ON THE CLASSIFICATION OF VECTOR BUNDLES ON PROJECTIVE CURVES

YURI A. DROZD AND GERT-MARTIN GREUEL

Abstract. We consider the "representation type" of the classification problem of vector bundles on a projective curve. We prove that this problem is always either finite, or tame, or wild and we completely describe those curves which are of finite, resp. tame, vector bundle type. We also give a complete list of indecomposable vector bundles for the finite and tame cases.

Introduction

Vector bundles over projective varieties, in particular, over projective curves have been widely studied. Nevertheless, rather little seems to be known about the classification of *all* vector bundles over some variety. Maybe, the only results here are those of Grothendieck [8] for the projective line and of Atiyah [1] for elliptic curves. On the other hand, the classification of vector bundles on projective curves is closely related to the study of Cohen-Macaulay modules on surface singularities, due to the work of Kahn [10]. Hence, it is important to have some ideas about the complexity of these classification problems.

This article is devoted to the study of vector bundles over projective curves from the viewpoint of representation theory. The latter often distinguishes the following three main cases of the classification problem:

- "finite," when indecomposable objects are completely defined by some "discrete" parameters (this is the case for the projective line);
- "tame," when indecomposable objects form "small" (usually only one-parameter) families (this is the case for elliptic curves);
- "wild," when there are families of indecomposable modules depending of any prescribed number of parameters.

Moreover, in the wild case the problem usually becomes "versal," i.e., containing other, more or less arbitrary, classification problems (cf. [4, 5]).

It turns out that this is the case also for vector bundles over projective curves. Namely, we prove that their classification problem is always either finite, or tame, or wild in the above sense (Theorem 6.4). Moreover, except for the cases of Grothendieck and Atiyah, the curves

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with finite and tame classification problems for vector bundles are of very special type. They are some sorts of "configurations of projective lines," i.e., their irreducible components are rational and their singular points are simple double points. Even among such configurations the finite and tame cases are distinguished by very restrictive conditions: their dual graphs (intersection diagrams) are either chains (in the finite case) or cycles (in the tame case).

The methods we use are the usual ones in representation theory. Namely, the techniques of "matrix problems," widely used, for instance, to prove the above "trichotomy" (cf. [4, 5]) or to determine the types of some classes of classification problems. Fortunately, after eliminating the wild cases, we come to a known matrix problem (the so called "Gelfand problem" in the version due to Bondarenko [2]). This gives us the possibility to obtain a complete list of indecomposable vector bundles for the finite and tame cases.

The reduction of the problem of classification of vector bundles or of torsion free sheaves on projective curves to matrix problems is similar to the approach of [5] or [7] in the local case of torsion free modules on curve singularities. However, the methods developed in this paper go further and should provide, among others, effective tools for studying all vector bundles on an arbitrary reduced projective curve, in particular, for the construction and investigation of appropriate moduli spaces.

We add two appendices. In the first one we specialize the list of indecomposable objects from [2] to our case, where it is a bit simpler than in general. The second one is devoted to a class of finite dimensional algebras which seem to be closely related to cyclic configurations of projective lines. We think that such relations could be interesting and useful both for algebraic geometry and for representation theory, as was the case, for instance, for "canonical algebras" and weighted projective lines (cf. [6]).

1. VB-type of a curve

Throughout this section we use the following notations:

Notations 1.1. 1. *C is an algebraic curve over an algebraically closed field $\mathbf{k}$, which we suppose to be* reduced *and* connected *but usually* singular *and even* reducible.

2. *$\mathcal{O} := \mathcal{O}_C$ denotes the structure sheaf of C and $\mathcal{K}$ denotes the sheaf of* rational functions *on C (its stalk at a point x is the full ring of quotients of $\mathcal{O}_x$).*
3. *$\mathsf{VB} = \mathsf{VB}(C)$ is the category of (finite dimensional) vector bundles on C or, equivalently, that of locally free (coherent) sheaves on C. (We identify vector bundles with the corresponding locally free sheaves and in our case it is more convenient to deal with sheaves.)*

4. *Let $\mathcal{M}$ be a sheaf of $\mathcal{O}$-modules. Call the* torsion part *of the sheaf $\mathcal{M}$ and denote by $\mathsf{t}(\mathcal{M})$ the kernel of the natural homomorphism $\mathcal{M} \to \mathcal{K}\otimes_{\mathcal{O}}\mathcal{M}$. The sheaf $\mathcal{M}$ is said to be* torsion free *if $\mathsf{t}(\mathcal{M}) = 0$ and* torsion *if $\mathsf{t}(\mathcal{M}) = \mathcal{M}$. In what follows we always identify a torsion free sheaf $\mathcal{M}$ with its image in $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{M}$.*

Obviously, $\mathcal{M}$ is torsion if and only if , for every point $x \in C$ and for every element $t \in \mathcal{M}_x$, there is a non-zero-divisor $a \in \mathcal{O}_x$ such that $at = 0$; $\mathcal{M}$ is torsion free if and only if , for every non-zero $t \in \mathcal{M}_x$ and for every non-zero-divisor $a \in \mathcal{O}_x$, $at \neq 0$. It is also clear that $\mathsf{t}(\mathcal{M})$ is the biggest torsion subsheaf of $\mathcal{M}$, while $\mathcal{M}/\mathsf{t}(\mathcal{M})$ is torsion free.

We are going to define the *vector bundle type* (*VB-type*) of a curve, i.e., its type with respect to the classification of vector bundles on it. We take into consideration that such classification involves evident "discrete" parameters, namely, rank and degree. However, if the curve has several irreducible components, these parameters become more complicated.

Definition 1.2. *Let C be a projective curve, $C = \cup_{i=1}^{s} C_i$ its decomposition into irreducible components, $\mathcal{B}$ a vector bundle over C and $\mathcal{B}_i$ the restriction of $\mathcal{B}$ onto C_i. The* vector-degree *of $\mathcal{B}$ is defined as the vector* $\operatorname{Deg}\mathcal{B} := (d_1, d_2, \dots, d_s)$, *where* $d_i = \deg \mathcal{B}_i$.

In particular, the mapping Deg defines an epimorphism $\operatorname{Pic}(C) \to \mathbb{Z}^s$. Choose a section of it, $\omega : \mathbb{Z}^s \to \operatorname{Pic}(C)$, and put $\mathcal{O}(\mathbf{d}) := \omega(\mathbf{d})$. Thus, we define $\mathbb{Z}^s$ as a group of "shifts" on the category of coherent sheaves (in particular, on that of vector bundles) by putting $\mathcal{M}(\mathbf{d}) := \mathcal{O}(\mathbf{d}) \otimes_{\mathcal{O}} \mathcal{M}$. Considering "representation types" of categories of sheaves, we should also take into account the action of this big discrete group.

If X is an algebraic variety, there is a natural notion of a family of vector bundles on a curve C with base X. Namely, such a family is just a vector bundle $\mathcal{V}$ on $X \times C$. For our purpose, a non-commutative analogue of this notion is also important.

Definition 1.3. 1. *Let Λ be a $\mathbf{k}$-algebra (not necessarily commutative). Denote by $\mathsf{VB}(C, \Lambda)$ the category of coherent flat sheaves of $\mathcal{O} \otimes \Lambda$-modules. The objects of this category are called* families of vector bundles over C with base Λ.

Given a family $\mathcal{M} \in \mathsf{VB}(C, \Lambda)$ and a finite dimensional[1] Λ-module N, we can construct the tensor product $\mathcal{M}(N) := \mathcal{M} \otimes_\Lambda N$, which is locally free over $\mathcal{O}$, i.e. is a vector bundle over C.

2. *A family $\mathcal{M} \in \mathsf{VB}(C, \Lambda)$ is said to be* strict *if the following conditions hold:*

[1] "*finite dimensional*" always means finite dimensional as a vector space over $\mathbf{k}$.

(a) *If N is an indecomposable finite dimensional Λ-module, then the sheaf $\mathcal{M}(N)$ is also indecomposable.*
(b) *If two finite dimensional Λ-modules N and N' are non-isomorphic, then the sheaves $\mathcal{M}(N)$ and $\mathcal{M}(N')$ are also non-isomorphic.*

For any morphism $f : C' \to C$ of curves and any family $\mathcal{M} \in \mathsf{VB}(C,\Lambda)$, the inverse image $f^*(\mathcal{M})$ belongs to $\mathsf{VB}(C',\Lambda)$. It is also quite obvious that if $\mathcal{M} \in \mathsf{VB}(C,\Lambda)$, then also $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{L} \in \mathsf{VB}(C,\Lambda)$ for every invertible sheaf $\mathcal{L}$ on C; in particular, $\mathcal{M}(\mathbf{d}) \in \mathsf{VB}(C,\Lambda)$ for every vector $\mathbf{d} \in \mathbb{Z}^s$. Moreover, if $\mathcal{M}$ is strict, so is $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{L}$ for each invertible sheaf $\mathcal{L}$; in particular, $\mathcal{M}(\mathbf{d})$ is strict for each $\mathbf{d}$. For every finite-dimensional Λ-module N, put $\mathcal{M}(\mathbf{d}, N) := \mathcal{M}(\mathbf{d})(N)$.

If $\Lambda = \mathbf{k}[X]$ for some affine variety X, then an object from $\mathsf{VB}(C,\Lambda)$ is obviously the same as a family of vector bundles on C with base X. However, our construction also produces some "subordinate" families of multiple dimensions. Note that for two different points $p \neq q$ of X the residue fields $\mathbf{k}(p)$ and $\mathbf{k}(q)$ are non-isomorphic as $\mathbf{K}[\,X\,]$-modules. Hence, for a strict family $\mathcal{M}$ over X, the fibres over p and q, i.e., the vector bundles $\mathcal{M}(p)$ and $\mathcal{M}(q)$ are also non-isomorphic.

Definitions 1.4. 1. *Call a curve C* vector bundle finite *or* VB-finite *if there is a finite set $\mathbf{M}$ of indecomposable vector bundles on C such that every indecomposable vector bundle on C is isomorphic to $\mathcal{B}(\mathbf{d})$ for some $\mathcal{B} \in \mathbf{M}$ and some vector $\mathbf{d} \in \mathbb{Z}^s$. (We shall see later that indeed $\operatorname{rk}(\mathcal{B}) = 1$ for every indecomposable vector bundle on a VB-finite curve.)*

2. *Call a curve C* VB-tame *if there is a non-empty set $\mathbf{M} = \{\, \mathcal{M}_i \,\}$ of strict sheaves $\mathcal{M}_i \in \mathsf{VB}(C,\Lambda_i)$ (note that the Λ_i may be different for different i) satisfying the following conditions:*
(a) *Each Λ_i is a commutative finitely generated integral smooth $\mathbf{k}$-algebra of Krull dimension 1.*
(b) *For each integer r and vector $\mathbf{d}$, the set $\mathbf{M}_{r,\mathbf{d}}$ is finite, where $\mathbf{M}_{r,\mathbf{d}} := \{\, \mathcal{M} \in \mathbf{M} \mid \operatorname{rk}(\mathcal{M}) = r\,, \operatorname{Deg} \mathcal{M} = \mathbf{d} \,\}$, where $\operatorname{Deg} \mathcal{M}$ is, by definition, $\operatorname{Deg}(\mathcal{M}/\mathbf{m}\mathcal{M})$ for some (and, hence, every) maximal ideal $\mathbf{m} \subset \Lambda_i$.*
(c) *For each integer r and vector $\mathbf{d}_0$, all but a finite number of locally free indecomposable sheaves on C of rank r and vector-degree $\mathbf{d}_0$ are isomorphic to those of the form $\mathcal{M}_i(\mathbf{d}, N)$, for some $\mathcal{M}_i \in \mathbf{M}$, $\mathbf{d} \in \mathbb{Z}^s$ and some finite dimensional Λ_i-module N.*

In this case call $\mathbf{M}$ a parametrizing set *for vector bundles over C. Denote $\nu(r)$ the minimal number of sheaves in $\mathbf{M}_{r,\mathbf{d}}$, where $\mathbf{M}$ runs through all such parametrizing sets and $\mathbf{d}$ runs through $\mathbb{Z}^s$. Then a VB-tame curve C is said to be:*

- bounded *if there is an integer m such that $\nu(r) \leq m$ for all ranks r;*

- unbounded *otherwise.*

3. *Call a curve* C VB-wild *if, for every finitely generated* $\mathbf{k}$*-algebra* Λ*, there is a strict sheaf* $\mathcal{M} \in \mathsf{VB}(C, \Lambda)$.
Hence, for wild curves, the classification of vector bundles is at least as complicated as the classification of the representations of all finitely generated $\mathbf{k}$*-algebras.*

Indeed, to prove wildness it is sufficient to check one "typical" algebra as the following result shows.

Proposition 1.5. *A curve* C *is VB-wild if there is a strict sheaf* $\mathcal{M} \in \mathsf{VB}(C, \Gamma)$*, where* Γ *is one of the following algebras:*

- $\mathbf{F} := \mathbf{k}\langle\, z_1, z_2 \,\rangle$*, the free algebra in two generators;*
- $\mathbf{k}[z_1, z_2]$*, the polynomial algebra in two generators;*
- $\mathbf{k}[[z_1, z_2]]$*, the power series algebra in two generators.*

Proof. It is well-known (cf. [4]) that if Γ is one of these algebras and Λ is an arbitrary finitely generated algebra, there is a *strict representation* of Γ over Λ, i.e., a Γ-Λ-bimodule V such that:

1. V is finitely generated and free as Λ-module.
2. If N is an indecomposable finite dimensional Λ-module, the Γ-module $V \otimes_\Lambda N$ is also indecomposable.
3. If N, N' are non-isomorphic finite dimensional Λ-modules, the Γ-modules $V \otimes_\Lambda N$ and $V \otimes_\Lambda N'$ are also non-isomorphic.

