Vom Fachbereich Mathematik der Technischen Universität Kaiserslautern
zur Verleihung des akademischen Grades Doktor der Naturwissenschaften
(Doctor rerum naturalium, Dr. rer. nat.) genehmigte

Dissertation

Large scale asymptotics for Markov processes in the analytic framework of Mosco-Kuwae-Shioya

von

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Datum der Disputation: 04.11.2022

Fachbereich Mathematik

AG Funktionalanalysis und Stochastische Analysis

D386

Danksagung

Meinem Betreuer, Professor Martin Grothaus, möchte ich meinen tiefsten Dank aussprechen. Das Vertrauen, die Geduld und die Förderung, die ich während der gesamten Zeit durch ihn erfuhr, waren ein Segen. Seine Tür stand für mich immer offen. Ich danke Professor Michael Röckner für seine Tätigkeit als Gutachter, seine wertvollen Hinweise sowie Anmerkungen, und ganz besonders für unsere persönliche Kommunikation im Anschluss an die Abgabe meiner Dissertation. Von unschätzbaren Wert in der finalen Phase meiner Promotion waren die Ratschläge und Rückmeldungen von Doktor Sema Coşkun, die einen großen Teil des Entwurfes Korrektur las und sich immer wieder meine Erzählungen über die Arbeit anhörte. Wenn ich mich gedanklich im Kreis zu bewegen drohte, gaben mir unsere Gespräche oft einen neuen Blickwinkel. Außerdem geht mein Dank an den "sixth floor" für die förderliche, kollegiale Arbeitsatmosphäre. Bei der Graduate School des Fachbereichs Mathematik möchte ich mich für die freundliche Beratung und Unterstützung bei organisatorischen Fragen zum Promotionsverfahren bedanken. Ermöglicht wurde dieses Unterfangen erst durch finanzielle Unterstützung von der Deutschen Forschungsgemeinschaft. Die Konzepte und die Fertigstellung dieser Arbeit entstanden Hand in Hand mit dem Projekt GR 1809/14-1.

Abstract

In this thesis, a new concept to prove Mosco convergence of gradient-type Dirichlet forms within the L^2-framework of K. Kuwae and T. Shioya for varying reference measures is developed. The goal is, to impose as little additional conditions as possible on the sequence of reference measure $(\mu_N)_{N\in\mathbb{N}}$, apart from weak convergence of measures. Our approach combines the method of Finite Elements from numerical analysis with the topic of Mosco convergence. We tackle the problem first on a finite-dimensional substructure of the L^2-framework, which is induced by finitely many basis functions on the state space $\mathbb{R}^d$. These are shifted and rescaled versions of the archetype tent function $\chi^{(d)}$. For $d = 1$ the archetype tent function is given by

$$\chi^{(1)}(x) := \big((-x+1) \wedge (x+1)\big) \vee 0, \quad x \in \mathbb{R}.$$

For $d \geq 2$ we define a natural generalization of $\chi^{(1)}$ as

$$\chi^{(d)}(x) := \Big(\min_{i,j\in\{1,\dots,d\}} \big(\{1 + x_i - x_j, 1 + x_i, 1 - x_i\}\big)\Big)_+, \quad x \in \mathbb{R}^d.$$

Our strategy to obtain Mosco convergence of $\mathcal{E}^N(u,v) = \int_{\mathbb{R}^d} \langle \nabla u, \nabla v\rangle_{\text{euc}} \, d\mu_N$ towards $\mathcal{E}(u,v) = \int_{\mathbb{R}^d} \langle \nabla u, \nabla v\rangle_{\text{euc}} \, d\mu$ for $N \to \infty$ involves as a preliminary step to restrict those bilinear forms to arguments u, v from the vector space spanned by the finite family $\{\chi^{(d)}(\frac{\cdot}{r} - \alpha) \,|\alpha \in Z\}$ for a finite index set $Z \subset \mathbb{Z}^d$ and a scaling parameter $r \in (0,\infty)$. In a diagonal procedure, we consider a zero-sequence of scaling parameters and a sequence of index sets exhausting $\mathbb{Z}^d$. The original problem of Mosco convergence, $\mathcal{E}^N$ towards $\mathcal{E}$ w.r.t. arguments u, v form the respective minimal closed form domains extending the pre-domain $C_b^1(\mathbb{R}^d)$, can be solved by such a diagonal procedure if we ask for some additional conditions on the Radon-Nikodym derivatives $\rho_N(x) = \frac{d\mu_N(x)}{dx}$, $N \in \mathbb{N}$. The essential requirement reads

$$\frac{1}{(2r)^d} \int_{[-r,r]^d} |\rho_N(x) - \rho_N(x+y)| \, dy \quad \overset{r\to 0}{\longrightarrow} \quad 0 \quad \text{in } L^1(dx), \text{ uniformly in } N \in \mathbb{N}.$$

