Geometrical properties of generalized single facility location problems

Stefan Nickel * Justo Puerto † Antonio M. Rodriguez-Chia ‡

August 23, 2000

Abstract

In this paper we deal with single facility location problems in a general normed space where the existing facilities are represented by sets. The criterion to be satisfied by the service facility is the minimization of an increasing function of the distances from the service to the closest point of each demand set. We obtain a geometrical characterization of the set of optimal solutions for this problem. Two remarkable cases — the classical Weber problem and the minmax problem with demand sets — are studied as particular instances of our problem. Finally, for the planar polyhedral case we give an algorithmic description of the solution set of the considered problems.

Keywords: Location Theory, Convex Analysis, Geometrical algorithms.

1 Introduction

The classical single facility location problem deals with the location of a point in a real normed space X in order to minimize some function depending on the distances to a finite number of given points (existing facilities or demand points).

The following question arises: Why do we have to consider points as existing facilities? A natural extension is to allow sets of points as existing facilities. This means that we cannot use anymore the natural distance induced by the norm in X. Therefore a new decision has to be made before to deal with the problem itself: Which kind of distance measure should be used? Two different ways of measuring distances can be considered. The first one takes into account the average behavior, so that any point in the set is visited according to a probability distribution. This approach leads us to the minimization of expected distances, as discussed for instance in the papers of Drezner and Wesolowsky [DW80] or Carrizosa, Conde, Muñoz and Puerto [CCMMP95]. The second interpretation measures the distances to the closest points in the sets. Therefore, rather than expected distances we have to consider the concept of infimal distance

*Fachbereich Mathematik. Universität Kaiserslautern and ITWM. Kaiserslautern. Germany.
†Facultad de Matematicas. Universidad de Sevilla. C/Tarfia s/n, 41012 Sevilla, Spain.
‡Facultad de Ciencias del Mar. Universidad de Cadiz. Poligono Rio San Pedro. Puerto Real, Cadiz, Spain.

to sets. This approach is quite general and includes as particular examples previous approaches in the literature, since infimal distances reduce to regular distances when points instead of sets are considered(see Boffey and Mesa [BM96] for a good review on the location of extensive facilities on networks).

It should be noted that this problem allows to model different real world situations better than the classical models. It applies in a natural way to two-level distribution models. Logistics companies usually distribute their products from a central department to medium-size warehouses in each one of the cities of their area of influence (using big trucks). Then, the warehouse delivers the products to final retailers in the city using its own vehicle fleet (small size trucks or vans which can circulate through the city). In this model, the plant is the facility to be located and the closest points to the plant in each of the cities are the optimal locations for the local warehouses. This would be also the case of the simultaneous location of a hub together with airports for a given set of cities. The hub would be the facility to be located and the airports should be located on each city at the closest point to the hub. It would be as well the case of the location of a recycle plant with respect to local garbage collection plants. Obviously, the cities locate their garbage plants as far away as possible (to avoid pollution and risks),but in their territory (county); and as close as possible to the recycle plant (to minimize transportation costs). In marketing positioning opinions of buyers come from sampling with thousands of interviews and are commonly clustered in a series of groups according to several measures (variables) which represent preferences. Opinions in the same cluster have similar behavior in terms of buying. Advertising campaigns search for the characteristics that are closest to all the groups. In doing that, the population in each group would be as close as possible to their preferences and this will rise the profit by selling. In this model clusters can be represented by the convex hull of their elements and the problem reduces to search for the location of the point minimizing a weighted distance to the clusters. Finally, the location model with infimal distances is also directly applicable in the location of a dam and distribution sub-stations of any liquid (water, gas, ...). The common elements in all these models are: 1) a facility must be located; 2) existing facilities have area and; 3) the closest points from the existing facilities to the new one are important (to minimize transportation cost or exposition to risk). It is worth noting that there are also economic reasons to consider points in the boundaries of the existing facilities: 1) the ground is cheaper and then the construction cost is smaller; and 2) it might have restrictions to get licenses to deliver inside the area of the existing facilities without having a representative in it.

The aim of this paper is to present a geometrical characterization of the set of optimal solutions of the general single facility location problem with infimal distances. To this end, we will use mainly convex analysis tools. We also address the very important cases of the Weber and minimax problem which are studied in detail. For the very particular case of $I\!R^2$ with polyhedral norms a constructive approach is developed.

The rest of the paper is organized as follows. First we introduce some basic tools and definitions which will be used throughout the paper. The next section gives a complete geometrical characterization of the set of optimal solutions. Then the relationship to some classical location problems is discussed. The next section is devoted to the interpretations in the planar case. The paper ends with some conclusions and extensions.

2 Basic tools and definitions

As we mentioned in the introduction everything takes place in a general vector space X equipped with several norms. The reason to consider this general framework is because it is the most general environment where one can formulate the problem and not more complicated tools have to be used to get the desired results. Let us denote by X_i^* the topological dual of X equipped with the norm $\|\cdot\|_i$ and by $\|\cdot\|_i^o$ its dual norm. The unit ball in X with the norm $\|\cdot\|_i$ (respectively X_i^*) is denoted by B_i (respectively B_i^o). The pairing between X and X^* will be indicated by $\langle\cdot,\cdot\rangle$. Nevertheless, for the ease of understanding the reader may replace the space X by $\mathbb{R}^n$ and everything will be more common. In this case the elements of the topological dual can be identified with itself and the pairing is the usual scalar product.

First, we restate some definitions which are needed throughout the paper. Let $B_i \subset X$ be a compact, convex set containing the origin in its interior, for $i \in \mathcal{M} := \{1, 2, \ldots, M\}$. The norm with respect to B_i is defined as

$$\gamma_i \;:\; X \;\to\; \mathbb{R} \quad , \quad \gamma_i(x) := \inf\{\, r > 0 \;:\; x \in rB_i \,\} \tag{1}$$

the polar set B_i° of B_i is given by

$$B_i^o := \{\, p \in \; X^* \;:\; \langle p, x\rangle \le 1 \quad \forall\; x \in B_i \,\} \tag{2}$$

the normal cone to B_i at x is given by

$$N_{B_i}(x) := \{p \in X^* \;:\; \langle p, y - x\rangle \le 0 \quad \forall\; y \in B_i\}. \tag{3}$$

The case where each γ_i with $i \in \mathcal{M}$ is a polyhedral norm in a finite dimensional space, which means B_i is a convex polytope with extreme points $Ext(B_i) := \{e_1^i, \ldots, e_{G_i}^i\}$ is studied in Section 5. In this case we define fundamental directions $d_1^i, \ldots, d_{G_i}^i$ as the halflines defined by 0 and $e_1^i, \ldots, e_{G_i}^i$.

The distance from a point x to a set A_i with the norm γ_i is defined as

$$d_i(x, A_i) = \inf\{\gamma_i(x - a_i) \;:\; a_i \in A_i\}$$

and the set of points $\mathrm{proj}_{A_i}(x) := \{a_i \in A_i \;:\; d_i(x, A_i) = \gamma_i(x - a_i)\}$ is called projection of x onto A_i with the norm γ_i. Notice that this set is not necessarily a singleton, and can even be empty if A_i is not closed or not compact.

Let f be a convex function $f : X \to \mathbb{R}$. A vector p is said to be a subgradient of f at a point x if

$$f(y) \ge f(x) + \langle p, y - x\rangle$$

for each y. The set of all subgradients of f at x is called the subdifferential of f at x and is denoted by $\partial f(x)$.

Given a closed set A_i we denote by $I_{A_i}(\cdot)$ its indicator function, that is,

$$I_{A_i}(x) = \begin{cases} 0 & \text{if } x \in A_i \\ +\infty & \text{otherwise} \end{cases}$$

and we denote by $\sigma_{A_i}(\cdot)$ the support function of the set A_i, i.e.

$$\sigma_{A_i}(p) = \sup_{x \in A_i} \langle p, x\rangle \quad \text{for any } p \in X^*.$$

Now, using [HUL93], we know that

$$\partial\gamma_i(x) = \begin{cases} B_i^o & \text{if } x = 0 \\ \{p_i \in B_i^o \ : \ \langle p_i, x\rangle = \gamma_i(x)\} & \text{if } x \neq 0 \end{cases} \tag{4}$$

$$\partial I_{A_i}(x) = N_{A_i}(x) \quad \forall x \in A_i \tag{5}$$

$$\partial\sigma_A(u) = \{a \in A \ : \ \langle u, a\rangle = \sup_{z\in A}\langle u, z\rangle\}. \tag{6}$$

Let f_1 and f_2 be two functions from X to $I\!R \cup \{+\infty\}$. Their infimal convolution is the function from X to $I\!R \cup \{+\infty\}$ defined by

$$\begin{aligned}(f_1 * f_2)(x) &:= \inf\{f_1(x_1) + f_2(x_2) \ : \ x_1 + x_2 = x\} \\ &= \inf_{y\in X}\{f_1(y) + f_2(x - y)\}.\end{aligned}$$

Another important concept that we need to recall is the concept of conjugate functions. Let f be a function from X to $I\!R \cup \{+\infty\}$ not identically equal to $+\infty$ and minorized by some affine function. The conjugate f^* of f is the function defined by

$$f^*(p) = \sup\{\langle p, x\rangle - f(x) : x \in \text{ dom } f\} \quad \text{for any } p \in X^*.$$

It is a well-known result from convex analysis that:

$$I^*_{A_i}(p) = \sigma_{A_i}(p) \text{ for any } p \in X^*. \tag{7}$$

Finally, we will denote by $ri(A)$ the relative interior of the set A, by $bd(A)$ the boundary of A, by $co(A)$ the convex hull of A and by $cone(A)$ the convex cone generated by the elements of the set A.

3 Geometrical characterization of optimal solutions

Let $\mathcal{A} = \{A_1, \ldots, A_M\}$ be a family of sets in X where each A_i, $i \in \mathcal{M}$ is a compact closed convex set. Let $\Phi(\cdot)$ be a monotone norm in $I\!R^M$. Recall that a norm Φ is said to be monotone on $I\!R^M$ if $\Phi(u) \leq \Phi(v)$ for every u, v verifying $|u_i| \leq |v_i|$ for each $i = 1, \ldots, M$ (see [BSW61]). We consider the following minimization problem

$$\inf_{x\in X} F(x) := \Phi(d_1(x, A_1), \ldots, d_M(x, A_M)). \tag{$P_\Phi(\mathcal{A})$}$$

Assume without loss of generality that there exist $i,\ j$ such that $A_i \cap A_j = \emptyset$. Indeed, if $\cap_{i=1}^M A_i \neq \emptyset$ then the solution set would be $\cap_{i=1}^M A_i \neq \emptyset$ with objective value of zero.

In what follows, we look for a sufficient condition which ensures the existence of optimal solutions of Problem $(P_\Phi(\mathcal{A}))$. In order to develop such a condition we will prove a previous lemma.

First of all, it is straightforward to see that the function $F = \Phi \circ d$ is convex on $I\!R^M$ provided that Φ is monotone (see Prop. 2.1.8 of Chapter IV in [HUL93]). Our first result states a sufficient condition which ensures that the set of optimal solutions of Problem $(P_\Phi(\mathcal{A}))$ is not empty. Thus, it is possible to replace the *inf* symbol by *min*.

To this end, we embed the optimization problem $(P_\Phi(\mathcal{A}))$ in a larger space in order to study existence properties of its optimal solution. Let us consider the normed space $(Y, |||\cdot|||)$ where $Y = X^M$ and for any $y \in Y$ $|||y||| = \Phi(\gamma_1(y_1), \ldots, \gamma_M(y_M))$.

Lemma 3.1 *If the diagonal set $D = \{y \in Y : y_1 = y_2 = \ldots = y_M\}$ is closed in Y the optimal solution set of Problem $(P_\Phi(\mathcal{A}))$ is non empty.*

Proof:
Since the sets A_i are compact for all $i \in \mathcal{M}$ then $m_0 = F(0) < +\infty$. Let us define the set $M_0 = \{y \in Y : F(y) \leq m_0\}$. The set M_0 is convex and closed since F is a continuous, convex function. Besides, M_0 is a bounded set. Indeed, assume that there exists $\{y_n\}_{n\in I\!N} \subset M_0$ such that $|||y^n||| \to \infty$. Since $|||y^n||| = \Phi(\gamma_1(y_1^n), \ldots, \gamma_M(y_M^n))$ and Φ is a monotone norm in $I\!R^M$, it must exist at least one i such that $\gamma_i(y_i^n) \to \infty$.

