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CLASSICAL NON-ABELIAN SOLITONS

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Abstract

In two space-time dimensions a class of classical multicomponent scalar field theories with discrete, in general non-Abelian global symmetry is considered. The corresponding soliton solutions are given for the cases of 2, 3, and 4 components.

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Recently renewed interest in solitons has arisen in connection with exceptional statistics occuring in low-dimensional quantum field theory. The nonperturbative approach to quantum solitons [1, 2, 3, 4, 5], based on the notion of a disorder variable [6, 7], does not make use of the well-known semiclassical quantisation procedure around classical soliton solutions [8]. In a recent article [9] the author introduced multicomponent scalar field models, treated nonperturbatively on a Euclidean space-time lattice. The exponentially decaying disorder correlation functions are connected with soliton fields showing non-Abelian braid group statistics. It is the aim of this note to present the corresponding classical soliton solutions, which do not seem to have appeared in the literature.

In the two-dimensional Minkowski space with its space-time points denoted by (t, x) the models considered in [9] look as follows. A set of n real scalar fields $\{\phi_k(t,x)\}$, $k = 1, \ldots, n$, is introduced with the classical Lagrangian density functional

$$L(\{\phi_k(t,x)\}) = \frac{1}{2}\sum_{k=1}^{n}\{(\partial_t\phi_k(t,x))^2 - (\partial_x\phi_k(t,x))^2\} - v(\{\phi_k(t,x)\}) \tag{1}$$

The interaction v is a fourth order polynomial, with $\lambda, \xi, b \in \mathbf{R}_+$,

$$v(\{\phi_k\}) = \lambda\{(|\phi|^2 - \xi)^2 + 4b \sum_{1\leq k<l\leq n} (\phi_k\phi_l)^2 \tag{2}$$

$$|\phi|^2 = \sum_{k=1}^{n} \phi_k^2. \tag{3}$$

The Lagrangian density (1) for given n is invariant under a corresponding global discrete group G transforming the fields. Moreover, the interaction (2) has a degenerate absolute minimum on which already a subgroup $G_0 \subset G$ acts transitively. The cases $n = 2, 3, 4$ are treated.

i) $n = 2$: We introduce the complex field $\phi = \phi_1 + i\phi_2$, then G_0 is the abelian group Z_4 and v has a fourfold degenerate absolute minimum at $\phi = \zeta_m$, $m = 1, \ldots 4$

$$\zeta_1 = -\zeta_3 = \sqrt{\xi}, \quad \zeta_2 = -\zeta_4 = i\sqrt{\xi} \tag{4}$$

ii) $n = 3$: We view $\phi = (\phi_1, \phi_2, \phi_3) \in \mathbf{R^3}$ as a real vector field, then G_0 is the octahedron group acting as a subgroup of the standard representation of $O(3)$ and v has a sixfold degenerate absolute minimum at $\phi = \zeta_m$, $m = 1, \ldots, 6$

$$\zeta_1 = -\zeta_3 = (\sqrt{\xi}, 0, 0), \quad \zeta_2 = -\zeta_4 = (0, \sqrt{\xi}, 0), \quad \zeta_5 = -\zeta_6 = (0, 0, \sqrt{\xi}) \tag{5}$$

iii) $n = 4$: We introduce the complex vector field $\phi = (\phi_1 + i\phi_3, \phi_2 + i\phi_4) \in \mathbf{C^2}$ and have $G_0 = \{\pm\sigma_0, \pm i\sigma_1, \pm i\sigma_2, \pm i\sigma_3\} \subset SU(2)$, with the unit matrix σ_0 and the Pauli matrices σ_k, $k = 1, 2, 3$. The interaction v has an eightfold degenerate absolute minimum at $\phi = \zeta_m$, $m = 1, \ldots, 8$

$$\begin{aligned} \zeta_1 &= -\zeta_5 = -i\zeta_3 = i\zeta_7 = (\sqrt{\xi}, 0), \\ \zeta_2 &= -\zeta_6 = -i\zeta_4 = i\zeta_8 = (0, \sqrt{\xi}) \end{aligned} \tag{6}$$

In the following we restrict to the distinguished value $b = 1$ in (2). Then (1) implies the classical field equations

$$\{(\partial_t)^2 - (\partial_x)^2\}\phi_k + 4\lambda\phi_k\{|\phi|^2 - \xi + 2(|\phi|^2 - \phi_k^2)\} = 0, \tag{7}$$

$$k = 1, \ldots, n,$$

and the Hamiltonian functional

$$H(\{\pi_k, \phi_k\}) = \int_{-\infty}^{\infty} dx \{\frac{1}{2}\sum_{k=1}^{n}[(\pi_k(t,x))^2 + (\partial_x\phi_k(t,x))^2] + v(\{\phi_k(t,x)\})\} \geq 0 \tag{8}$$

where π_k is the momentum canonically conjugate to ϕ_k.