As an intermediate step towards a setting with an infinite-dimensional state space, we let E be a Suslin space and analyse the Mosco convergence of $\mathcal{E}^N(u,v) = \int_E \int_{\mathbb{R}^d} \langle \nabla_x u(z,x), \nabla_x v(z,x)\rangle_{\text{euc}} \, d\mu_N(z,x)$ with reference measure μ_N on $E \times \mathbb{R}^d$ for $N \in \mathbb{N}$. The form $\mathcal{E}^N$ can be seen as a superposition of gradient-type forms on $\mathbb{R}^d$. Subsequently, we derive an abstract result on Mosco convergence for classical gradient-type Dirichlet forms $\mathcal{E}^N(u,v) = \int_E \langle \nabla u, \nabla v\rangle_H \, d\mu_N$ with reference measure μ_N on a Suslin space E and a tangential Hilbert space $H \subseteq E$. The preceding analysis of superposed gradient-type forms can be used on the component forms $\mathcal{E}_k^N$, which provide the decomposition $\mathcal{E}^N = \sum_k \mathcal{E}_k^N$. The index of the component k runs over a suitable orthonormal basis of admissible elements in H. For the asymptotic form $\mathcal{E}$ and its component forms $\mathcal{E}^k$, we have to assume $\mathcal{D}(\mathcal{E}) = \bigcap_k \mathcal{D}(\mathcal{E}^k)$ regarding their domains, which is equivalent to the Markov uniqueness of $\mathcal{E}$. The abstract results are tested on an example from statistical mechanics. Under a scaling limit, tightness of the family of laws for a microscopic dynamical stochastic interface model over $(0,1)^d$ is shown and its asymptotic Dirichlet form identified. The considered model is based on a sequence of weakly converging Gaussian measures $(\mu_N)_{N\in\mathbb{N}}$ on $L^2((0,1)^d)$, which are perturbed by a class of physically relevant non-log-concave densities.

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Chapter 1 Introduction

1.1 Motivation

The abstract framework, presented by Kuwae and Shioya in [28], elaborates the functional analytic ideas of Mosco [23] concerning the convergence of spectral structures on a Hilbert space. Their adaptation of the topic accommodates a set-up of varying Hilbert spaces. The concept has found application in the field of partial differential equations and in probability theory. Mosco convergence often stands at the beginning of a further discussion on the probabilistic side. To convey the idea, μ shall denote a probability measure on the Borel σ-algebra $\mathcal{B}(E)$ of a topological Hausdorff space. Given an μ-symmetric Hunt process $X = (\Omega, \mathscr{F}, (X_t)_{t\geq 0}, (P_x)_{x\in E})$ with state space E and transition function $p_t(x, A) := P_x(\{X_t \in A\})$, $x \in E$, $A \in \mathcal{B}(E)$, $t \geq 0$, the measure μ is an invariant distribution of X. Extending the linear operator

$$\tilde{p}_t : u \mapsto \int\limits_E u(y)\,\mathrm{d}p_t(\cdot, \mathrm{d}y),$$

which acts on the bounded, measurable functions on E, to a symmetric contraction operator T_t on $L^2(E, \mu)$ for $t \geq 0$, the relation of X and its Dirichlet form $\mathcal{E}$ is determined by the equations

$$\begin{aligned} \mathcal{D}(\mathcal{E}) \;&=\; \Big\{ u \in L^2(E,\mu) \,\Big|\, \sup_{t>0} \frac{1}{t} \int_E u\,(u - T_t u)\,\mathrm{d}\mu \;<\infty \Big\} \\ \text{and} \quad \mathcal{E}(u,v) \;&=\; \lim_{t\to 0} \frac{1}{t} \int_E u\,(v - T_t v)\,\mathrm{d}\mu. \end{aligned}$$

The family $(T_t)_{t\geq 0}$ forms a strongly continuous contraction semigroup on $L^2(E, \mu)$. The matter of convergence of a sequence of such processes, indexed by a parameter $N \in \mathbb{N}$ which runs to infinity, can be approached via Mosco convergence. Given a countable family of Hunt processes

$$\big\{X^N = (\Omega_N,\, \mathscr{F}^N,\, (X_t^N)_{t\geq 0},\, (P_x^N)_{x\in E})\big\}, \quad N \in \mathbb{N},$$

and corresponding semigroups $(T_t^N)_{t>0}$ on $L^2(E, \mu_N)$, we assume the μ_N-symmetry of X^N, while X is as above. Moreover, the weak convergence of measures $(\mu_N)_{N\in\mathbb{N}}$ towards μ on their common state space E is a basic condition, under which the approximation problem is tackled. The equilibrium laws are defined $\widetilde{P}_N(B) := \int_E P_x^N(B)\,\mathrm{d}m_N(x)$ for $B \in \mathscr{F}^N$, $N \in \mathbb{N}$, and $\widetilde{P}(B) := \int_E P_x(B)\,\mathrm{d}m(x)$ for $B \in \mathscr{F}$. The convergence of the finite-dimensional distributions of equilibrium fluctuations, i.e.