Therefore, if a sheaf $\mathcal{M} \in \mathsf{VB}(C, \Gamma)$ is strict, so is also $\mathcal{M} \otimes_\Lambda V \in \mathsf{VB}(C, \Lambda)$. □

Note also that this definition of tameness (namely, the condition $\mathbf{M} \neq \emptyset$) implies that "tame" excludes "finite," i.e., we have a real trichotomy.

First of all, consider VB-types of smooth projective curves. It is known that each vector bundle on $\mathbb{P}^1$ is isomorphic to $\mathcal{O}(n)$ for some n [8]. Hence, $\mathbb{P}^1$ is VB-finite. On the other hand, the classification of vector bundles on elliptic curves [1] implies that all elliptic curves are VB-tame and bounded (indeed, in this case $\nu(r) = 1$ for each r). So the next proposition accomplishes the answer.

Proposition 1.6. *Any smooth projective curve* C *of genus* $g > 1$ *is VB-wild.*

Proof. We shall even construct a sheaf $\mathcal{M} \in \mathsf{VB}(C, \mathbf{F})$, where $\mathbf{F} = \mathbf{k}\langle\, z_1, z_2 \,\rangle$, such that $\mathcal{M}(N) \simeq \mathcal{M}(N') \otimes_{\mathcal{O}} \mathcal{L}$ for some line bundle $\mathcal{L}$ if and only if $N \simeq N'$ and $\mathcal{L} \simeq \mathcal{O}$.

For any two points $x \neq y$ of C

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}(x), \mathcal{O}(y)) \simeq \mathrm{H}^0(C, \mathcal{O}(y - x)) = 0\,.$$

On the other hand,

$$\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O}(x), \mathcal{O}(y)) \simeq \mathrm{H}^1(C, \mathcal{O}(y - x)),$$

as $\mathcal{E}xt^1_{\mathcal{O}}(\mathcal{O}(x),\mathcal{O}(y)) = 0$. Using the Riemann–Roch theorem for the divisor $y-x$, we get

$$\dim \mathrm{H}^1(C,\mathcal{O}(y-x)) = g-1 \geq 1 .$$

Now consider the divisor $2x$ for some point x. The Riemann-Roch theorem together with the Clifford theorem [9, Theorem IV.5.4] gives that $\dim \mathrm{H}^0(C,\mathcal{O}(2x)) \leq 2$. Hence, all functions from this space are of the form $\lambda+\mu f_0$ for some fixed f_0 and $\lambda,\mu\in\mathbf{k}$. In particular, there is at most one pair (y,z) of points distinct from x such that $2x\sim y+z$.

Choose 5 different points $x_1,\dots,x_5$ in such a way that $2x_i \not\sim x_j+x_k$ for $x_j\neq x_i,\ x_k\neq x_i$, and consider the class of locally free sheaves $\mathcal{A}$ admitting an exact sequence:

$$0\longrightarrow \mathcal{A}_1\longrightarrow\mathcal{A}\longrightarrow\mathcal{A}_2\longrightarrow 0, \tag{1}$$

where

$$\mathcal{A}_1 = r_1\mathcal{O}(x_1)\oplus r_2\mathcal{O}(x_2)\oplus r_3\mathcal{O}(x_3)$$

and

$$\mathcal{A}_2 = r_4\mathcal{O}(x_4)\oplus r_5\mathcal{O}(x_5) .$$

Let $\xi\in\mathrm{Ext}_{\mathcal{O}}(\mathcal{A}_2,\mathcal{A}_1)$ be the element corresponding to the sequence (1). As there are no homomorphisms from the subsheaf to the factor-sheaf, one can easily check that two elements $\xi,\xi'\in\mathrm{Ext}_{\mathcal{O}}(\mathcal{A}_2,\mathcal{A}_1)$ lead to isomorphic modules $\mathcal{A}$ and $\mathcal{A}'$ if and only if there are automorphisms $\alpha:\mathcal{A}_1\overset{\sim}{\to}\mathcal{A}_1$ and $\beta:\mathcal{A}_2\overset{\sim}{\to}\mathcal{A}_2$ such that $\alpha\xi=\xi\beta$ (we mean here the Yoneda multiplication). Choose some non-zero elements $\xi_{ij}\in\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O}(x_j),\mathcal{O}(x_i))$. Put $\mathcal{S}=\mathcal{O}\otimes\mathbf{F}$, where $\mathbf{F}=\mathbf{k}\langle\, z_1,z_2\,\rangle$, the free $\mathbf{k}$-algebra in two generators, $\mathcal{S}(x)=\mathcal{S}\otimes_{\mathcal{O}}\mathcal{O}(x)$ for $x\in C$. Then $\mathrm{Ext}^1_{\mathcal{S}}(\mathcal{S}(x),\mathcal{S}(y))\simeq\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O}(x),\mathcal{O}(y))\otimes\mathbf{F}$. Consider the exact sequence of locally free $\mathcal{S}$-modules

$$0\longrightarrow\mathcal{S}(x_1)\oplus\mathcal{S}(x_2)\oplus\mathcal{S}(x_3)\longrightarrow\mathcal{M}\longrightarrow\mathcal{S}(x_4)\oplus\mathcal{S}(x_5)\longrightarrow 0$$

corresponding to the element of the Ext-space given by the matrix

$$\begin{pmatrix}\xi_{14} & \xi_{15}\\ \xi_{24} & z_1\xi_{25}\\ \xi_{34} & z_2\xi_{35}\end{pmatrix} .$$

If N is any finite dimensional $\mathbf{F}$-module, then the locally free $\mathcal{O}$-module $\mathcal{M}(N)$ corresponds to the element of Ext-space given by the matrix

$$\begin{pmatrix}\xi_{14}I & \xi_{15}I\\ \xi_{24}I & \xi_{25}Z_1\\ \xi_{34}I & \xi_{35}Z_2\end{pmatrix} .$$

Here I denotes the identity matrix of size $\dim_{\mathbf{k}} N$, while Z_1 and Z_2 are the matrices describing the action of z_1 and z_2, respectively, on the module N. Then an easy straightforward calculation shows that $\mathcal{M}(N)\simeq\mathcal{M}(N')$ if and only if $N\simeq N'$.

Suppose now that $\mathcal{M}(N) \simeq \mathcal{M}(N') \otimes_{\mathcal{O}} \mathcal{L}$, where $\mathcal{L} = \mathcal{O}(D)$ for some divisor D on C. Then for each $i \in \{1,2,3\}$, there are $j,k \in \{1,2,3,4,5\}$ such that

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}(x_i), \mathcal{O}(D+x_j)) = \mathrm{H}^0(C, \mathcal{O}(D+x_j-x_i)) \neq 0$$

and

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}(D+x_i), \mathcal{O}(x_k)) = \mathrm{H}^0(C, \mathcal{O}(-D+x_k-x_i)) \neq 0\,.$$

The first inequality implies that $\deg D \geq 0$, while the second one implies $\deg D \leq 0$. Hence, $\deg D = 0$. But then both $D + x_j - x_i$ and $-D + x_k - x_i$ are equivalent to zero, whence $2x_i \sim x_k + x_j$. The choice of these points implies that $x_j = x_k = x_i$ and $D \sim 0$, i.e., we return to the just considered case. $\square$

Remark: The bundles constructed in this proof are semi-stable in the sense of Mumford (cf. [11]).

2. Shifting bimodules

The study of vector bundles on singular curves is closely related to "bimodule problems" considered in [3, 4]. Here we recall and make precise some of the corresponding definitions and, in addition, modify them to take into account more complicated sets of "discrete parameters".

In what follows "*category*" usually means a category over an algebraically closed field $\mathbf{k}$. This means that all Hom-spaces are vector spaces over $\mathbf{k}$ and the product of morphisms is $\mathbf{k}$-bilinear. Given two categories A_1 and A_2, an A_1-A_2-*bimodule* is, by definition, a functor $\mathsf{U} : \mathsf{A}_1^\circ \otimes \mathsf{A}_2 \to \mathsf{Vect}$, the category of $\mathbf{k}$-vector spaces. Here $\otimes$ denotes, as usually, $\otimes_{\mathbf{k}}$. If $\mathsf{A}_1 = \mathsf{A}_2 = \mathsf{A}$ call U an A-*bimodule.* Usually we suppose that our categories are additive. Whenever some category C is not additive, we consider its *additive hull*, i.e., the smallest additive category $\mathsf{A} = \mathrm{add}\,\mathsf{C}$ containing C, and identify C-bimodules with A-bimodules.

Let U be an A-bimodule. Define the category $\mathsf{El}(\mathsf{U})$ of *elements of the bimodule* U as follows. Its object set is

$$\mathrm{Ob}\,\mathsf{El}(\mathsf{U}) := \bigcup_{A \in \mathrm{Ob}\,\mathsf{A}} \mathsf{U}(A,A)$$

and morphisms from $u \in \mathsf{U}(A,A)$ to $v \in \mathsf{U}(B,B)$ are morphisms $f \in \mathsf{A}(A,B))$ such that $fu = vf$. Here (and later on) we write fu instead of $\mathsf{U}(1,f)u$ and vf instead of $\mathsf{U}(f,1)v$. Note that both these elements belong to $\mathsf{U}(A,B)$.

Call a *shift* in a category A any self-equivalence $\mathsf{A} \xrightarrow{\sim} \mathsf{A}$. Define now a *shifting category* to be a triple $(\mathsf{A}, \Sigma, \rho)$, where A is a category, Σ is a group and ρ is a homomorphism of Σ to the group of shifts in A.

As usually, we write $\sigma(a)$ instead of $\rho(\sigma)(a)$, where a is an object or a morphism of A. We call Σ the *group of shifts* in A.

Let U be an A-bimodule and σ be some shift in A. A *shift in* U *compatible with* σ is, by definition, an isomorphism of A-bimodules $\sigma : \mathsf{U} \xrightarrow{\sim} \mathsf{U}^\sigma$, where U^σ denotes the bimodule obtained from U by pulling it back via σ, i.e. $\mathsf{U}^\sigma(a_1, a_2) := \mathsf{U}(\sigma(a_1), \sigma(a_2))$.

Given a shifting category $(\mathsf{A}, \Sigma, \rho)$, define a *shifting* A-*bimodule* as a pair (U, ρ_u), where ρ_u maps each element $\sigma \in \Sigma$ to a shift of U compatible with $\rho(\sigma)$ and $\rho_u(\sigma\tau) = \rho_u(\sigma)\rho_u(\tau)$ for each $\sigma, \tau \in \Sigma$. Again we write $\sigma(u)$ instead of $\rho_u(\sigma)(u)$. Note that in this case the category $\mathsf{El}(\mathsf{U})$ also becomes a shifting category with the same group Σ of shifts.

Let C be another $\mathbf{k}$-category. Then we can consider the tensor product $\mathsf{A} \otimes \mathsf{C}$ and the $\mathsf{A} \otimes \mathsf{C}$-bimodule $\mathsf{U} \otimes \mathsf{C}$. We say that the elements of $(\mathsf{U} \otimes \mathsf{C})(A \otimes P, A \otimes P)$, where $A \in \operatorname{Ob} \mathsf{A}$, $P \in \operatorname{Ob} \mathsf{C}$, are *based on* A. We are mostly interested in the case when $\mathsf{C} := \mathsf{Pr}\,\Lambda$ is the category of finitely generated (right) projective modules over some $\mathbf{k}$-algebra Λ. Then we write U^Λ instead of $\mathsf{U} \otimes \mathsf{C}$ and $\mathsf{El}(\mathsf{U}, \Lambda)$ instead of $\mathsf{El}(\mathsf{U} \otimes \mathsf{C})$. If, moreover, Λ is a commutative domain and $u \in \mathsf{U}^\Lambda(A \otimes P, A \otimes P)$, where $\operatorname{rk}_\Lambda P = r$, we say that u is an *element of rank* r *based on* A.

Note that any functor $\theta : \mathsf{C} \to \mathsf{C}'$ induces the functor $\theta_* : \mathsf{El}(\mathsf{U} \otimes \mathsf{C}) \to \mathsf{El}(\mathsf{U} \otimes \mathsf{C}')$. In particular, given a Λ'-Λ-bimodule N (i.e., left Λ- and right Λ'-), which is finitely generated and projective over Λ', we get the functor $\mathsf{El}(\mathsf{U}, \Lambda) \to \mathsf{El}(\mathsf{U}, \Lambda')$ induced by the tensor product $\_ \otimes_\Lambda N$. The image of an element $u \in \mathsf{El}(\mathsf{U}, \Lambda)$ under this functor will be denoted by $u(N)$.

Denote by vect the category of finite dimensional vector spaces over $\mathbf{k}$. Then $\mathsf{A} \otimes \mathsf{vect} \simeq \mathsf{A}$ for each additive category A and we always identify these categories. Hence, any functor $N : \mathsf{C} \to \mathsf{vect}$ gives rise to the functor $N_* : \mathsf{El}(\mathsf{U} \otimes \mathsf{C}) \to \mathsf{El}(\mathsf{U})$. In particular, if $\mathsf{C} = \mathsf{Pr}\,\Lambda$, such a functor is given by some finite dimensional Λ-module and we shall identify both. For this reason, in the general case, we also call functors $\mathsf{C} \to \mathsf{vect}$ C-*modules* (more precisely, they should be called "finite dimensional modules," but we never deal with other ones). Denote the category of C-modules by C-mod.