On the other hand, for any $a \in A_i$ $\gamma_i(y_i^n - a) \geq \gamma_i(y_i^n) - \gamma_i(a) \geq \gamma_i(y_i^n) - \max_{a\in A_i} \gamma_i(a) \overset{n\to\infty}{\longrightarrow} \infty$. Hence, since Φ is a monotone norm in $I\!R^M$ then $F(y^n) = \Phi(\gamma_1(y_1^n - a_1), \ldots, \gamma_M(y_M^n - a_M)) \to \infty$ which contradicts the definition of M_0. Thus M_0 is bounded and it must exist $K > 0$ such that $|||y||| \leq K$ for any $y \in M_0$. Therefore, the problem to be solved is:

$$\inf\{F(y) : y \in M_0 \cap D\},$$

being $D = \{y \in Y : y_1 = y_2 = \ldots = y_M\}$. By hypothesis, D is closed, and thus $M_0 \cap D$ is a non-empty, bounded, closed, convex set. Now, by Proposition 38.12 in Ekeland and Temar [ET76] the problem has an optimal solution and we can replace the inf symbol by the min one. □

Remark 3.1 Sufficient conditions which ensure that L is closed are for instance that X is a finite dimension space or that the topology induced by γ_i for some i is finer than the remainders. It is worth noting that no additional assumptions on Φ nor the shape of the demand sets are needed to ensure existence of optimal solutions. In the rest of the paper we will assume that the optimal solution exists, which is, for example, the case if the assumptions of Lemma 3.1 are fulfilled. However, it might be that we know that the optimal solution exists even if Lemma 3.1 is not applicable.

Recall that for an unconstrained minimization problem with objective function f convex, x is an optimal solution if and only if $0 \in \partial f(x)$. Because we look for the characterization of the solution set of $(P_\Phi(\mathcal{A}))$ and the objective function F is a convex function, we are interested in obtaining a precise description of its subdifferential set.

Our main objective in this section will be to characterize the set of optimal solutions of $(P_\Phi(\mathcal{A}))$. In order to do that we are going to study the subdifferential of the objective function.

First of all, we have that

$$d_i(x, A_i) = \inf_{a_i \in A_i} \gamma_i(x - a_i) = (I_{A_i} * \gamma_i)(x),$$

then by Corollary VI.4.5.5 in [HUL93], we obtain the following representation of the subdifferential of $d_i(\cdot, A_i)$

$$\partial d_i(x, A_i) = \partial I_{A_i}(a_i) \cap \partial \gamma_i(x - a_i) \quad \text{for any } a_i \in \mathrm{proj}_{A_i}(x).$$

Notice that when $x \in A_i$ $\mathrm{proj}_{A_i}(x) = \{x\}$ and then $\partial \gamma_i(0) = B_i^o$.

Thus using (4) and (5) we obtain that

$$\partial d_i(x, A_i) = N_{A_i}(a_i) \cap \{p_i \in B_i^o : \langle p_i, x-a_i\rangle = \gamma_i(x-a_i)\} \quad \text{for any } a_i \in \mathrm{proj}_{A_i}(x). \quad (8)$$

Remark 3.2 It is also possible to obtain $\partial d_i(x, A_i)$ in a different way using the concept of level sets. The level set $L_i^{\leq}(r)$ of the function d_i with value $r > 0$ is

$$L_i^{\leq}(r) = \{x \in X : d_i(x, A_i) \leq r\}.$$

Notice that we can write $L_i^{\leq}(r) = A_i + rB_i$. Then, for any $y = a_i + rz$ with $a_i \in A_i$ and $z \in B_i$ we have,

$$N_{L_i^{\leq}(r)}(y) = N_{A_i}(a_i) \cap N_{B_i}(z).$$

Since, it holds that $N_{L_i^{\leq}(r)}(y) = cone(\partial d_i(y, A_i))$ then, one easily obtains that

$$\partial d_i(y, A_i) = N_{A_i}(a_i) \cap \{p \in B_i^o : \langle p, z\rangle = \gamma_i(z)\}.$$

Remark 3.3

1. If A_i is a strictly convex set then $\mathrm{proj}_{A_i}(x) = \{a_i\}$ and
$$\partial d_i(x, A_i) = N_{A_i}(a_i) \cap \{p_i \in B_i^o \; : \; \langle p_i, x - a_i\rangle = \gamma_i(x - a_i)\}.$$
2. If A_i is not a strictly convex set the set $\mathrm{proj}_{A_i}(x)$ is not necessarily a singleton. Nevertheless, we can obtain the subdifferential $\partial d_i(x, A_i)$ without any problem. This is due to the fact that its expression does not depend on the particular choice of the minimal element on the projection (see Corollary VI.4.5.5. and Theorem VI.4.5.1. in [HUL93]).

Our next result characterizes the subdifferential of the objective function of the $(P_\Phi(\mathcal{A}))$.

Lemma 3.2 *Let $x \in X$, $x^* \in \partial F(x)$ iff there exist $a_i \in \mathrm{proj}_{A_i}(x)$, $p_i \in N_{A_i}(a_i) \cap B_i^o \quad \forall i \in \mathcal{M}$ and $\lambda = (\lambda_1, \ldots, \lambda_M) \geq 0$ such that*

1. $x \in \bigcap_{i=1}^{M} a_i + N_{B_i^o}(p_i),$
2. $\Phi^o(\lambda) = 1$ *and* $\sum_{i=1}^{M} \lambda_i d_i(x, A_i) = F(x),$
3. $x^* = \sum_{i=1}^{M} \lambda_i p_i.$

Proof:

First of all, we consider $t, s \in I\!R^M_+$ such that $t - s \in I\!R^M_+$ and $\lambda \in \partial\Phi(t)$, then by the monotonicity of Φ and the subgradient inequality we have that

$$0 \leq \Phi(t) - \Phi(s) \leq \langle \lambda, t - s \rangle.$$

Since this inequality holds for all $t \in R^M_+$ such that $t - s \in I\!R^M_+$,this implies that $\lambda \geq 0$ (see [PF95]).

Hence, defining the function $\Phi^+(t) := \Phi(t^+)$ where $t^+ = (t_1^+, \ldots, t_M^+)$ with $t_i^+ = \max\{0, t_i\}$ for $i = 1, \ldots, M$ and using that Φ is a norm, we have that whenever $t \neq 0$

$$\partial\Phi^+(t) = \left\{ (\lambda_1, \ldots, \lambda_M) \in I\!R^M_+ \; : \; \Phi^o(\lambda) = 1; \; \sum_{i=1}^M \lambda_i t_i^+ = \Phi(t^+) \right\}.$$

On the other hand, since $d(x) \gneqq 0$ for any $x \in X$ and $d(x) = d^+(x)$ then by Theorem VI.4.3.1 in [HUL93], we know the subdifferential of the composition of nondecreasing convex functions with convex ones, is given by

$$\partial F(x) = \partial\Phi^+(d(x)) = \left\{ \sum_{i=1}^M \lambda_i p_i \; : \; (\lambda_1, \ldots, \lambda_M) \in \partial\Phi^+(d(x)), \; p_i \in \partial d_i(x, A_i) \right\}$$

where $d(x) = (d_1(x, A_1), \ldots, d_M(x, A_M))$. Therefore, we have that λ and p verify

1. $\lambda = (\lambda_1, \ldots, \lambda_M), \; \lambda_i \geq 0, \; \Phi^o(\lambda) = 1, \; \displaystyle\sum_{i=1}^M \lambda_i d_i(x, A_i) = F(x).$
2. $p_i \in N_{A_i}(a_i) \cap \{p \in B_i^o : \langle p, x - a_i \rangle = \gamma_i(x - a_i)\}$ where $\quad a_i \in \mathrm{proj}_{A_i}(x) \quad \forall i \in \mathcal{M}.$

Finally, using the well-known equivalence between

$$\hat{p} \in \{p \in B^o \; : \; \langle p, x - a \rangle = \gamma(x - a)\} \quad \text{iff} \quad x \in a + N_{B^o}(\hat{p})$$

where B^o is the polar set of B the unit ball of γ, we obtain the characterization of this theorem. $\square$

In order to obtain a characterization of the set of optimal solutions of the Problem $(P_\Phi(\mathcal{A}))$ we need to introduce some additional concepts.

Definition 3.1 *Given $p = (p_1, \ldots, p_M)$ with $p_i \in B_i^o$ and $I \subseteq \mathcal{M}$ let*

$$C_I(p) := \bigcap_{i \in I} \partial d_i^*(p_i),$$

where d_i^* is the conjugate function of $d_i(x, A_i)$, and for any $\lambda = (\lambda_1, \ldots, \lambda_M) \geq 0$ let

$$D_I(\lambda) := \{x \; : \; \sum_{i \in I} \lambda_i d_i(x, A_i) = F(x)\}.$$

The sets $C_I(p)$ were previously used in Durier and Michelot (1985) (see Lemma 3.1) for characterizing optimal solution sets of optimization problems with objective function given by sum of convex functions. They call these sets elementary convex sets when the convex functions are norms. For this reason and since we consider distances to sets rather than norms to points, we will call the sets $C_I(p)$ generalized elementary convex sets (g.e.c.s.).

Remark 3.4 It should also be noted that $d_i(x, A_i) = (I_{A_i} * \gamma_i)(x)$. Therefore, by Corollary X.2.1.3 in [HUL93], $d_i^* = I_{A_i}^* + \gamma_i^*$. But by (7) $I_{A_i}^*$ is the support function of A_i, i.e. $I_{A_i}^* = \sigma_{A_i}$, and the conjugate of the norm γ_i is the indicator function of its unit dual ball, i.e. $\gamma_i^* = I_{B_i^o}$. Hence,

$$\partial d_i^*(p_i) = \partial(I_{A_i}^* + \gamma_i^*)(p_i) = \partial I_{A_i}^*(p_i) + \partial\gamma_i^*(p_i) = \partial\sigma_{A_i}(p_i) + N_{B_i^o}(p_i).$$

Different generalizations of these sets can be found in the literature, see for instance Puerto and Fernández [PF95] and Muriel and Carrizosa [MC95].

First to all, it is straightforward to see that the g.e.c.s. are convex. Indeed, they are defined by a finite intersection of convex sets (recall that subdifferential sets are convex).

Another interesting property of this family of sets is that the function $d_i(x, A_i)$ is linear within $\partial d_i^*(p_i)$. This result is proved in the next lemma.

Lemma 3.3 *For each $p_i \in B_i^o$ $d_i(x, A_i)$ is a linear function within $\partial d_i^*(p_i)$.*

Proof:

By Fenchel identity we have

$$x \in \partial d_i^*(p_i) \text{ iff } p_i \in \partial d_i(x, A_i).$$

Thus, applying (8), for any $x \in \partial d_i^*(p_i)$ we get

$$d_i(x, A_i) = \langle p_i, x - a_i\rangle \text{ for any } a_i \in \text{ proj}_{A_i}(x).$$

This concludes the proof. $\square$

In order to give more insights into the geometry of our g.e.c.s. we need to impose additional hypothesis to the space X. Let us assume that X fulfills the Krein-Milman property.

A first consequence of Lemma 3.3 is that there always exists an optimal solution of $(P_\Phi(\mathcal{A}))$ in the set of extreme points of the g.e.c.s.. Notice that we use in this result that these convex sets are given by the convex hull of their extreme points. This property extends the intersection point result obtained in $\mathbb{R}^2$ by Wendell and Hurter [WH73] for the l_1-norm, by Thisse, Ward and Wendell [TWW84] for the polyhedral norm case and by Durier and Michelot for the Fermat-Weber problem with linear cost.

On the second hand, we give a geometrical description of g.e.c.s. that will be used in Section 5. Let us denote by $\mathcal{Y}_i$ the set of all the faces of any dimension of the set A_i with $i \in \mathcal{M}$. That is to say, $\mathcal{Y}_i$ contains faces of any dimension and extreme points. Recall that Y_i is a exposed face of A_i if $Y_i = H_i \cap A_i$ for some supporting hyperplane H_i to A_i.