$$\begin{aligned} &\lim_{N\to\infty} \int\limits_{\Omega_N} f_1(X_{t_1}^N) \cdot f_2(X_{t_1+t_2}^N) \cdot \dots \cdot f_k(X_{t_1+t_2+\dots+t_k}^N)\,\mathrm{d}\widetilde{P}_N \\ &\qquad = \lim_{N\to\infty} \int\limits_E T_{t_1}^N(f_1 \cdot T_{t_2}^N(\dots T_{t_{k-1}}^N(f_{k-1} \cdot T_{t_k}^N f_k)\dots))\,\mathrm{d}\mu_N(x) \\ &\qquad = \int\limits_E T_{t_1}(f_1 \cdot T_{t_2}(\dots T_{t_{k-1}}(f_{k-1} \cdot T_{t_k} f_k)\dots))\,\mathrm{d}\mu(x) \end{aligned}$$

$$= \int_{\Omega_N} f_1(X_{t_1}) \cdot f_2(X_{t_1+t_2}) \cdot \dots \cdot f_k(X_{t_1+t_2+\dots+t_k}) \, \mathrm{d}\widetilde{P}$$

with $f_1, \dots, f_k \in C_b(E)$, $t_1, \dots, t_k \in [0, \infty)$, $k \in \mathbb{N}$, is equivalent to Mosco convergence of the corresponding sequence of Dirichlet forms towards the corresponding asymptotic form. This is due to the theorem of Mosco-Kuwae-Shioya, as stated in [28, Theorem 2.4]. In the symmetric case, often a standard argumentation via the Lyons-Zheng decomposition can additionally show the tightness of equilibrium laws on a suitable path space. Closed symmetric forms of gradient-type

$$\mathcal{E}(u, v) = \int_E \langle \nabla u, \nabla v \rangle_H \, \mathrm{d}\mu, \qquad u, v \in \mathcal{D}(\mathcal{E}), \tag{1.1.1}$$

appear as standard examples for Dirichlet forms on $L^2(E, \mu)$ in the classic textbooks [20] of Ma, Röckner and [38] of Fukushima, Oshima, Takeda. Gradient forms present the central objects in this text. In the line above, we consider a Hilbert space $(H, \langle \cdot, \cdot \rangle)$ which is densely and continuously included in E, playing the role of a tangential space. Mosco convergence for a sequence of closed symmetric forms is formulated in terms of two conditions, (a) of [23, Definition 2.1] respectively (F1') of [28, Definition 2.11]), and (b) of [23, Definition 2.1] respectively (F2) of [28, Definition 2.11]). We call them (M1) and (M2).

Concerning the topic of Mosco convergence of gradient-type Dirichlet forms there is a vastness of open questions. When searching the literature for general results on Mosco convergence in this context, it is striking that the class of problems seems to divide into two groups. For the first one, where μ is log-concave and E is a real separable Hilbert space, there is an impressive theory, developed in [29], [36] and [37] among others. A probability measure m on E is called log-concave if for every pair U, V of open sets in E the inequality

$$\log m\big((1-t)U + tV\big) \geq (1-t) \log m(U) + t \log m(V), \quad t \geq 0,$$

holds true. The abstract result of [37] seals the deal for many cases in which μ and its weak approximations μ_N, $N \in \mathbb{N}$, are log-concave. The form $\mathcal{E}$ as above can be identified as the Mosco limit of

$$\mathcal{E}^N(u, v) = \int_E \langle \nabla u, \nabla v \rangle_{H_N} \, \mathrm{d}\mu_N, \qquad u, v \in \mathcal{D}(\mathcal{E}^N), \tag{1.1.2}$$

for N tending to infinity, if $(H_N)_{N \in \mathbb{N}}$ approximates H in a suitable sense. Quite surprisingly, besides the weak convergence of measures, the log-concavity of each individual μ_N is the only condition which needs to be imposed. A similar result seems hopeless in the other category of problems, which are characterized by the lack of a log-concavity assumption. Schematic guides to deal with Mosco convergence are rather rare to find. Hence, taking the weak convergence of the invariant measures as the only fixed assumption, the asymptotic analysis for gradient forms becomes a challenging and interesting topic on its own right. It is the commitment of this survey. Our motivation lies in the expansion of available theory in the field of Dirichlet forms, closing the significant gap between these two categories a little bit. The criterion for Mosco convergence we derive in the general analysis part of this text is fertile enough to allow for non-log-concave measures. To make our abstract results more palpable, it comes with a perturbation theory. The following problems in particular inspired the set-up of this survey.

[42] investigates the instance, where $H = E = L^2((0,1), \mathrm{d}t)$ and $H_N = E_N$ is the linear span of indicator functions $\mathbf{1}_{[2^{-N}(i-1), 2^{-N}i)}$, $i = 1, \dots, 2^N$. The authors prove Mosco convergence for the sequence of gradient forms $(\mathcal{E}^N)_N$, defined as in (1.1.2). The respective reference measure is chosen as

$$\mathrm{d}\mu_N(h) \propto \exp(-V(h))\, \mathrm{d}\tilde{\mu}_N(h), \qquad V : E \ni h \mapsto \int_0^1 f(h(t))\, \mathrm{d}t, \tag{1.1.3}$$

where $f : \mathbb{R} \to \mathbb{R}$ is of bounded variation and $\tilde{\mu}_N$ denotes the image measure under the orthogonal projection $E \to E_N$ of the law $\tilde{\mu}$ of a Brownian bridge between 0 and 0 in the interval $[0,1]$. The difficulty, as the authors point out, lies in the fact that measure of (1.1.3) is not log-concave, due to the non-convexity of the perturbing potential. The asymptotic form is a perturbed version of the standard gradient form on E in the Gaussian case,

$$\mathcal{E}(u,v) = \int_E \langle \nabla u, \nabla v \rangle_E \exp(-V)/Z \,\mathrm{d}\tilde{\mu}, \quad u, v \in \mathcal{D}(\mathcal{E}). \tag{1.1.4}$$

The domain of $\mathcal{E}$ coincides with the Sobolev space $H^{1,2}(E, \tilde{\mu})$ and $Z := \int \exp(-V)\, \mathrm{d}\tilde{\mu}$. Taking similar density functions, and analysing the convergence of gradient forms in a frame, where the approximating Gaussian measures are not images of one and the same measure under orthogonal projections, the problem becomes much more involved.

