THE PERTURBATIVE APPROACH TO THE CRITICAL BEHAVIOUR OF TWO-MATRIX MODELS IN THE LIMIT $N \to \infty$

S.BALASKA*, J. MAEDER†, W. RÜH L‡

Department of Physics, University of Kaiserslautern, P.O.Box 3049

67653 Kaiserslautern, Germany

ABSTRACT

We construct representations of the Heisenberg algebra by pushing the perturbation expansion to high orders. If the multiplication operators $B_{1,2}$ tend to differential operators of order $l_{2,1}$, respectively, the singularity is characterized by (l_1, l_2). Let $l_1 \geq l_2$. Then the two cases A : "l_2 does not divide l_1" and B : "l_2 divides l_1" need a different treatment. The universality classes are labelled $[p, q]$ where $[p, q]$=$[l_1$,$l_2]$ in case A and $[p, q]$=$[l_1 + 1$,$l_2]$ in case B.

*On leave of absence from Laboratoire de Physique Théorique, Université d'Oran, 31100 Es-Sénia, Algeria, E-mail: balaska@physik.uni-kl.de

†E-mail: maeder@physik.uni-kl.de

‡E-mail: ruehl@physik.uni-kl.de

1. Introduction

Chains of random matrices describe in the double scaling limit $N \to \infty$ two-dimensional gravity coupled to matter which in turn is represented by rational conformal field theories [1,2,3]. A connection of the one-matrix models with the theory of Korteweg-de Vries equations was first established in [2]. In the series of papers [4−6] the identification of the matter fields with the critical Ising model, Lee-Yang edge singularity, and the tricritical Ising model was achieved by comparison of the critical exponents (field dimensions). A systematic and constructive approach to arbitrar y matrix models [7−9] was made possible by shifting the interest to representations of a Heisenberg algebra in terms of Gelfand-Dikii quasi-differential operators [10]. This is a special aspect of Korteweg-de Vries equation theory. All this progress in matrix model research was obtained by application of the technique of orthogonal polynomials which was developed a long time ago [11,12]. More details of the history of matrix models can be found in review articles (e.g. [13]).

After the completion of the work [7−9] interest concentrated on the verification of the universality classes [p,q] where p and q are coprime, in different realizations. From the one-matrix models the sequence of classes [$2l+1$,2], $l \geq 1$ was obtained and only these. Two-matrix and some three-matrix models were studied in [6],in appendix C of [7], [8,9] and [14−16]. But the universality classes explicitly constructed were not many : [4,3], [5,4], and [8,3]. The claim that all [p,q] classes appear in two-matrix models was put forward (see the abstract of [9]). This assertion is easily verified by the construction proposed by us.

Our perturbative approach is not based on an investigation of the Heisenberg algebra. On the contrary we construct the representations of this algebra by a perturbative solution of the Schwinger-Dyson equations (synonymous: recursion relations). Generally high orders of perturbation theory are needed and a systematic treatment of this expansion is therfore crucial.

The Schwinger-Dyson equations (section 2) involve two multiplication operators B_1 and B_2 and two differentiation operators A_1 and A_2. We postulate that B_1 and B_2 tend to differential operators in the scaling domain

$$B_1 \to a^{-l_2\gamma} \sum_{n=0}^{\infty} a^{-n\gamma} Q^{(n)}, \quad Q^{(0)} = Q \tag{1}$$

$$B_2 \to a^{-l_1\gamma} \sum_{n=0}^{\infty} a^{-n\gamma} P^{(n)}, \quad P^{(0)} = P \tag{2}$$

If $L_1(L_2)$ is the degree of the potential $V_1(V_2)$ in the two-matrix model action, then necessarily

$$1 \leq l_i \leq L_i \tag{3}$$

All $\{Q^{(n)}\}$, $\{P^{(n)}\}$ are differential operators of order

$$\mathrm{ord}\, Q^{(n)} = \mathrm{ord}\, Q + n = l_2 + n \tag{4}$$

$$\mathrm{ord}\, P^{(n)} = \mathrm{ord}\, P + n = l_1 + n \tag{5}$$

We shall always assume $l_1 \geq l_2$. γ is the "string susceptibility exponent" and is negative,

$$\frac{1}{N} = a^{2-\gamma} \tag{6}$$

and N tends to infinity.

If the ansatz (1), (2) is satisfied we speak of a "singularity of type (l_1,l_2)" in the action of the model. We can choose this singularity "maximal" by setting

$$l_1 = L_1, l_2 = L_2 \tag{7}$$

We are convinced that all types of singularities that exist can be found if we let L_1, L_2 run over all naturals.

After solution of the Schwinger-Dyson equations the commutator

$$[B_1, A_1] = [B_2, B_1] = [A_2, B_2] \tag{8}$$

is diagonal. Setting it equal the unit operator gives one nonlinear differential equation. The perturbative expansion starts from a linear system of equations for the deviations of B_1 , B_2 from their limiting difference operator. This linear sys tem has corank l_2-2. It uses two functions u_1 , v_{-1} as basis functions. Use of biorthogonal systems of polynomials introduces a gauge degree of freedom that affects the whole perturbation theory. The "susceptibility function" u is gauge invariant

$$u = u_1 + v_{-1} + a^{-2\gamma} u_1 v_{-1} \tag{9}$$

(this form for u is valid in our preferred normalization, see section 3). The nonvanishing corank $l_2 - 2$ permits the introduction of $l_2 - 2$ new gauge invariant basis functions $\{w_i\}_{i=1}^{l_2-2}$. Higher order perturbation theory gives one nonlinear differential equation for each one, and these equations are ident ical to those obtained from the commutator (8) if we postulate that it be diagonal.

After we study several examples in detail we can summarize our results as follows. There are two series of singularity types :

(A) l_2 does not divide l_1 (remember $l_1 \geq l_2$). Then only the leading orders in (1),(2) are relevant and in the sense of quasi-differential operators [10]

$$(-)^{l_1} P = Q_+^{l_1/l_2} \tag{10}$$

The susceptibility exponent is

$$\gamma = \frac{-2}{p+q-1} \tag{11}$$

where the universality class is

$$[p, q] = [l_1, l_2] \tag{12}$$

(B) l_2 divides l_1, but $l_2 \neq 2$. Then

$$(-)^{l_1} P = Q^{l_1/l_2} \tag{13}$$

and

$$[B_2, B_1] \to a^{-(l_1+l_2+1)\gamma} [\tilde{P}, Q] + \text{ higher order terms} \tag{14}$$

where

$$\tilde{P} = (-)^{l_1} P^{(1)} - \sum_{1\le m\le M} Q^{M-m} Q^{(1)} Q^{(m-1)} \quad (M = \tfrac{l_1}{l_2}) \tag{15}$$

We show that

$$\tilde{P} = c(l_1, l_2)\, Q_+^{\frac{l_1+1}{l_2}} \tag{16}$$

with

$$c(l_1, l_2) = \frac{(l_1 l_2 - l_1 - l_2)}{l_2} \tag{17}$$

and correspondingly that

$$\gamma = \frac{-2}{p+q-1} \tag{18}$$

with

$$[p, q] = [l_1 + 1, l_2] \tag{19}$$

Some comments are in order in this context. First there is no universality class for singularities $(2m, 2)$. We will later in (section 5) give an argument why this is so. Next a universality class with

$$p = nq + 1 \ , \ n \in \mathbb{N} \tag{20}$$

can be realized by a singularity either in (A) or in (B). Finally universality classes where p and q have a common divisor but this divisor is not q itself, are contained in (A). The symmetric case $l_1 = l_2$ belongs to series (B).

