Green's functions usually have a singularity at zero time, so problems arise if the source s is a singular function as well. E.g., if s contains white noise, defining mathematically equations (1) and (2) requires specification of a stochastic calculus [0, 0]. For our purposes, however, a different approach is more natural. Namely, the problem of stochastic calculus may be regarded as the well-known problem of multiplication of generalised functions [0]. As was noted by Bogoliubov [0], this is exactly the mathematical problem that underlies divergences and the need for renormalisations in quantum field theory. With the goal in mind of characterising Feyman paths as solutions to SDEs, the common mathematics underlying these two problems (i.e. renormalisations in QFT and stochastic calculus in SDEs) appears very encouraging indeed.

Consider, for example, the equation,

$$i\partial_t\psi(t) = \varepsilon\psi(t)W'(t), \quad G(t) = -i\theta(t), \tag{4}$$

where ε is a constant and W' is the derivative of the Wiener process. It can only be defined in the sense of the theory of generalised functions,

$$\int dt\, W'(t)\varphi(t) = -\int dt\, W(t)\varphi'(t), \tag{5}$$

where $\varphi(t)$ is a "good" function. It would suffice for φ to have finite support and be continuously differentiable, so that φ' is a function of finite variation. It is then clear that for $\psi W'$ to be defined, it would suffice for ψ to be continuously differentiable, but this is certainly not consistent with (4). Hence the product $\psi(t)W'(t)$ is undefined and so is equation (4).

The standard way around this problem is to define $\int \psi W' dt = \int \psi dW$ as a stochastic integral so that the equation $\psi = \psi(0) - i\varepsilon \int_0^t \psi dW$ is defined [0]. Instead, we replace an *undefined* integral equation, $\psi = \psi_0 - i\varepsilon \int_{-\infty}^t \psi W' dt'$, by a *regularised* integral equation,

$$\psi(t) = \psi_0 + \int dt' \, G_{\text{reg}}(t - t')\psi(t')\varepsilon(t')W'(t'). \tag{6}$$

$G_{\text{reg}}(t)$ is a *causally regularised* retarded Green's function, which is (i) causal, $G_{\text{reg}}(t) = 0, t \leq 0$, (ii) a given number of times continuously differentiable, and (iii) in a certain sense close to $G(t)$. For example, $(k \geq 1)$

$$G_{\text{reg}}(t) = -i\theta(t)\left(1 - \mathrm{e}^{-\Gamma t}\right)^{k+1} \tag{7}$$

is k times continuously differentiable. Γ here is an arbitrary positive parameter, and the final limit $\Gamma \to \infty$ is implied. We have also introduced a truncating function $\varepsilon(t)$ into the white noise factor W' to have a consistent *in*-field formulation: $\varepsilon(t) \leq \varepsilon$ is infinitely differentiable, and a negative T' exists such that $\varepsilon(t) = \varepsilon$ if $t > T'$ and $\varepsilon(t) = 0$ if $t < 2T'$ (for $t < 2T'$, (6) thus reduces to $\psi = \psi_0$). Equation (6) is now consistent with the assumption of ψ being k times continuously differentiable. If this is indeed the case, the factor $G_{\text{reg}}(t-t')\psi(t')\varepsilon(t')$ at a given $t > 2T'$ is (at least) continuously differentiable and has a finite support $\{t' : \ 2T' < t' < t\}$, so that the integral on the right of (6) is defined; then, continuity of its k-th derivative by t follows from the fact that W' is, loosely speaking, no more singular than a δ-function.

An indication that in the limit $\Gamma \to \infty$ a solution to equation (6) approaches a solution to the Ito [0, 0] differential equation (4) may be seen from the following considerations. As a generalised function, $\varepsilon(t')W'(t')$ may be approximated by a discrete sum of δ-functions,

$$\varepsilon(t')W'(t') \approx \sum_{k=1}^{\infty} \tilde{\varepsilon}_k \delta(t - t_k), \tag{8}$$

where $t_k = 2T' + k\Delta t$, and Δt is a discretisation scale. Equation (6) then has a unique solution,

$$\psi(t) = \psi_0 + \sum_{k=1}^{\infty} \tilde{\varepsilon}_k G_{\text{reg}}(t - t_k)\psi(t_k), \tag{9}$$

where $\psi(t_k)$ may be found recurrently,

$$\psi(t_m) = \psi_0 + \sum_{k=1}^{m-1} \tilde{\varepsilon}_k G_{\text{reg}}(t_m - t_k)\psi(t_k). \tag{10}$$