Definition 2.1. *Let* U *be a shifting bimodule with the group of shifts* Σ. *Call an element* $u \in \mathsf{El}(\mathsf{U} \otimes \mathsf{C})$ strict *if it satisfy the following conditions:*

1. *The element* $u(N)$ *is indecomposable (in* $\mathsf{El}(\mathsf{U})$*) for each indecomposable* C*-module* N.
2. *For each two* C*-modules* N, N' *and for each shift* $\sigma \in \Sigma$, *the elements* $u(N)$ *and* $\sigma(u)(N')$ *are isomorphic (in* $\mathsf{El}(U)$*) if and only if* $\sigma = 1$ *and* $N \simeq N'$ *(as* Λ*-modules).*

Definition 2.2. *Let* U *be a shifting bimodule with the group* Σ *of shifts. Suppose we have given, for each* $\mathbf{k}$*-algebra* Λ*, a full subcategory* $\mathsf{El}'(\mathsf{U},\Lambda) \subseteq \mathsf{El}(\mathsf{U},\Lambda)$ *satisfying the following conditions:*

1. $u(N) \in \mathsf{El}'(\mathsf{U},\Lambda')$ *for each element* $u \in \mathsf{El}'(\mathsf{U},\Lambda)$ *and for each* Λ'*-*Λ*-bimodule* N *which is finitely generated and projective over* Λ'*.*
2. $\sigma(u) \in \mathsf{El}'(\mathsf{U},\Lambda)$ *for each shift* $\sigma \in \Sigma$ *and for each element* $u \in \mathsf{El}'(\mathsf{U},\Lambda)$*.*

Then call the family of subcategories $\{\, \mathsf{El}'(\mathsf{U},\Lambda)\,\}$ *a* correct family *and the elements* $u \in \mathsf{El}'(\mathsf{U},\Lambda)$ *the* correct elements *(with respect to this correct family).*

Definitions 2.3. *The* representation type *of a shifting* A*-bimodule* U *(with the group* Σ *of shifts) supplied by a correct family of subcategories is defined as follows. This bimodule is called to be:*

- correctly finite *if there exists a finite set of indecomposable correct elements* $\mathbf{M} \subseteq \mathsf{El}'(\mathsf{U})$ *such that each indecomposable correct element is isomorphic to* $\sigma(u)$ *for some* $\sigma \in \Sigma$ *and some* $u \in \mathbf{M}$*.*
- correctly tame *if there exists a set* $\mathbf{M}$ *consisting of strict elements* $u \in \mathsf{El}'(\mathsf{U},\Lambda_u)$ *such that:*
  1. *Each* Λ_u *is a commutative domain, finitely generated as* $\mathbf{k}$*-algebra and of Krull dimension* 1 *(note that it may depend on* u*).*
  2. *For each object* $A \in \operatorname{Ob}\mathsf{A}$ *and for each natural number* r*, the set* $\mathbf{M}_{A,r} = \{u \in M | u$ *is an element of rank* r *based on* $A\}$ *is finite.*
  3. *For each object* $A \in \operatorname{Ob}\mathsf{A}$*, all indecomposable correct elements from* $\mathsf{U}(A,A)$*, except possibly for a finite number, are isomorphic to* $\sigma(u)(N)$ *for some element* $u \in \mathbf{M}$*, some shift* $\sigma \in \Sigma$ *and some (finite dimensional)* Λ_u*-module* N*.*

  In this case we call $\mathbf{M}$ *a* parametrizing set *for correct elements of* U*.*

  Moreover, if U *is correctly tame, call it:*
  - bounded *if there exists a parametrizing set* $\mathbf{M}$ *for correct elements of* U *such that* $|\mathbf{M}_{A,r}| \le C$ *for some constant* C *and all possible* A *and* r*.*
  - unbounded *if there is no such parametrizing set.*
- correctly wild *if there exists a correct strict element* $u \in \mathsf{El}'(\mathsf{U},\Lambda)$ *for each finitely generated* $\mathbf{k}$*-algebra* Λ*.*

In the "simplest" case, when all *elements are considered as correct, we omit the word "correct" at all and speak about* finite, tame *(bounded or unbounded) or* wild *shifting bimodule.*

Recall once more that to prove U being correctly wild, we only have to find a correct strict element in $\mathsf{El}'(\mathsf{U},\mathbf{F})$, where $\mathbf{F} := \mathbf{k}\langle\, z_1, z_2 \,\rangle$ is a free (non-commutative) $\mathbf{k}$-algebra with 2 generators.

In most cases we deal with so called *bipartite bimodules* [3]. They are defined as follows. If U is an A_1-A_2-bimodule, we can consider it as a bimodule over the direct product $\mathsf{A} := \mathsf{A}_1 \times \mathsf{A}_2$ by putting $\mathsf{U}((a_1, a_2), (b_1, b_2)) := \mathsf{U}(a_1, b_2)$ for $a_i, b_i \in \mathsf{A}_i$. Call this bimodule a *bipartite* A_1-A_2-bimodule. In this case an element of U based on a pair (A_1, A_2), where $A_i \in \operatorname{Ob} \mathsf{A}_i$, is indeed an element of $\mathsf{U}(A_1, A_2)$. All the other above definitions are modified just in the same way. Note that in [4] only bipartite bimodules were considered.

3. Relation between vector bundles and bimodules

Consider now the case of singular curves. We introduce, in addition to Notations 1.1, the following

Notations 3.1. 1. $\nu : \tilde{C} \to C$ *denotes the* normalization *of* C. *(Note that* $\tilde{C}$ *can be reducible or, which is equivalent, non-connected.)*

2. $S = S(C)$ *denotes the set of* singular points *of* C *and we set* $\tilde{S} := \nu^{-1}(S)$.
3. *Set* $\tilde{\mathcal{O}} = \nu_*(\mathcal{O}_{\tilde{C}})$ *; we identify* $\mathcal{O}$ *with its natural image in* $\tilde{\mathcal{O}}$.
4. *Let* $\mathcal{J}$ *be the* conductor *of* $\mathcal{O}$ *in* $\tilde{\mathcal{O}}$*, i.e., the biggest sheaf of* $\tilde{\mathcal{O}}$*-ideals contained in* $\mathcal{O}$.
5. *Set* $\mathcal{F} = \mathcal{O}/\mathcal{J}$ *and* $\tilde{\mathcal{F}} = \tilde{\mathcal{O}}/\mathcal{J}$.
6. *For any torsion free sheaf* $\mathcal{B}$ *of* $\mathcal{O}$*-modules, put* $\tilde{\mathcal{B}} = \tilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{B}/\,\mathsf{t}(\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{B})$ *(cf. 1.1) and* $\overline{\mathcal{B}} = \mathcal{B}/\mathcal{J}\mathcal{B}$*. In particular,* $\mathcal{F} = \overline{\mathcal{O}}$ *and* $\tilde{\mathcal{F}} = \overline{\tilde{\mathcal{O}}}$.
   As $\mathcal{B}$ *is torsion free, the canonical map* $\mathcal{B} \to \tilde{\mathcal{B}}$ *is a monomorphism and we always consider* $\mathcal{B}$ *as a subsheaf in* $\tilde{\mathcal{B}}$*. Note also that if* $\mathcal{B}$ *is a vector bundle, then* $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{B}$ *has no torsion part, hence, coincides with* $\tilde{\mathcal{B}}$.

Lemma 3.2. *For every torsion free sheaf* $\mathcal{B}$ *of* $\mathcal{O}$*-modules* $\tilde{\mathcal{B}}$ *is naturally isomorphic to the* $\tilde{\mathcal{O}}$*-subsheaf in* $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{B}$ *generated by* $\mathcal{B}$.
(Remind that $\mathcal{K}$ denotes the sheaf of rational functions on C.)

Proof. By definition of the torsion part, we have an exact sequence

$$0 \longrightarrow \mathsf{t}(\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{B}) \longrightarrow \tilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{B} \longrightarrow \mathcal{K} \otimes_{\mathcal{O}} (\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{B}) \simeq \mathcal{K} \otimes_{\mathcal{O}} \mathcal{B}.$$

The image of $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{B}$ is hence isomorphic to $\tilde{\mathcal{B}}$ and obviously coincide with the $\tilde{\mathcal{O}}$-subsheaf of $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{B}$ generated by $\mathcal{B}$. □

Note that $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are sky-scraper sheaves, zero outside S and with finite dimensional fibres. Hence, we may (and will) identify them with the finite dimensional $\mathbf{k}$-algebras $\bigoplus_{x \in S} \mathcal{F}_x$ and $\bigoplus_{x \in S} \tilde{\mathcal{F}}_x$ respectively. Just in the same way we identify the sky-scraper sheaf $\overline{\mathcal{B}}$ with the $\mathcal{F}$-module $\bigoplus_{x \in S} \overline{\mathcal{B}}_x$.

When considering families of torsion free sheaves, we have to impose some conditions, which guarantee that they are "uniformly embedded" into their $\tilde{\mathcal{O}}$-closures. Thus, we give the following definition for such families (in [7] they are called *δ-constant*).

Definition 3.3. *Let* Λ *be a* $\mathbf{k}$*-algebra (not necessarily commutative). Denote by* $\mathsf{TF}(C,\Lambda)$ *the category whose objects are coherent sheaves of* $\mathcal{O}\otimes\Lambda$*-modules* $\mathcal{B}$ *satisfying the following conditions:*

1. $\mathcal{B}$ *is torsion free over* $\mathcal{O}$.
2. $\tilde{\mathcal{B}}$ *is flat over* $\tilde{\mathcal{O}}\otimes\Lambda$.
3. $\tilde{\mathcal{B}}/\mathcal{B}$ *is flat over* Λ.

Such sheaves are called families of torsion free sheaves *on* C *with base* Λ.

Lemma 3.4. *If* $\mathcal{B}\in\mathsf{TF}(C,\Lambda)$, *then it is flat over* Λ *and, for every* Λ*-module* N, *the sheaf* $\mathcal{B}(N)=\mathcal{B}\otimes_\Lambda N$ *is also torsion free over* $\mathcal{O}$; *moreover, the natural homomorphism* $\mathcal{B}(N)\to\tilde{\mathcal{B}}(N)$ *is an embedding and induces an isomorphism* $\widetilde{\mathcal{B}(N)}\simeq\tilde{\mathcal{B}}(N)$.

Proof. Put $\mathcal{T}=\tilde{\mathcal{B}}/\mathcal{B}$, which is a torsion sheaf over $\mathcal{O}$. Consider the exact sequence $0\to\mathcal{B}\to\tilde{\mathcal{B}}\to\mathcal{T}\to 0$. As $\tilde{\mathcal{B}}$ and $\mathcal{T}$ are both Λ-flat, so is $\mathcal{B}$. Tensoring by N over Λ, we get again an exact sequence:

$$0\longrightarrow\mathcal{B}(N)\longrightarrow\tilde{\mathcal{B}}(N)\longrightarrow\mathcal{T}(N)\longrightarrow 0\,.$$

As $\tilde{\mathcal{B}}$ is flat over $\tilde{\mathcal{O}}\otimes\Lambda$ and $(\tilde{\mathcal{B}}\otimes_\Lambda N)\otimes_{\tilde{\mathcal{O}}}\mathcal{X}\simeq\tilde{\mathcal{B}}\otimes_{\tilde{\mathcal{O}}\otimes\Lambda}(N\otimes\mathcal{X})$ for any sheaf of $\tilde{\mathcal{O}}$-modules $\mathcal{X}$, the sheaf $\tilde{\mathcal{B}}(N)=\tilde{\mathcal{B}}\otimes_\Lambda N$ is flat over $\tilde{\mathcal{O}}$, hence, torsion free. Therefore, $\mathcal{B}(N)$ is also torsion free. Moreover, as the image of $\mathcal{B}(N)$ obviously generates $\tilde{\mathcal{B}}(N)$, the latter coincides with $\widetilde{\mathcal{B}(N)}$ in view of Lemma 3.2. □

Using this notion, we are able to define *TF-finite, TF-tame* and *TF-wild* curves just in the same way as we have defined the corresponding VB-types. Nevertheless, we shall see that indeed the TF-type of a curve coincides with its VB-type.

Definitions 3.5. 1. *Let* Λ *be a* $\mathbf{k}$*-algebra,* $\mathcal{A}$ *a flat coherent sheaf of* $\tilde{\mathcal{O}}\otimes\Lambda$*-modules and* $\mathcal{M}$ *a coherent* $\mathcal{F}\otimes\Lambda$*-subsheaf in* $\overline{\mathcal{A}}:=\mathcal{A}/\mathcal{J}\mathcal{A}$. *Call this subsheaf* correct *if it satisfies the following conditions:*

   (a) $\overline{\mathcal{A}}/\mathcal{M}$ *is flat over* Λ.
   (b) $\mathcal{M}$ *is flat over* $\mathcal{F}\otimes\Lambda$.
   (c) *The natural homomorphism* $\tilde{\mathcal{F}}\otimes_{\mathcal{F}}\mathcal{M}\to\overline{\mathcal{A}}$ *is an isomorphism.*

   If only condition (a) holds and $\tilde{\mathcal{F}}\mathcal{M}=\overline{\mathcal{A}}$, *call* $\mathcal{M}$ semi-correct.

2. *Define the category* $\mathsf{C}^s=\mathsf{C}^s(C,\Lambda)$ *as follows: objects are pairs* $(\mathcal{A},\mathcal{M})$, *where* $\mathcal{A}$ *is a flat coherent sheaf of* $\tilde{\mathcal{O}}\otimes\Lambda$*-modules of constant rank on* $\tilde{C}$ *and* $\mathcal{M}$ *is a semi-correct submodule in* $\overline{\mathcal{A}}$, *and a morphism* $(\mathcal{A},\mathcal{M})\to(\mathcal{A}',\mathcal{M}')$ *is a morphisms* $f:\mathcal{A}\to\mathcal{A}'$ *such that the induced mapping* $\overline{f}:\overline{\mathcal{A}}\to\overline{\mathcal{A}}'$ *maps* $\mathcal{M}$ *to* $\mathcal{M}'$.