A second relevant problem emerges from question which comes up in [32, Remark 5.2]. There it is conjectured that for a sequence of weakly converging Gaussian measures $(\mu_N)_N$, with limit μ and continuous densities $(\rho_N)_N$, uniformly converging to a density ρ, a Mosco convergence result for gradient forms with reference measures $\rho_N \mu_N$, $N \in \mathbb{N}$, and asymptotic measure $\rho\mu$ is expected to hold. Moreover, it is stated that the exact conditions on $(\rho_N)_N$ for such a claim to be true are unclear. With our perturbation theory we can bring a little more light into that obscurity. The basic idea of [32] to address convergence in infinite dimension is the disintegration, just as this survey does. However, the method developed in this text, how the disintegrating densities are processed is very different and brings improvement. The mild assumptions we ask for allow for a useful perturbation theory. We assume in essence

$$p_N(z, s_0) - \frac{1}{2r} \int_{-r}^{r} \rho_N(z, s_0 + s)\, \mathrm{d}s \quad \overset{r \to 0}{\longrightarrow} \quad 0, \quad \text{in a suitable } L^1\text{-sense}, \tag{1.1.5}$$

uniformly in N, for the disintegration of $(\rho_N)_N$ along lines. The state space on which this approach works is quite general a Suslean locally convex vector space, as in the classic Dirichlet form set up of [22], [17], etc.

It is the common practice for a good reason to test how a newly derived, abstract result behaves when fed with some relevant example from physics. So we consider a problem inspired by [44]. It is addressed in the last chapter of this text.

1.2 An example from statistical mechanics

The Laplace operator is a mathematical tool to quantise the stiffness of a physical surface, such as a membrane, or the interface separating two coexisting phases of a medium in thermodynamic equilibrium. Therefore, Gaussian measures, whose covariances are determined by the Laplace operator Δ, or alternatively its mixed power $\Delta^l + \Delta^k$, for some $l, k \in \mathbb{N}_0$, $l \leq k$, appear naturally in the stochastic modelling of interfaces. Let's assume we want to describe the dynamics of a $(d+1)$-dimensional interface, with $d \in \mathbb{N}$, which tends to maintain a minimal surface tension, despite being subjected to random fluctuations. In a simple approach, we consider a process $X_t : [0,1]^d \to \mathbb{R}$, $t \geq 0$, and set-up the linear equation

$$\mathrm{d}X_t = \mathrm{d}W_t + (\Delta^l + \Delta^k)X_t \,\mathrm{d}t, \quad t \geq 0. \tag{1.2.1}$$

A suitable state space E has yet to be specified and the choice will depend on k and on d. We want $H := L^2((0,1)^d, \mathrm{d}z)$ to be densely included in E to have the SDE driven by an E-valued Brownian motion $(W_t)_{t\geq 0}$, whose covariance is given by the scalar product $\langle \cdot, \cdot \rangle$ of H, i.e.

$$\mathbb{E}\big[h_1(W_s)h_2(W_t)\big] = \min\{s,t\}\langle h_1, h_2\rangle, \quad h_1, h_2 \in E',\, s,t \geq 0. \tag{1.2.2}$$

The right-hand side of (1.2.2) has to be read in the sense of $E' \subset H' = H \subset E$. To pin the interface at the boundary of $[0,1]^d$, we define $(\Delta, \mathcal{D}(\Delta))$ to be the Friedrich's extension on H of

$$\Delta u = \sum_{i=1}^{d} \partial_i \partial_i u, \quad u \in C^2_{\mathrm{comp}}((0,1)^d),$$

operating on the space of twice continuously differentiable functions, compactly supported in $(0,1)^d$. The right choice for a state space, on which (1.2.1) is well-defined, is closely connected to the question, whether a Gaussian measure with covariance $(\Delta^l + \Delta^k)^{-1}$ exists. The mapping

$$C(u) := \exp\big(-\frac{1}{2}\big|(\Delta^l + \Delta^k)^{-\frac{1}{2}}u\big|_H^2\big), \quad u \in C^2_{\mathrm{comp}}((0,1)^d),$$

can be extended to a continuous function on the Hilbert space $\big(\mathcal{D}(\Delta^{\frac{k}{2}}), \langle \Delta^{\frac{k}{2}} \cdot, \Delta^{\frac{k}{2}} \cdot \rangle\big)$. So, the question can be answered, for example, with the Bochner-Minlos Theorem, as stated in [24, Theorems 1.5.2 & 1.5.3]. If H^s, $s \in [0,\infty)$, denotes the Hilbert space $\big(\mathcal{D}(\Delta^{\frac{s}{2}}), \langle \Delta^{\frac{s}{2}} \cdot, \Delta^{\frac{s}{2}} \cdot \rangle\big)$ and s_0 is large enough such that the embedding $H^{s_0+k} \hookrightarrow H$ is of Hilbert-Schimdt type, then $(\Delta^l + \Delta^k)^{-1}$ is the covariance operator of a centered Gaussian measure μ on $E := (H^{s_0})'$. In other words, there exists a Gaussian measure on $(E, \mathcal{B}(E))$ with characteristic function

$$\int_E \exp(ih(\varphi))\,\mathrm{d}\mu(h) = C(\varphi), \quad \varphi \in H^{s_0}.$$