We study the double scaling limit with $N \to \infty$ and only one coupling tuned to a critical value. Nevertheless the complete set of nonlinear differential equations for u and $\{w_i\}$ is derived. Only the "integration constants" are ze ro and x is the single variable in the equations that is kept.

In order to make this work self-contained we introduce matrix models and discuss their analysis by the orthogonal polynomial method in section 2. Two elementary and easily proved propositions which are not new but play a crucial role in the perturbative a pproach are given the form of "Lemmas". In section 3 we develop the perturbative analysis of the Dyson-Schwinger equations systematically, the decisive observation being that the corank of the linear approximation is $l_2 - 2$. This method is complet e in the sense that any singularity (l_1, l_2) can be treated with it. In the subsequent two sections we study examples from the series (A) : $(4,3)$, $(5,4)$ and $(6,4)$; and from the series (B) : $(3,3)$ and $(6,3)$. The two examples $(4,3)$ and $(5,4)$ (where l_1 and l_2 are coprime) serve only the purpose to prove that our method reproduces exactly the known results. Moreover we show in section 5 that singularities $(2m, 2)$ do not fulfill the commutator equation. Expectation values, in p articular their role in tests of gauge invariance, are studied in section 6.

2. The orthogonal polynomial approach to matrix models

Matrix models based on $N \times N$ hermitean or unitary or other matrices as dynamical variables have a history of about twenty years. It has turned out that a class of "solvable" matrix models exist which are built on a finite number r of such matrices

$$\{M^{(\alpha)}\} \tag{21}$$

which are bilinearly coupled to a chain. In the case of hermitean matrix models (to which we shall restrict our attention) this means an action

$$S(M^{(1)}, M^{(2)}, \ldots, M^{(r)}) = \mathrm{Tr}\left\{\sum_{\alpha=1}^{r} V_\alpha(M^{(\alpha)}) - \sum_{\alpha=1}^{r-1} c_\alpha M^{(\alpha)} M^{(\alpha+1)}\right\} \tag{22}$$

where each V_α is a polynomial of degree L_α (only $L_\alpha \geq 3$ is of interest)

$$V_\alpha(x) = \sum_{k=1}^{L_\alpha} \frac{g_k^{(\alpha)}}{k} x^k \tag{23}$$

The action is stable if and only if

$$\begin{cases} L_\alpha \text{ is even, all } \alpha \\ g_{L_\alpha}^{(\alpha)} \text{ is positive, all } \alpha \end{cases} \tag{24}$$

In the stable case the partition function of the model is

$$Z = \int \prod_\alpha dM^{(\alpha)} e^{-S} \tag{25}$$

$$dM^{(\alpha)} = \prod_{i\leq j} d(\mathrm{Re} M_{ij}^{(\alpha)}) \prod_{k<l} d(\mathrm{Im} M_{kl}^{(\alpha)}) \tag{26}$$

If the action is symmetric under

$$M^{(\alpha)} \leftrightarrow M^{(r+1-\alpha)} \tag{27}$$

we call the model "$\mathbb{Z}_2$-symmetric", otherwise "asymmetric". A model may have a $\mathbb{Z}_2$-symmetric critical point which can be approached in an asymmetric fashion such as an Ising model with external field H at the critical point.

When we claim that such matrix models (2) are "solvable", we mean solvability in the scaling domain of a critical point which implies the existence of a full asymptotic series

$$\langle \Omega \rangle \sim \sum_{m=0}^{\infty} a^{-m\gamma} \omega_m(z) \tag{28}$$

with $a^{-\gamma} \to 0$ in the scaling domain, z a scale invariant variable and computability of all ω_m by analytic methods, for any "observable" Ω. These observables include thermodynamic objects such as $\log Z$ or correlation functions of the "Mehta-accessible" class (see section 6). Our aim is to develop an algorithm so far, that for a given action all singularities can be extracted, the universality class can be described for each case and non-leading orders in the scaling behaviour can be calculated to any precision desired.

The full richness of structure develops only in asymmetric two-matrix and many-matrix models, which have not been studied carefully before. The symmetric models can often be obtained thereafter by a reduction of degrees of freedom. In any case we apply the method of orthogonal polynomials (by Bessis and Mehta, [11,12]). By a theorem of Mehta the action (22) can be expressed in terms of eigenvalues

$$\{\lambda_i^{(\alpha)}\}_1^N \ \text{ of } \ M^{(\alpha)} \tag{29}$$

$$S(\{\lambda_i^{(\alpha)}\}) = \sum_{i=1}^{N}\left[\sum_{\alpha=1}^{r} V_\alpha(\lambda_i^{(\alpha)}) - \sum_{\alpha=1}^{r-1} c_\alpha \lambda_i^{(\alpha)}\lambda_i^{(\alpha+1)}\right] \tag{30}$$

The partition function goes into

$$Z = C_r(N)\int \prod_{\alpha=1}^{r} d\mu(\Lambda^{(\alpha)})\Delta(\lambda^{(1)})\Delta(\lambda^{(r)})\exp[-S(\{\lambda_i^{(\alpha)}\})] \tag{31}$$

where the measure is

$$d\mu(\Lambda^{(\alpha)}) = \prod_{i=1}^{N} d\lambda_i^{(\alpha)}, \tag{32}$$

and

$$\Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j) \tag{33}$$

is the Vandermonde determinant. For the solvability it is crucial that both $\Delta(\lambda)$ appear in the numerator.

The biorthogonal polynomials

$$\{\Pi_m(\lambda), \tilde{\Pi}_n(\mu)\}_{m,n=0}^{\infty} \tag{34}$$

are defined by

$$\deg \Pi_m = \deg \tilde{\Pi}_m = m \tag{35}$$

and

$$\int \prod_{\alpha=1}^{r} d\lambda^{(\alpha)} \Pi_m(\lambda^{(1)})\tilde{\Pi}_n(\lambda^{(r)})\exp\{-\sum_{\alpha=1}^{r} V_\alpha(\lambda^{(\alpha)}) + \sum_{\alpha=1}^{r-1} c_\alpha \lambda^{(\alpha)}\lambda^{(\alpha+1)}\} = \delta_{mn} \tag{36}$$

through the Schmidt orthogonalization procedure. But this procedure determines only $2m+1$ of the $2m+2$ coefficients in Π_m, $\tilde{\Pi}_m$. In fact we have the

First Lemma:

Let

$$\Pi_m(\lambda) = s_m\lambda^m + \mathrm{O}(\lambda^{m-1}) \tag{37}$$

$$\tilde{\Pi}_m(\mu) = \tilde{s}_m\mu^m + \mathrm{O}(\mu^{m-1}) \tag{38}$$

then the orthogonalization procedure determines only

$$s_m\tilde{s}_m \tag{39}$$

but not s_m and $\tilde{s}_m$ separately. □

This leads to the following corollary: We can of course require that

$$s_m = \tilde{s}_m > 0 \tag{40}$$

as in the $\mathbb{Z}_2$-symmetric case, but expectation values of observables ought to depend only on $s_m \tilde{s}_m$ and not on such arbitrary definitions. They do, as we shall see, but in a nontrivial fashion (related with the gauge invariance of Yang-Mills theory observables).