3. *Let* $\mathsf{C} = \mathsf{C}(C, \Lambda)$ *be the full subcategory of* $\mathsf{C}^s(C, \Lambda)$ *consisting of all pairs* $(\mathcal{A}, \mathcal{M})$ *with a correct submodule* $\mathcal{M}$ *.*
   We write $\mathsf{C}^s(C)$ *and* $\mathsf{C}(C)$ *instead of* $\mathsf{C}^s(C, \mathbf{k})$ *and* $\mathsf{C}(C, \mathbf{k})$ *correspondingly.*
4. *Define a functor* $F : \mathsf{TF}(C, \Lambda) \to \mathsf{C}^s(C, \Lambda)$ *mapping any sheaf* $\mathcal{B} \in \mathsf{TF}(C, \Lambda)$ *to the pair* $(\tilde{\mathcal{B}}, \overline{\mathcal{B}})$ *and any morphism* $g : \mathcal{B} \to \mathcal{B}'$ *to the morphism* $\tilde{g} : \tilde{\mathcal{B}} \to \tilde{\mathcal{B}}'$ *.*

One can easily check that $\overline{\mathcal{B}}$ is indeed a semi-correct submodule in $\tilde{\mathcal{B}}/\mathcal{J}\tilde{\mathcal{B}}$ and $\overline{\tilde{g}}(\overline{\mathcal{B}}) \subseteq \overline{\mathcal{B}}'$. Note only that the definitions imply immediately that $\mathcal{J}\tilde{\mathcal{B}} = \mathcal{J}\mathcal{B}$. Moreover, if $\mathcal{B}$ is a flat sheaf, the submodule $\overline{\mathcal{B}}$ is indeed correct. This construction allows to formulate the following result.

Proposition 3.6. *The functor* F *establishes an equivalence of the categories* $\mathsf{TF}(C, \Lambda)$ *and* $\mathsf{C}^s(C, \Lambda)$*. Moreover, the restriction of* F *to* $\mathsf{VB}(C, \Lambda)$ *establishes an equivalence of* $\mathsf{VB}(C, \Lambda)$ *and* $\mathsf{C}(C, \Lambda)$*.*

Proof. Define the inverse functor G as follows: for any object $\mathcal{P} = (\mathcal{A}, \mathcal{M})$ of C^s let $\mathcal{B} = G\mathcal{P}$ be the preimage of $\mathcal{M} \subseteq \overline{\mathcal{A}}$ in $\mathcal{A}$. It is a coherent subsheaf in $\mathcal{A}$ such that $\tilde{\mathcal{O}}\mathcal{B} = \mathcal{A}$. In particular, $\mathcal{B}$ is torsion free and $\tilde{\mathcal{B}} \simeq \mathcal{A}$. As $\mathcal{A}/\mathcal{B} \simeq \overline{\mathcal{A}}/\mathcal{M}$ is flat over Λ, $\mathcal{B} \in \mathsf{TF}(C, \Lambda)$. If $\mathcal{P}' = (\mathcal{A}', \mathcal{M}')$ is another pair and $f : \mathcal{P} \to \mathcal{P}'$ is a morphism from $\mathsf{C}^s(C, \Lambda)$, then, by construction, $f(G\mathcal{A}) \subseteq G\mathcal{A}'$. Therefore, we obtain a functor $G : \mathsf{C}^s \to \mathsf{VB}$, inverse to F.

Let now the pair $\mathcal{P}$ be correct, that is, $\mathcal{M} \simeq \mathcal{B}/\mathcal{J}\mathcal{B}$ is flat over $\mathcal{F} \otimes \Lambda$. As it is also coherent, it is a projective $\mathcal{F} \otimes \Lambda$-module. Fix a point $x \in S$ and put $M = \mathcal{M}_x/\mathfrak{m}\mathcal{M}_x$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{F}_x$. M is a flat coherent, hence, projective Λ-module. Then $P = \mathcal{O}_x \otimes M$ is a projective $\mathcal{O}_x \otimes \Lambda$-module such that $P/\mathfrak{m}P \simeq \mathcal{B}_x/\mathfrak{m}\mathcal{B}_x$. Hence, there is a homomorphism $f : P \to \mathcal{B}_x$ such that $\operatorname{Im} f + \mathfrak{m}\mathcal{B}_x = \mathcal{B}_x$. As $\mathcal{B}_x$ is finitely generated as $\mathcal{O}_x \otimes \Lambda$-module, then f is an epimorphism. Moreover, as $\overline{\mathcal{A}}_x \simeq \tilde{\mathcal{F}} \otimes_{\mathcal{F}} \mathcal{M}_x$, also $\mathcal{A}_x/\mathfrak{m}\mathcal{A}_x \simeq \tilde{P}/\mathfrak{m}\tilde{P}$, where $\tilde{P} = \tilde{\mathcal{O}}_x \otimes M = \tilde{\mathcal{O}}_x \otimes_{\mathcal{O}_x} P$. As both $\tilde{P}$ and $\mathcal{A}_x$ are finitely generated projective $\tilde{\mathcal{O}}_x \otimes \Lambda$-modules, they are isomorphic and the commutative diagram

$$\begin{array}{ccc}
\mathcal{O}_x \otimes M & \xrightarrow{f} & \mathcal{B}_x \\
\big\downarrow & & \big\downarrow \\
\tilde{\mathcal{O}}_x \otimes M & \xrightarrow{\sim} & \mathcal{A}_x
\end{array}$$

shows that f is a monomorphism, hence, isomorphism. Therefore, $\mathcal{B}_x$ is projective (thus flat) $\mathcal{O}_x \otimes \Lambda$-module for every point $x \in S$. As for all other points y, $\mathcal{B}_y = \mathcal{A}_y$ is also flat, the whole sheaf $\mathcal{B}$ is flat over $\mathcal{O} \otimes \Lambda$, that is, belongs to $\mathsf{VB}(C, \Lambda)$. □

For each vector bundle $\tilde{\mathcal{A}}$ of constant rank r on $\tilde{C}$, we can always choose an open affine subvariety $C' \subset \tilde{C}$ such that $\tilde{S} \subset C'$ and the restriction of $\tilde{\mathcal{A}}$ on C' is trivial: $\tilde{\mathcal{A}}|_{C'} \simeq r\mathcal{O}_{\tilde{C}}|_{C'}$. Using this, we can (and do) always suppose that $\tilde{\mathcal{A}}|_{C'} = r\mathcal{O}_{\tilde{C}}|_{C'}$. Therefore, setting $\mathcal{A} = \nu_*\tilde{\mathcal{A}}$, we get $\mathcal{A}|_{\nu(C')} = r\tilde{\mathcal{O}}|_{\nu(C')}$ and hence $\overline{\mathcal{A}} = r\tilde{\mathcal{F}}$. Such agreement is compatible with tensor products if we always identify $\tilde{\mathcal{O}}\otimes_{\tilde{\mathcal{O}}}\tilde{\mathcal{O}}$ with $\tilde{\mathcal{O}}$ via the natural isomorphism. A correct subsheaf in $\overline{\mathcal{A}}$ is then given by r elements $v_1, v_2, \dots, v_r$ of $r\tilde{\mathcal{F}}$ linearly independent over $\mathcal{F}$, namely, $\mathcal{M} = \sum_{i=1}^r \mathcal{F}v_i$. We often write $(\mathcal{A}, v_1, v_2, \dots, v_r)$ instead of $(\mathcal{A}, \mathcal{M})$.

For instance, a line bundle $\mathcal{L}$ over C is given by a line bundle $\tilde{\mathcal{L}}$ over $\tilde{C}$ and an invertible element v of the algebra $\tilde{\mathcal{F}}$. If $\mathcal{B}$ is the torsion free sheaf corresponding to a semi-correct pair $\mathcal{P} = (\mathcal{A}, \mathcal{M})$ then their tensor product $\mathcal{B}\otimes_{\mathcal{O}}\mathcal{L}$ is given by the pair $\mathcal{P}^{\mathcal{L}} := (\mathcal{A}\otimes_{\tilde{\mathcal{O}}}\mathcal{L}, v\mathcal{M})$. This is the way how the Picard group $\mathrm{Pic}(C)$ acts on the category $\mathsf{C}^s(C)$. Of course, if $\mathcal{P} \in \mathsf{C}$, also $\mathcal{P}^{\mathcal{L}} \in \mathsf{C}$.

In particular, we get a rule for tensor products of line bundles. Note that the bundles corresponding to the pairs $(\tilde{\mathcal{L}}, v)$ and $(\tilde{\mathcal{L}}', v')$ are isomorphic if and only if $\tilde{\mathcal{L}} \simeq \tilde{\mathcal{L}}'$ and $v' = \theta v$ for some invertible element $\theta \in \mathcal{F}$ (take into account that both $\tilde{\mathcal{L}}|_{C'}$ and $\tilde{\mathcal{L}}'|_{C'}$ coincide with $\nu_*\mathcal{O}_{\tilde{C}}|_{C'}$). This immediately gives the following corollary.

Corollary 3.7. $\mathrm{Pic}(C) \simeq \mathrm{Pic}(\tilde{C}) \times (\tilde{\mathcal{F}}^*/\mathcal{F}^*)$.

The vector-degree Deg defines a homomorphism $\mathrm{Pic}(C) \to \mathbb{Z}^s$ and it is evident that $\mathrm{Deg}\,\mathcal{B} = \mathrm{Deg}\,\tilde{\mathcal{B}}$. In what follows, we choose a section $\omega : \mathbb{Z}^s \to \mathrm{Pic}(C)$ in such a way that $\omega(\mathbf{e}_i) = \mathcal{O}(p_i)$ for some point $p_i \notin S$, where $\mathbf{e}_i := (0, \dots, 0, 1, 0, \dots 0)$ (1 at the i-th place). Put, as above, $\mathcal{O}(\mathbf{d}) := \omega(\mathbf{d})$ and $\mathcal{A}(\mathbf{d}) := \mathcal{A} \otimes_{\mathcal{O}} \mathcal{O}(\mathbf{d})$ for every sheaf $\mathcal{A}$ of $\mathcal{O}$-modules. Then $\mathsf{VB}(C)$ becomes a shifting category with the group $\Sigma = \mathbb{Z}^s$ of shifts. The same is true for C and C^s: if $\mathcal{P} = (\mathcal{A}, \mathcal{M})$, put $\mathcal{P}(\mathbf{d}) := (\mathcal{A}(\mathbf{d}), \mathcal{M})$. Note that under the restriction imposed above, $\mathcal{A}(\mathbf{d})|_{nu(C')} = \mathcal{A}|_{\nu(C')}$, hence, $\overline{\mathcal{A}}(\mathbf{d}) = \overline{\mathcal{A}}$. Certainly, the equivalence $\mathsf{VB}(C) \simeq \mathsf{C}$ preserves these shifts.

The category C^s has the advantage that it can be easily reinterpreted with the help of some bimodule category.

Denote by A the category of locally free (coherent) $\tilde{\mathcal{O}}$-sheaves and by B the category of projective (finitely generated) $\mathcal{F}$-modules. Define a bipartite B-A-bimodule U by putting, for $B \in \mathsf{B}$ and $\mathcal{A} \in \mathsf{A}$,

$$\mathsf{U}(B, \mathcal{A}) = \mathrm{Hom}_{\mathcal{F}}(B, \overline{\mathcal{A}}).$$

This is also a shifting bimodule with the same group $\Sigma = \mathbb{Z}^s$ of shifts. Namely, shifting by $\mathbf{d}$ acts on A as above, acts on B trivially and maps an element $u \in \mathsf{U}(B, \mathcal{A})$, i.e., a homomorphism $B \to \overline{\mathcal{A}}$, to the same element considered as a homomorphism $B \to \overline{\mathcal{A}}(\mathbf{d}) = \overline{\mathcal{A}}$ (we denote it by $u(\mathbf{d})$).

Define two correct families of elements of the bimodule U. The first one is denoted by El_c and called the family of *correct* elements consisting of all elements $u \in \mathsf{U}^\Lambda(B,\mathcal{A}) = \mathrm{Hom}_{\mathcal{F}\otimes\Lambda}(B,\overline{\mathcal{A}})$ satisfying the following conditions:

1. $B \simeq F \otimes P$ for some free $\mathcal{F}$-module F and some projective Λ-module P.
2. $\mathrm{Ker}\, u \subseteq (\mathrm{rad}\,\mathcal{F})B$.
3. $\mathrm{Coker}\, u$ is flat as Λ-module.
4. The induced map $u_{\tilde{\mathcal{F}}} : \tilde{\mathcal{F}} \otimes_{\mathcal{F}} B \to \overline{\mathcal{A}}$ is an isomorphism.

The second family is denoted by El_{sc} and called the family of *semi-correct* elements consisting of all elements satisfying conditions 2 and 3 above and the following one:

4′. $u_{\tilde{\mathcal{F}}}$ is an epimorphism.

Proposition 3.8. *For each correct (semi-correct) element $u \in \mathsf{U}^\Lambda(P,\mathcal{A})$, the image $\mathrm{Im}\, u$ is a correct (semi-correct) submodule in $\overline{\mathcal{A}}$ and the mapping $u \to (\mathcal{A}, \mathrm{Im}\, u)$ induces a functor $F : \mathsf{El}_c(\mathsf{U}^\Lambda) \to \mathsf{C}(C,\Lambda)$ (correspondingly $\mathsf{El}_{sc}(\mathsf{U},\Lambda) \to \mathsf{C}^s(C,\Lambda)$) having the following properties:*

1. *It is* dense*, i.e., each object from C^s is isomorphic to FA for some object A from TF.*
2. *It is* full*, i.e., all induced maps $\mathsf{C}^s(A,A') \to \mathsf{TF}(FA,FA')$ are surjective.*
3. *It* reflects isomorphisms*, i.e., $A \simeq A'$ if and only if $FA \simeq FA'$.*
4. *It* preserves indecomposability*, i.e., A is indecomposable if and only if FA is indecomposable.*
5. *It is* compatible with shifts*, i.e., $F(u(\mathbf{d})) \simeq \sigma(F(u))(\mathbf{d})$ for each $\sigma \in \mathbb{Z}^n$.*

Usually a functor having properties *1*, *3* and *4* is called a *representation equivalence*. We call a functor having properties *1–4* a *full representation equivalence* and a functor having all five properties a *full shifting representation equivalence*. Note that the condition *4* is a formal consequence of the conditions *1* and *3*.