Definition 3.2 *Given a family of sets $\mathcal{Y} = \{Y_1, Y_2, \ldots, Y_M\}$ where each $Y_i \in \mathcal{Y}_i$, $p = (p_1, \ldots, p_M)$ with $p_i \in B_i^o \cap N_{A_i}(y_i)$ for any $y_i \in Y_i$ and $I \subseteq \mathcal{M}$ let*

$$C(Y_i, p_i) := \{x \;:\; \mathrm{proj}_{A_i}(x) \subseteq Y_i\,;\ \textit{and exists}\ \ a_i \in \mathrm{proj}_{A_i}(x);\ \langle p_i, x - a_i\rangle = d_i(x, A_i)\},$$

$$C_I(\mathcal{Y}, p) := \bigcap_{i \in I} C(Y_i, p_i).$$

Remark 3.5 We use in the definition of the set $C(Y_i, p_i)$ the existence of a particular $a_i \in \mathrm{proj}_{A_i}(x)$. Nevertheless, the definition does not depend on this a_i, because by the convexity of A_i if $\hat{y}_i \in ri(Y_i)$ and $p_i \in N_{A_i}(\hat{y}_i)$ then $p_i \in N_{A_i}(y_i)$ for any $y_i \in ri(Y_i)$ (notice that $N_{A_i}(y_i)$ is constant in $ri(Y_i)$). Therefore, we have $\langle p_i, a - a_i\rangle \leq 0 \quad \forall a \in A_i$. In particular, for all $a \in \mathrm{proj}_{A_i}(x)$ we obtain that $\langle p_i, x - a\rangle \geq \langle p_i, x - a_i\rangle$ and that means that $d_i(x, A_i) = \langle p_i, x - a\rangle$ for all $a \in \mathrm{proj}_{A_i}(x)$.

We can describe our g.e.c.s. in an alternative way using these sets. The following theorem shows this characterization.

Theorem 3.1 *Let A be a closed, compact convex set, $\mathcal{Y}$ denote the set of all its faces and let $\gamma(\cdot)$ be a norm with unit ball B. For any $p \in B^o$ there exists $Y \in \mathcal{Y}$ such that $p \in N_A(y)$ for any $y \in Y$ and $N_{B^o}(p) + \partial\sigma_A(p)$=C(Y,p).*
Conversely, for any $Y \in \mathcal{Y}$ such that $p \in B^o \cap N_A(y)$ for any $y \in Y$ then $C(Y, p) = N_{B^o}(p) + \partial\sigma_A(p)$.

Proof:

Let $x \in N_{B^o}(p) + \partial\sigma_A(p)$. Then there exists $q \in N_{B^o}(p)$ and $a(x) \in \partial\sigma_A(p)$ such that $x = a(x) + q$. Since $q \in N_{B^o}(p)$ $\langle v, q\rangle \leq \langle p, q\rangle$ $\forall v \in B^o$. Therefore, $\gamma(q) = \gamma(x - a(x)) = \langle p, x - a(x)\rangle$. Using that $\partial\sigma_A(p) = \{y \in A : \langle p, y\rangle = \sup_{z \in A}\langle p, z\rangle\}$ we have that $\langle p, a(x)\rangle = \sup_{a \in A}\langle p, a\rangle$. Thus,

$$\gamma(x - a) = \sup_{v \in B^o}\langle v, x - a\rangle \geq \langle p, x - a\rangle \geq \langle p, x - a(x)\rangle = \gamma(x - a(x))\ \forall a \in A.$$

Hence, $d(x, A) = \gamma(x - a(x))$. Now, it suffices to consider $Y = \{a \in A : \langle p, a\rangle = \sigma_A(p)\}$ and we have $N_{B^o}(p) + \partial\sigma_A(p) \subseteq C(Y, p)$.

Conversely, $x \in C(Y, p)$ if and only if there exists $a(x) \in Y$ such that $d(x, A) = \gamma(x - a(x)) = \langle p, x - a(x)\rangle$. But, $\gamma(x - a(x)) = \sup_{v \in B^o}\langle v, x - a(x)\rangle$. Therefore, $\langle v - p, x - a(x)\rangle \leq 0 \quad \forall v \in B^o$. That is to say, $q := x - a(x) \in N_{B^o}(p)$. Hence, $x = a(x) + q$ with $a(x) \in Y$ and $q \in N_{B^o}(p)$. In addition, $p \in N_A(y)$ for any $y \in Y$ then $\langle p, a(x)\rangle \geq \langle p, a\rangle$ $\forall a \in A$, that is, $\langle p, a(x)\rangle = \sup_{a \in A}\langle p, a\rangle$. That means that $a(x) \in \partial\sigma_A(p)$ and also implies that $Y = \{a \in A : \langle p, a\rangle = \sigma_A(p)\}$ which concludes the proof. □

Example 3.1 *(See Figure 1) Consider $\mathbb{R}^2$ with the rectilinear l_1-norm and a set $A_1 := co\{(1,1), (1,-1), (-1,-1), (-1,1)\}$. Let $Y_{11} := co\{(1,1), (1,-1)\}$ and $p_1 = (1,0)$ then*

$$C(Y_{11}, p_1) = \{x \in \mathbb{R}^2 \;:\; x_1 \geq 1,\ 1 \leq x_2 \leq -1\}.$$

For $Y_{12} = \{(-1,-1)\}$, $p_2 = (-1,-1)$, we have

$$C(Y_{12}, p_2) = \{x \in \mathbb{R}^2 \;:\; x_1 \leq -1,\ x_2 \leq -1\}.$$

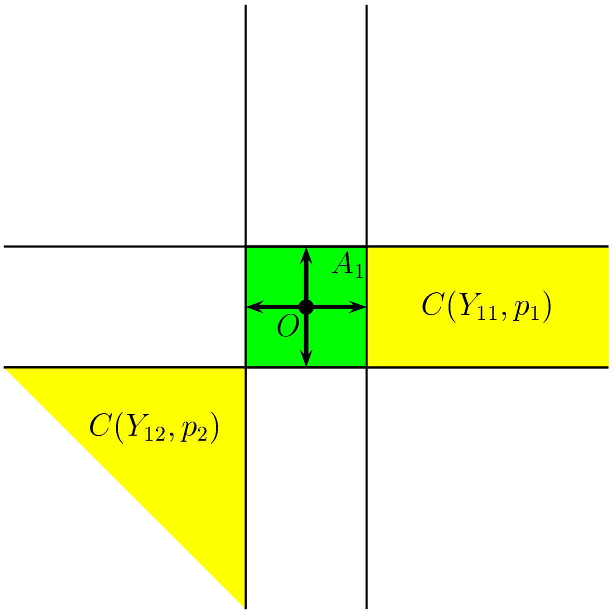

Figure 1: Illustration of Definition 3.2

It should be noted that if the unit balls are polytopes we can obtain the generalized elementary convex sets (g.e.c.s.) as intersection of cones generated by fundamental directions of these balls pointed on the faces or vertices of each demand set (see Section 5 for details on the construction of g.e.c.s.).

Let $M_\Phi(\mathcal{A})$ be the set of optimal solutions of $(P_\Phi(\mathcal{A}))$.

Lemma 3.4 *$x \in M_\Phi(\mathcal{A})$ iff there exist: 1) a non empty index set $I \subseteq \mathcal{M}$; 2) $\lambda = (\lambda_1, \ldots, \lambda_M)$ with $\lambda_i > 0$ $i \in I$, and $\lambda_i = 0$ $i \notin I$ satisfying $\Phi^o(\lambda) = 1$; and, 3) $p_i \in B_i^o$ $i \in I$, with $\sum_{i \in I} \lambda_i p_i = 0$ such that $p_i \in N_{A_i}(y_i) \cap B_i^o$ for any $y_i \in \partial\sigma_{A_i}(p_i)$ satisfying*

$$x \in C_I(p) \cap D_I(\lambda).$$

Proof:

$x \in M_\Phi(\mathcal{A})$ iff $0 \in \partial F(x)$. Therefore applying Lemma 3.2 and the definitions of $C_I(p)$ and $D_I(\lambda)$ the thesis of the theorem follows.

□

It should be noted that Lemma 3.4 proves

$$C_I(p) \cap D_I(\lambda) \subseteq M_\Phi(\mathcal{A})$$

for any choice of p and λ verifying the hypotheses. Therefore, we have now to prove that there exists a particular choice of these sets such that the inclusion becomes an identity and therefore that both sets are equal.

Theorem 3.2

1. *If $M_\Phi(\mathcal{A}) \neq \emptyset$, then there exist: 1) a non empty index set $I \subseteq \mathcal{M}$; 2) $\lambda = (\lambda_1, \dots, \lambda_M)$ $\lambda_i > 0$ $i \in I$, $\lambda_i = 0$ $i \notin I$ satisfying $\Phi^o(\lambda) = 1$, and; 3) $p = (p_i)_{i \in I}$ such that $p_i \in B_i^o \cap N_{A_i}(y_i)$ for any $y_i \in \partial\sigma_{A_i}(p_i)$ $\forall i \in I$ with $\sum_{i=1}^M \lambda_i p_i = 0$ satisfying*
$$M_\Phi(\mathcal{A}) = C_I(p) \cap D_I(\lambda).$$

2. *If there exist 1) a non empty set of indexes $I \subseteq \mathcal{M}$; 2) $\lambda = (\lambda_1, \dots, \lambda_M)$ $\lambda_i > 0$ $i \in I$, $\lambda_i = 0$ $i \notin I$ satisfying $\Phi^o(\lambda) = 1$; and; 3) $p = (p_i)_{i \in I}$ $p_i \in B_i^o \cap N_{A_i}(y_i)$ for any $y_i \in \partial\sigma_{A_i}(p_i)$ $\forall i \in I$ such that $\sum_{i \in I} \lambda_i p_i = 0$ and $C_I(p) \cap D_I(\lambda) \neq \emptyset$ then*
$$M_\Phi(\mathcal{A}) = C_I(p) \cap D_I(\lambda).$$

Proof:
First of all, if (I, p, λ) exists satisfying the conditions of the theorem, by Lemma 3.4 we have that
$$C_I(p) \cap D_I(\lambda) \subseteq M_\Phi(\mathcal{A}).$$
Conversely, let $x \in M_\Phi(\mathcal{A})$ then by Lemma 3.4 (I, λ, p) exists such that the conditions of this theorem are fulfilled. In addition, there exists $a_i(x) \in \mathrm{proj}_{A_i}(x)$ such that
$$F^* := F(x) = \sum_{i=1}^M \lambda_i \langle p_i, x - a_i(x) \rangle = -\sum_{i=1}^M \lambda_i \langle p_i, a_i(x) \rangle.$$

On the other hand, since F^* is minimum then $\langle p_i, a_i(x) \rangle = sup_{a_i \in A_i} \langle p_i, a_i \rangle$ and $a_i(x) \in \partial\sigma_{A_i}(p_i)$.