The sum in (9) is in fact finite, so that $\psi(t)$ inherits all the "goodness" of $G_{\text{reg}}(t)$. It is now easy to see that if $\Gamma\Delta t \gg 1$, the integral in (6) coincides with the partial sum $-i \sum \psi(t_k)\varepsilon(t_k)\left[W(t_{k+1}) - W(t_k)\right]$; so that in the limit $\Delta t \to 0$ we recover the Ito integral [0, 0] $-i \int_{2T'}^t dW(t')\psi(t')\varepsilon(t')$. From the practical point of view, this "proves" the hypothesis that in the limit $\Gamma \to \infty$ the Ito calculus is recovered, since any real calculation implies time discretisation. In order to prove it mathematically, one should commute the limits: first $\sum_{k=1}^{\infty} \tilde{\varepsilon}_k \delta(t-t_k) \to \varepsilon(t')W'(t')$ and second $\Gamma \to \infty$, while the above considerations imply the opposite order of the limits.

Later we shall see [0] that the causal regularisation is chosen by the normally-ordered form of the interaction in the quantum problem, so that this regularisation is the correct one if an SDE related to a quantum problem is considered (see also [0]). However if an SDE is considered by itself, irrespective of its possible quantum interpretation, alternative regularisations leading to different stochastic calculus are equally possible. For example, regularising the source by smearing it in time makes it infinitely differentiable so that equation (4) becomes mathematically defined. Then normal calculus holds at all stages of the regularisation procedure; this means that in the end the Stratonovich calculus must be recovered [0].

For purposes of this paper, however, details of how equation (1) is regularised are irrelevant. For simplicity we just assume that G and s are infinitely differentiable functions. Note that this means that the integral equation (2) rather than the differential equation (1) has to be considered: the latter only emerges when regularisations are removed.

3. Classical stochastic self-action problem.

3.1. Characterisation of the stochastic sources and fields.

In order to have a fully defined equation (2), the functional dependence of the source on the full (*local* or *microscopic*) field, $\psi(r,t)$, is needed. Such a characterisation of the source is usually given by the physical model considered. In general, the source is a *stochastic* variable whose properties depend on the local field $\psi(r,t)$. *Solving* equation (1) then means finding $\psi(r,t)$ as a stochastic variable depending on the *in*-field $\psi_0(r,t)$. This constitutes a classical stochastic self-action problem.

3.1.1. Microscopic characterisation of the stochastic source. More formally, the dependence of the local source on the local field is determined by a probability distribution, $P(s|\psi)$, of the function $s(r,t)$ conditioned on the *full* field $\psi(r,t)$. We will start from considering real fields and will generalize the results to complex fields at the end. We introduce a characteristic functional $S(\alpha|\psi)$ corresponding to $P(s|\psi)$ as,

$$S(\alpha|\psi) = \overline{\mathrm{e}^{\alpha s}}\big|_{\psi} = \int Ds\, \mathrm{e}^{\alpha s} P(s|\psi), \tag{11}$$

Here, $\alpha(r,t)$ is an arbitrary "good" real function and $\int Ds$ denotes a functional integration over the trajectories $s(r,t)$. We use a condensed notation in which $\alpha s =$

$\int dx \alpha(x) s(x)$, where $x = \{r, t\}$ and $\int dx = \int dr dt$. Differentiating S over $\alpha(x)$ produces multi-space-time averages of the source, conditioned on the full field,

$$\overline{s(x_1)\cdots s(x_n)}\Big|_{\psi} = \frac{\delta^n}{\delta\alpha(x_1)\cdots\delta\alpha(x_n)} S(\alpha|\psi)\Big|_{\alpha=0}. \tag{12}$$

To specify the functional dependence of the source on ψ it is convenient to introduce generalised *susceptibilities* $\chi^{(m,n)}(x_1, \cdots, x_m; x'_1, \cdots, x'_n)$. They are coefficients in the series expressing cumulants of the local source $s(x)$ in terms of the powers of the local field $\psi(x)$.