Proof. For each pair $(\mathcal{A},\mathcal{M})$ from C^s the $\mathcal{F}$-module $U := \mathcal{M}/(\mathrm{rad}\,\mathcal{F})\mathcal{M}$ is semi-simple, hence, isomorphic to a direct sum of $U_i \otimes P_i$, where U_i are simple $\mathcal{F}$-modules and P_i are projective Λ-modules. Then $U_i \simeq B_i/(\mathrm{rad}\,\mathcal{F})B_i$, where B_i is a projective $\mathcal{F}$-module, and there is an epimorphism $u : B := \oplus_i B_i \otimes P_i \to \mathcal{M}$ such that $\mathrm{Ker}\, u \subseteq (\mathrm{rad}\,\mathcal{F})B$. Obviously u is a semi-correct (correct if $\mathcal{M}$ is a correct submodule) element of $\mathsf{U}^\Lambda(B,\mathcal{A})$ and $F(u) \simeq (\mathcal{A},\mathcal{M})$. Moreover, as any homomorphism can be lifted to projective covers, the functor F is full and as $\mathrm{Ker}\, u \subseteq (\mathrm{rad}\,\mathcal{F})B$, it reflects isomorphisms. Hence, it is a full representation equivalence. The compatibility with shifts follows immediately from the definition of F. □

Due to condition *2* of the definition of correct elements, if an element $u \in \mathsf{U}(B,\mathcal{A})$ is correct, the number of indecomposable summands in B is not more that $\dim_{\mathbf{k}} \overline{\mathcal{A}}$. Hence, if $\mathcal{A}$ is fixed, there are only finitely many possibilities for B. On the other hand, if $\mathcal{A} \in \mathsf{A}$ is of rank r, then $\operatorname{Deg} \mathcal{A}(\mathbf{e}_i) = \operatorname{Deg} \mathcal{A} + r\mathbf{e}_i$. Hence, any correct element u has a shift lying in $\mathsf{U}(B,\mathcal{A})$ with $0 \le \operatorname{Deg} \mathcal{A} < (r,r,\dots,r)$. Here an inequality for vectors means inequality for all components. Together with Proposition 3.8 it implies the following corollary.

Corollary 3.9. *A curve C is VB-finite, VB-tame or VB-wild if and only if the bimodule $\mathsf{U}(C)$ is correctly finite, correctly tame or correctly wild respectively.*
(The same is true for TF-types of curves and semi-correct types of the corresponding bimodules.)

Certainly, if the normalization $\tilde{C}$ is VB-wild, so is the curve C. It follows by Proposition 1.6, that whenever C is not VB-wild, the irreducible components of $\tilde{C}$ are either rational or elliptic curves. We show that the latter case is still impossible.

Proposition 3.10. *If a singular curve C is not VB-wild, then all irreducible components of $\tilde{C}$ are rational curves.*

Proof. Let $C_1, C_2, \dots, C_t$ be the irreducible components of $\tilde{C}$, $\mathcal{O}_k := \mathcal{O}_{C_k}$. Suppose there is a component C_1 which is elliptic. As C is singular and connected, there is a point $e \in C$ which lies on $\nu(C_1)$ and such that $\mathcal{O}_e \neq \tilde{\mathcal{O}}_e$. Consider the case when also $e \in \nu(C_2)$ for another component C_2 (other cases are even simpler to handle). Find 4 different points $x_1, \dots, x_4$ on $C_1 \setminus \tilde{S}$ such that $2x_i \not\sim x_j + x_k$ for $x_j \neq x_i,\, x_k \neq x_i$, and a point $y \in C_2 \setminus \tilde{S}$. Consider the element u from $\mathsf{El}(\mathsf{U},\mathbf{F})$, where $\mathbf{F} = \mathbf{k}\langle\, z_1, z_2 \,\rangle$, defined as follows:

$$u \in \mathsf{U}(B,A) \quad \text{where}$$

$$A := \bigoplus_{i=1}^{4} (\tilde{\mathcal{O}}(x_i + iy) \otimes \mathbf{F}),$$

$$B := 4\mathcal{F} \otimes \mathbf{F}.$$

In this case $u : B \to A$ can be given by a set of 4×4 matrices u_{pk} with entries from $\mathcal{O}_{kp}/\mathcal{J} \otimes \mathbf{F}$, as such a matrix defines a homomorphism $B_p \to A_p$. Here p runs through $S(C)$ and $k = 1,2,\dots t$. We set all components equal to the identity matrices except of u_{e2} which is

$$\begin{pmatrix} z_1 & z_2 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

It is not difficult to verify that we obtain a strict element, hence C is VB-wild. Moreover, if $\mathcal{M} \in \mathsf{VB}(C,\mathbf{F})$ is the corresponding sheaf, one can verify, just as in the proof of Proposition 1.6, that $\mathcal{M}(N) \simeq \mathcal{M}(N') \otimes_{\mathcal{O}} \mathcal{L}$ for some line bundle $\mathcal{L}$ if and only if $N \simeq N'$ and $\mathcal{L} \simeq \mathcal{O}$. □

4. Rationally composed curves and special bimodules

From now on we suppose that all components of $\tilde{C}$ are rational curves (i.e., isomorphic to $\mathbb{P}^1$). In this case we say that the curve C itself is *rationally composed.* To find VB-types on rationally composed curves, we introduce the following class of bipartite bimodules.

First of all, consider the category L_0 such that $\operatorname{Ob} \mathsf{L}_0 = \mathbb{Z}$ (the integers) and the set of morphisms is generated by morphisms $x_n : n \to n+1$ and $y_n : n \to n+1$ subject to the relations: $x_{n+1}y_n = y_{n+1}x_n$ for all $n \in \mathbb{Z}$. Let L be its additive hull. It is well-known that L is equivalent to the category of vector bundles over the projective line $\mathbb{P}^1$, the object n corresponding to the sheaf $\mathcal{O}_{\mathbb{P}^1}(n)$. There is a natural shift σ on L mapping n to $n+1$ and we consider L as a shifted category with the group of shifts $\mathbb{Z} \simeq \{\, \sigma^k \mid k \in \mathbb{Z} \,\}$.

Data for a special bimodule consist of:

1. *A finite dimensional commutative algebra* $\mathbf{R}$ *and its subalgebra* $\mathbf{S}$.
   Let $\mathbf{R} := \prod_{k=1}^{r} \mathbf{R}_i$ and $\mathbf{S} := \prod_{j=1}^{s} \mathbf{S}_j$ be their decompositions into direct product of local algebras. We denote by $\mathbf{J}$ the set of all pairs (k,j) ($1 \le k \le r$, $1 \le j \le s$) such that the projection of $\mathbf{S}_j$ onto $\mathbf{R}_k$ is non-zero.
2. *An equivalence relation* $\sim$ *on the set of local components of* $\mathbf{R}$ *(indeed on the index set* $\{\,1,2,\dots,r\,\}$*).*
3. *A mapping* $\kappa : \mathbf{J} \to \mathbb{P}^1$ *(the projective line over* $\mathbf{k}$*).*
   We denote $\kappa(k,j) := (\xi_{kj} : \eta_{kj})$.

We impose the following restrictions on these data:

1. $\mathbf{S}$ *contains no non-zero ideal of* $\mathbf{R}$.
2. *Whenever* $j \ne j'$ *and* $k \sim k'$ *are such that* $(k,j) \in \mathbf{J}$ *and* $(k',j') \in \mathbf{J}$*, then* $\kappa(k',j') \ne \kappa(k,j)$.
3. *For each* $k = 1,2,\dots,r$ *there exists a* j *such that* $(k,j) \in bJ$.
4. *If* $\mathbf{J} = \mathbf{J}' \cup \mathbf{J}''$ *such that* $(k,j) \in \mathbf{J}'$ *implies* $(k',j') \notin \mathbf{J}''$ *for each* j' *and each* $k' \sim k$ *as well as* $(k'',j) \notin \mathbf{J}''$ *for each* k''*, then either* $\mathbf{J}'$ *or* $\mathbf{J}''$ *is empty.*[2]

We denote such data (somewhat ambiguously) by $[\,\mathbf{R},\mathbf{S},\kappa\,]$ and call them "*special data.*"

To special data $[\,\mathbf{R},\mathbf{S},\kappa\,]$ we associate a shifting bimodule $\mathbf{U}[\,\mathbf{R},\mathbf{S},\kappa\,]$ (called a *special bimodule*) in the following way.

[2]This condition, some sort of connectedness, is not restrictive indeed, but we prefer to impose it to simplify the definitions. In any case, we never need "non-connected" data.

Let $\mathbf{C}$ be the set of equivalence classes of the relation $\sim$, $\mathbf{C} := \{ c_1, c_2, \dots, c_t \}$. For each class c_i put $\mathbf{R}(i) := \prod_{k\in c_i} \mathbf{R}_k$. Consider the category $\mathsf{A} := \mathsf{L}^t$ whose indecomposable objects are in one-to-one correspondence with the pairs (n, i), where $n \in \mathbb{Z}$, $1 \le i \le t$ and we identify them with such pairs. If some object of A decomposes into a direct sum: $A = \oplus_k (n_k, i_k)$, put $|A| := \oplus_k \mathbf{R}(i_k)$ considered as a projective module over the algebra $\mathbf{R}$. Note that the endomorphism ring of the object (n, i) coincides with $\mathbf{k}$ and the complete set of morphisms in A is generated by the morphisms $x_{ni}, y_{ni} : (n, i) \to (n+1, i)$ originated in the morphisms x_n, y_n of L. The category A is a shifting one with the group of shifts $\Sigma := \mathbb{Z}^t = \langle\, \sigma_i \,|\, 1 \le i \le t \,\rangle$: the shift σ_i maps (n, i') to $(n + \delta_{ii'}, i')$.

Let now $\mathsf{B} := \mathsf{Pr}\,\mathbf{S} \simeq \prod_{j=1}^{s} \mathsf{Pr}\,\mathbf{S}_j$. Its indecomposable objects are in one-to-one correspondence with the indices $j = 1, \dots, s$ and we identify them. Here the endomorphism ring of the object j is $\mathbf{S}_j$ and there are no morphisms between different indecomposable objects. We consider B as a shifting category with the same groups of shifts Σ but with all shifts acting trivially.

Now define the special (shifted and bipartite) B-A-bimodule $\mathsf{U} := \mathsf{U}[\,\mathbf{G}, \varphi, \xi\,]$ with the same group Σ of shifts in the following way:

$$\begin{aligned}
\mathsf{U}(j, (n, i)) &:= \mathrm{Hom}_{\mathbf{S}}(\mathbf{S}_j, \mathbf{R}(i))\,;\\
x_{ni} f(n) &:= \xi_{ij} f(n+1)\,;\\
y_{ni} f(n) &:= \eta_{ij} f(n+1)\,;\\
\sigma_{i'} f(n) &:= f(n + \delta_{ii'})\,.
\end{aligned}$$

Here $f(n)$ denotes a homomorphism $f \in \mathrm{Hom}_{\mathbf{S}}(\mathbf{S}_j, \mathbf{R}(i))$ considered as an element of $\mathsf{U}(j, (n, i))$ (and we regard such homomorphisms with different indices n as different elements of U). Note that if $(i, j) \notin \mathbf{J}$, we have $\mathsf{U}(j, (n, i)) = 0$.

One can easily check that replacing the pair (ξ_{ij}, η_{ij}), for some arrow (i, j), by $(\lambda\xi_{ij}, \lambda\eta_{ij})$, with $\lambda \in \mathbf{k}\text{-}\{\,0\,\}$, leads to an isomorphic shifting bimodule, so U really depends only on κ as a mapping to the projective line. Moreover, for each fixed i we can replace all values $\xi(k, j)$ for the arrows with source $k \in c_i$ by $\gamma\xi(k, j)$, where γ is any collineation of $\mathbb{P}^1$ given by an invertible matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$: we only have to change the action of x_{ni} and y_{ni} (for each n) on all spaces $\mathsf{U}(\mathbf{S}_j, \mathbf{R}(i))$ to $ax_{ni} + by_{ni}$ and $cx_{ni} + dy_{ni}$ respectively. This also leads to an isomorphic shifted bimodule.

Let now Λ be a $\mathbf{k}$-algebra. The objects from $\mathsf{A} \otimes \Lambda$ are direct sums of the shape $A = \bigoplus_k A_k \otimes P_k$, where $A_k \in \mathrm{Ob}\,\mathsf{A}$, $P_k \in \mathsf{Pr}\,\Lambda$. Put $|A| := \bigoplus_k |A_k| \otimes P_k$ and consider it as a projective $\mathbf{R} \otimes \Lambda$-module. On the other hand, objects from $\mathbf{B} \otimes \Lambda$ are indeed projective $\mathbf{S} \otimes \Lambda$-modules. Hence, to each element $u \in \mathsf{U}^{\Lambda}(B, A)$ we can associate a homomorphism $|u| \in \mathrm{Hom}_{\mathbf{S}\otimes\Lambda}(B, |A|)$.