We follow the usual line and introduce differentiation matrices

$$\begin{aligned} \Pi'_m &= \sum_n (A_1)_{mn} \Pi_n && (41) \\ \tilde{\Pi}'_m &= \sum_n (A_r)_{nm} \tilde{\Pi}_n && (42) \end{aligned}$$

and r multiplication matrices B_α

$$\begin{aligned} \lambda\, \Pi_m &= \sum_n (B_1)_{mn} \Pi_n(\lambda) && (43) \\ \mu\, \tilde{\Pi}_m &= \sum_n (B_r)_{nm} \tilde{\Pi}_n(\mu) && (44) \end{aligned}$$

$$(B_2, B_3, \ldots, B_{r-1} \text{ are not given here})$$

so that

$$[B_1, A_1] = [A_r, B_r] = 1 \tag{45}$$

By arguments typical for the derivation of Dyson-Schwinger equations one can derive a set of "equations of motion"

$$\begin{aligned} A_1 + c_1 B_2 &= V'_1(B_1) \\ &\vdots \\ c_{\alpha-1} B_{\alpha-1} + c_\alpha B_{\alpha+1} &= V'_\alpha(B_\alpha), \quad 2 \leq \alpha \leq r-1 \\ &\vdots \\ A_r + c_{r-1} B_{r-1} &= V'_r(B_r). && (46) \end{aligned}$$

If these equations are satisfied then

$$\begin{aligned} [B_1, A_1] &= c_1 [B_2, B_1] \\ \cdots &= c_\alpha [B_{\alpha+1}, B_\alpha] = \cdots \\ &= c_{r-1} [B_r, B_{r-1}] \ = [A_r, B_r] && (47) \end{aligned}$$

can be shown to hold easily. Moreover the following restrictions on the domain of A_1, A_r, B_α, where these matrices have nonzero entries, can be shown to be consistent with the Dyson-Schwinger equations:

$$\begin{aligned} (A_1)_{m,n} &= 0 \text{ except for } -\prod_{\alpha=1}^{r} (L_\alpha - 1) \leq n - m \leq -1 && (48) \\ (A_r)_{m,n} &= 0 \text{ except for } 1 \leq n - m \leq \prod_{\alpha=1}^{r} (L_\alpha - 1) && (49) \\ (B_\alpha)_{m,n} &= 0 \text{ except for } -\prod_{\beta>\alpha}^{r} (L_\beta - 1) \leq n - m \leq \prod_{\beta<\alpha}^{r} (L_\beta - 1) && (50) \end{aligned}$$

Our strategy is to solve the Dyson-Schwinger equations (46) perturbatively under the constraints (48)-(50) and impose the commutator equations (45) at the end. This approach is considerably simplified by the following

Second Lemma:

If (46) and (48)-(50) hold, then the commutators (47) are diagonal.

To prove this lemma we remind the reader that the chain (47) of equations follows from (46). On the other hand from (48)- (50) one can conclude that

$[B_1, A_1]$ is a lower left triangular matrix and

$[A_r, B_r]$ is an upper right triangular matrix.

Equality of both commutators proves that they are diagonal.

Then we can express the equation (45) by

$$\begin{aligned} (B_1A_1)_{nn} &= c_1(B_2B_1)_{nn} = \cdots \\ &= n + \text{ const. indep. of } n \end{aligned} \tag{51}$$

3. The perturbative expansion in the double scaling domain

Those coupling constants that appear as expansion coefficients in the potentials V_α (22), (23) are fixed to critical values. The corresponding critical action is then multiplied with

$$\frac{N}{g} \tag{52}$$

and tuned as

$$N \to \infty, \quad g \to g_c. \tag{53}$$

The matrix labels n, m become continuous variables in this limit

$$\begin{aligned} \frac{n}{N} &= \xi \\ 0 &< \xi \leq 1 \quad \text{(for the labels of the eigenvalues).} \end{aligned} \tag{54}$$

With the help of the string susceptibility exponent γ $(0 < -\gamma < \frac{1}{2})$ one defines

$$\frac{1}{N} = a^{2-\gamma} \tag{55}$$

$$\xi = \frac{g_c}{g}(1 - a^2x) \tag{56}$$

$$g = g_c(1 - a^2z) \tag{57}$$

$$z = (g_c - g)N^{\frac{2}{2-\gamma}}\,; \tag{58}$$

z is the single scale invariant variable, later we will introduce the "integration constants" h_s that play a similar role, and γ is as in (11), (18).

For the two-matrix models, be they symmetric or not, we proceed as follows: We set

$$(B_1)_{n,n+k} = r_k(n) \tag{59}$$

$$(B_2)_{n,n+k} = s_k(n) \tag{60}$$

and define

$$
\begin{aligned}
r_k(n-\tfrac{k}{2}) &= \rho_k + a^{-2\gamma} u_k(x) && (61)\\
s_k(n-\tfrac{k}{2}) &= \sigma_k + a^{-2\gamma} v_k(x). && (62)
\end{aligned}
$$

The translation of the arguments of r_k, s_k symmetrizes the support of the matrices B_1, B_2 around the diagonal and saves labour (and terms) in the perturbative expansions. For the coefficients $\{\rho_k, \sigma_k\}$ we make the ansatz

$$
\begin{aligned}
\sum_{k=-(l_2-1)}^{1} \rho_k z^k &= z\left(1-\frac{1}{z}\right)^{l_2} && (63)\\
\sum_{k=-1}^{l_1-1} \sigma_k z^k &= \frac{1}{z}(1-z)^{l_1} && (64)
\end{aligned}
$$

which is in agreement with the support conditions (48)-(50) if we set $l_1 = L_1$, $l_2 = L_2$ so that the singularity is maximal. Moreover we set

$$
c_1 = 1 \tag{65}
$$

in (46). The normalizations (63)-(65) can be changed; these renormalizations will be discussed at the end of this section.

Inserting (63)-(65) into the Dyson-Schwinger equations (46) gives critical coupling constants

$$
\begin{aligned}
g_k^{(1)} &\to g_k^{(l_1,l_2)} && (66)\\
g_k^{(2)} &\to g_k^{(l_2,l_1)} && (67)
\end{aligned}
$$

which can be presented in the explicit form

$$
\begin{aligned}
(-1)^k g_k^{(l_1,l_2)} &= \binom{l_1}{k} - \sum_{l_1 \geq m > k} \binom{l_1}{m}\binom{l_2(m-1)}{m-k} \\
&+ \sum_{l_1 \geq m > m' > k} \binom{l_1}{m}\binom{l_2(m-1)}{m-m'}\binom{l_2(m'-1)}{m'-k} \\
&\pm \ \ldots && (68)
\end{aligned}
$$

The Dyson-Schwinger equations can then be given as two systems of equations for the $\{v_k(x)\}$ and $\{u_{-k}(x)\}$

$$
\begin{aligned}
v_k(x) &= S_k([u_{-m}(x)], \{g_r^{(l_1,l_2)}\}), \quad 0 \leq k \leq l_1 - 1 && (69)\\
u_{-k}(x) &= R_{-k}([v_m(x)], \{g_r^{(l_2,l_1)}\}), \quad 0 \leq k \leq l_2 - 1. && (70)
\end{aligned}
$$