We can regard (1.2.1) as an equation on E. There exists a conservative μ-symmetric diffusion process $\big((X_t)_{t\geq 0}, (\mathbf{P}_z)_{z\in E}\big)$ with state space E, solving (1.2.1) weakly. It is fully characterized by its transition semigroup of Markov kernels

$$p_t(z, A) := \mathbf{P}_z(\{X_t \in A\}), \quad A \in \mathcal{B}(E),\, z \in E,\, t \geq 0.$$

Symmetric, conservative transition semigroups, such as this one, are the primary objects we are interested in. The relevant semigroup of (1.2.1), the Mehler semigroup, can be comfortably handled, by means of Dirichlet form methods, using the results of [25] and [18]. The explicit formula for the semigroup reads

$$p_t(z, A) = \int_E \mathbf{1}_A\big(e^{t(\Delta^l + \Delta^k)}z + \sqrt{\mathrm{id} - e^{2t(\Delta^l + \Delta^k)}}z'\big)\,\mathrm{d}\mu(z')$$

for $A \in \mathcal{B}(E)$, $z \in E$ and $t \geq 0$. The associated Dirichlet form is the standard gradient form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(E,\mu)$. On

$$\mathcal{F}C_b^1(E') := \Big\{ F\big(h_1(\cdot), \dots, h_m(\cdot)\big) \,\Big|\, F \in C_b^1(\mathbb{R}^m),\, h_1, \dots, h_m \in E',\, m \in \mathbb{N} \Big\}$$

we obtain the representation

$$\mathcal{E}(u,v) = \int_E \langle \nabla u, \nabla v \rangle \,\mathrm{d}\mu, \quad u, v \in \mathcal{F}C_b^1(E').$$

The last chapter of this text mainly focuses on an approximation problem related to a centred, non-degenerate Gaussian measure μ which is supported on H and has a general covariance operator. So,

$$\int_H \langle h_1, k \rangle \langle h_2, k \rangle \,\mathrm{d}\mu(k) = \langle A^{-1} h_1, h_2 \rangle, \quad h_1, h_2 \in H, \tag{1.2.3}$$

where A is a positive definite, self-adjoint operator on H. The inverse A^{-1} is necessarily a trace class operator on H, in this setting. Important examples in the context of interface models include the cases $A = -\Delta$ with $d = 1$, or $A = -\Delta - \Delta^2$ with $d \in \{1, 2, 3\}$. Our motivation is to generalize the following fact. Let $(\mu_N)_{N \in \mathbb{N}}$ be finite-dimensional, centred Gaussian approximations for μ in the sense of weak convergence of measures on H. By this, we mean

$$\lim_{N \to \infty} \int_H F \,\mathrm{d}\mu_N = \int_H F \,\mathrm{d}\mu, \quad F \in C_b(H),$$

and additionally $\operatorname{supp}[\mu_N] = V_N$ for some subspace $V_N \subset H$ with $\dim(V_N) < \infty$, $N \in \mathbb{N}$. Further, let A_N be the symmetric, positive operator on V_N such that

$$\int_{V_N} \langle h_1, k \rangle \langle h_2, k \rangle \,\mathrm{d}\mu_N(k) = \langle A_N^{-1} h_1, h_2 \rangle, \quad h_1, h_2 \in V_N.$$

As a consequence of $\mu_N \Rightarrow \mu$, the corresponding Mehler semigroups,

$$p_t^N(z, B) = \int_{V_N} \mathbf{1}_B\big(e^{-tA_N} z + \sqrt{\mathrm{id} - e^{-2tA_N}}\, z'\big) \,\mathrm{d}\mu_N(z'),$$

$B \in \mathcal{B}(V_N)$, $z \in V_N$, $t \geq 0$ converge towards

$$p_t(z, B) = \int_H \mathbf{1}_B\big(e^{-tA} z + \sqrt{\mathrm{id} - e^{-2tA}}\, z'\big) \,\mathrm{d}\mu(z'), \quad B \in \mathcal{B}(H),\, z \in H,\, t \geq 0,$$

for $N \to \infty$. The notion of convergence, we are referring to, is the weak convergence of the Markov kernels, meaning

$$\begin{aligned} \lim_{N \to \infty} \int_{V_N} G(z) \int_{V_N} F(y) p_t^N(z, \mathrm{d}y) \,\mathrm{d}\mu_N(z) \\ = \int_H G(z) \int_H F(y) p_t(z, \mathrm{d}y) \,\mathrm{d}\mu(z), \quad G, F \in C_b(H),\, t \geq 0. \end{aligned} \tag{1.2.4}$$

This equation immediately leads to the convergence in law for the weak solution of

$$\mathrm{d}X_t = \mathrm{d}W_t^N + A_N X_t \,\mathrm{d}t, \quad t \geq 0,$$

towards the weak solution of

$$\mathrm{d}X_t = \mathrm{d}W_t + AX_t\,\mathrm{d}t, \quad t \geq 0,$$

if started in equilibrium. W^N denotes a Brownian motion on V_N with covariance given by $\langle \cdot, \cdot \rangle$. The tightness of the equilibrium laws follows from general concept for symmetric Dirichlet forms, based on the Lyons-Zheng decomposition.