Call an element $u \in \mathsf{U}^{\Lambda}(B,A)$ *correct* if it satisfy the following conditions:

1. $B \simeq F \otimes P$ *for some* free S*-module* F *and some* $P \in \mathsf{Pr}\,\Lambda$.
2. $\operatorname{Ker}|u| \subseteq (\operatorname{rad}\mathsf{S})B$.
3. $\operatorname{Coker}|u|$ *is flat as* Λ*-module.*
4. *The induced homomorphism* $|u|_{\mathbf{R}} : \mathbf{R} \otimes_{\mathbf{S}} B \to |A|$ *is an isomorphism.*

If u satisfies the conditions 2 and 3 above and the following one:

$4'$. $|u|_{\mathbf{R}}$ *is an epimorphism,*

we call it *semi-correct.* Note that condition 3 is empty if $\Lambda = \mathbf{k}$ (i.e., for elements of U themselves).

Denote by $\mathsf{El}_c(\mathsf{U},\Lambda)$ and $\mathsf{El}_{sc}(\mathsf{U},\Lambda)$ respectively the full subcategories of $\mathsf{El}(\mathsf{U},\Lambda)$ consisting of all correct and of all semi-correct elements. Evidently, both families are correct families in the sense defined in Section 2. Therefore, we have for the bimodule U the notion of *correctly* (or *semi-correctly*) *finite*, *tame* (*bounded* or *unbounded*) or *wild.*

Special bimodules naturally arise when we apply the procedure of Section 3 to rationally composed curves. Namely, let C be such a curve, $\tilde{C}$ be its normalization and $C_1, C_2, \dots, C_t$ be the irreducible components of $\tilde{C}$. Fix isomorphisms $C_k \simeq \mathbb{P}^1$. Then $\mathsf{VB}(\tilde{C}) \simeq \mathsf{L}^t$. Put $\mathbf{R} := \tilde{\mathcal{F}} = \tilde{\mathcal{O}}/\mathcal{J}$ and $\mathbf{S} := \mathcal{F} = \mathcal{O}/\mathcal{J}$. Let $S = S(C) = \{\, p_1, p_2, \dots, p_s \,\}$ be the set of singular points of C and $\tilde{S} = \{\, q_1, q_2, \dots, q_r \,\}$ the preimage of S in $\tilde{C}$. Then $\mathbf{S} = \prod_{j=1}^{s} \mathbf{S}_j$, where $\mathbf{S}_j = \mathcal{F}_{p_j}$, $\mathbf{R} = \prod_{k=1}^{r} \mathbf{R}_k$, where $\mathbf{R}_k = \tilde{\mathcal{F}}_{q_k}$, and all algebras $\mathbf{S}_j$, $\mathbf{R}_k$ are local. Define the equivalence relation on the indices $1, 2, \dots, r$ by putting $k \sim l$ if and only if $\nu(q_k) = \nu(q_l)$ where $\nu : \tilde{C} \to C$ is the normalization map. Note that in this case $(k,j) \in \mathbf{J}$ means that $\nu(q_k) = p_j$. In this case put $\kappa(k,j) := q_k$. Now the following result is quite obvious.

Proposition 4.1. *In the situation above* $[\mathbf{R},\mathbf{S},\kappa]$ *are special data, the special bimodule* $\mathsf{U}[\mathbf{R},\mathbf{S},\kappa]$ *is isomorphic to the bimodule* $\mathsf{U}(C)$ *corresponding to the curve* C *via Proposition 3.8 and the notions of correct and semi-correct elements for these bimodules coincide.*

5. Representation type of special bimodules

Now we are going to describe the representation types of special bimodules and hence the VB-types of rationally composed curves.

Step 5.1. *If the algebra* $\mathbf{R}$ *is non-semi-simple, then the shifting bimodule* $\mathbf{U} = \mathbf{U}[\,\mathbf{R},\mathbf{S},\kappa\,]$ *is correctly (hence also semi-correctly) wild.*

Proof. Note first that there are no morphisms in A from (i,n) to (i,n') if $n > n'$. Hence, verifying the second condition of the definition of a strict element from Section 2 we may always suppose that $\sigma = 1$.

Let $\mathbf{F} := \mathbf{k}\langle z_1, z_2 \rangle$ be a free $\mathbf{k}$-algebra with 2 generators, $B := \mathbf{S}^2 \otimes \mathbf{F}$ and

$$A := \bigoplus_{i=1}^{t} ((i,n) \oplus (i,n+1)) \otimes \mathbf{F}$$

for some (arbitrary) $n \in \mathbb{Z}$. Then

$$\begin{aligned} \mathsf{U}^{\mathbf{F}}(B,A) &\simeq \bigoplus_{(i,j)\in \mathbf{J}} U_{ij} \otimes \mathbf{F}, \quad \text{where} \\ U_{ij} &= \operatorname{Hom}_{\mathbf{S}}(\mathbf{S}_j^2, \mathbf{R}(i))(n) \oplus \operatorname{Hom}_{\mathbf{S}}(\mathbf{S}_j^2, \mathbf{R}(i))(n+1) . \end{aligned}$$

Put $\mathbf{R}_{ij} := \mathbf{R}(i)\mathbf{S}_j$. It is a direct summand of $\mathbf{R}(i)$ and evidently, $\operatorname{Hom}_{\mathbf{S}}(\mathbf{S}_j, \mathbf{R}(i)) \simeq \mathbf{R}_{ij}$. We identify the elements of $U_{ij} \otimes \mathbf{F}$ with the 2×2 matrices with the entries in $\mathbf{R}_{ij} \otimes \mathbf{F}$,

$$u_{ij} := \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where the first row corresponds to the (n)-component and the second one to the $(n+1)$-component.

Choose some index i_0 such that $\mathbf{R}(i_0)$ is non-semi-simple and some minimal ideal $I \subseteq \operatorname{rad} \mathbf{R}_{i_0 j_0}$ for some j_0. As $\mathbf{k}$ is algebraically closed, $I = \mathbf{k} \cdot \alpha$ for some element α. As $\mathbf{S} \not\supset I$, $\mathbf{S} \cap I = \{ 0 \}$. For simplicity, change κ such that $\kappa(i_0, j_0) = (1:0)$.

Consider the element $u \in \mathsf{U}^{\mathbf{F}}(B,A)$ whose components are the following:

$$\begin{aligned} u_{i_0 j_0} &= \begin{pmatrix} 1 & \alpha \\ \alpha z_1 & 1 + \alpha z_2 \end{pmatrix}, \\ u_{ij} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for } (i,j) \neq (i_0, j_0) . \end{aligned}$$

It is obviously correct as $|u|$ coincides modulo radical with the natural embedding $\mathbf{S}^2 \otimes \mathbf{F} \to \mathbf{R}^2 \otimes \mathbf{F}$. We prove that it is strict.

Let N be an m-dimensional $\mathbf{F}$-module, Z_k ($k = 1, 2$) be the matrices describing the action of z_k on the vector space N. Then $u(N)$ can be identified with the family of $2m \times 2m$ matrices $u_{ij}(Z_1, Z_2)$ with entries in $\mathbf{R}_{ij}$, where:

$$\begin{aligned} u_{i_0 j_0}(Z_1, Z_2) &= \begin{pmatrix} E & \alpha E \\ \alpha Z_1 & 1 + \alpha Z_2 \end{pmatrix}, \\ u_{ij}(Z_1, Z_2) &= \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \quad \text{for } (i,j) \neq (i_0, j_0), \end{aligned}$$

(E is the identity matrix). If $u(N') := \{ u_{ij}(Z'_1, Z'_2) \}$ is the element corresponding to another $\mathbf{F}$-module N', any morphism $f : u(N) \to$

$u(N')$ is given by a family of matrices $\{ f_{ij}, g_j \mid (i,j) \in \mathbf{J},\, 1 \le j \le t \}$,

$$\begin{aligned} f_{ij} &= \begin{pmatrix} a_i & 0 \\ \xi_{ij} b_i + \eta_{ij} c_i & d_i \end{pmatrix}, \\ g_j &= \begin{pmatrix} s_{1j} & s_{2j} \\ s_{3j} & s_{4j} \end{pmatrix}, \end{aligned}$$

where a_i, b_i, c_i, d_i are $m \times m$ matrices over $\mathbf{k}$ and s_{kj} $(k = 1,2,3,4)$ are $m \times m$ matrices over $\mathbf{S}_j$ satisfying the relations:

$$f_{ij} u_{ij}(Z_1, Z_2) = u_{ij}(Z_1', Z_2') g_j \text{ for all } (i,j) \in \mathbf{J}\,. \tag{2}$$

In particular, for $(i,j) = (i_0, j_0)$ these relations give for the first row:

$$\begin{aligned} a_{i_0} &= s_{1j_0} + \alpha s_{3j_0}\,, \\ \alpha a_{i_0} &= s_{2j_0} + \alpha s_{4j_0}\,, \end{aligned}$$

whence $a_{i_0} = s_{1j_0}$, $s_{2j_0} = 0 = \alpha s_{3j_0}$ and $\alpha a_{i_0} = \alpha s_{4j_0}$ as $\mathbf{S} \cap \mathbf{R}\alpha = \{0\}$. Now the relations (2) for $(i,j) = (i_0, j_0)$ and the second row give

$$\begin{aligned} b_{i_0} + \alpha d_{i_0} Z_1 &= \alpha Z_1' s_{1j_0} + s_{3j_0}\,, \\ \alpha b_{i_0} + d_{i_0} + \alpha d_{i_0} Z_2 &= s_{4j_0} + \alpha Z_2' s_{4j_0}. \end{aligned}$$

This gives $b_{i_0} = s_{3j_0}$, whence both of them are zero as $\alpha s_{3j_0} = 0$; $d_{i_0} = s_{4j_0}$, whence both of them equal a_{i_0}; at last $d_{i_0} Z_1 = Z_1' s_{1j_0}$ and $d_{i_0} Z_1 = Z_1' s_{4j_0}$. Thus

$$f_{i_0, j_0} = g_{j_0} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \quad \text{where} \quad a = a_{1i_0}$$

and $aZ_k = Z_k' a$ $(k = 1,2)$, i.e., $a \in \operatorname{Hom}_{\mathbf{F}}(N, N')$.

The relations (2) for other values of (i,j) easily imply that

$$f_{ij} = g_j = \begin{pmatrix} a & 0 \\ h_{ij} & a \end{pmatrix}, \quad \text{where} \quad h_{ij} = \xi_{ij} b_i + \eta_{ij} c_i\,.$$

We have only to take into account the "connectedness" condition *3* from the definition of special data. Now it is evident that the functor $u \mapsto u(N)$ maps indecomposable to indecomposable and non-isomorphic to non-isomorphic. Therefore, U is correctly wild. $\square$

For the rest of the section suppose $\mathbf{R}$ (and hence $\mathbf{S}$) to be semi-simple. In this case condition *2* from the definition of correct or semi-correct elements means that $|u|$ is a monomorphism. For each $i = 1, \dots, t$, let m_i be the number of such j that $(k,j) \in \mathbf{J}$ for some $k \in c_i$. On the other hand, for each $j = 1, \dots, s$, let l_j be the number of such k that $(k,j) \in \mathbf{J}$. Note that we always have $l_j > 1$. Otherwise, there were a unique index k such that $(k,j) \in \mathbf{J}$ with $\dim \mathbf{R}_k = 1$. But then $\mathbf{S}_j \subseteq \mathbf{R}_k$, hence, $\mathbf{S}_j = \mathbf{R}_k$, which contradicts condition 1 of the definition of special data. On the other hand, as $\mathbf{S}_j\mathbf{S}_{j'}=0$ for $j\neq j'$, $(k,j)\in\mathbf{J}$ implies $(k,j')\notin\mathbf{J}$ for $j\neq j'$.

Step 5.2. *If $l_{j_0}>2$ for some j_0, the shifting bimodule $\mathbf{U}$ is correctly (hence also semi-correctly) wild.*

Proof. Suppose there are 3 indices k_1,k_2,k_3 such that $(k_q,j_0)\in\mathbf{J}$ for $q=1,2,3$. Put $B:=\mathbf{S}^4\otimes\mathbf{F}$ and

$$A:=\left(\bigoplus_{i=1}^{t}\oplus_{m=0}^{3}(i,n+m)\right)\otimes\mathbf{F}$$

for some (arbitrary) $n\in\mathbb{Z}$. Note that now $\operatorname{Hom}_{\mathbf{S}}(\mathbf{S}_j,\mathbf{R}_k)\simeq\mathbf{k}$ for $(k,j)\in\mathbf{S}$. Hence, the elements from $\mathsf{U}^{\mathbf{F}}(B,A)$ can be regarded as sets of 4×4 matrices $\{\,u_{kj}\,|\,(k,j)\in\mathbf{J}\,\}$, u_{kj} having entries in $\mathbf{R}_k\otimes\mathbf{F}$.

Now take u such that all its components are identity matrices except of the next two ones:

$$u_{k_1j_0}=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},$$
$$u_{k_2j_0}=\begin{pmatrix}1&1&z_1&z_2\\0&1&1&0\\0&0&1&1\\0&0&0&1\end{pmatrix},$$

Just as in Step 5.1, some easy straightforward calculation, which we omit, shows that u is indeed a strict element. □

Suppose now that $l_j\leq 2$ (hence $l_j=2$) for each j. Then the pair $\mathbf{S}\subset\mathbf{R}$, together with the equivalence relation $\sim$, can be completely described by its *diagram* $\Delta=\Delta[\,\mathbf{R},\mathbf{S},\sim]$. The vertices of Δ are just the indices $i=1,\dots,t$, its edges are the indexes $j=1,\dots,s$ and an edge j is incident to a vertex i if and only if $(k,j)\in\mathbf{J}$ for some $k\in c_i$. In case when j is incident to a unique i (then, of course, $\dim\mathbf{R}(i)=2$), consider j as a *loop* at the vertex i.