Let $\bar{x} \neq x$, we have
$$\begin{aligned} F^* = -\textstyle\sum_{i=1}^M \lambda_i \langle p_i, a_i(x) \rangle &\le -\textstyle\sum_{i=1}^M \lambda_i \langle p_i, a_i(\bar{x}) \rangle \quad \forall a_i(\bar{x}) \in \mathrm{proj}_{A_i}(\bar{x}) \\ &= \textstyle\sum_{i=1}^M \lambda_i \langle p_i, \bar{x} - a_i(\bar{x}) \rangle \quad \forall a_i(\bar{x}) \in \mathrm{proj}_{A_i}(\bar{x}). \end{aligned}$$

Since $d_i(\bar{x}, A_i) = \sup\limits_{p_i \in B_i^o} \langle p_i, \bar{x} - a_i(\bar{x}) \rangle = \gamma_i(\bar{x} - a_i(\bar{x}))$ using that $\Phi(\cdot)$ is a norm and $\Phi^o(\lambda) = 1$, we obtain
$$F^* \le \sum_{i=1}^M \lambda_i \langle p_i, \bar{x} - a_i(\bar{x}) \rangle \le \sum_{i=1}^M \lambda_i d_i(\bar{x}, A_i) \le F(\bar{x}). \tag{9}$$

Hence, if $\bar{x} \in M_\Phi(\mathcal{A})$ all these inequalities are equalities, that is,
$$\sum_{i=1}^M \lambda_i \langle p_i, \bar{x} - a_i(\bar{x}) \rangle = \sum_{i=1}^M \lambda_i d_i(\bar{x}, A_i) \quad \forall a_i(\bar{x}) \in \mathrm{proj}_{A_i}(\bar{x})$$

and
$$\sum_{i=1}^M \lambda_i \langle p_i, a_i(x) \rangle = \sum_{i=1}^M \lambda_i \langle p_i, a_i(\bar{x}) \rangle.$$

This together with the relationships existing between each term leads us to deduce that for all $i \in I$ it holds: i)$\langle p_i, \bar{x} - a_i(\bar{x})\rangle = d_i(\bar{x}, A_i)$ and; ii) $\langle p_i, a_i(\bar{x})\rangle = \langle p_i, a_i(x)\rangle$. From the condition i), we obtain:

$$d_i(\bar{x}, A_i) = \gamma_i(\bar{x} - a_i(\bar{x})) = \langle p_i, \bar{x} - a_i(\bar{x})\rangle \quad \text{for any } i \in I.$$

Therefore, $p_i \in \partial\gamma_i(\bar{x} - a_i(\bar{x}))$ which is equivalent to $\bar{x} - a_i(\bar{x}) \in \partial\gamma_i^*(p_i)$ for any $i \in I$.

From the condition ii), and since $a_i(x) \in \partial\sigma_{A_i}(p_i)$ for any $i \in I$ we deduce that $a_i(\bar{x}) \in \partial\sigma_{A_i}(p_i)$ for any $i \in I$. Hence,

$$\bar{x} \in a_i(\bar{x}) + \partial\gamma_i^*(p_i) \subset \partial\sigma_{A_i}(p_i) + \partial\gamma_i^*(p_i) := C_i(p_i) \quad \text{for any } i \in I,$$

and we get that $\bar{x} \in C_I(p)$.

Moreover, when $\bar{x} \in M_\Phi(\mathcal{A})$ we have using the last inequality in (9) $F(\bar{x}) = \sum_{i\in I} \lambda_i d_i(\bar{x}, A_i)$ then $\bar{x} \in D_I(\lambda)$. Hence, $\bar{x} \in \bigcap_{i\in I} C_i(p_i) \cap D_I(\lambda)$ and the proof is complete. □

The last part of this section is devoted to prove some properties of the optimal solution set $M_\Phi(\mathcal{A})$ of $P_\Phi(\mathcal{A})$. The first property states the relationship between the problem $P_\Phi(\mathcal{A})$ and a specific Weber problem.

Let us denote by $F_\Omega^*(A)$ and $M_\Omega(A)$,respectively, the optimal value and the set of optimal solutions of the following Weber problem

$$F_\Omega^*(A) = \min_{x\in X} \sum_{i=1}^{M} \omega_i \gamma_i(x - a_i) \qquad (P_\Omega(A))$$

where $A = \{a_1, \ldots, a_M\}$ and $\Omega = \{\omega_1, \ldots, \omega_M\}$. Finally, let F^* denote the optimal value of Problem $P_\Phi(\mathcal{A})$.

Theorem 3.3 *For each Φ monotone norm such that $M_\Phi(\mathcal{A}) \neq \emptyset$ it holds:*

1. *There exists a set of nonnegative weights $\Omega = \{\omega_1, \ldots, \omega_M\}$ and a set of points $A = \{a_1, \ldots, a_M\}$ with $a_i \in A_i$ $i \in \mathcal{M}$ such that*

$$M_\Omega(A) \cap M_\Phi(\mathcal{A}) \neq \emptyset \text{ and } F^* = F_\Omega^*(A).$$

2. *If $\Omega = \{\omega_1, \ldots, \omega_M\}$ is given such that $D(\Omega) = \{x \in X : \sum_{i=1}^{M} \omega_i d_i(x, A_i) = F^*\} \neq \emptyset$ then $P_\Omega(A)$ and $P_\Phi(\mathcal{A})$ share optimal solutions.*

Proof:

Let $x^* \in M_\Phi(\mathcal{A})$ then there exists (I, p, λ) satisfying the hypotheses of Theorem 3.2 such that $x^* \in C_I(p) \cap D_I(\lambda)$. In particular, $x^* \in C_I(p)$ what implies that $x^* \in \bigcap_{i\in I} C_i(p_i)$. Therefore, for each $i \in I$ there exists $a_i \in \text{proj}_{A_i}(x^*) \subseteq \partial\sigma_{A_i}(p_i)$ and $p_i \in B_i^o \cap N_{A_i}(y_i)$ for any $y_i \in \partial\sigma_{A_i}(p_i)$ $\forall i \in I$ such that

$$x^* \in a_i + N_{B_i^o}(p_i) \quad \forall i \in I \text{ and } \sum_{i\in I} \lambda_i p_i = 0.$$

In addition, since $x^* \in D_I(\lambda)$

$$F^* = \Phi(d(x^*)) = \sum_{i\in I} \lambda_i d_i(x^*, A_i) = \sum_{i\in I} \lambda_i \gamma_i(x^* - a_i).$$

Therefore, if we take $\Omega = \{\omega_1, \ldots, \omega_M\}$ with $\omega_i = \lambda_i \; \forall i \in I$, $\omega_i = 0 \; i \notin I$ and $A = \{a_1, \ldots, a_M\}$ then $x^* \in M_\Omega(A)$. Hence,

$$M_\Omega(A) \cap M_\Phi(\mathcal{A}) \neq \emptyset \text{ and } F^* = F^*_\Omega(A).$$

□

After that, we state a localization result on the plane which gives us a set where we always can find an optimal solution. This property is a straightforward extension of the well-known hull property of Wendell and Hurter [WH73] which also holds for expected distances (see Carrizosa et al (1995) [CCMMP95]).

Let us assume that all the norms γ_i are equal to γ.

Corollary 3.1 *In $\mathbb{R}^2$ there exists at least an optimal solution to $P_\Phi(\mathcal{A})$ belonging to $co(\bigcup_{i=1}^k A_i)$.*

If γ_i are strict norms a more precise relation can be shown.

Corollary 3.2 *If $\gamma_i(\cdot) \quad \forall i \in \mathcal{M}$ are strict norms and there exist three demand sets which can not be met by a line then Ω exist such that $M_\Omega(A) \subseteq M_\Phi(\mathcal{A})$ and $F^* = F^*_\Omega(A)$.*

Proof:

It is well-known that if $\gamma_i(\cdot)$ is a strict norm and the existing facilities are not collinear then for any set of weights the classic Weber problem has a unique optimal solution. Since under the hypotheses of this corollary any family of points $A = \{a_1, \ldots, a_k\}$ with $a_i \in A_i$ can not be collinear, Theorem 3.2 leads us to

$$M_\Omega(A) \subseteq M_\Phi(\mathcal{A}).$$

□

Remark 3.6 It is important to remark that this corollary is only a sufficient condition and that in general inclusion cannot be ensured. The following examples show that: 1) the same result can be obtained without the hypotheses of Corollary 3.2; 2) there is not a general inclusion relationship between the set of optimal solutions.

Consider a problem with three existing facility sets. Each one of them is a unit circle in $\mathbb{R}^2$ and their relative positions are given in Figure 2.

Take $\Phi = l_1$ the rectilinear norm in $\mathbb{R}^3$ and $\gamma_1 = \gamma_2 = \gamma_3 = l_2$ the Euclidean norm in $\mathbb{R}^2$. The optimal solution set $M_\Phi(\mathcal{A})$ is given by the segment drawn in thick line, that is, the diameter of A_2 on the line through the three centers. Consider now the Weber problem $P_\Omega(A)$ with existing facility set A given by any point in the diameter of the central circle and the points in each one of the external circles nearest to the

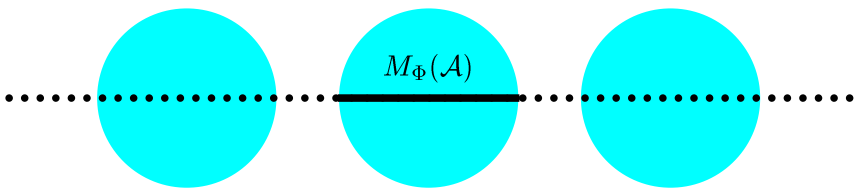

Figure 2: Illustration of Remark 3.6

central one, and weights $\omega_1 = \omega_3 = 1 \quad \omega_2 = 3$. The optimal solution set $M_\Omega(A)$ is the point of the central circle. Obviously, $M_\Omega(A) \subset M_\Phi(\mathcal{A})$ and the objective value of both problems coincide. However, the hypothesis of Corollary 3.2 does not hold. Moreover, if we would have taken weights $\omega_1 = \omega_3 = 1$ and $\omega_2 = 0$ the optimal objective value of both problem would have been the same but the solution set $M_\Omega(A)$ would be the segment joining the points in the external circles. Notice than in this case $M_\Phi(\mathcal{A}) \subset M_\Omega(A)$.

4 Relationships with two classical problems: some important examples

We consider a set $\mathcal{A} = \{A_1, \ldots, A_M\}$ where each A_i is a compact, convex set, $W = \{\omega_1, \ldots, \omega_M\}$ is a set of positive weights and $\gamma_i(\cdot)$ with $i \in \mathcal{M}$ a set of norms in X with unit ball B_i.

4.1 The Weber problem with infimal distances

The Weber problem with infimal distances for the data $\mathcal{A}$ and W is:

$$\min_{x \in X} G(x) := \sum_{i=1}^{M} \omega_i d_i(x, A_i) \tag{10}$$

Recall that $d_i(x, A_i) = \inf_{a \in A_i} \gamma_i(x - a_i)$.

Our main goal will be to characterize the set of optimal solutions $M_W(\mathcal{A})$ of (10). The following results are particular cases of Lemma 3.2 and Theorem 3.2 taking $\Phi = l_1$-norm in $I\!R^M$ and $\gamma_i' = \omega_i \gamma_i$ for all $i \in \mathcal{M}$. Therefore, the proofs are omitted here.

Lemma 4.1 *It holds that $x^* \in \partial G(x)$ for some $x \in X$ iff there exist $a_i \in \mathrm{proj}_{A_i}(x)$ $i = 1, 2, \ldots, k$, $p_i \in N_{A_i}(a_i) \cap B_i^o$ such that*

1. $x \in \bigcap_{i=1}^{M} C_i(p_i)$, *where* $C_i(p_i) = (a_i + N_{B_i^o}(p_i))$,
2. $x^* = \sum_{i=1}^{M} \omega_i p_i$.

Theorem 4.1 *1. If $M_W(\mathcal{A}) \neq \emptyset$ then there exist $(I, \lambda = W, p)$ satisfying the hypotheses of Theorem 3.2 verifying $\sum_{i\in I} \omega_i p_i = 0$ such that $M_W(\mathcal{A}) = \bigcap_{i\in I}(a_i + N_{B_i^\circ}(p_i))$*

2. If there exist $(I, \lambda = W, p)$ satisfying the hypotheses of Theorem 3.2 verifying $\sum_{i\in I} \omega_i p_i = 0$ such that

$$\bigcap_{i\in I}(a_i + N_{B_i^\circ}(p_i)) \neq \emptyset$$

then $M_W(\mathcal{A}) = \bigcap_{i\in I}(a_i + N_{B_i^\circ}(p_i))$.