We first observe that there exist no equations for $v_{-1}(x)$ and $u_1(x)$, so that all $\{v_k(x), u_{-k}(x)\}$ are expressed in terms of these functions and their derivatives (plus the $\{w_k(x)\}$). Second we look at the linear approximation to (69), (70)

$$v_k(x) - \sum_{m=-1}^{l_2-1} \Omega_{km}^{(l_1,l_2)} u_{-m} = 0 \tag{71}$$

$$u_{-m}(x) - \sum_{k=-1}^{l_1-1} \Omega_{mk}^{(l_2,l_1)} v_k = 0 \tag{72}$$

and find that its corank is $l_2 - 2$ (remember $l_2 \leq l_1$). The general form of the matrix $\Omega^{(l_1,l_2)}$ is

$$\Omega_{km}^{(l_1,l_2)} = (-1)^{k+m} \sum_{n=k+m+2}^{l_1} (n-1)(-1)^n g_n^{(l_1,l_2)} \binom{l_2(n-2)}{n-(k+m+2)}. \tag{73}$$

Obviously $\Omega_{km}^{(l_1,l_2)}$ depends on only $k+m$. Due to the nonvanishing corank ($l_2 > 2$) we need $l_2 - 2$ additional functions $\{w_k\}$ which in the perturbative expansion of the Schwinger-Dyson equations appear each at order $a^{-k\gamma}$. Explicitly this expansion looks as

$$\begin{aligned} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_{l_1-1} \end{pmatrix} &= S^{(0)} \begin{pmatrix} u_1 \\ v_{-1} \end{pmatrix} + a^{-\gamma} S^{(1)} w_1 + a^{-2\gamma} S^{(2)} \begin{pmatrix} u_1'' \\ v_{-1}'' \\ u_1^2 \\ u_1 v_{-1} \\ v_{-1}^2 \\ w_2 \end{pmatrix} \\ &+ a^{-3\gamma} S^{(3)} \begin{pmatrix} w_1'' \\ w_1 u_1 \\ w_1 v_{-1} \end{pmatrix} + \mathrm{O}(a^{-4\gamma}) \end{aligned} \tag{74}$$

and correspondingly

$$\begin{pmatrix} u_0 \\ u_{-1} \\ u_{-2} \\ \vdots \\ u_{-l_2+1} \end{pmatrix} = R^{(0)} \begin{pmatrix} u_1 \\ v_{-1} \end{pmatrix} + \ldots \tag{75}$$

The matrices $S^{(n)}, R^{(n)}$ have rational numbers as entries.

On the other hand the Dyson-Schwinger equations imply that these new basis functions satisfy (to leading order) nonlinear differential equations

$$D_s(\{w_k\}; u_1, v_{-1}) = 0, \qquad 1 \leq s \leq l_2 - 2. \tag{76}$$

The introduction of the $\{w_k\}$ is not unambiguous, e.g. one can add to w_2 any linear combination of the functions appearing in the same column in (74) as w_2 itself or any higher order function such as $a^{-\gamma}.w_1''$. The set of all possible choices can be reduced by the requirement of gauge invariance (section 6) .

Since we have recursion relations for $s_m, \tilde{s}_m$ (37), (38)

$$\frac{s_m}{s_{m+1}} = r_1(m) \tag{77}$$

$$\frac{\tilde{s}_m}{\tilde{s}_{m+1}} = s_{-1}(m+1) \tag{78}$$

the First Lemma implies that all observables may depend only on

$$r_1(n-\tfrac{1}{2})s_{-1}(n+\tfrac{1}{2}) = 1 + a^{-2\gamma}(u_1(x)+v_{-1}(x)) + a^{-4\gamma}u_1(x)v_{-1}(x) \tag{79}$$

and other gauge invariant functions. Thus we are led to define as in (9)

$$u(x) = u_1(x) + v_{-1}(x) + a^{-2\gamma}u_1(x)v_{-1}(x). \tag{80}$$

We emphasize that (79), (80) depend on our normalization (63), (64)

$$\rho_1 = \sigma_{-1} = 1 \tag{81}$$

and must be changed if we renormalize these quantities (see below). Thus we conclude that also the functions $\{w_k\}_1^{l_2-2}$ must be introduced in such a way that the free energy, all expectation values, the leading orders of the differential operators in (8) and the equations (76) depend only on u and $\{w_k\}$.

Now we rescale N such that

$$a^{-\gamma} \to \Lambda a^{-\gamma}. \tag{82}$$

It is obvious from (61), (62) and (74), (75), (80) that to all functions u_k, v_k, u, w_k and their derivatives can be ascribed a dimension (or "degree", "grade") under (82)

$$\dim\{u_k, v_k, u\} = 2 \qquad (\text{all } k) \tag{83}$$

$$\dim\{w_k\} = k+2 \tag{84}$$

$$\dim\left\{\frac{d}{dx}\right\} = 1. \tag{85}$$

Let us consider c_1 as a free parameter in the Dyson-Schwinger equations but maintain the normalizations (63), (64). Then the critical coupling constants (68) are replaced by

$$g_k^{(l_1,l_2)} \to c_1 g_k^{(l_1,l_2)} \tag{86}$$

but the perturbative solution of the Dyson-Schwinger equations is otherwise unchanged. The asymptotic expressions (1), (2) for B_1, B_2 remain the same and the commutator equation is now

$$c_1\left[\sum_n a^{-n\gamma}P^{(n)}, \sum_m a^{-m\gamma}Q^{(m)}\right] = \begin{cases} 1 & (\text{case A}) \\ a^{-\gamma} & (\text{case B}) \end{cases} \tag{87}$$

Using the concept of dimension in the above mentioned sense and the dimensional homogeneity of the differential equations (76), (87), we can show that c_1 can be tuned to one by an appropriate rescaling.

We can do more. Renormalizing (63), (64) by replacing the r.h.s. by

$$d_1 z\left(1-\frac{1}{z}\right)^{l_2} \quad \text{resp.} \quad d_2\frac{1}{z}(1-z)^{l_1} \tag{88}$$

and keeping c_1 as in (65), we obtain a new gauge invariant function for which (9), (80) is a special case

$$u(x)=d_2u_1(x)+d_1v_{-1}(x)+a^{-2\gamma}u_1(x)v_{-1}(x). \tag{89}$$

This eliminates one degree of freedom in the renormalization. The remaining degree of freedom can be eliminated by a rescaling just as c_1. Thus we have shown that no relevant free parameters are left over in the characterization of the universality classes.

In case (A) the commutator

$$[P,Q]=\textstyle\sum_{s=0}^{l_2-2}\{r_s,\partial^s\}, \quad \partial=\frac{d}{dx}$$
$$(P=P^{(0)},Q=Q^{(0)}) \tag{90}$$

does not vanish identically and can therefore be set equal to 1. Now the second Lemma asserts that

$$r_s=0, \quad s\in\{1,2,\ldots,l_2-2\} \tag{91}$$

whereas

$$2r_0=1 \tag{92}$$

holds by fiat. The equations (91) are integrable to

$$\int r_s\,dx=h_s, \quad s\in\{1,2,\ldots,l_2-2\}. \tag{93}$$

The "constants of integration" h_s play the role of external scaling fields. Equation (92) can be integrated to

$$2\int r_0\,dx=x. \tag{94}$$

The differential equations determine u and all w_k as functions of x and h_s. This is the standard approach [7].