$$\begin{aligned}\overline{s(x)}\Big|_{\psi} = \chi^{(1,0)}(x) &+ \int dx' \chi^{(1,1)}(x; x')\psi(x') \\ &+ \frac{1}{2}\int dx' dx'' \chi^{(1,2)}(x; x', x'')\psi(x')\psi(x'') + \ldots, \end{aligned} \tag{13}$$

$$\begin{aligned}\overline{s(x)s(x')}\Big|_{\psi} = \overline{s(x)}\Big|_{\psi}\ \overline{s(x')}\Big|_{\psi} &+ \chi^{(2,0)}(x, x') + \int dx'' \chi^{(2,1)}(x, x'; x'')\psi(x'') + \ldots, \\ &+ \frac{1}{2}\int dx'' dx''' \chi^{(2,2)}(x, x'; x'', x''')\psi(x'')\psi(x''') + \ldots, \\ &\vdots \end{aligned} \tag{14}$$

With them the characteristic functional can be given a simple form

$$S(\alpha|\psi) = \exp \sum_{m,n=0}^{\infty} \frac{1}{m!n!} \alpha^m \chi^{(m,n)} \psi^n, \tag{15}$$

where $\alpha\chi^{(1,1)}\psi = \int dx dx'\, \alpha(x)\chi^{(1,1)}(x; x')\psi(x')$ etc. The susceptibilities should obey the causality condition,

$$\chi^{(m,n)}(x_1, \cdots, x_m; x'_1, \cdots, x'_n) = 0 \quad \text{for} \quad \max(t'_1, \cdots, t'_n) > \max(t_1, \cdots, t_m); \tag{16}$$

this condition can also be formulated as "the latest argument of a susceptibility is always an output one"; hence $\chi^{(0,n)} = 0$ for all n.

The quantity $\chi^{(1,0)}(x)$ is a given source (non-random). For $m = 1$ we find susceptibilities proper: the linear one, $\chi^{(1,1)}$, and the nonlinear ones, $\chi^{(1,n)}$ for $n > 1$. If $\chi^{(m,n)} = 0$ for $m > 1$, the dependence of the source on the local field is not stochastic, and equation (1) is not stochastic either; it is linear, if $\chi^{(1,n)} = 0$ for $n > 1$, and otherwise nonlinear. Non-zero $\chi^{(m,n)}$ for $m > 1$ introduce stochasticity: non-zero $\chi^{(m,0)}$ are cumulants of a given random source, while $\chi^{(m,n)}$ for both $m > 1$ and $n > 0$ describe how the statistics of the source depend on the field.

In practical problems, the susceptibilities are commonly local (e.g., $\chi^{(1,1)}(x; x') \sim \delta(x - x')$), so that the source at the point x depends only on the local field at the same point. The causality condition (16) is then satisfied automatically. Locality gives more physical sense to the equations, but may lead to mathematical problems. In this paper, we assume that these problems are overcome by regularisations.

For stochastic sources proper the susceptibilities are not arbitrary: for example, we may not have $\chi^{(2,0)}(x,x') = -\delta(x-x')$. Furthermore, to have only a finite number of nonzero susceptibilities we have to assume that the local source is Gaussian if conditioned on the full field [0]. These restrictions can be eased by introducing pseudostochastics [0], which, loosely speaking, means allowing the probability to become nonpositive. In this paper, we try to avoid this so that the equations we consider are stochastic in the strict meaning of the term.

3.1.2. Macroscopic characterisation of the stochastic source and field. Macroscopically, the source should be characterised by its dependence on the *in*-field ψ_0. We therefore introduce the probability distribution $\Pi(s|\psi_0)$ of the source function s at a given *in*-field ψ_0. If this probability distribution is known, solving for the field is straightforward. To describe the field, we introduce the characteristic functional of the multi-space-time field averages,

$$\Phi(\zeta) = \Phi(\zeta|\psi_0) = \overline{\mathrm{e}^{\zeta\psi}}\Big|_{\psi_0}, \tag{17}$$

so that

$$\overline{\psi(x_1)\cdots\psi(x_n)} = \overline{\psi(x_1)\cdots\psi(x_n)}\Big|_{\psi_0} = \frac{\delta^n}{\delta\zeta(x_1)\cdots\delta\zeta(x_n)}\Phi(\zeta)\Big|_{\zeta=0}. \tag{18}$$

Since $\psi = Gs + \psi_0$, the field statistics are effectively those of the source, and we have

$$\Phi(\zeta) = \int Ds\, \mathrm{e}^{\zeta(Gs+\psi_0)}\Pi(s|\psi_0) = \mathrm{e}^{\zeta\psi_0}\Sigma(\zeta G|\psi_0), \tag{19}$$

where $[\zeta G](x) = \int dx'\zeta(x')G(x',x)$. Here we have introduced the characteristic functional $\Sigma(\alpha|\psi_0)$ corresponding to $\Pi(s|\psi_0)$:

$$\Sigma(\alpha|\psi_0) = \overline{\mathrm{e}^{\alpha s}}|_{\psi_0} = \int Ds\, \mathrm{e}^{\alpha s}\Pi(s|\psi_0). \tag{20}$$