Example 4.1 *(See Figure 3) Consider a 3-sets configuration with the l_1-norm in $\mathbb{R}^2$. The demand sets are $A_1 := co\{(0,1), (-1,2), (1,2)\}$, $A_2 := co\{(2,-0.5), (2,0.5), (3,0.5), (3,-0.5)\}$ and $A_3 := co\{(-2,-2), (-2,-1), (-3,-1), (-3,-2)\}$ with $\omega_1 = \omega_2 = \omega_3 = 1$.*

We see that the g.e.c.s are those sets delimited by the lines drawn in Figure 3 (these sets are characterized in $\mathbb{R}^2$ with more detail in Section 5). The optimal solution is described by $p_1 = (0,-1)$, $p_2 = (-1,0)$ and $p_3 = (1,1)$.

$$C_I\Big((p_1, p_2, p_3)\Big) = co\{(0,-0.5), (0,0.5)\}.$$

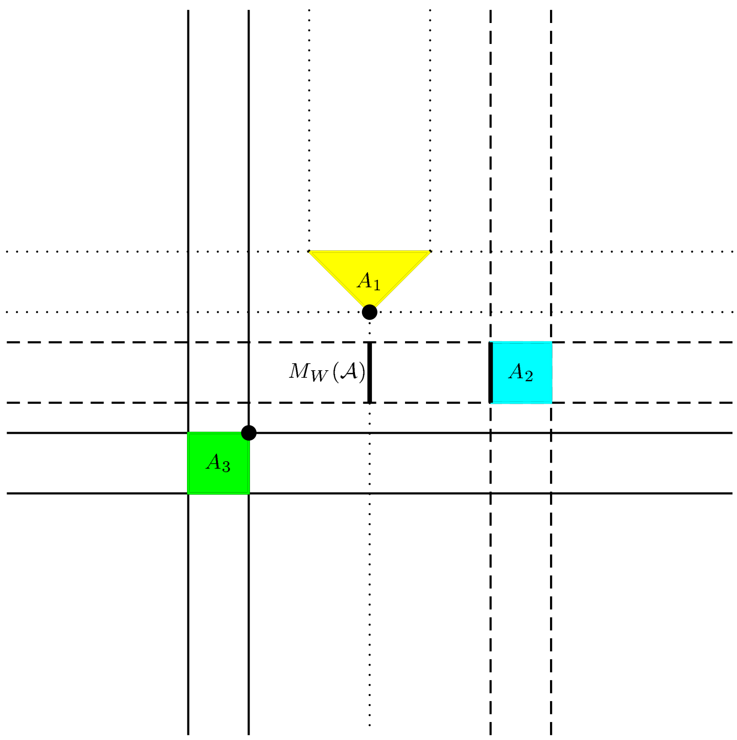

Figure 3: Illustration of Example 4.1

Corollary 4.1 *The Weber problem with infimal distances always has an optimal solution in the set of extreme points of the g.e.c.s.*

Let us assume for the last result in this section that $\gamma_i = \gamma$ for all $i = 1, \dots, M$. We can derive a majority theorem similar to the one valid for the classical case with points as existing facilities.

Corollary 4.2 *If γ is a norm in X, $M_W(\mathcal{A}) \neq \emptyset$ and there exists $A_i \in \mathcal{A}$ such that $w_i \geq \sum_{i \neq j} w_j$ then an optimal solution exists in A_i.*

Proof:
Let $x^* \in M_W(\mathcal{A})$ and assume that $x^* \notin A_i$. Let $x = proj_{A_i}(x^*)$ then we have:

$$G(x^*) = \sum_{i=1}^{M} \omega_i d(x^*, A_i) \leq G(x) = \sum_{j \neq i} \omega_j d(x, A_j) \leq w_i \gamma(x - x^*) + \sum_{j \neq i} \omega_j d(x^*, A_j) = G(x^*).$$

Hence, x is also an optimal solution. □

4.2 Minimax problem with infimal distances.

Let $\mathcal{A} = \{A_1, \dots, A_M\}$ where each A_i is a closed, compact, convex set, $W = \{\omega_1, \dots, \omega_M\}$ is a set of positive weights and $\gamma_i(\cdot)$ with $i \in \mathcal{M}$ a set of norms in X with unit ball B_i.

$$\min_{x \in X} H(x) := \max_{1 \leq i \leq M} \omega_i d_i(x, A_i) \tag{11}$$

where $d_i(x, A_i) = \inf_{a \in A_i} \gamma_i(x - a_i)$. Denote by $M_W^{l_\infty}(\mathcal{A})$ the set of optimal solutions of Problem (11). Define for $I \subseteq \mathcal{M}$ the following set:

$$AS_I(\alpha) = \{x \in X : \omega_i d_i(x, A_i) = \alpha,\ \forall i \in I; \omega_i d_i(x, A_i) \leq \alpha,\ \forall i \notin I\}.$$

Then the following theorem gives us the characterization of the optimal solution set of Problem (11). Let us denote as it is usual $C_i(p_i) = (a_i + N_{B_i^o}(p_i))$.

Theorem 4.2

1. *If $M_W^{l_\infty}(\mathcal{A}) \neq \emptyset$ then there exist $I \subseteq \mathcal{M}$, $I \neq \emptyset$; $\alpha > 0$ and $p = (p_i)_{i \in \mathcal{M}}$ $p_i \in N_{A_i}(y_i) \cap B_i^o$ with $y_i \in \partial \sigma_{A_i}(p_i)$ $\forall i \in I$ verifying $\sum_{i \in I} \omega_i p_i = 0$ such that*

$$M_W^{l_\infty}(\mathcal{A}) = \bigcap_{i \in I} C_i(p_i) \cap AS_I(\alpha)$$

2. *If there exist $I \subseteq \mathcal{M}$, $I \neq \emptyset$; $\alpha > 0$; $p = (p_i)_{i \in I}$ $p_i \in N_{A_i}(y_i) \cap B_i^o$ with $y_i \in \partial \sigma_{A_i}(p_i)$ $\forall i \in I$, verifying $\sum_{i \in I} \omega_i p_i = 0$ such that*

$$\bigcap_{i \in I} C_i(p_i) \cap AS_I(\alpha) \neq \emptyset$$

then

$$M_W^{l_\infty}(\mathcal{A}) = \bigcap_{i \in I} C_i(p_i) \cap AS_I(\alpha).$$

Proof:

The proof consists of applying the general Theorem 3.2 for $\Phi = l_\infty$-norm in $I\!R^M$ and $\gamma_i' = \omega_i\gamma_i$ for all $i \in \mathcal{M}$. Being $\Phi = l_\infty$-norm implies that the dual norm $\Phi^o = l_1$-norm. Therefore, $\gamma^o(\lambda) = 1$ if and only if $\sum_{i\in I}\lambda_i = 1$. Hence, $\alpha = F(x) = \max_{1\le i\le M} w_i d_i(x, a_i) = \sum_{i\in I}\lambda_i w_i d_i(x, A_i)$ with $\sum_{i\in I}\lambda_i = 1$ is equivalent to $w_i d_i(x, A_i) = \alpha$ for any $i \in I$ and $w_i d_i(x, A_i) \le \alpha$ for any $i \notin I$. In other words, $x \in D_I(\lambda)$ if and only if $x \in AS_I(\alpha)$ for $\alpha = F(x)$ and the proof follows. □

Remark 4.1 The value α which defines the optimal solution set $AS_I(\alpha)$ is the optimal objective value of Problem (11).

Example 4.2 *(See Figure 4) Consider a problem with* $\Phi(x_1, x_2, x_3) = \max\{|x_1|, |x_2|, |x_3|\}$ *and the following set of demand sets* $\mathcal{A} = \{A_1 := co\{(5,1),(5,-1),(3,-1),(3,1)\}, A_2 := co\{(1,-3),(0,-5),(-1,-3)\}, A_3 := co\{(-3,3),(-3,5),(-5,4)\}\}$ *with weights* $W = (1,1,1)$ *and* $\gamma_i = l_\infty$*-norm in* $I\!R^2$ *for* $i = 1,2,3$.

The problem to be solved is:

$$\min_{x\in I\!R^2} F(x) := \max_{i=1,2,3} d_i(x, A_i).$$

Take $I = \{2,3\}$, $p = \{p_1 = (1,0), p_2 = (0,1), p_3 = (0,-1)\}$, *and* $\alpha = 1$ *then:*

$$C_I(p) = \{x \in I\!R^2 \ : \ x_1 - x_2 \le 2,\ x_1 - x_2 \ge 1,\ x_1 + x_2 \le 0,\ x_1 + x_2 \ge -1\}$$

Now, we have for $\alpha = 1$ *that* $AS_I(1) = \{(0,0)\}$ *which equals* $D_I((0,0.5,0.5))$. *Notice that for* $\lambda = (0,0.5,0.5)$ *we have* $\Phi^o(\lambda) = 1$. *Indeed, this set is*

$$D_I((0,0.5,0.5)) = \{x : \max_{i=1,2,3} d_i(x, A_i) = \frac{1}{2}\langle(0,1), x-a_2\rangle + \frac{1}{2}\langle(0,-1), x-a_3\rangle\} = \{(0,0)\}.$$

Then

$$M_W^{l_\infty}(\mathcal{A}) = C_I(p) \cap AS_I(\alpha) = \{(0,0)\}.$$

In Figure 4 the dark-shaded square contains $co(\mathcal{A})$ *and by Corollary 3.1 we know that an optimal solution of the problem can be found in* $co(\mathcal{A})$. *Therefore, we can restrict the search for g.e.c.s. which fulfill the hypothesis of Theorem 4.2 to this square. Thus, reducing the overall effort.*

5 The planar case: Interpretations.

In order to obtain the solution set of the Problem $P_\Phi(\mathcal{A})$, the importance of the sets $C(\mathcal{Y}, p)$ should be noted, since in these sets the infimal distance function is linear.

In this section, we restrict ourselves to $I\!R^2$ and total polyhedrality. This reduction allows us to describe in an easy and understandable way the geometrical characterization given in the previous sections. Although it is possible to describe these sets using their theoretical expansion in terms of subdifferential sets, we use in this section

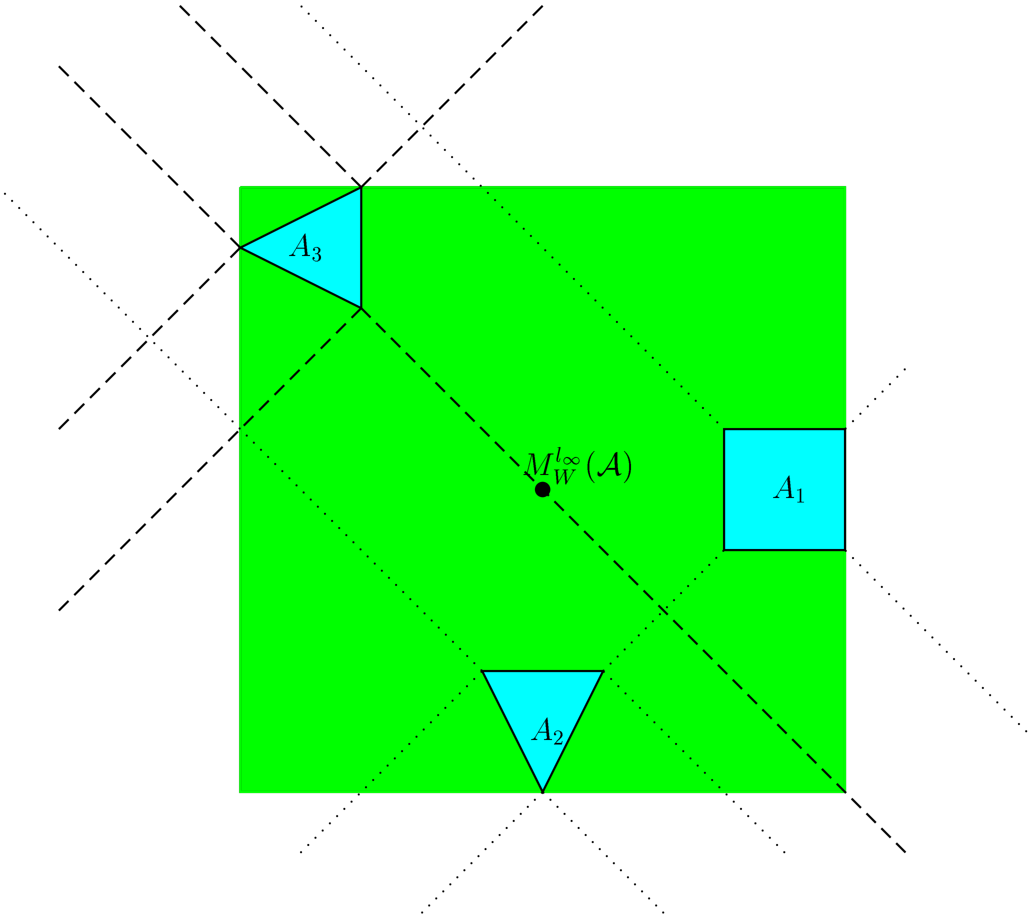

Figure 4: Illustration of Example 4.2

a different approach. We look for an efficient algorithmic to find the g.e.c.s. in $I\!R^2$ whenever polyhedral norms are used to measure the distances.