Now, what we are searching for is a perturbation result for the convergence of transition semigroups as in (1.2.4). This task motivates us to look at the disturbed gradient forms

$$\mathcal{E}^N(u,v) = \int_{V_N} \langle \nabla u, \nabla v \rangle \rho_N \,\mathrm{d}\mu_N, \quad u,v \in C_b^1(V_N),\ N \in \mathbb{N},$$

and a potential limit gradient form

$$\mathcal{E}(u,v) = \int_H \langle \nabla u, \nabla v \rangle \rho \,\mathrm{d}\mu, \quad u,v \in \mathcal{F}C_b^1(H).$$

The convergence of semigroups, associated to their respective closures, can provide a challenging problem. If the perturbing densities ρ_N, $N \in \mathbb{N}$, respectively ρ are not not log-concave, then the most customary methods found in the literature do not apply. In our case ρ_N and ρ are bounded measurable functions from H into $\mathbb{R}$, which are not continuous. The family of disturbing densities, which are treated in this text, read

$$\Big\{ h \mapsto \exp\Big(\int_{(0,1)^d} f(h(z))\,\mathrm{d}z \Big) \,\Big|\, f : \mathbb{R} \to \mathbb{R} \text{ is a function of bd. variation} \Big\}. \tag{1.2.5}$$

We formulate a notion, in which sense a sequence $(f_N)_{N\in\mathbb{N}}$ of functions of bounded variation must converge towards f, such that, upon defining the corresponding sequence of densities $(\rho_N)_N$, the desired perturbation result for the convergence of semigroups can be derived.

The most important example, we have in mind, is the so called $(\nabla\varphi + \Delta\varphi)$ interface model, investigated in [44], where the weak measures convergence for the respective static Gaussian models is proven. It is most common in the context of scaling limits for interface models to take the Euclidean space, $\mathbb{R}^{k_N}$, as an effective state space for a microscopic point of view with approximation order N instead of V_N, if $\dim(V_N) = k_N$. Thus, we consider probability measures and Dirichlet forms on $\mathbb{R}^{k_N}$ rather than on V_N first, and later take the respective image structures under an injective linear map $\Lambda_N(x) \in H$, $x \in \mathbb{R}^{k_N}$, with $\mathrm{Im}(\Lambda_N) = V_N$. This map is called the height map. In [44], the height map is defined as follows for $N \in \mathbb{N}$. It starts with a set of grid points $G_N := (\frac{1}{N}, \frac{N-1}{N})^d \cap \frac{1}{N}\mathbb{Z}^d$. The approximation order k_N equals $(N-3)^d$, the size of the set G_N. Now, we take any ordering $p_1, \ldots, p_{k_N}$ of the elements of G_N. The height $\Lambda_N(x) := c_N (h(z))_{z\in[0,1]^d}$, for some state $x \in \mathbb{R}^{k_N}$, is given by the piecewise-linear, continuous interpolation $h : [0,1]^d \to \mathbb{R}$ of the sample

$$\big\{ (p_i, x_i) \,\big|\, i = 1, \ldots, k_N \big\} \cup \Big\{ (p, 0) \,\Big|\, p \in \tfrac{1}{N}\mathbb{Z}^d \setminus (\tfrac{1}{N}, \tfrac{N-1}{N})^d \Big\} \subset [0,1]^d \times \mathbb{R}.$$

The positive number c_N is a suitable scaling constant, depending on N and on d. In other words, by defining suitable bases functions, we can write

$$h(z) := c_N \sum_{i=1}^{k_N} x_i \chi^{p_i}_{\frac{1}{N}}(z), \quad z \in [0,1]^d.$$

The function $\chi^p_{\frac{1}{N}}$ is the tent function at the node $p \in G_N$ with scaling parameter $\frac{1}{N}$. The reader encounters the class of tent functions first in Section 3.1.1, as they also play an essential role for the abstract convergence theory of Chapter 4 and Chapter 3. Having chosen a suitable height map Λ_N, the next step is to define the relevant Hamiltonian functions of the model. We extend the ordering $p_1, \dots, p_{k_N}$ of the elements of G_N to an ordering $p_1, \dots, p_{\bar{k}_N}$ of all elements in the set $(0,N)^d \cap \frac{1}{N}\mathbb{Z}^d$, where $\bar{k}_N = (N-1)^d$. Then, we define the relation $p \sim q :\iff |p-q|_1 = \frac{1}{N}$ for $p, q \in \frac{1}{N}\mathbb{Z}^d$ and denote by $\bar{x}$ the vector in $\mathbb{R}^{\bar{k}_N}$ with $\bar{x}_i = 0$, $k_N < i \leq \bar{k}_N$, for given $x \in \mathbb{R}^{k_N}$. The relevant Hamiltonian functions are given by

$$\mathcal{H}_N^{\nabla}(x) := -\frac{1}{2}\sum_{i=1}^{k_N} \sum_{\substack{j=1,\dots,\bar{k}_N: \\ p_i \sim p_j}} x_i(\bar{x}_j - x_i),$$

$$\mathcal{H}_N^{\Delta}(x) := \frac{N^2}{2}\sum_{i=1}^{\bar{k}_N} \Big(\sum_{\substack{j=1,\dots,\bar{k}_N: \\ p_i \sim p_j}} (\bar{x}_j - \bar{x}_i)\Big)^2,$$

$$\mathcal{H}_N^{\nabla+\Delta}(x) := \mathcal{H}_N^{\nabla}(x) + \mathcal{H}_N^{\Delta}(x), \quad x \in \mathbb{R}^{k_N}.$$