In the complex case, our definitions must be generalised to allow one to consider averages containing both the fields and sources and their complex conjugates. Thus the notation $\Pi(s|\psi_0)$ used for the real case, must be replaced for complex fields by $\Pi(s,s^*|\psi_0,\psi_0^*)$; the functional integration in the complex case will be denoted as $\int D\psi D\psi^*$ and $\int DsDs^*$, etc. The characteristic functionals S, Σ and Φ are defined as,

$$S(\alpha,\alpha^\dagger|\psi,\psi^*) = \int DsDs^*\, \mathrm{e}^{\alpha^\dagger s+\alpha s^*}P(s,s^*|\psi,\psi^*), \tag{21}$$

$$\Sigma(\alpha,\alpha^\dagger|\psi_0,\psi_0^*) = \int DsDs^*\, \mathrm{e}^{\alpha^\dagger s+\alpha s^*}\Pi(s,s^*|\psi_0,\psi_0^*), \tag{22}$$

$$\Phi(\zeta,\zeta^\dagger) = \Phi(\zeta,\zeta^\dagger|\psi_0,\psi_0^*) = \int DsDs^*\, \mathrm{e}^{\zeta^\dagger(Gs+\psi_0)+\zeta(G^*s^*+\psi_0^*)}\Pi(s,s^*|\psi_0,\psi_0^*), \tag{23}$$

where $\alpha(x), \alpha^\dagger(x), \zeta(x)$ and $\zeta^\dagger(x)$ are arbitrary "good" functions. (The dagger "†" is here just a convenient notation, and does not represent Hermitian conjugation.) One can also consider $\alpha,\alpha^\dagger$ and $\zeta,\zeta^\dagger$ as pairs of complex-conjugated functions, $\alpha^\dagger = \alpha^*$, $\zeta^\dagger = \zeta^*$.

3.2. Relation between $\Pi(s|\psi_0)$ and $P(s|\psi)$.

Our goal now is to find a formal solution to the stochastic self-action problem, equation (2), in the form of a relation between the microscopic and macroscopic probability distributions, $P(s|\psi)$ and $\Pi(s|\psi_0)$. Assume discretisation of the time axis, $s(t), \psi(t) \to s(t_k), \psi(t_k)$, where $t_k = k\Delta t, k = -\infty, \cdots, \infty$; a final limit of $\Delta t \to 0$ is implied. We omit the spatial variable as irrelevant. With discretisation, averages of the source at the given field are given by a multiple integration ($t, t', \cdots t''$ are among the t_k),

$$\left.\overline{s(t)s(t')\cdots s(t'')}\right|_{\psi} = \int s(t)s(t')\cdots s(t'')P(s|\psi)\prod_{k=-\infty}^{\infty} ds(t_k). \tag{24}$$

The distribution $P(s|\psi)$ is causal, ie, $s(t_k)$ depends on $\psi(t_m)$ only for $m \leq k$. This allows one to introduce reduced probability distributions,

$$P_m\left(s_{\leq t_m}\left|\psi_{\leq t_m}\right.\right) = \int P(s|\psi)\prod_{k=m+1}^{\infty} ds(t_k), \tag{25}$$

where $s_{\leq t_m} = \{s(t_k): \ k \leq m\}$, and $\psi_{\leq t_m} = \{\psi(t_k): \ k \leq m\}$. The fact that P_m depends on $\psi_{\leq t_m}$ and not on the whole ψ leads to the following causality condition for the averages,

$$\frac{\delta}{\delta\psi(t_m)}\left.\overline{s(t)s(t')\cdots s(t'')}\right|_{\psi} = 0 \ \text{ for } \ t_m > \max(t, t', \cdots t''). \tag{26}$$

The unravelling of the source statistics in time is described by the conditional probability distribution,

$$P_m\left(s(t_m)\left|s_{\leq t_{m-1}}, \psi_{\leq t_m}\right.\right) = \frac{P_m(s_{\leq t_m}\left|\psi_{\leq t_m}\right.)}{P_{m-1}(s_{\leq t_{m-1}}\left|\psi_{\leq t_{m-1}}\right.)}. \tag{27}$$