We will do that with the following scheme. Having already proved that the g.e.c.s. are the sets of point projecting onto faces of the existing facilities we will characterize the maximal projection domains using the norm associated with each facility. To do that we, first, characterize the projection onto lines, then onto segments and, finally, onto cones. After that, we can characterize the projection onto convex polygons, since they can be seen as segments plus corners (cones).

Let γ be a polyhedral norm with unit ball B. In the following we say that a point x projects onto the line, r, with the direction d if $\overline{x} \in \mathrm{proj}_r(x)$ and $x = \overline{x} + \lambda d$ with $\lambda > 0$.

Lemma 5.1 *Let π_1 be an open halfspace determined by a line r. The projection of the point belonging to π_1 onto r can be:*

1. *unique, then all points of π_1 project with the same fundamental direction.*

2. *not unique, then all points of π_1 project with two consecutive fundamental directions.*

Proof:

1. If the projection is unique, it is obvious that each point projects with only one fundamental direction. We must prove that all the points in π_1 project with the same fundamental direction. Let $x, y \in \pi_1$ and d_1, d_2 be two fundamental directions such that x projects with d_1 and y with d_2 and we assume without loss of generality that $\gamma(d_1) = \gamma(d_2) = 1$. Now, we consider the points:

$$\begin{aligned} x &= \overline{x} + \lambda_1 d_1 \in r, \\ x &= z + \lambda_2 d_2 \in r, \\ y &= \omega + \mu_1 d_1 \in r, \\ y &= \overline{y} + \mu_2 d_2 \in r, \end{aligned}$$

   Since $\overline{x}$ and $\overline{y}$ are the unique projections of x and y onto r respectively, we have that $\lambda_1 < \lambda_2$ and $\mu_1 > \mu_2$.

   We know that the triangles $x\overline{x}z$ and $y\overline{y}w$ have equal angles then

$$1 > \frac{\lambda_1}{\lambda_2} = \frac{\mu_1}{\mu_2} > 1,$$

   which is a contradiction.

2. Consider $x \in \pi_1$ whose projection onto r is not unique. This means, that x projects onto r with more than one fundamental direction. It should be noted that if x projects onto r with two fundamental directions then there exists a facet of B parallel to r. Moreover, if x projects with three fundamental directions then a facet of B is determined by three fundamental directions, and this is impossible.

   Finally, in order to prove that every point of π_1 projects onto r with the same two fundamental directions, we only have to see that the unit ball B in every point of r has the same facet parallel to r and included in π_1. Thus, every point of π_1 projects with the same two fundamental directions which are defined by the parallel facet.

□

In the following proposition we determine the direction projections according to the two cases analyzed in Lemma 5.1.

Proposition 5.1 *Let π_1 be an open halfspace determined by a line r and let $\overline{AB}$ be a segment included in r. Then*

1. *If $x \in \pi_1$ projects with the fundamental direction d_1 onto r and $\overline{x} = \mathrm{proj}_r(x)$ then*

$$\exists p \in B^o \quad d(x, r) = \langle p, x - \overline{x} \rangle \quad \forall x \in \pi_1,$$

   and if $x \in \pi_1$ and $\mathrm{proj}_r(x) \in \overline{AB}$ then

$$x \in \overline{AB} + \mu d_1 \quad \mu \geq 0.$$

2. *If $x \in \pi_1$ projects with the directions d_1 and d_2 onto r then*

$$\exists q \in B^o \quad d(x,r) = \langle q, x - \overline{x}\rangle \quad \forall x \in \pi_1 \text{ and any } \overline{x} \in \operatorname{proj}_r(x).$$

Moreover, if $x \in \pi_1$ and $\operatorname{proj}_r(x) \cap \overline{AB} \neq \emptyset$ then

$$x \in \overline{AB} + cone(d_1, d_2).$$

Proof:

1. If $x \in \pi_1$, using Lemma 5.1 we have that any point of π_1 projects onto r with the same fundamental direction, d_1. By definition of $\partial\gamma$ we know that there exists $p(d_1) \in \partial\gamma(d_1) \subseteq B^o$ such that $\langle p(d_1), d_1\rangle = \gamma(d_1)$. Thus, if $x \in \pi_1$ and $\overline{x} = \operatorname{proj}_r(x)$ we have that $x - \overline{x} = \lambda_x d_1$ with $\lambda_x > 0$ and

$$d(x,r) = \gamma(x-\overline{x}) = \gamma(\lambda_x d_1) = \lambda_x\gamma(d_1) = \lambda_x\langle p(d_1), d_1\rangle = \langle p(d_1), \lambda_x d_1\rangle = \langle p(d_1), x-\overline{x}\rangle$$

Obviously, the set of points included in π_1 whose projection belongs to the line segment $\overline{AB}$ are $\overline{AB} + \mu d_1$ with $\mu > 0$, because these points project onto r with the direction d_1.

2. In this case for any $x \in \pi_1$ we have that the projection of x onto r is not unique. Using Lemma 5.1 we have that every point of π_1 projects with the same consecutive fundamental directions, d_1 and d_2. Without loss of generality we can assume that $\gamma(d_1) = \gamma(d_2) = 1$. (See Figure 5).

Let $\overline{x}_1, \overline{x}_2 \in \operatorname{proj}_r(x)$ such that $x = \overline{x}_1 + \lambda d_1$ and $x = \overline{x}_2 + \lambda d_2$ with $\lambda > 0$. Since d_1 and d_2 are two consecutive fundamental directions there exists $p(d_1, d_2) \in \partial\gamma(d_1) \cap \partial\gamma(d_2) \subseteq B^o$ such that;

$$\gamma(\theta d_1 + (1-\theta)d_2) = \langle p(d_1,d_2), \theta d_1 + (1-\theta)d_2\rangle \quad \forall \theta \in [0,1].$$

Moreover, if $\overline{x} \in \operatorname{proj}_r(x)$ we have that, there exists $\theta_o \in [0,1]$, such that, $\overline{x} = \theta_o\overline{x}_1 + (1-\theta_o)\overline{x}_2$, that means that $x = \overline{x} + \lambda(\theta_o d_1 + (1-\theta_o)d_2)$. Therefore,

$$\begin{aligned} d(x,r) &= \gamma(x-\overline{x}) = \gamma\Big(\lambda(\theta_o d_1 + (1-\theta_o)d_2)\Big) = \lambda\gamma\Big(\theta_o d_1 + (1-\theta_o)d_2\Big) \\ &= \lambda\langle p(d_1,d_2), \theta_o d_1 + (1-\theta_o)d_2\rangle = \langle p(d_1,d_2), x-\overline{x}\rangle. \end{aligned}$$

Finally, since every point of π_1 projects with d_1 and d_2, we have that the set of points whose projection has not empty intersection with the segment $\overline{AB}$ is $\overline{AB} + cone(d_1, d_2)$ (see Figure 5).

□

Theorem 5.1 *Let h_1 and h_2 be two halflines with the same origin O and contained in the lines r_1 and r_2 respectively. Let π_1, π_2 be the two open halfspaces determined by r_1 and r_2 such that $h_1 \cap \pi_2 = \emptyset$ and $h_2 \cap \pi_1 = \emptyset$. The following statements hold:*

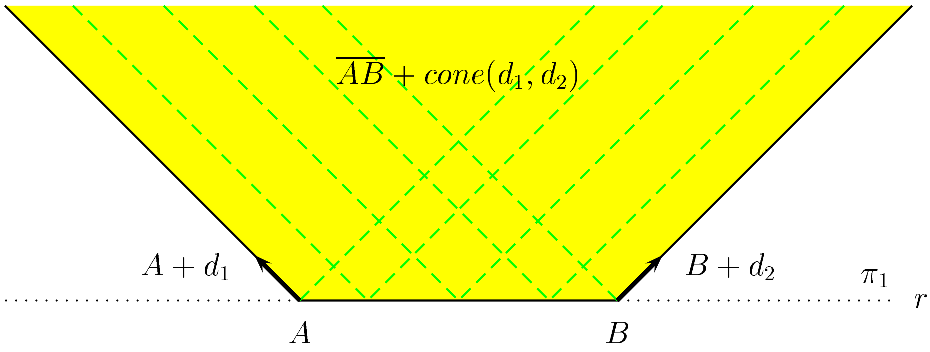

Figure 5: Set of points belonging to π_1, whose projection onto r with l_∞-norm has not empty intersection with the line segment $\overline{AB}$

1. *If $x \in \pi_1$ and $\overline{x}_1 \in \mathrm{proj}_{r_1}(x) \cap (h_1 \setminus \{O\})$ then there exists $p_1 \in B^o$ verifying:*

$$d(x, co(h_1, h_2)) = \langle p_1, x - \overline{x}_1 \rangle.$$

   (The analogous result holds for π_2.)

2. *If $x \in \pi_1 \cup \pi_2$ and $\mathrm{proj}_{r_i}(x) \cap (h_i \setminus \{O\}) = \emptyset$ with $i = 1, 2$ then $\mathrm{proj}_{co(h_1,h_2)}(x) = O$ and there exists $p_x \in B^o$ verifying*

$$d(x, co(h_1, h_2)) = \langle p_x, x - O \rangle.$$

Where $co(h_1, h_2)$ is the convex hull of h_1 and h_2.

Proof:

1. It is a straightforward consequence of Proposition 5.1.

2. Let be $x \in \pi_1 \cup \pi_2$ and $\overline{x} \in \mathrm{proj}_{co(h_1,h_2)}(x)$. Since $x \notin co(h_1, h_2)$, using the convexity of γ we have that $\overline{x} \in h_1 \cup h_2$. Now, we have to prove that $\overline{x} = O$. Let us assume that $\overline{x} \in (h_1 \cup h_2) \setminus \{O\}$.

   Since $\mathrm{proj}_{r_i}(x) \cap (h_i \setminus \{O\}) = \emptyset$ for $i = 1, 2$ using the convexity of γ we have that $\gamma(x - O) < \gamma(x - y)$ $\forall y \in h_i \setminus \{O\}$, $\quad i = 1, 2$. This contradicts that $\overline{x} \in (h_1 \cup h_2) \setminus \{O\}$. Thus, we obtain that $\overline{x} = O$.

   Therefore, there exists a cone (probably degenerated to a line), $cone(D_O)$, generated by the fundamental directions which are used to project by the points whose unique projection onto $co(h_1, h_2)$ is O. That is, if $x \in O + cone(D_O)$ then $\mathrm{proj}_{co(h_1,h_2)}(x) = O$. Thus, for all $x \in O + cone(D_O)$ there exists $p_x \in \partial\gamma(x - O)$ verifying that

$$d(x, co(h_1, h_2)) = \gamma(x - O) = \langle p_x, x - O \rangle.$$

The result follows by contradiction. □

Corollary 5.1 *(See Figure 6) The function $d(x, co(h_1, h_2))$ is linear in the following sets*

1. *$h_i + cone(D_i)$, where D_i is the set of fundamental directions of projection of π_i onto r_i with $i = 1, 2$.*

2. *$O + cone(d_r, d_{r+1})$ being d_r and d_{r+1} two consecutives fundamental directions of D_O and where D_O is the set of consecutive fundamental directions verifying that $|D_1 \cap D_O| = |D_2 \cap D_O| = 1$ and that $O + cone(D_O) \subseteq cl(\pi_1 \cup \pi_2)$.*

Remark 5.1 It should be noted that D_O might have only one element. In this case, $O + cone(d_r, d_{r+1})$ is a cone degenerated to a line.