On the other hand we have in all cases studied that

$$D_s(w_k,u)=\int r_s\,dx=0. \tag{95}$$

Since we have tuned only one coupling constant g, the integration constants are automatically zero. Of course we can also introduce effective actions, e.g.

$$S_{\text{eff}}=\mathrm{Tr}\left\{Q^{\frac{l_1+l_2}{l_2}}+\sum_{s=0}^{l_2-2}t_sQ^{\frac{s+1}{l_2}}\right\} \tag{96}$$

for case (A), where

$$t_0 \sim x, \quad t_s \sim h_s \quad (s \geq 1) \tag{97}$$

Thus the known picture (as presented in [7], say) is completely reproduced by our perturbative approach.

4. Examples from the series (A)

In this and the subsequent section we study the differential operator limits of B_1 and B_2. For this purpose we need to know the numbers

$$P_n = \sum_k k^n \rho_k \tag{98}$$

$$\Sigma_n = \sum_n k^n \sigma_k \tag{99}$$

and the functions

$$U_n = \sum_k k^n u_k \tag{100}$$

$$V_n = \sum_k k^n v_k \tag{101}$$

From (63),(64) results

$$P_n = 0, \quad n < l_2 \tag{102}$$

$$P_{l_2} = l_2! \tag{103}$$

$$P_{l_2+1} = (1 - \frac{l_2}{2})(l_2+1)! \tag{104}$$

and

$$\Sigma_n = 0, \quad n < l_1 \tag{105}$$

$$\Sigma_{l_1} = (-)^{l_1} l_1! \tag{106}$$

$$\Sigma_{l_1+1} = (-)^{l_1+1}(1 - \frac{l_1}{2})(l_1+1)! \tag{107}$$

In terms of these quantities the asymptotic expansions of B_1 and B_2 are

$$a^{l_2\gamma} B_1 = \sum_{r=0}^{\infty} q_r(x)\partial^r = \sum_{n=0}^{\infty} a^{-n\gamma} Q^{(n)} \tag{108}$$

$$a^{l_1\gamma} B_2 = \sum_{s=0}^{\infty} p_s(x)\partial^s = \sum_{n=0}^{\infty} a^{-n\gamma} P^{(n)} \tag{109}$$

where

$$q_r(x) = \frac{1}{r!}[a^{-(r-l_2)\gamma} P_r + \sum_{n=0}^{\infty} a^{-(n+2-l_2)\gamma} \frac{U_{n+r}^{(n)}(x)}{2^n n!}] \tag{110}$$

$$p_s(x) = \frac{1}{s!}[a^{-(s-l_1)\gamma}\,\Sigma_s + \sum_{n=0}^{\infty} a^{-(n+2-l_1)\gamma}\,\frac{V^{(n)}_{n+s}(x)}{2^n n!}] \tag{111}$$

For the series (A) only $Q = Q^{(0)}$ and $P = P^{(0)}$ are of interest.

In the case $(l_1, l_2) = (4, 3)$ which describes the critical Ising model, we have only one function $w = w_1$ which we define by (exact to all orders)

$$u_0 = u_1 + v_{-1} + a^{-\gamma} w_1 \tag{112}$$

It turns out that this definition of w_1 can be chosen in all models (see, however, (197)). We obtain from (74), (75)

$$\begin{aligned}
U_0 &= 3w\,a^{-\gamma} + O(a^{-2\gamma}) && (113)\\
U_1 &= 3u + O(a^{-\gamma}) && (114)\\
U_2 &= O(1) && (115)\\
V_0 &= (\tfrac{2}{3}u'' + 2u^2)\,a^{-2\gamma} + O(a^{-3\gamma}) && (116)\\
V_1 &= 4wa^{-\gamma} + O(a^{-2\gamma}) && (117)\\
V_2 &= 8u + O(a^{-\gamma}) && (118)\\
V_3 &= O(1) && (119)
\end{aligned}$$

This implies

$$\begin{aligned}
Q &= \partial^3 + \frac{3}{2}\{u, \partial\} + 3w && (120)\\
P &= \partial^4 + 2\{u, \partial^2\} + 2\{w, \partial\} - \frac{1}{3}u'' + 2u^2 && (121)
\end{aligned}$$

and

$$\begin{aligned}
D_1(w, u) &= w'' + 6wu \\
&= \int r_1 dx = h_1 = 0 && (122)
\end{aligned}$$

These results are equivalent with those known [7].

The case $(l_1, l_2) = (5, 4)$ corresponds to the tricritical Ising model. We have two functions w_1, w_2 which we define by (112) and

$$u_{-1} = -5u_1 + v_{-1} + 2w_1\,a^{-\gamma} + (6u_1^2 + v_{-1}^2 + w_2)a^{-2\gamma} \tag{123}$$

it follows then from (74),(75)

$$\begin{aligned}
U_0 &= (u'' + 6u^2 + 4w_2)\,a^{-2\gamma} + O(a^{-3\gamma}) && (124)\\
U_1 &= 4w_1\,a^{-\gamma} + O(a^{-2\gamma}) && (125)\\
U_2 &= 8u + O(a^{-\gamma}) && (126)\\
U_3 &= O(1) && (127)\\
V_0 &= (-\tfrac{5}{4}w_1'' - 5uw_1)\,a^{-3\gamma} + O(a^{-4\gamma}) && (128)\\
V_1 &= (-\tfrac{5}{2}u'' - 10u^2 - 5w_2)\,a^{-2\gamma} + O(a^{-3\gamma}) && (129)\\
V_2 &= -10w_1\,a^{-\gamma} + O(a^{-2\gamma}) && (130)\\
V_3 &= -30u + O(a^{-\gamma}) && (131)\\
V_4 &= O(1) && (132)
\end{aligned}$$

The two differential operators come out as

$$Q = \partial^4 + \{2u, \partial^2\} + \{2w_1, \partial\} + 4w_2 + 6u^2 \tag{133}$$

$$-P = \partial^5 + \frac{5}{2}\{u, \partial^3\} + \frac{5}{2}\{w_1, \partial^2\} + \frac{5}{2}\{w_2 - \tfrac{1}{4}u'' + 2u^2, \partial\} + 5uw_1 \tag{134}$$

After the substitutions

$$u \to -\frac{1}{2}u \tag{135}$$

$$w_1 \to \frac{1}{2}w \tag{136}$$

$$4w_2 + 2u^2 \to v \tag{137}$$

these differential operators and the differential equations (92), (95) can be shown to be identical with those of Ginsparg et al. [7].