The Gaussian measures on $\mathbb{R}^{k_N}$ corresponding to that Hamiltonian functions are defined

$$d\mu_N^{\nabla}(x) := \frac{1}{Z_N^{\nabla}} \exp\big(-\mathcal{H}_N^{\nabla}(x)\big)dx \quad \text{with} \quad Z_N^{\nabla} := \int_{\mathbb{R}^{k_N}} \exp\big(-\mathcal{H}_N^{\nabla}(x)\big)dx,$$

and μ_N^{Δ}, Z_N^{Δ}, $\mu_N^{\nabla+\Delta}$, $Z_N^{\nabla+\Delta}$ analogously. Let μ^{∇} denote the Gaussian measure on $L^2((0,1), \mathrm{d}z)$ with covariance given by (1.2.3) for the case $d=1$ and $A=\Delta$. Further, let μ^{Δ} and $\mu^{\nabla+\Delta}$ denote the Gaussian measures on $L^2((0,1)^d, \mathrm{d}z)$ with covariance given by (1.2.3) for the cases $d \in \{1,2,3\}$ and $A = \Delta^2$, respectively $A = \Delta + \Delta^2$. The weak convergence of the image measures

$$\begin{aligned} \mu_N^{\nabla} \circ \Lambda_N^{-1} &\Rightarrow \mu^{\nabla}, \quad \text{in case } d=1, \\ \mu_N^{\Delta} \circ \Lambda_N^{-1} &\Rightarrow \mu^{\Delta}, \quad \text{in case } d=1,2,3, \\ \mu_N^{\nabla+\Delta} \circ \Lambda_N^{-1} &\Rightarrow \mu^{\nabla+\Delta}, \quad \text{in case } d=1,2,3, \end{aligned}$$

for a suitable choice of $(c_N)_N$, as stated in [44, Theorem 2.1] represent the standard examples, the reader should have in mind when reading Chapter 5.

1.3 Outline

We endeavour to find new methods and tools in the topic of Mosco convergence. The idea for our approach in an abstract setting can be briefly summarized as follows. In the standard setting, property (M2) is an immediate consequence of the weak convergence of underlying invariant measures. Thinking of the convergence of symmetric, non-negative definite bilinear forms on a finite-dimensional vector space V, also (M1) is satisfied automatically in this case. That brings up the idea to look at a class of finite-dimensional subspaces of the relevant L^2-spaces, which is apt to provide property (M1) via a diagonal approximation procedure. In Section 2.1 the basic topological notions related to converging abstract Hilbert spaces are recalled from the literature and suitably adapted for the specific needs of this survey. In particular, the strong and the weak topology on the disjoint union $\mathcal{H}$ of these spaces is defined. At the end of the section we introduce a specific set-up and notation which is used throughout the whole text. Lemma 2.1.3 gives a comparison criterion for the convergence on $\mathcal{H}$. It is used in the course of this text to prove the weak measure convergence for distorting densities. The perturbation results addressed in subsequent chapters consider density functions which are not necessarily continuous. So, the weak measure convergence of $(\mu_N)_N$ does not trivially imply that of $(\rho_N \mu_N)_N$.

The central theorem of Mosco-Kuwae-Shioya is discussed in Section 2.2.1. A simplification of (M1) is found in Lemma 2.2.3. Then, the concept of compatible classes is explained. It constitutes our main contribution in the abstract setting and generalizes the the idea with finite-dimensional subspaces mentioned above. In a sense, compatible classes are substructures of $\mathcal{H}$, on which (M1) is assumed to be very easy to show. Intuitively, a compatible class represents a collection of subspaces which lie in all the disjoint Hilbert spaces of $\mathcal{H}$ and have some nice properties. For example, that property could be a finite dimension, or a uniform Lipschitz constant etc. The idea how to achieve the verification of (M1) on the whole of $\mathcal{H}$ is a diagonal approximation procedure. It is addressed in Theorem 2.2.8 and Theorem 2.2.9.

Section 3.1.1 realizes a particular scheme to fill the concept of a compatible class with life. The name *Finite Elements* was chosen due to the similarity with methods from numerics. Finite Elements accommodate the class of piecewise linear functions on $\mathbb{R}^d$ w.r.t. an equidistant triangulation, the so-called the Coxeter-Freudenthal-Kuhn triangulation. For an element of the resulting function space, the calculation of the weak gradient and its squared norm becomes a particularly easy expression and is derived in Theorem 3.1.6. Proposition 3.1.8 builds the bridge between Finite Elements and the convergence theory of Dirichlet forms. Theorem 3.1.12 and Theorem 3.1.13 provide the convergence result for gradient forms on $\mathbb{R}^d$ which is obtained via Finite Elements.