In turn,

$$P_m\left(s_{\leq t_m}\left|\psi_{\leq t_m}\right.\right) = \prod_{k=-\infty}^{m} P_k\left(s(t_k)\left|s_{\leq t_{k-1}}, \psi_{\leq t_k}\right.\right). \tag{28}$$

Since we always observe a system only for finite times, P_m with m large enough is actually as good as $P(s|\psi)$, so that we can write

$$P(s|\psi) = \prod_{k=-\infty}^{\infty} P_k\left(s(t_k)\left|s_{\leq t_{k-1}}, \psi_{\leq t_k}\right.\right). \tag{29}$$

The advantage of the unravelled representation (29) for $P(s|\psi)$ is that it is perfectly designed to accept the dynamical relation between the source and the field, equation (2). With time discretisation, (2) is understood as

$$\psi(t_m) = \psi_0(t_m) + \Delta t\sum_{k=-\infty}^{\infty} G(t_m - t_k)s(t_k). \tag{30}$$

Note that since $G(t)$ is both causal and regular (or regularised), $G(0) = 0$ and hence the latest source value to contribute to $\psi(t_m)$ is $s(t_{m-1})$. Hence $\psi(t_m)$ depends on $s_{\leq t_{m-1}}$ and not on $s_{\leq t_m}$. (Implying that ψ_0 is either non-random or uncorrelated with s.) Then, both $s_{\leq t_{m-1}}$ and $\psi_{\leq t_m}$ in the conditional probability $P_m\left(s(t_m)\,\middle|\,s_{\leq t_{m-1}}, \psi_{\leq t_m}\right)$ are predetermined and we can write

$$\Pi_m\left(s(t_m)\,\middle|\,s_{\leq t_{m-1}}, (\psi_0)_{\leq t_m}\right) = P_m\left(s(t_m)\,\middle|\,s_{\leq t_{m-1}}, (Gs+\psi_0)_{\leq t_m}\right). \tag{31}$$

Here $\Pi_m\left(s(t_m)\,\middle|\,s_{\leq t_{m-1}}, (\psi_0)_{\leq t_m}\right)$ is the probability distribution for the source at t_m, conditioned on its own prehistory and the *in*-field. Relation (31) solves the self-action problem for the source, expressing its *macroscopically* observable statistics in terms of the *microscopic* relations characterising the system. For the multi-time probability distribution of the source, conditioned on the *in*-field, we have,

$$\Pi(s|\psi_0) = \prod_{m=-\infty}^{\infty} \Pi_m\left(s(t_m)\,\middle|\,s_{\leq t_{m-1}}, (\psi_0)_{\leq t_m}\right), \tag{32}$$

and hence,

$$\Pi(s|\psi_0) = P(s|Gs+\psi_0). \tag{33}$$

In the complex case the derivation is the same, but the result should be written as

$$\Pi(s, s^*|\psi_0, \psi_0^*) = P(s, s^*|Gs+\psi_0, G^*s^*+\psi_0^*) \tag{34}$$

to comply with the convention regarding notation introduced in section above.

This derivation reveals the physical content of the regularisation of the retarded Green's function G. Formally, it allowed us to untangle interactions in the system at zero time delays, assuming that the source at a particular time depends on the field at earlier times (causality) but not at the same time (regularisation). This means that by regularising G we eliminate the system's self-action at zero times (and distances); in other words, we cancel the "self-action of the point charge". Consequently, the causal regularisation of G is effectively a classical "renormalisation of charge". These considerations also make it physically clear why regularisation of G leads to an *Ito* SDE. In Ito calculus, the stochastic increment is by definition uncorrelated with the field at the same time. This property, which can be regarded as a "maximal-degree causality", is exactly the one enforced mathematically by the causal regularisation of G.

4. Causal Wick's theorems.

Our aim is now to rewrite relation (33) in terms of the characteristic functionals. To this end, we first rewrite it using the shift operator,

$$\Pi(s|\psi_0) = \exp\left(\frac{\delta}{\delta\psi}Gs\right) P(s|\psi)\Big|_{\psi=\psi_0}. \tag{35}$$