Now we come to the case $(l_1, l_2) = (6, 4)$ which was not studied before. In this case there is a common divisor of l_1 and l_2 : 2. Once again we can define w_1 and w_2 as in (112),(123) but we prefer to define w_2 b y

$$u_{-1} = -5u_1 + v_{-1} + 2w_1\, a^{-\gamma} + (\tfrac{9}{2}u_1^2 - \tfrac{1}{2}v_{-1}^2 - 3u_1v_{-1} + w_2)a^{-2\gamma} \tag{138}$$

because this minimizes the terms in the differential equations that we have to give explicitly. The differential operators are

$$Q = (\partial^2 + 2u)^2 + 2\{w_1, \partial\} - 4w_2 \tag{139}$$

$$P = (\partial^2+2u)^3+3\{w_1,\partial^3\}-3\{w_2,\partial^2\}+\{-\tfrac{1}{2}w_1''+6uw_1,\partial\}-2w_2''+6w_1^2-12uw_2 \tag{140}$$

By explicit calculation we found that

$$P = Q_+^{\frac{3}{2}} \tag{141}$$

Moreover we get the differential equations

$$\begin{aligned} h_2 &= D_2(w_1, w_2, u) \\ &= -w_1^{(4)} - 8uw_1'' - 12u'w_1' - 4u''w_1 - 24w_1w_2 && (142) \\ h_1 &= D_1(w_1, w_2, u) \\ &= w_2^{(4)} + 8uw_2'' + 4u'w_2' + 12w_2^2 - 10w_1w_1'' - 5w_1'^{\,2} - 24uw_1^2 && (143) \\ x &= \int [2r_0 + \tfrac{1}{2}r_2'' + 2u'(D_2(w_1, w_2, u) - h_2)]dx \\ &= 8w_1w_2'' - 8w_1'w_2' - 16w_1^3 - 2h_2u && (144) \end{aligned}$$

To make $2r_0$ integrable we had to add the term $2u'(D_2 - h_2)$. But this is also necessary in the (5,4) example (see [7]).

5. Examples from the series (B)

The simplest non trivial example is the symmetric case

$$(l_1, l_2) = (3, 3)$$

There exists one function $w_1 = w$ which we introduce by (112). In this example we obtain from perturbation theory

$$-\ P^{(0)} = Q^{(0)} = \partial^3 + \frac{3}{2}\{u, \partial\} + 3w \tag{145}$$

$$-\ P^{(1)} - Q^{(1)} = \partial^4 + 2\{u, \partial^2\} + 2\{w, \partial\} - \frac{1}{3}u'' + 2u^2 \tag{146}$$

and

$$D_1(w, u) = w'' + 6wu = 0 \tag{147}$$

$P^{(1)}$ and $Q^{(1)}$ are separately not gauge invariant.

Thus we have reproduced the differential operators and equations (120)-(122). The commutator

$$\left[\sum_n a^{-n\gamma} P^{(n)}, \sum_m a^{-m\gamma} Q^{(m)} \right] = \left[-P^{(1)} - Q^{(1)}, Q^{(0)} \right] a^{-\gamma} + O(a^{-2\gamma}) \tag{148}$$

gives an additionnal factor $a^{-\gamma}$ that enlarges the total power of $a^{-\gamma}$ in $[B_2, B_1]$ to

$$3 + 3 + 1 = 7 = 4 + 3 \tag{149}$$

as is necessary for the universality class $[4, 3]$.

Now we turn to the rather complicated case

$$(l_1, l_2) = (6, 3)$$

Again we introduce $w_1 = w$ by (112). The differential operators are

$$Q^{(0)} = \partial^3 + \frac{3}{2}\{u, \partial\} + 3w \tag{150}$$

$$Q^{(1)} = -\frac{1}{2}\partial^4 - \frac{1}{4}\{u_1 + 7v_{-1}, \partial^2\} - \{w, \partial\} + \frac{1}{24}(u_1'' + 13v_{-1}'') + 3u_1^2 - u^2 \tag{151}$$

$$P^{(0)} = (Q^{(0)})^2 \tag{152}$$

$$\begin{aligned} P^{(1)} = {} & 2\partial^7 + \frac{1}{2}\{17u_1 + 11v_{-1}, \partial^5\} + 7\{w, \partial^4\} \\ & + \frac{1}{2}\{-\tfrac{1}{12}(281u_1'' + 161v_{-1}'') + 43u_1^2 + 56u_1v_{-1} + 19v_{-1}^2, \partial^2\} \\ & + \{\tfrac{1}{12}(83u_1'''' + 47v_{-1}'''') - \tfrac{1}{8}(155u_1u_1'' + 119u_1''v_{-1} + 107u_1v_{-1}'' + 71v_{-1}v_{-1}'') \\ & - \tfrac{9}{4}(15(u_1')^2 + 22u_1'v_{-1}' + 7(v_{-1}')^2) \\ & + 26u_1^3 + 42u_1^2v_{-1} + 24u_1v_{-1}^2 + 8v_{-1}^3 + 30w^2\ ,\ \partial\} \\ & + \frac{11}{6}w'''' + 3uw'' - \frac{1}{2}w'(49u_1' + 31v_{-1}') - \frac{1}{4}w(15u_1'' + 3v_{-1}'') \\ & + 18w(3u_1^2 + 4u_1v_{-1} + 2v_{-1}^2) \end{aligned} \tag{153}$$

Both $P^{(1)}$ and $Q^{(1)}$ are gauge dependent. We consider the commutator

$$\begin{aligned}
&\Big[\sum_n a^{-n\gamma}P^{(n)}\ ,\ \sum_m a^{-m\gamma}Q^{(m)}\Big] = \\
&\Big[\sum_n a^{-n\gamma}P^{(n)} - (\sum_m a^{-m\gamma}Q^{(m)})^2\ ,\ \sum_m a^{-m\gamma}Q^{(m)}\Big] = \\
&\Big[\ P^{(1)} - Q^{(0)}Q^{(1)} - Q^{(1)}Q^{(0)}, Q^{(0)}\ \Big]\, a^{-\gamma} + O(a^{-2\gamma})
\end{aligned} \tag{154}$$

The first differential operator in (154) can be calculated and is

$$P^{(1)} - (Q^{(0)}Q^{(1)} + Q^{(1)}Q^{(0)}) = 3(Q^{(0)})_+^{\frac{7}{3}} \tag{155}$$

The factor 3 can be eliminated by rescaling as was discussed in section 3. The additional factor $a^{-\gamma}$ changes the total power of $a^{-\gamma}$ in $[B_2, B_1]$ to

$$6 + 3 + 1 = 7 + 3 \tag{156}$$

as is necessary for the universality class $[7,3]$. Thus this class has been proven to arise.

We remember that one-matrix models lead to universality classes $[2m+1,2]$, and we could imagine that such classes can also be produced from two-matrix models with singularities

$$(l_1, l_2) = (2m, 2)$$

However , if we insert an ansatz for such singularity into the Dyson-Schwinger equations, the perturbative solution cannot be made to satisfy the commutator equation

$$[B_2, B_1] = 1 \tag{157}$$

at any order in $a^{-\gamma}$ as we shall show in the sequel.

Of course a quadratic potential V_2 can be eliminated from the partition function by performing a Gaussian integral. In the Dyson-Schwinger equations this comes out as follows. If

$$A_1 + cB_2 = V_1'(B_1) \tag{158}$$

$$A_2 + cB_1 = g_1^{(2)} + g_2^{(2)}B_2\ ,\quad (g_2^{(2)} \neq 0) \tag{159}$$

then B_2 can be eliminated by substituting (159) into (158)

$$\begin{aligned}
A_1 + \frac{c}{g_2^{(2)}}A_2 &= \frac{c}{g_2^{(2)}}(g_1^{(2)} - cB_1) + V_1'(B_1) \\
&= \tilde{V}(B_1)
\end{aligned} \tag{160}$$

Now A_1 is strictly lower and A_2 strictly upper triangular and under transposition

$$B_1 \to B_1^T \tag{161}$$

they exchange their role in (160). In the one-matrix model there is only one orthogonal system $\{\Pi_m(\lambda)\}$ (section 2) implying that B_1 is symmetric. In this case (160) reduces to

$$A_1 + A_1^T = \tilde{V}'(B_1) \tag{162}$$

Via the antisymmetric matrix

$$C = A_1 - \frac{1}{2}\tilde{V}'(B_1) \tag{163}$$

and the double scaling limit

$$\begin{aligned} B_1 &\to a^{-2\gamma}Q &(164)\\ C &\to a^{-(2m+1)\gamma}P &(165)\end{aligned}$$

the universality class $[2m+1, 2]$ is obtained.