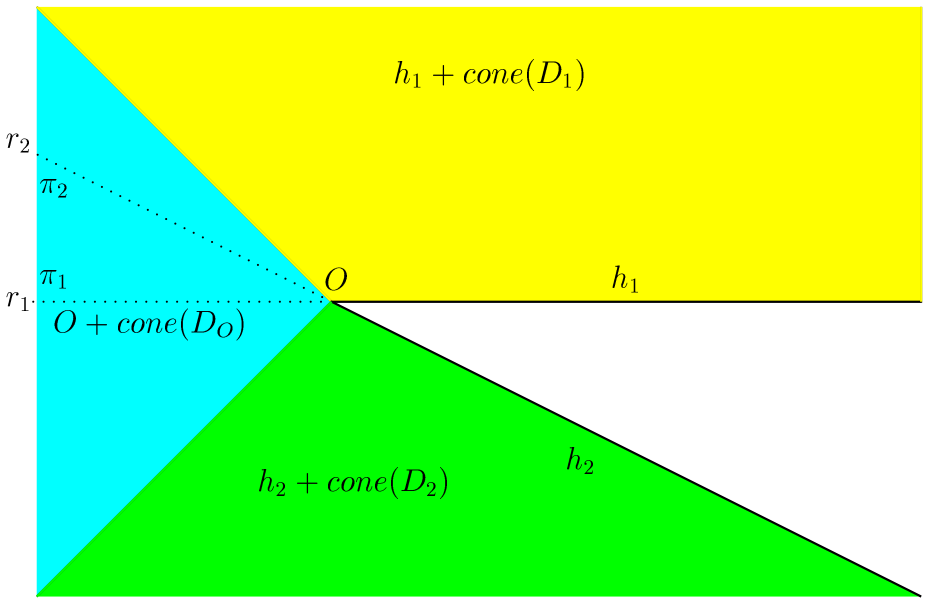

Figure 6: Different sets where the distance to $co(h_1, h_2)$, using the l_∞-norm, is a linear function.

Proof:

1. It is a straightforward consequence of Theorem 5.1.

2. It should be noted that the set of points included in $\pi_1 \cup \pi_2$ whose unique projection onto $co(h_1, h_2)$ is O, is the set

$$P_O = \Big\{ x \ : \ x \in cl\Big((\pi_1 \cup \pi_2) \setminus (h_1 + cone(D_1) \cup h_2 + cone(D_2)) \Big) \Big\}$$

where we denote by cl the topological closure.

Therefore, P_O is a pointed cone at O, generated by the set of fundamental directions, D_O, included between D_1 and D_2 and such that $O+cone(D_O) \subseteq cl(\pi_1 \cup \pi_2)$. Thus, if d_j and d_{j+1} (two consecutive fundamental directions) belong to D_O we have that there exists $p_j \in \partial\gamma(d_j) \cap \partial\gamma(d_{j+1})$ such that

$$d(x, O) = \langle p_j, x - O\rangle \quad \forall x \in O + cone(d_j, d_{j+1}).$$

$\square$

In the previous result we have characterized the sets where the inf-distance to a cone is linear. Now, in the following corollary, we extend these results to the inf-distance to a polygon.

Corollary 5.2 *(See Figure 7) Let A be a polygon, where $F_1, \ldots, F_L$ are its facets and $O_1, \ldots, O_L$ are its vertices. Let r_j be the line containing the facet F_j and π_j the open halfspace defined by r_j and not containing A, with $j = 1, \ldots, L$. We have that there exist $p(D_j), p(O_j, s) \in B^o$ for all $j = 1, \ldots, L$, such that*

$$d(x, A) = \begin{cases} \langle p(D_j), x - \overline{x}_j\rangle & \forall x \in F_j + cone(D_j) \text{ and } \overline{x}_j \in \text{proj}_A(x) \\ \langle p(O_j, s), x - O_j\rangle & \forall x \in O_j + cone(d_s, d_{s+1}) \text{ with } d_s, d_{s+1} \in D_{O_j} \end{cases}$$

where D_j and D_{O_j} with $i = 1, \ldots, L$ are defined like in the previous lemma.

Remark 5.2 Since $\partial d(x, A) \neq \emptyset$ we can choose $p(D_j) \in N_A(\bar{x})$ for any $\bar{x} \in \text{proj}_A(x) \cap ri(F_j)$ such that $\langle p(D_j), x - \bar{x}\rangle = d(x, A)$ and therefore we obtain that $C(F_j, p(D_j)) = F_j + cone(D_j)$. In the same way, there exists $p(O_j, s) \in N_A(O_j)$ such that $C(O_j, p(O_j, s)) = O_j + cone(d_s, d_{s+1})$.

After these results, we characterize the maximal sets $C(\mathcal{Y}, p)$ by means of an algorithm that constructs these sets. In fact, the algorithm gives us a methodology to build the maximal domain of linearity of the function inf-distance to each set of the family $\mathcal{A}$. Hence, we can obtain $C(\mathcal{Y}, p)$ as intersection of the sets obtained from the algorithm. Before starting with the description of the algorithm we need the following result.

Lemma 5.2 *Let π_1 be an open halfspace determined by a line r. If $\Big((\overline{x} - d_1) + B\Big) \cap cl(\pi_1) \subseteq r$ with $\overline{x} \in r$ then the points of π_1 projects onto r at least with d_1.*

Proof:

We can assume without loss of generality that every fundamental direction d verifies that $\gamma(d) = 1$.

We have that there exists a fundamental direction d_1, such that $\Big((\overline{x} - d_1) + B\Big) \cap cl(\pi_1) \subseteq r$ with $\overline{x} \in r$. Then, two cases can occur;

1. $\Big((\overline{x} - d_1) + B\Big) \cap cl(\pi_1) = (\overline{x} - d_1) + d_1$

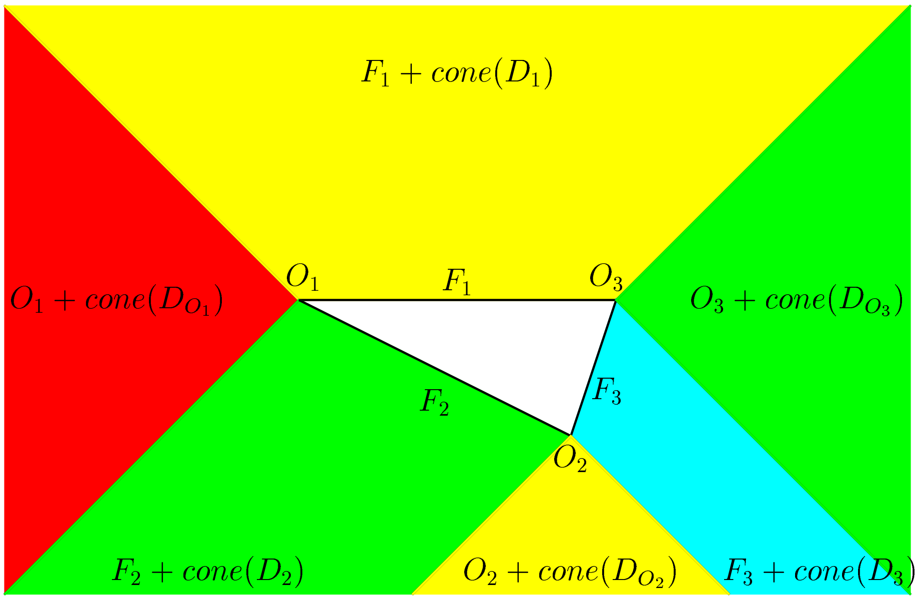

Figure 7: Different sets where the distance to this triangle, using the l_∞-norm, is a linear function.

2. $\Big((\overline{x} - d_1) + B\Big) \cap cl(\pi_1) = \theta((\overline{x} - d_1) + d_1) + (1-\theta)((\overline{x} - d_1) + d_2)$ with $\theta \in [0,1]$ and d_2 a consecutive fundamental direction of d_1.

Now, we consider a fundamental direction d, such that $d \neq d_1$ in Case 1. and $d \neq \theta d_1 + (1-\theta)d_2 \quad \forall \theta \in [0,1]$ in Case 2. Then again two cases can occur;

1. $\forall \lambda > 0$ we have that $(\overline{x} - d_1) + \lambda d \notin r$

2. $\exists \lambda > 0$ such that $(\overline{x} - d_1) + \lambda d \in r$

The first case implies that any point of π_1 does not project onto r with the direction d.

In the second case (see Figure 8), let $x = \overline{x} + d_1$, and $y = (\overline{x} - d_1) + \lambda d \in r$. Since, $((\overline{x} - d_1) + B) \cap cl(\pi_1) \neq (\overline{x} - d_1) + d$ it follows that $\lambda > 1$.

We have that $x = \overline{x} + d_1$ or equivalently $x = \overline{x} + d_1 - \lambda d + \lambda d$. Moreover, since $\overline{x} \in r$ and $(\overline{x} - d_1) + \lambda d \in r$ then $\overline{x} - (-d_1 + \lambda d)$ also belongs to r. Thus, x is equal to an element of r, $\overline{x} - (-d_1 + \lambda d)$, plus λd. That means that the distance from r to x with direction d is λ. We know that $\lambda > 1$ and the distance from r to x with d_1 is 1. Therefore x does not project with d. $\square$

In the following algorithm we will make use of the previous lemmas to obtain the domains of linearity in $O(L+G)$ time.

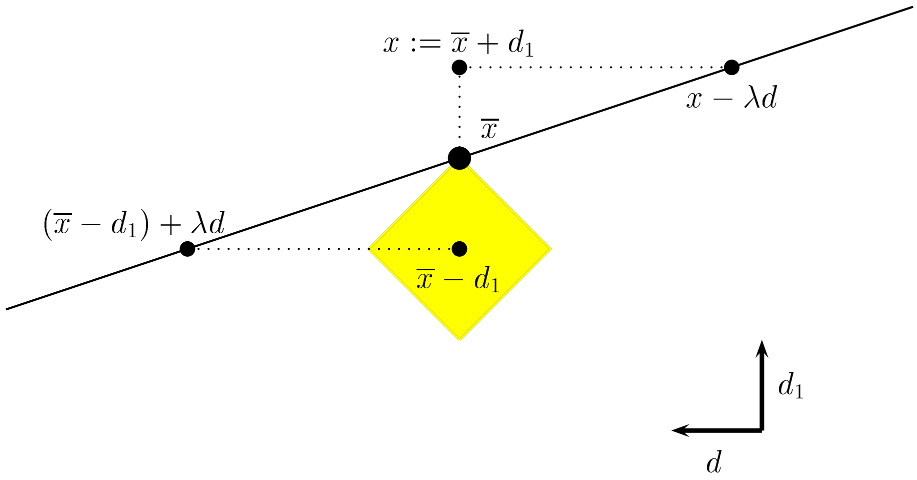

Figure 8: Illustration of the proof of Lemma 5.2.