Here, we again use condensed notation, $\frac{\delta}{\delta\psi}Gs = \int dx dx' \frac{\delta}{\delta\psi(x)}G(x;x')s(x')$; note that the spatial dependence of the fields is restored. Then

$$\begin{aligned}\Sigma(\alpha|\psi_0) &= \int Ds \exp\left(\alpha s + \frac{\delta}{\delta\psi}Gs\right) P(s|\psi)\Big|_{\psi=\psi_0} \\ &= \exp\left(\frac{\delta}{\delta\psi}G\frac{\delta}{\delta\alpha}\right)\int Ds\, \mathrm{e}^{\alpha s}P(s|\psi)\Big|_{\psi=\psi_0} = \exp\left(\frac{\delta}{\delta\psi}G\frac{\delta}{\delta\alpha}\right) S(\alpha|\psi)\Big|_{\psi=\psi_0}. \end{aligned}\tag{36}$$

For the functional $\Phi(\zeta)$, relations (36) and (19) result in,

$$\Phi(\zeta) = \exp\left(\frac{\delta}{\delta\psi}G\frac{\delta}{\delta\alpha}\right)\exp\left(\zeta\psi\right) S(\alpha|\psi)\Big|_{\alpha=0,\psi=\psi_0}. \tag{37}$$

It is worth noting why we have to write, $\frac{\delta}{\delta\psi}(\cdots)|_{\psi=\psi_0}$, rather than just $\frac{\delta}{\delta\psi_0}$. The problem is that ψ_0 is a solution to a free equation, ie, $\frac{\delta}{\delta\psi_0}$ is a derivative with constraints, whereas when applying the shift operator, $\exp\left(\frac{\delta}{\delta\psi}Gs\right)$, to $P(s|\psi)$ in order to turn it into $P(s|Gs+\psi)$, one has to assume that ψ is arbitrary. Since $P(s|\psi)$ is indeed defined for an arbitrary ψ, the whole situation is consistent.

In the complex case, relations expressing Σ and Φ over S are found to be

$$\Sigma(\alpha,\alpha^\dagger|\psi_0,\psi_0^*) = \exp\left(\frac{\delta}{\delta\psi}G\frac{\delta}{\delta\alpha^\dagger} + \frac{\delta}{\delta\psi^\dagger}G^*\frac{\delta}{\delta\alpha}\right) S(\alpha,\alpha^\dagger|\psi,\psi^\dagger)\Big|_{\psi=\psi_0,\psi^\dagger=\psi_0^*}, \tag{38}$$

$$\begin{aligned}\Phi(\zeta,\zeta^\dagger) &= \exp\left(\frac{\delta}{\delta\psi}G\frac{\delta}{\delta\alpha^\dagger} + \frac{\delta}{\delta\psi^\dagger}G^*\frac{\delta}{\delta\alpha}\right) \\ &\quad\times \exp\left(\zeta^\dagger\psi + \zeta\psi^\dagger\right) S(\alpha,\alpha^\dagger|\psi,\psi^\dagger)\Big|_{\alpha=\alpha^\dagger=0,\psi=\psi_0,\psi^\dagger=\psi_0^*}. \end{aligned}\tag{39}$$

In these relations, $\psi, \psi^\dagger$ is a pair of "good" functions which can be either arbitrary or complex conjugated. We have assumed that $S(\alpha,\alpha^\dagger|\psi,\psi^*)$ may be regarded as an analytic function of ψ and ψ^* separately; in all practical examples $S = \exp(\text{Polynomial of } \psi,\psi^*)$ (cf equations 13–14), so that this assumption is valid. (It is easy to see that assuming $\psi, \psi^\dagger$ "good" is consistent only with a regular $G(x;x')$, so that relations (36-39) imply regularisations.)

Relations (36–39) are the main result of the present paper. Eqs. (36) and (38) are startlingly reminiscent of relations known in QFT that express Wick's theorem for bosonic operators as a differential operation [0], and we shall call them the *causal Wick's theorems* (for real and complex fields, respectively). The RHS of equations (37) and (39) will be referred to as *generating formulae for causal diagram series*. In the following paper [0] we shall see that they play the same role in deriving diagram series for the SDEs as Wick's theorem proper plays in deriving diagram series for interacting bosonic fields. The assumptions they are based on are: (i) causality, (ii) equation (2) and (iii) the condition $G(r,r',0) = 0$. Note that the last condition is meaningful only for regularised SDEs, so that regularisations are inherent to our approach rather than, as in QFT, introduced *post-factum* to rescue it.

5. Conclusion

We have shown that the choice of stochastic calculus, necessary for a proper mathematical definition of an SDE, may be regarded as a result of a regularisation of this SDE. We have shown furthermore that the formal solution to an arbitrary regularized stochastic differential equation can be written in a form similar to Wick's theorem in QFT called therefore the causal Wick's theorem.

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