Instead we solve the Dyson-Schwinger equations (158), (159) perturbatively as they stand. First we set

$$L_1 = l_1 = 4\ ,\ m = 2 \tag{166}$$

Our algorithm simplifies by the equalities to all orders

$$\begin{aligned} u_0(x) &= v_0(x) &(167)\\ u_{-1}(x) &= v_{-1}(x) &(168)\end{aligned}$$

Due to the fact that no functions $\{w_k\}$ are needed, we obtain only even powers in the expansions (1), (2)

$$B_1 \to a^{-2\gamma}\sum_{n=0}^{\infty} a^{-2n\gamma}Q^{(2n)} \tag{169}$$

$$B_2 \to a^{-4\gamma}\sum_{n=0}^{\infty} a^{-2n\gamma}P^{(2n)} \tag{170}$$

We find

$$\begin{aligned} Q^{(0)} &= \partial^2 + 2u &(171)\\ Q^{(2)} &= \frac{1}{12}\partial^4 + \frac{1}{4}\{u,\partial^2\} - \frac{1}{8}u'' - \frac{1}{4}(u_1 - v_{-1})^2 &(172)\\ P^{(0)} &= (Q^{(0)})^2 &(173)\\ P^{(2)} &= \frac{2}{3}\partial^6 + \frac{1}{12}\{41u_1 + 11v_{-1}, \partial^4\}\\ &\quad +\{-\tfrac{1}{8}(23u_1'' + 3v_{-1}'') + 16u_1^2 + 15u_1v_{-1} + v_{-1}^2, \partial^2\}\\ &\quad +\frac{1}{4}(5u_1''' + v_{-1}''') - \frac{1}{2}(15u_1'^2 + 9u_1'v_{-1}')\\ &\quad +\frac{1}{8}(-23u_1u_1'' + 5u_1v_{-1}'' - 5v_{-1}u_1'' + 7v_{-1}v_{-1}'')\\ &\quad +\frac{1}{4}(27u_1^3 + 67u_1^2v_{-1} + 37u_1v_{-1}^2 - 3v_{-1}^3) &(174)\end{aligned}$$

An almost trivial observation is that all $P^{(2n)}$, $Q^{(2n)}$ are symmetric operators. Consequently the commutator

$$\begin{aligned}
&\Big[\sum_n a^{-2n\gamma}P^{(2n)}\ ,\ \sum_m a^{-2m\gamma}Q^{(2m)}\Big] = \\
&\Big[\,P^{(2)} - Q^{(0)}Q^{(2)} - Q^{(2)}Q^{(0)}, Q^{(0)}\,\Big]\, a^{-2\gamma} + O(a^{-4\gamma}) = \\
&\sum_{k=0}^{5}\{r_k^{(2)}, \partial^k\}\, a^{-2\gamma} + O(a^{-4\gamma}) \qquad (175)
\end{aligned}$$

is antisymmetric

$$r_0^{(2)} = r_2^{(2)} = r_4^{(2)} = 0 \qquad (176)$$

This symmetry argument is valid to any order (and for any m)

$$r_0^{(2n)} = 0\ ,\ \text{all n} \qquad (177)$$

and therefore the commutator relation cannot be satisfied.

Only as a side remark we add that the three differential equations

$$r_1^{(2)} = r_3^{(2)} = r_5^{(2)} = 0 \qquad (178)$$

have the unique solution

$$u_1 = v_{-1} \qquad (179)$$

which implies

$$B_1 = B_1^T \qquad (180)$$

6. Expectation values and gauge invariance

The basis functions $\{w_k\}$ have hitherto been introduced (see (112), (123),(138)) by the requirement that the relevant objects, namely differential operators and equations are gauge invariant. This fixes them only at low orders. Expectation values are a simple device to extend this definition to higher orders.

For any two-matrix model we obtain for a finite N using (31), (37),(48) and (77), (78) that the partition function can be expressed as

$$Z = C(N)N!(s_0\tilde{s}_0)^{-N}\prod_{k=1}^{N-1}(r_1(N-k-1)s_{-1}(N-k))^k \qquad (181)$$

so that the free energy is

$$\begin{aligned}
F &= \log Z = \log(C(N)N!) - N\log(s_0\tilde{s}_0) \\
&+ a^{-2\gamma}(1-a^2z)^{-2}\int_{a^{-2}}^{z} dx(z-x)\log(1+a^{-2\gamma}u(x)) \\
&+ \text{ corrections from Euler's summation formula.} \qquad (182)
\end{aligned}$$

The latter corrections in (182) form an infinite series. The function $u(x)$ increases for $x \to \infty$ and one can convince oneself that

$$a^{-2\gamma} u(a^{-2}) = O(1) \tag{183}$$

so that the perturbative expansion of the integral in (182) fails at the lower end. The corrections due to Euler's summation formula contain terms

$$u^{(n)}(a^{-2}), \quad n \geq 0. \tag{184}$$

Therefore we set

$$F(z) = F(0) + (F(z) - F(0)) \tag{185}$$

and expand only the difference perturbatively. A similar caveat must be applied to the expectation values.

Expectation values

$$\langle G(M^{(1)}, M^{(2)}) \rangle \tag{186}$$

with respect to the partition function (25) or (31) can apparently be obtained by parametric differentiation, if G is a polynomial of the variables

$$\mathrm{Tr}(M^{(1)})^a, \quad \mathrm{Tr}(M^{(2)})^b, \quad \mathrm{Tr}(M^{(1)} M^{(2)}). \tag{187}$$

After reduction of the partition function by means of Mehta's theorem, these variables correspond to

$$\sum_i \lambda_i^a, \quad \sum_i \mu_i^b, \quad \sum_i \lambda_i \mu_i, \quad (\lambda_i = \lambda_i^{(1)}, \mu_i = \lambda_i^{(2)}). \tag{188}$$

First we show that the expectation value of the third variable with a reduced partition function is not gauge invariant.

Let us denote the trace over the upper left $N \times N$ submatrix of an infinite ($\mathbb{N} \times \mathbb{N}$) matrix by Tr_N. Then insertion of

$$\sum_i \lambda_i \mu_i$$

into (31) gives for fixed N

$$\mathrm{Tr}_N(B_1 B_2)$$

which can be evaluated to yield

$$\begin{aligned} & N \quad \binom{l_1 + l_2}{l_1} - \left[\frac{(l_1 + l_2 - 1)!}{l_1! l_2!} (l_1 l_2 - l_1 - l_2) + 1 \right] \\ & - \quad a^{-\gamma} \int_{a^{-2}}^{z} dx \left\{ \sum_k [\rho_{-k} v_k(x) + \sigma_k u_{-k}(x)] \right\} + \mathrm{O}(a^{-3\gamma}) \end{aligned} \tag{189}$$

for any maximal singularity (l_1, l_2). For $(l_1, l_2) = (4, 3)$ the integral is to leading order (using still (112))

$$14 u_1(x) - 43 v_{-1}(x) - 17 w_1(x) a^{-\gamma} \tag{190}$$

which is not gauge invariant. Therefore only the first two expressions in (188) make sense.