ALGORITHM 5.1

Preprocessing:

- *For existing facility $A \in \mathcal{A}$ we denote by $-n_1, \ldots, -n_L$ the negative normal vectors to the facets of A. They are sorted in counterclockwise order.*
- *For each fundamental direction d_i of the unit ball B_A, we build $B_A - d_i$ and denote by Cd_i the cone generated by the two facets d_i^O and d_i^C which start in the origin (of $B_A - d_i$). Also the d_i (and therefore also the Cd_i) are assumed to be sorted in counterclockwise order. Moreover, we assume that we have the elements in a circular list, i.e. $G + 1 = 1$.*

A test routine: **bool IsActive(Cd_i, $-n_j$)**

1. *IF $\langle -n_j, d_i^O \rangle \geq 0$ and $\langle -n_j, d_i^C \rangle \geq 0$ then return TRUE*
2. *else return FALSE*

The main algorithm:

1. *i := 1*
2. *WHILE NOT IsActive(Cd_i, $-n_1$) i := i + 1. (* Find the active projections for $-n_1$ *)*
3. *ActiveCones := $\{Cd_i\}$*
4. *IF (i=1) AND IsActive(Cd_G, $-n_1$) then ActiveCones := ActiveCones $\cup$ $\{Cd_G\}$.*

5. *IF IsActive*(Cd_{i+1}, $-n_1$)
   then ActiveCones := *ActiveCones* $\cup$ $\{Cd_{i+1}\}$, $i := i + 1$.
6. *ActiveDirs*($-n_1$) := *ActiveCones*
7. *FOR* j := *2 TO L DO*
   (a) *FOR all cones* $Cd \in$ *ActiveCones DO*
      i. *IF NOT IsActive* (Cd, $-n_j$)
         then ActiveCones := *ActiveCones* $\setminus$ $\{Cd\}$.
         (* *Note, that we have maximally* 2 *active cones* *)
   (b) *IF* $|ActiveCones| = 1$ *then*
      IF IsActive(Cd_{i+1}, $-n_j$)
      then ActiveCones := *ActiveCones* $\cup$ $\{Cd_{i+1}\}$
   (c) *IF ActiveCones* $= \emptyset$ *then*
      i. *WHILE NOT IsActive*(Cd_i, $-n_j$) $i := i + 1$.
      ii. *ActiveCones* := $\{Cd_i\}$
      iii. *IF IsActive*(Cd_{i+1}, $-n_1$)
         then ActiveCones := *ActiveCones* $\cup$ $\{Cd_{i+1}\}$, $i := i + 1$.
   (d) *ActiveDirs*($-n_j$) := *ActiveCones*
8. *FOR* $j := 1$ *TO* $L - 1$
   (a) $ActiveDirs(p_j) := Cone(last(ActiveDirs(-n_j)), first(ActiveDirs(-n_{j+1})))$.
9. $ActiveDirs(p_L) := Cone(last(ActiveDirs(-n_L)), first(ActiveDirs(-n_1)))$.
10. *END*

The running time of the algorithm is $O(L{+}G)$ and the $ActiveDirs(-n_j)$ and $ActiveDirs(p_j)$ contain the directions spanning the maximal linearity domains.

Remark 5.3 It should be noted that in the previous algorithm we have used that the projection of the points with the same fundamental direction is a connected set. If we would not have this property then it may occur that $(F_i{+}cone(D_i)) \cap (F_j{+}cone(D_j)) \neq \emptyset$ with $j > i{+}1$, that means that there may exist points in $I\!R^2$ projecting with two non consecutive fundamental directions. This is impossible by the convexity of the norm γ.

Once we have developed an algorithm to compute the maximal domain of linearity of the inf-distance to any polygon, we can obtain the domain of linearity of any problem where the demand sets are polygons as intersection of the maximal domain of linearity of the inf-distance to each demand set. These maximal domains of linearity are called cells and they are the natural extension of the elementary convex sets when we consider a problem with demand points.

In order to solve a general problem with polygons as demand sets we are going to develop an algorithm to compute the optimal solution of this problem.

ALGORITHM 5.2 */* Solving the Problem $(P_\Phi(\mathcal{A}))$ in $\mathbb{R}^2$ */*
Input:

1. *Demand sets $A_i \subseteq \mathbb{R}^2$, $i \in \mathcal{M}$.*
2. *Polyhedral gauges $\gamma_i : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $i \in \mathcal{M}$.*
3. *The objective function $F(x) = \Phi(d_1(x, A_1), \ldots, d_M(x, A_M))$.*

STEP 1 COMPUTE the planar graph generated by the cells and let V be its set of vertices.

STEP 2 CHOOSE $x^o \in V$, and let $L^1 := \{x^o\}$, $L^* := \{x^o\}$.

STEP 3 WHILE $L^1 \neq \emptyset$

STEP 3.1 CHOOSE $x^1 \in L^1$ and set $V := V \setminus \{x^1\}$ and bool:=$false$.
STEP 3.2 IDENTIFY $\mathcal{L}(x^1)$ the set of adjacent nodes to x^1 in V.
STEP 3.3 WHILE $\mathcal{L}(x^1) \neq \emptyset$

- *CHOOSE $x^2 \in \mathcal{L}(x^1)$ and set $\mathcal{L}(x^1) := \mathcal{L}(x^1) \setminus \{x^2\}$*
- *IF $F(x^2) = F(x^1)$ THEN $L^* := L^* \cup \{x^2\}$ and $L^1 := L^1 \cup \{x^2\}$*
- *ELSEIF $F(x^2) < F(x^1)$ THEN set $x^1 := x^2$, $L^1 := \{x^1\}$, $L^* := \{x^1\}$, $\mathcal{L}(x^1) := \emptyset$ and bool $:= true$.*

STEP 3.4 ENDWHILE
STEP 3.5 IF bool $= false$ then $L^1 := L^1 \setminus \{x^1\}$

STEP 4 ENDWHILE

Output:

The optimal solution are the points in the cell defined by L^*.

Step 1 is done by means of sweep line technique and described in more detail in [Wei99] and [NPRCW99]. In order to use the sweep line technique we need to consider a bounded region on the plane. The following result shows that we can apply this technique because we only have to look for the optimal solutions in a bounded region.

Lemma 5.3 *The optimal solution of Problem $(P_\Phi(\mathcal{A}))$ in $\mathbb{R}^2$ is bounded.*

Proof:
Assume that x is an optimal solution of Problem $(P_\Phi(\mathcal{A}))$. Then, there is no $y \in \mathbb{R}^2$ such that

$$d_i(y, A_i) < d_i(x, A_i) \quad \forall i \in \mathcal{M}$$

that means, especially for $y = 0$ there exists $j_x \in \{1, 2, \ldots, M\}$ such that

$$d_{j_x}(0, A_{j_x}) \geq d_{j_x}(x, A_{j_x}),$$

that is,

$$\inf_{a_{j_x} \in A_{j_x}} \gamma_{j_x}(-a_{j_x}) \geq \inf_{a_{j_x} \in A_{j_x}} \gamma_{j_x}(x - a_{j_x}). \tag{12}$$

On the other hand, using the triangular inequality we have that

$$\gamma_{j_x}(x) \leq \gamma_{j_x}(a_{j_x} + x - a_{j_x}) \leq \gamma_{j_x}(a_{j_x}) + \gamma_{j_x}(x - a_{j_x}),$$

hence

$$\inf_{a_{j_x}\in A_{j_x}} \gamma_{j_x}(x) \leq \inf_{a_{j_x}\in A_{j_x}} \Big(\gamma_{j_x}(a_{j_x} + x - a_{j_x})\Big) \leq \sup_{a_{j_x}\in A_{j_x}} \gamma_{j_x}(a_{j_x}) + \inf_{a_{j_x}\in A_{j_x}} \gamma_{j_x}(x - a_{j_x}).$$

Thus, using the inequality (12) and the existence of the constant M_{j_x} (A_{j_x} is bounded), such that, $\sup_{a_{j_x}\in A_{j_x}} \gamma_{j_x}(a_{j_x}) = M_{j_x}$, we have the following inequality

$$\gamma_{j_x}(x) \leq M_{j_x} + \inf_{a_{j_x}\in A_{j_x}} \gamma_{j_x}(-a_{j_x}). \tag{13}$$

Now, if we denote by $l_2(\cdot)$ the standard l_2-norm by elementary calculations using the law of sines we have that

$$l_2(x) \leq \gamma_{j_x}(x) r^{j_x}_{\max} \quad \text{and} \quad \gamma_{j_x}(\pm a_{j_x}) \leq \frac{l_2(\pm a_{j_x})}{r^{j_x}_{\min}}$$

where

$$\begin{aligned}
r^m_{\max} &:= \max_{j=1,\dots,G_m} \{l_2(e^m_i)\}\ \forall m \in \mathcal{M} \\
r^m_{\min} &:= \min_{i=1,\dots,G_m} \left\{ l_2\left(\frac{\langle e^m_{i+1}, e^m_{i+1} - e^m_i\rangle}{\langle e^m_{i+1} - e^m_i, e^m_{i+1} - e^m_i\rangle} e^m_i - \frac{\langle e^m_i, e^m_{i+1} - e^m_i\rangle}{\langle e^m_{i+1} - e^m_i, e^m_{i+1} - e^m_i\rangle} e^m_{i+1} \right) \right\}\ \forall m \in \mathcal{M}.
\end{aligned}$$

Therefore, from (13), we have that

$$\begin{aligned}
l_2(x) &\leq \gamma_{j_x}(x) r^{j_x}_{\max} \\
&\leq \frac{r^{j_x}_{\max}}{r^{j_x}_{\min}} \left(\sup_{a_{j_x}} l_2(a_{j_x}) + \inf_{a_{j_x}} l_2(-a_{j_x}) \right).
\end{aligned}$$

□

Example 5.1 *Let A_1, A_2 and A_3 be the demand sets defined as follows:* $A_1 = co\{(4.5, 10), (10.5, 10), (10.5, 13.5), (4.5, 13.5)\}$, $A_2 = co\{(19.5, 15), (23.5, 17), (24, 15)\}$ *and* $A_3 = co\{(18.5, 4), (18.5, 6), (20.5, 6), (18.5, 6)\}$. *We consider that $\gamma_1 = l_1-$norm, $\gamma_2 = \gamma_3 = l_\infty-$norm. The problem that we want to solve is given by the following formulation:*

$$\min_{x\in \mathbb{R}^2} 2d_1(x, A_1) + d_2(x, A_2) + d_3(x, A_3)$$

In order to solve this problem, we compute the generalized elementary convex sets following Algorithm 5.1 (see Figure 9). After that, we know where every elementary convex set is placed. Using Algorithm 5.2 we get as optimal solution the shaded region $M_\Phi(\mathcal{A})$.

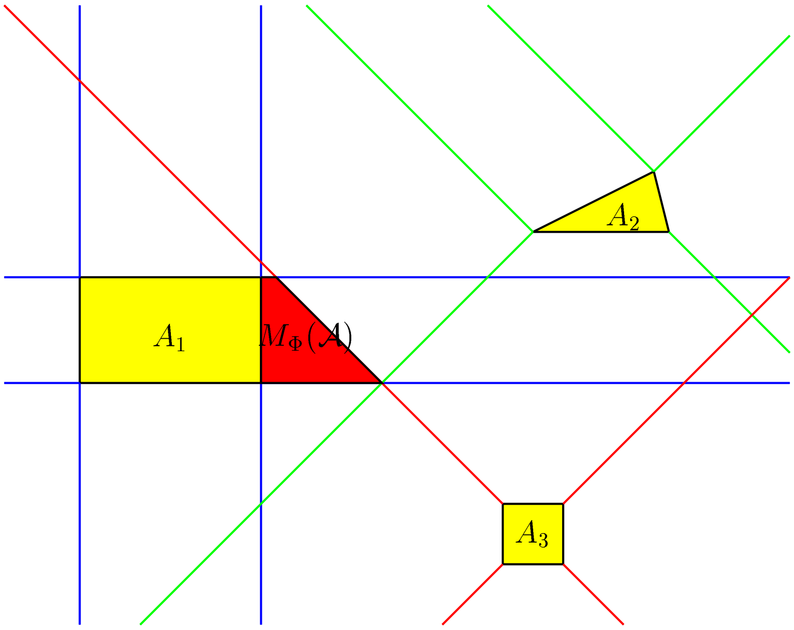

Figure 9: Illustration for Example 5.1. The generalized elementary convex sets.

6 Concluding remarks

There exists another natural extension that can be addressed: the location of a regional facility with respect to existing facilities that are sets.

Let us consider a fix set B closed, compact and convex. The problem consists of determining the translation vector x such that minimizes the following problem:

$$\min_{x\in X} \Phi(d_1(x+B, A_1), \ldots, d_M(x+B, A_M))$$

where $d_i(x+B, A) = \inf_{b\in B} \inf_{a_i \in A_i} \gamma_i(x+b-a_i)$.

Now, it is straightforward to see that

$$\inf_{b\in B} \inf_{a_i\in A_i} \gamma_i(x+b-a) = \inf_{c_i\in B-A_i} \gamma_i(x-c_i).$$

Therefore, we reduce this problem to the first one by only consider a new family $\mathcal{A}' = \{B - A_1, \ldots, B - A_M\}$.

Similar results can also be obtained when the norms γ_i associated with each set A_i are replaced by gauges.

Acknowledgements

The authors want to thank an anonymous referee for his/her careful reading which helps to improve the presentation of the paper. We also would to like to thank the Spanish Direccion General de Investigación for partial support through grant number PB98-0707.

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