Of course, given any graph Δ, non-oriented but possibly with loops and (or) multiple edges, we can restore some pair $\mathbf{S}\subset\mathbf{R}$ and an equivalence relation $\sim$ (with semi-simple $\mathbf{R}$ and all $l_j=2$) such that Δ is just their diagram. So, to obtain a special data of this kind, we need only a graph and a function κ. Therefore we call such data and the corresponding bimodule *graphical.*

Step 5.3. *Graphical data with the diagram Δ are:*

1. *Finite (hence also semi-correctly and correctly finite) if Δ is a Dynkin diagram of type* A*, i.e., a chain.*

2. *Tame (hence also semi-correctly and correctly tame) if* Δ *is an extended Dynkin diagram of type* $\tilde{\mathrm{A}}$*, i.e., a cycle (possibly, one vertex with one loop). Moreover, in this case they are correctly (hence, semi-correctly and "absolutely") unbounded.*
3. *Correctly (hence also semi-correctly and "absolutely") wild otherwise.*

Note that in the first two cases $m_i \le 2$ for each $i = 1, 2, \dots, r$ (in particular, each equivalence class c_i consists of at most two elements). Hence, the graph Δ determines the bimodule U up to isomorphism (as any pair of points of a projective line can be moved to any other pair by a collineation).

Proof. Case *1* is very simple. Put $\mathbf{R}_{kj} := \mathbf{R}_k \mathbf{S}_j$ for $(k, j) \in \mathbf{J}$. All these spaces are one-dimensional, hence we may identify them with $\mathbf{k}$. Moreover, $s = t - 1$ and we can arrange the indices in such a way that the edge i is incident to the vertices i and $i + 1$ for each $i = 1, \dots, t-1$. Then one checks immediately that the indecomposable elements of $\mathsf{El}_{sc}(\mathsf{U})$ are in one-to-one correspondence with the finite sequences ν of integers of the form:

$$\nu = (m, r; \delta_0, \delta_1; n_1, n_2, \dots, n_r),$$

where $1 \le m \le t$, $0 \le r \le t - m + 1$ and both δ_0 and δ_1 are either 0 or 1, while n_i is arbitrary; moreover, if $m = 1$, then $\delta_0 = 1$, if $r = t - m + 1$, then $\delta_1 = 1$, and if $r = 0$, then $\delta_0 = \delta_1 = 0$. Namely, the element $u = u(\nu)$ corresponding to such a sequence lies in $\mathsf{U}(B, A)$, where

$$\begin{aligned} A &:= \oplus_{i=1}^{r} (m + i - 1, n_i), \\ B &:= \oplus_{i=1+\delta_0}^{r+1-\delta_1} \mathbf{S}_{m+i-1} \end{aligned} \tag{3}$$

and all components of u are equal to 1. Certainly, such an element equals $\sigma_m^{n_0} \sigma_{m+1}^{n_1} \dots \sigma_{m+r-1}^{n_{m+r-1}} u'$, where u' corresponds to the sequence ν' with the same values of m, r, δ_0, δ_1 but with all $k_i' = 0$. So the bimodule U is finite. Note that the element $u(\nu)$ is semi-correct if and only if either $\delta_0 = 0$ or $m = 0$ and, moreover, either $\delta_1 = 0$ or $r = t - m$. This element is correct if and only if $m = 1$ and $r = t$.

Case *2*. Here $s = t$ and we can arrange the indices in such a way that the edge i is incident to the vertices i and $i + 1$ (we define the vertex $t + k$ to be the same as the vertex k). This case fits into the framework of "bunches of chains" (cf. [2] or Appendix A).

Namely, in our case the underlying index set $\mathbf{I}$ is just the set of pairs $\{ (i, i), (i, i + 1) \mid i = 1, 2, \dots, t \}$. We set, for each pair $(i, j) \in \mathbf{I}$, $E_{ij} = \mathbb{Z}$ (with natural ordering) and $F_{ij} = \{ o_{ij} \}$ (a single element). To distinguish the elements from different E_{ij} we write them in the form $n(i, j)$ with $n \in \mathbb{Z}$, $(i, j) \in \mathbf{I}$. The equivalence relation on the union of all E_{ij} and F_{ij} is given by the rule: $n(i, j) \sim n(i, j')$

and $o_{ij} \sim o_{i'j}$ for all possible values of i, i', j, j'. Then one can verify that the bimodule corresponding to this bunch of chains coincides with the graphical bimodule corresponding to the graph Δ. Hence, we can use the results of [2] (cf. also Appendix A) to obtain a complete list of indecomposable elements. Taking into account the shape of this bunch, we can rearrange strings and bands defined in Appendix A as follows.

String representations are very much like the representations of the previous case. They correspond to *string data*, i.e., sequences of integers:

$$\nu = (m, r; \delta_0, \delta_1; n_0, n_1, \dots, n_{r-1}),$$

where $1 \le m \le t$, $r \ge 0$ and both δ_0 and δ_1 is either 0 or 1, while n_i are arbitrary. The corresponding elements $u(\nu)$ lies in $\mathsf{U}(A, B)$, where A and B are defined by the same formulae (3) if we put $(nt+j, k) := (j, k)$ and $\mathbf{S}_{nt+j} := \mathbf{S}_j$ for each n. The non-zero components of u are only those belonging to $\mathsf{U}(\mathbf{S}_{m+i-1}, (m+i-1, n_i))$, except for $i = 1$ if $\delta_0 = 1$, and $\mathsf{U}(\mathbf{S}_{m+i}, (m+i-1, n_i))$, except for $i = r$ if $\delta_1 = 1$. All these components are equal to 1. A string element is semi-correct if and only if $r > 0$ and $\delta_0 = \delta_1 = 0$. It is never correct.

Band representations correspond to *band data*, i.e., triples $\mathbf{b} = (\nu, d, \lambda)$, where d is a positive integer, $\lambda \in \mathbf{k}^*$ and ν is a sequence of integers $(n_1, n_2, \dots, n_{tr})$ which is *t-aperiodic*, i.e., is not a multiple self-concatenation of a shorter sequence whose length is also divisible by t.

The corresponding element $u_{\mathbf{b}} = u(\nu, d, \lambda)$ lies in $\mathsf{U}(A, B)$ for

$$A := \oplus_{i=1}^{tr} d(i, n_i),$$
$$B := \oplus_{i=1}^{tr} d\mathbf{S}_i,$$

Its non-zero components are those belonging to $\mathsf{U}(\mathbf{S}_i, (i, n_i))$ and $\mathsf{U}(\mathbf{S}_{i+1}, (i, n_i))$, which are given by unit matrices of dimension d, and the component belonging to $\mathsf{U}(\mathbf{S}_1, (tr, n_{tr}))$, which is given by the Jordan cell of dimension d with the eigenvalue λ. All band elements are correct.

All string elements are pairwise non-isomorphic. The isomorphic band elements are those corresponding to the triples (ν, d, λ) and (ν', d, λ), where ν' is a *t-cyclic* permutation of ν, i.e., $\nu' = (n_{tl+1}, n_{tl+2}, \dots, n_{tl})$.

As string and band elements exhaust all indecomposable elements of U, this bimodule is tame. Moreover, the band elements $u(\nu, 1, T) \in \mathsf{El}(\mathsf{U}, \mathbf{k}[T, T^{-1}])$ form a parametrizing family of elements of U. But their number grows with r. For instance, consider the band elements corresponding to sequences $\nu_{r,l}$ of length rt having all components 0 except the first and the $(tl+1)$-st ones, which are equal to 1. If $l \le [\frac{r}{2}]$ these band elements are pairwise non-isomorphic even up to shift and there are $[\frac{r}{2}]$ of them. Hence, U is correctly unbounded.

Case *3*. Suppose that Δ is neither a chain nor a cycle. Then it contains a vertex i_0 incident either to at least three edges or to a loop and to at least one more edge. We consider the former situation (the latter can be treated in the same way). Let j_k $(k = 1, 2, 3)$ be three edges incident to the vertex i_0. Denote the second end of the edge j_k by i_k (some of i_k may be equal). We put again $B := \mathbf{S} \otimes \mathbf{F}$,

$$A := \left(\bigoplus_{i=1}^{t} \oplus_{k=0}^{3} (i, n+k) \right) \otimes \mathbf{F}$$

for some (arbitrary) n and take u, all whose components are identity matrices except of the next two ones:

$$\begin{aligned} u_{i_0 j_1} = u_{i_0 j_2} &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\ u_{i_0 j_3} &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & z_1 \\ 1 & 1 & 0 & z_2 \end{pmatrix}. \end{aligned}$$

Again some easy straightforward calculation, which we omit, shows that u is a strict element. $\square$

Evidently, Steps 5.1–5.3 cover all possibilities, thus, defining correct (semi-correct) type of any special bimodule. Moreover, Cases 1 and 2 of Step 5.3 give a complete description of indecomposable elements in the finite and tame cases.

6. VB-types of rationally composed curves

We are able now to determine VB-type of a rationally composed curve C. Keep the notations introduced at the end of Section 4, before Proposition 4.1. First of all, it follows from Step 5.1 of Section 5 that whenever the algebra $\mathbf{R} = \tilde{\mathcal{F}}$ is non-semi-simple, the curve C is VB-wild. The same holds if some point $p \in S(C)$ has at least 3 preimages in $\tilde{S}$ (Step 5.2). Therefore, we can suppose that $\tilde{\mathcal{F}}$ is semi-simple, hence, coincides with $\prod_{k=1}^{r} \mathbf{k}(q_i)$, and $\nu^{-1}(p_j)$ consists of 2 points for each $j = 1, 2, \ldots, s$. Call such a curve C a *projective configuration.* In other words, a rationally composed curve is a projective configuration if and only if all its singular points are simple nodes (simple self-intersections). Such configurations are defined by their *dual graphs*, which coincide with the diagrams of the corresponding special bimodules. Recall the corresponding definition.

Definitions 6.1. *If C is a projective configuration, its* dual graph *is the graph $\Delta(C)$ whose vertices are the irreducible components of C, the edges are the singular points of C and an edge corresponding to the point p_j is incident to the vertex corresponding to the component C_i if and only if $p_j \in C_i$.*

Note that the graph $\Delta(C)$ is non-oriented, but may have loops and multiple edges. A loop appears if a singular point p_j belong to a unique component C_i (in this case the edge corresponding to p_j is only incident to the vertex corresponding to C_i). As we always suppose C to be connected, the graph $\Delta(C)$ is connected as well.

It is convenient to consider $\mathbb{P}^1$ also as a projective configuration. As it has only 1 component and no singular points, its dual graph has one vertex and no edges at all.

Now Step 5.3 of Section 5 can be reformulated as follows.

Proposition 6.2. *Let C be a projective configuration of. Then:*

1. *C is TF-finite (hence, VB-finite) if and only if $\Delta(C)$ is a Dynkin diagram of type A, i.e., a chain. (For instance, this is the case if $C = \mathbb{P}^1$.) Moreover, in this case all indecomposable vector bundles on C are of rank 1 and are determined up to isomorphism by their vector-degrees.*
2. *C is TF-tame (hence, VB-tame) if and only if $\Delta(C)$ is an extended Dynkin digram of type $\tilde{\mathrm{A}}$, i.e., a cycle. (For instance, this is the case if C is irreducible and has only one simple node.) Moreover, in this case it is VB-unbounded (hence, TF-unbounded).*
3. *In all other cases C is VB-wild (hence, TF-wild).*

For the sake of convenience, we give a complete list of indecomposable vector bundles on a projective configuration whose diagram is a cycle with t vertices. Such a bundle is given by a band datum (cf. Case 2 of Step 5.3) $\mathbf{b} = (\nu, d, \lambda)$. Namely, the corresponding locally free sheaf $\mathcal{B} = \mathcal{B}_{\mathbf{b}}$ can be constructed as follows.

Put $\mathcal{A}^i := d\mathcal{O}_i(n_i)$, where $\mathcal{O}_i := \mathcal{O}_k$ for $i = lt + k$. Consider the sheaf $\mathcal{A} := \oplus_{i=1}^{tr} \mathcal{A}^i$. For each singular point x_k and each $i \equiv k \pmod t$ denote by $\langle\, \mathbf{e}_{i1}, \mathbf{e}_{i2}, \dots, \mathbf{e}_{id} \,\rangle$ a $\mathbf{k}$-basis of $\mathcal{A}^i_{x_k}/\mathcal{J}_{x_k}\mathcal{A}^i_{x_k}$ and by $\langle\, \mathbf{e}'_{i1}, \mathbf{e}'_{i2}, \dots, \mathbf{e}'_{id} \,\rangle$ a $\mathbf{k}$-basis of $\mathcal{A}^{i+1}_{x_k}/\mathcal{J}_{x_k}\mathcal{A}^{i+1}_{x_k}$. Then $\mathcal{B}$ is the subsheaf of $\mathcal{A}$ such that $\mathcal{B}_y = \mathcal{A}_y$ for all $y \notin S$, while $\mathcal{B}_{x_k}$ is the $\mathcal{O}_{x_k}$-submodule generated by the elements $\mathbf{v}_{ij}$, where i runs through all indexes between 1 and tr such that $i \equiv k \pmod t$, while $1 \le j \le d$. These elements are defined by the following rules:

$$\mathbf{v}_{ij} := \begin{cases} \mathbf{e}_{ij} + \mathbf{e}'_{ij} & \text{if } i \ne tr\,, \\ \mathbf{e}_{ij} + \lambda\mathbf{e}'_{ij} + \mathbf{e}'_{i(j-1)} & \text{if } i = tr,\ j \ne 1\,, \\ \mathbf{e}_{ij} + \lambda\mathbf{e}'_{ij} & \text{if } i = tr,\ j = 1\,. \end{cases}$$

Now Step 2 of Section 5 implies the following result:

Corollary 6.3. *If C is a projective configuration, whose dual graph is a cycle, then any indecomposable vector bundle on C is isomorphic to a vector bundle of the form $\mathcal{B}_{\mathbf{b}}$ for some band datum $\mathbf{b}$.*

Altogether we come to the theorem completely describing VB-types of projective curves.