Expectation values of monomials can be reduced easily (see [19])

$$\begin{aligned}\langle \mathrm{Tr}(M^{(1)})^a \mathrm{Tr}(M^{(1)})^b\rangle &= \mathrm{Tr}_N(B_1^a)\mathrm{Tr}_N(B_1^b) + \mathrm{Tr}_N(B_1^a \Pi_N B_1^b) && (191)\\ \langle \mathrm{Tr}(M^{(1)})^a \mathrm{Tr}(M^{(2)})^b\rangle &= \mathrm{Tr}_N(B_1^a)\mathrm{Tr}_N(B_2^b) + \mathrm{Tr}_N(B_1^a \Pi_N B_2^b) && (192)\end{aligned}$$

where Π_N is the projection matrix

$$(\Pi_N)_{rs} = \begin{cases} \delta_{rs} & \text{if } r \geq N \text{ and } s \geq N \\ 0 & \text{else.} \end{cases} \qquad (193)$$

Of course for the simple cases of one variable (187) we have from (191), (192)

$$\begin{aligned}\langle \mathrm{Tr}(M^{(1)})^a\rangle &= \mathrm{Tr}_N(B_1)^a && (194)\\ \langle \mathrm{Tr}(M^{(2)})^b\rangle &= \mathrm{Tr}_N(B_2)^b. && (195)\end{aligned}$$

Consider (194) with $a = 1$. Then

$$\begin{aligned}\mathrm{Tr}_N B_1 &= N\rho_0 - a^{-\gamma}\int_{a^{-2}}^{z} dx u_0(x)\\ &+ \text{ corrections from Euler's summation formula.} && (196)\end{aligned}$$

We recognize that instead of (112) we better use

$$\begin{aligned}u_0(x) &= u_1 + v_{-1} + a^{-\gamma}w_1 + a^{-2\gamma}u_1 v_{-1}\\ &= u + a^{-\gamma}w_1. && (197)\end{aligned}$$

The same argument can be applied to $M^{(2)}$

$$\begin{aligned}\mathrm{Tr}_N B_2 &= N\sigma_0 - a^{-\gamma}\int_{a^{-2}}^{z} dx v_0(x)\\ &+ \text{ corrections from Euler's summation formula.} && (198)\end{aligned}$$

so that v_0 must depend only on u and w_1. Indeed we find for the maximal singularity $(4,3)$ from (197)

$$v_0 = u - w_1 a^{-\gamma} - (\tfrac{1}{12}u'' + u^2)a^{-2\gamma} + \tfrac{2}{5}(\tfrac{1}{6}w_1'' + uw_1)a^{-3\gamma} + \mathrm{O}(a^{-4\gamma}) \qquad (199)$$

and for $(3,3)$

$$v_0 = u - w_1 a^{-\gamma} - (\tfrac{1}{12}u'' + u^2)a^{-2\gamma} + 2(\tfrac{1}{6}w_1'' + uw_1)a^{-3\gamma} + \mathrm{O}(a^{-4\gamma}). \qquad (200)$$

Finally we consider the cases $a = 2$ in (194) and $b = 2$ in (195).

To sum up the results we introduce abbreviations

$$\begin{aligned}\phi_1(x) &= u(x) && (201)\\ \phi_2(x) &= w_1(x) && (202)\\ \phi_3(x) &= \tfrac{1}{12}u''(x) + u^2(x) && (203)\\ \phi_3(x) &= \tfrac{1}{6}w_1''(x) + u(x)w_1(x) && (204)\end{aligned}$$

and

$$\Phi_n(z) = \int_{a^{-2}}^{z} \phi_n(x)\,dx. \tag{205}$$

We write for (194), (195)

$$\gamma_1 N + \gamma_0 + \sum_{n\geq 1} c_n \Phi_n(z) a^{-n\gamma} \tag{206}$$

and obtain the coefficients in the following table

sing	a	b	γ_1	γ_0	c_1	c_2	c_3	c_4
(4,3)	1		-3	0	-1	-1	0	0
	2		15	-3	4	2	-1	$-\frac{22}{5}$
		1	-4	0	-1	1	1	$-\frac{2}{5}$
		2	28	-6	6	-4	-5	-10
(3,3)	1		-3	0	-1	-1	0	0
	2		15	-3	4	2	-1	2
		1	-3	0	-1	1	1	-2
		2	15	-3	4	-2	-3	18

Table 1. Expansion coefficients for the expectation values of (194), (195) for the singularities $(4,3)$ and $(3,3)$.

The general expression for γ_1 is

$$\gamma_1 = \begin{cases} (-1)^a \binom{a\, l_2}{a}, & \text{respectively} \\ (-1)^b \binom{b\, l_1}{b}. \end{cases} \tag{207}$$

7. References

1. E. Brézin, V.A. Kazakov, *Phys. Lett.* **B236** (1990) 144.
2. D.J. Gross, A.A. Migdal, *Phys. Rev. Lett.* **64** (1990) 127.
3. M.R. Douglas, S.H. Shenker, *Nucl. Phys.* **B335** (1990) 635.
4. V.A. Kazakov, *Phys. Lett.* **A119** (1986) 140.
5. E. Brézin, M.R. Douglas, V.A. Kazakov, S.H. Shenker *Phys. Lett.* **B237** (1990) 43.
6. Č. Crnković, P. Ginsparg, G. Moore *Phys. Lett.* **B237** (1990) 196 .
7. P. Ginsparg, M. Goulian, M.R. Plesser *Nucl. Phys.* **B342** (1990) 539.
8. M.R. Douglas, *Phys. Lett.* **B238** (1990) 176.
9. M.R. Douglas, *1990 Cargèse workshop on : " Random Surfaces and Quantum Gravity ".* Eds. O. Alvarez, E. Marinari, P. Windey. *NATO ASI Series B : Physics* **Vol. 262** (1992) 77.

10. I.M. Gelfand, L.A. Dikii, *Russian Math. Surveys* **30,5** (1975) 77.
11. M.L. Mehta. *Commun. Math. Phys.* **79** (1981) 327.
12. S. Chadha, G. Mahoux, M.L. Mehta *J. Phys. A*, **14** (1981) 579.
13. P. di Francesco, P. Ginsparg, J. Zinn-Justin, *Physics Reports* **254** (1995) 1-133.
14. T. Tada, M. Yamaguchi, *Phys. Lett.* **B250** (1990) 38.
15. T. Tada, *Phys. Lett.* **B259** (1991) 442.
16. J.-M. Daul, V.A. Kazakov, I.K. Kostov, *"Rational Theories of 2d-Gravity from the Two-Matrix Model "*, **CERN-TH 6834/93**.
17. M. Kreuzer, R. Schimmrigk, *Phys. Lett.* **B248** (1990) 51.
18. M. Kreuzer, *Phys. Lett.* **B254** (1991) 81.
19. T. Banks, M.R. Douglas, N. Seiberg, S.H. Shenker, *Phys. Lett.* **B238** (1990) 279.