In the cases $n = 2, 3, 4$ considered there are two classes of stationary classical solutions $\phi(t,x) = \varphi(x)$ of (7) having finite energy: α) the obvious vacuum solutions $\varphi(x) \equiv \zeta_m$, β) soliton solutions $\varphi(x)$ satisfying $\varphi(+\infty) = \zeta_l$, $\varphi(-\infty) = \zeta_m$, $l \neq m$, to be presented. The conserved topological current

$$j_k^\mu(t,x) = \varepsilon^{\mu\nu}\partial_\nu\phi_k(t,x) \tag{9}$$

implies for a given solution $\varphi(x)$ the "topological charge" $\varphi(+\infty) - \varphi(-\infty)$, which is a vector in $\mathbf{C}$, $\mathbf{R^3}$ and $\mathbf{C^2}$ respectively, for the cases considered.

Due to the symmetry group G_0 the individual minima in the degenerate absolute minimum are physically equivalent. Hence it suffices to list only solutions having the boundary value $\varphi(+\infty) = \zeta_1$, with ζ_1 selected arbitrarily. We label the different soliton types by the respective maps T defined by

$$\varphi(-\infty) = T\varphi(+\infty). \tag{10}$$

Then the topological charge can be written as

$$\varphi(+\infty) - \varphi(-\infty) = (1 - T)\varphi(+\infty). \tag{11}$$

If $\varphi(+\infty)$ has a stabilizer group $H \in G_0$, as in *ii*), T is any representative of the coset decomposition of G_0 with respect to H.

Listing the soliton solutions the shorthand

$$y = (x-s)\sqrt{2\lambda\xi} \tag{12}$$

with $s \in \mathbf{R}$ is used.

i)

$$\varphi(x) \in \mathbf{C} \quad ; \quad \varphi(-\infty) = e^{i\alpha}\zeta_1 \tag{13}$$

There are 3 different soliton types

$$e^{i\alpha} = -1 \quad : \quad \varphi(x) = \sqrt{\xi}\tanh y \tag{14}$$

$$e^{i\alpha} = \pm i \quad : \quad \varphi(x) = \frac{1}{2}\sqrt{\xi}\{1+\tanh y \pm i(1-\tanh y)\} \tag{15}$$

ii)

$$\varphi(x) \in \mathbf{R^3} \quad ; \quad \varphi_k(-\infty) = \sum_{l=1}^{3} D_{kl}(\zeta_1)_l, \quad k = 1,2,3. \tag{16}$$

There are 5 different soliton types. Denoting by $D^{(l)}(\alpha)$ the rotation around the l - axis with the rotation angle α we find

$$D = D^{(3)}(\pi) \quad : \quad \varphi(x) = \sqrt{\xi}(\tanh y, 0, 0) \tag{17}$$

$$D = D^{(3)}(\pm\frac{\pi}{2}) \quad : \quad \varphi(x) = \frac{1}{2}\sqrt{\xi}(1+\tanh y, \pm(1-\tanh y), 0) \tag{18}$$

$$D = D^{(2)}(\pm\frac{\pi}{2}) \quad : \quad \varphi(x) = \frac{1}{2}\sqrt{\xi}(1+\tanh y, 0, \mp(1-\tanh y)) \tag{19}$$

iii)

$$\varphi(x) \in \mathbf{C^2} \quad ; \quad \varphi_k(-\infty) = \sum_{l=1}^{2} u_{kl}(\zeta_1)_l, \quad k = 1,2. \tag{20}$$

There are 7 different soliton types

$$u = -\sigma_0 \quad : \quad \varphi(x) = \sqrt{\xi}(\tanh y, 0) \tag{21}$$

$$u = \pm i\sigma_1 \quad : \quad \varphi(x) = \frac{1}{2}\sqrt{\xi}(1+\tanh y, \pm i(1-\tanh y)) \tag{22}$$

$$u = \pm i\sigma_2 \quad : \quad \varphi(x) = \frac{1}{2}\sqrt{\xi}(1+\tanh y, \mp(1-\tanh y)) \tag{23}$$

$$u = \pm i\sigma_3 \quad : \quad \varphi(x) = \frac{1}{2}\sqrt{\xi}(1+\tanh y \pm i(1-\tanh y), 0) \tag{24}$$

The energy $E(\varphi)$ of these solutions follows from (8): $E = \frac{4}{3}\xi(2\lambda\xi)^{1/2}$ for (14), (17), (21), whereas $E = \frac{2}{3}\xi(2\lambda\xi)^{1/2}$ for the other ones.

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