Theorem 6.4. *Let C be a projective curve.*

1. *If C is a projective configuration whose dual graph is a Dynkin diagram of type A, then it is TF-finite (hence, VB-finite).*
2. *If C is a smooth elliptic curve, then it is VB-tame, bounded.*
3. *If C is a projective configuration whose dual graph is an extended Dynkin diagram of type $\tilde{\mathrm{A}}$, then it is both TF- and VB-tame, unbounded.*
4. *In all other cases C is VB-wild (hence, TF-wild).*

Appendix A. Bunches of chains

Here we recall some definitions and results related to the *bunches of chains* considered by Bondarenko in [2]. We rearrange the definitions to make them more convenient for our purpose and consider only the case of *chains* (not semi-chains) as we need only this one and it is technically much easier.

Definition A.1. *A* bunch of chains $\mathbf{C} = \{\mathbf{I}, E_i, F_i, \sim\}$ *is defined by the following data:*

1. *A set $\mathbf{I}$ of indexes.*
2. *Two chains (i.e., linear ordered sets) E_i and F_i given for each $i \in \mathbf{I}$.*
   Put $\mathbf{E} := \bigcup_{i\in\mathbf{I}} E_i$, $\mathbf{F} := \bigcup_{i\in\mathbf{I}} F_i$ and $|\mathbf{C}| := \mathbf{E} \cup \mathbf{F}$.
3. *An equivalence relation $\sim$ on $|\mathbf{C}|$ such that each equivalence class consists of at most 2 elements.*

We also write $a - b$ if $a \in E_i$, $b \in F_i$ or vice versa (with the same index i). Moreover, we consider the ordering on $|\mathbf{C}|$, which is just the union of all orderings on E_i and F_i (i.e., $a < b$ means that a, b belong to the same chain E_i or F_i and $a < b$ in this chain).

If a bunch of chains $\mathbf{C} = \{\mathbf{I}, E_i, F_i, \sim\}$ is given, define the corresponding category $\mathsf{A} = \mathsf{A}(\mathbf{C})$ and the corresponding A-bimodule $\mathsf{U} = \mathsf{U}(\mathbf{C})$ as follows:

- The objects of A are the equivalence classes of $|\mathbf{C}|$ with respect to $\sim$.
- If x, y are two such equivalence classes, a basis of the morphism space $\mathsf{A}(x, y)$ consists of elements p_{ab} with $b \in x,\ a \in y,\ b < a$ and, if $x = y$, the identity morphism 1_x.
- The multiplication is given by the rule: $p_{ab}p_{bc} = p_{ac}$ if $c < b < a$, while all other possible products are zeros.

- A basis of $\mathsf{U}(x,y)$ consists of elements u_{ab} with $a \in y \cap \mathbf{E}$, $b \in x \cap \mathbf{F}$.
- The action of A on U is given by the rule: $p_{ca}u_{ab} = u_{cb}$ if $a < c$; $u_{ab}p_{bd} = u_{ad}$ if $d < b$, while all other possible products are zeros.

The category of representations of the bunch $\mathbf{C}$ is then defined as the category $\mathsf{El}(\mathsf{U})$ of the elements of this bimodule. One can easily verify that this definition gives just the same representations as the definition from [2]. Note that in [2] a more general situation was investigated, but we need only this case, which is essentially simpler than the general one. The following result is the specialization of the description of the representations given in [2] to our case, though it can be obtained directly using the same recursive procedure. First define some combinatorial objects called "strings" and "bands."

Definitions A.2. *Let $\mathbf{C} = \{\,\mathbf{I}, E_i, F_i, \sim\,\}$ be a bunch of chains.*

1. *A $\mathbf{C}$-word is a sequence $w = a_0r_1a_1r_2a_2\dots r_ma_m$, where $a_k \in |\mathbf{C}|$ and each r_k is either $\sim$ or $-$, such that:*
   (a) *$a_{k-1}r_ka_k$ in $|\mathbf{C}|$.*
   (b) *If $r_k = \sim$, then $r_{k+1} = -$ and vice versa.*
   Possibly $m = 0$, i.e., $w = a$ for some $a \in \mathbf{C}$.
2. *If $a_m = a_0$, $r_1 = \sim$ and $r_m = -$ call the word w a $\mathbf{C}$*-cycle. Note that in this case m is always even.
3. *Call a $\mathbf{C}$-word* full *if, whenever a_0 is not a unique element in his equivalence class, then $r_1 = \sim$ and whenever a_m is not a unique element in his equivalence class, then $r_m = \sim$.*
4. *Call a $\mathbf{C}$-cycle $w = a_0r_1a_1r_2a_2\dots r_ma_m$* aperiodic *if the sequence $a_0r_1a_1r_2a_2\dots r_m$ cannot be written as a multiple self-concatenation $vv\dots v$ of a shorter sequence v.*
5. *We say that an equivalence class x* occurs *in a word w if w contains a subword a in case $x = \{\,a\,\}$ is a singleton, or either a subword $a \sim b$ or a subword $b \sim a$ in case $x = \{\,a, b\,\}$ with $a \neq b$. In the former case we say that this occurrence corresponds to the occurrence of a, while in the latter case we say that it corresponds to both the occurrence of a and to the occurrence of b. Denote by $\nu(x, w)$ the number of occurrences of x in w.*

Definition A.3. *For a $\mathbf{C}$-word $w = a_0r_1a_1r_2a_2\dots r_ma_m$ call its $\sim$-subword any subword of the form $v = a \sim b$ as well as that of the form $v = a$, where $a \in \mathbf{C}$ is unique in its equivalence class. In the latter case put $|v| = \{\,a\,\}$, while in the former case put $|v| = \{\,a, b\,\}$. Denote by $[w]$ the collection of all $\sim$-subwords of w.*

Note that if w is a cycle it contains no entries $a \in \mathbf{C}$ such that a is unique in its equivalence class.

Definition A.4. *For any full $\mathbf{C}$-word $w = a_0r_1a_1r_2a_2\dots r_ma_m$ define the corresponding string representation $u = u_s(w)$ of the bunch $\mathbf{C}$ as follows.*

1. *$u \in \mathsf{U}(A,A)$, where $A = \bigoplus_{v\in[w]} |v|$.*
2. *Suppose there is a subword $v_1 - v_2$ in w with $v_i \in [w]$. Let a be the right end of the word v_1 and b be the left end of the word v_2. Then $\mathsf{U}(A,A)$ has a direct summand $\mathsf{U}(|v_1|,|v_2|) \oplus \mathsf{U}(|v_2|,|v_1|)$ and we define the corresponding components of u to be $(0,u_{ab})$ if $a \in \mathbf{E}$ and $(u_{ba},0)$ if $a \in \mathbf{F}$.*
3. *All other components of u are defined to be zero.*

Definition A.5. *For any triple (w,d,λ), where w is an aperiodic $\mathbf{C}$-cycle and d is a positive integer and $\lambda \in \mathbf{k}^* = \mathbf{k} \setminus \{0\}$, define the corresponding band representation $u = u_b(w,d,\lambda)$ of the bunch $\mathbf{C}$ as follows.*

1. *$u \in \mathsf{U}(A,A)$, where $A = \bigoplus_{v\in[w]} d|v|$.*
2. *Suppose there is a subword $v_1 - v_2$ in w with $v_i \in [w]$. Let a be the right end of the word v_1 and b be the left end of the word v_2. Then $\mathsf{U}(A,A)$ has a direct summand*
$$\begin{aligned}&\mathsf{U}(d|v_1|,d|v_2|) \oplus \mathsf{U}(d|v_2|,d|v_1|) \simeq \\ &\qquad \mathrm{Mat}(d\times d, \mathsf{U}(|v_1|,|v_2|) \oplus \mathrm{Mat}(d\times d, \mathsf{U}(|v_2|,|v_1|)))\end{aligned}$$
and we define the corresponding components of u to be $(0,u_{ab}I)$ if $a \in \mathbf{E}$ and $(u_{ba}I,0)$ if $a \in \mathbf{F}$, where I denotes the identity matrix.
3. *Let now v_1 be the last and v_2 be the first $\sim$-subword in w (they may coincide), a be the right end of the word v_1 and b be the left end of the word v_2. Then $\mathsf{U}(A,A)$ has a direct summand*
$$\begin{aligned}&\mathsf{U}(d|v_1|,d|v_2|) \oplus \mathsf{U}(d|v_2|,d|v_1|) \simeq \\ &\qquad \mathrm{Mat}(d\times d, \mathsf{U}(|v_1|,|v_2|) \oplus \mathrm{Mat}(d\times d, \mathsf{U}(|v_2|,|v_1|)))\end{aligned}$$
and we define the corresponding components of u to be $(0,u_{ab}J)$ if $a \in \mathbf{E}$ and $(u_{ba}J,0)$ if $a \in \mathbf{F}$, where J denotes the Jordan cell of dimension d with the eigenvalue λ.
4. *All other components of u are defined to be zero.*

Theorem A.6. 1. *All representations $u_s(w)$ and $u_b(w,d,\lambda)$ defined above are indecomposable and each indecomposable representation of $\mathbf{C}$ is isomorphic to one of these representations.*

2. *The only possible isomorphisms between these representations are the following:*
   (a) *$u_s(w) \simeq u_s(w')$ if $w = a_0r_1a_1\dots r_ma_m$ and $w' = a_mr_ma_{m-1}\dots r_1a_0$, the reversed word.*
   (b) *$u_b(w,d,\lambda) \simeq u_b(w',d,\lambda')$ if $w = a_0r_1a_1\dots r_ma_m$, $w' = a_{2k}r_{2k+1}a_{2k+2}\dots r_{2k}a_{2k}$ is a cyclic permutation of w, and $\lambda' = \lambda$ for k even, while for k odd $\lambda' = \lambda^{-1}$.*
   (c) *$u_b(w,d,\lambda) \simeq u_b(w',d,\lambda')$ if $w = a_0r_1a_1\dots r_ma_m$, $w' = a_{2k+1}r_{2k+1}a_{2k}\dots r_{2k+2}a_{2k+1}$ is a cyclic permutation of the reversed word, and $\lambda' = \lambda$ for k odd, while for k even $\lambda' = \lambda^{-1}$.*

Corollary A.7. *For any bunch of chains* $\mathbf{C}$ *the bimodule* $\mathsf{U}(\mathbf{C})$ *is tame (finite if there are no* $\mathbf{C}$*-bands at all). Moreover, a parametrizing set for its elements consists of all band representations* $u_b(w,1,T) \in \mathsf{El}(\mathsf{U}, \mathbf{k}[T,T^{-1})$.

Appendix B. Relation to finite dimensional algebras

There exists an amazing correspondence between rationally composed curves and some finite dimensional algebras. Suppose C is a rationally composed curve such that the algebra $\tilde{\mathcal{F}}$ is semi-simple. Then C can be completely described by its normalization $\tilde{C}$, the set S of its singular points (just a finite set), its preimage $\tilde{S} \subset \tilde{C}$ and the projection $\nu : \tilde{S} \to S$. Define the corresponding algebra $\mathbf{A} := \mathbf{A}(C)$ by its diagram $\Gamma := \Gamma(C)$ (quiver) and relations as follows.

Let $\tilde{C} = \cup_{i=1}^{t} C_i$, where $C_i \simeq \mathbb{P}^1$, $S = \{ p_1, p_2, \dots, p_s \}$ and $\nu^{-1}(p_j) \cap C_i = \{ q_{ijk} \mid 1 \le k \le m_{ij} \}$ for each j. Then Γ has $2t+s$ vertices $\{ a_i, b_i, c_j \mid 1 \le i \le t,\ 1 \le j \le s \}$. There are two arrows $x_i, y_i : a_i \to b_i$ and m_{ij} arrows $z_{ijk} : c_j \to a_i$. The defining relations for these arrows are:

$$\xi_{ijk} x_i z_{ijk} = \eta_{ijk} y_i z_{ijk}, \ \text{ where } q_{ijk} = (\xi_{ijk} : \eta_{ijk}) .$$

Then the following theorem holds.

Theorem B.1. *The algebra* $\mathbf{A}(C)$ *is tame (wild) if and only if the curve* C *is tame (wild).*

The proof is quite easy and straightforward, so we only sketch it. For each $i = 1, 2, \dots, t$ the subalgebra generated by x_i, y_i is a Kronecker algebra, for which all representations are known. So we can reduce all these arrows and then get a bimodule problem for the remainder. The observation is that this bimodule problem “almost coincides” with that corresponding to the curve C, as in Section 4. At least, it is not simpler, which implies the “only if” part of the theorem. But if C is not VB-wild, the points q_{ijk} can always be chosen as $(1:0)$ or $(0:1)$. In this case the resulting algebra $\mathbf{A}$ is special biserial, hence, tame [12].

We hope that there should be another explanation of this fact, which does not involve explicit calculations, but at the moment we cannot even predict of which kind such explanation could be.

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Department of Mechanics and Mathematics, Kyiv Taras Shevchenko University, 252033 Kyiv, Ukraine

E-mail address: drozd@uni-alg.kiev.ua

Fakultät für Mathematik, Universität Kaiserslautern, 66351 Kaiserslautern, Deutschland

E-mail address: greuel@mathematik.uni-kl.de