Theorem 3.2.9 manifests an asymptotic result for the superposition of d-dimensional gradient Dirichlet forms. The mixing measures my have disjoint support, even from the asymptotic one. In this sense Section 3.2.2 differs a bit from the standard setting, in which we not only assume the weak measure convergence but also that the topological support of the approximating measures is contained in the asymptotic one. The intention of superposing standard gradient forms is, of course, to reach to a infinite-dimensional result. That is achieved in Proposition 4.1.1 and Theorem 4.1.3 for classical gradient forms. Subsequently, the perturbation results in this setting are derived in Section 4.2. The model in [44] provides a nice opportunity to test it. We consider the scaling limit of a perturbed Gaussian interface model over $(0,1)^d$. Chapter 5 starts with a tightness result for the dynamical model, which is standard Dirichlet form theory. Then, the role of the height map Λ_N, which appears naturally in scaling problems for interface models, is discussed. In how far the weak convergence of measures $\mu_N \circ \Lambda_N^{-1} \Rightarrow \mu$ actually depends on the choice of the height map, is a relevant question. In the literature concerning interface models a variety of choices for Λ_N ia available. We find a useful class of height maps such that $\mu_N \circ \Lambda_N^{-1} \Rightarrow \mu$ is indeed indifferent of $(\Lambda_N)_{N \in \mathbb{N}}$, as long it belongs to that class. The most common height maps are of that class. We consider densities ρ_N, ρ from the family of (1.2.5) and the corresponding sequence $(f_N)_N$ of functions of bounded variance must converge in a specific sense, generalizing the notion of uniform convergence. Finally, the convergence of the perturbed transition semigroups is shown by Theorem 5.2.6. In the analysis of Chapter 5, it doesn't play a role, which sequence of Gaussian measures is chosen in particular. Only their weak measure convergence is sufficient. The converging covariances allow to choose the directions in which to disintegrate in such a way, that the verification of Mosco convergence in the unperturbed case becomes even trivial.

Chapter 2 The theory of Mosco-Kuwae-Shioya

2.1 Converging sequences of Hilbert spaces

In this section the basic terminology for a sequence of converging Hilbert spaces is introduced. Perpetually throughout this text it is assumed that the reader is well familiar with the definitions and concepts given in the following pages. The framework originates from a survey by Kazuhiro Kuwae and Takashi Shioya from 2003, [28], who topologised the disjoint union $\mathcal{H}$ of a not necessarily countable family of Hilbert spaces specifying all convergent nets and thus setting up a convergence class on $\mathcal{H}$. Retreating to a countable family of Hilbert spaces, the setting is the same as in [31] or [33] and represents a particular realization of the framework of Kuwae and Shioya. This text is committed to presenting these preliminaries in a minimalist style, i.e. focusing on what is absolutely needed for the purpose of our study on Mosco convergence in the upcoming chapters. The proof of Lemma 2.1.1 can be found in either of the referenced articles [28], [31] or [33].

All abstract Hilbert spaces are assumed to be real and separable. A sequence of converging Hilbert spaces comprises linear maps

$$\Phi_N : \mathcal{C} \to H_N$$

indexed by the parameter $N \in \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, where $\mathcal{C}$ is a dense linear subspace of a Hilbert space $(H, (\cdot,\cdot)_H)$ and the image space $(H_N, (\cdot,\cdot)_{H_N})$ is Hilbert as well. Apart from that, the asymptotic equations

$$\begin{aligned} \Phi_\infty \varphi = \varphi \quad & \text{and} \\ \lim_{N\to\infty} (\Phi_N \varphi, \Phi_N \varphi)_{H_N} = (\varphi,\varphi)_H \quad & \text{for every } \varphi \in \mathcal{C} \end{aligned} \tag{2.1.1}$$

are assumed to hold. The norm on H, or H_N for $N \in \mathbb{N}$, is denoted by $\|\cdot\|_H$, respectively $\|\cdot\|_{H_N}$. Occasionally the symbols H_∞, $(\cdot,\cdot)_\infty$, and $\|\cdot\|_\infty$ are used equivalently for H, $(\cdot,\cdot)$ or $\|\cdot\|$, respectively. Two different topologies are introduced on the disjoint union

$$\mathcal{H} := H \sqcup \Big(\bigsqcup_{N\in\mathbb{N}} H_N \Big),$$

which in precise notation is better written as the set of tuples

$$\mathcal{H} = \big\{ [u, N] \,\big|\, N \in \overline{\mathbb{N}},\, u \in H_N \big\}.$$

However, we prefer to write u instead of $[u, N]$ for an element $[u, N] \in \mathcal{H}$ and usually do so, unless the identification of the Hilbert space to which the element u belongs is ambiguous.

We now explain how the topologies on $\mathcal{H}$ are introduced in this text. Our motivation is, to do the same as in [31] or [33], both of which put the spotlight on such sequences $([u_k, N_k])_{k\in\mathbb{N}}$ in $\mathcal{H}$ for which $(N_k)_{k\in\mathbb{N}}$ is strictly increasing. This makes perfect sense as the Theorem of Mosco-Kuwae-Shioya, the reason why we need a notion of convergence on $\mathcal{H}$, only uses the term of convergence for exactly those sequences, anyway. On top of that, the notation of [28] can be largely simplified by this approach, because the formalism there is quite abundant if one deals with a countable family of Hilbert spaces only, where the asymptotic Hilbert space is uniquely defined. We want to introduce a notion of convergence as in [31] or [33], which comes from a topology on $\mathcal{H}$.

We proceed as follows. Inspired by [31, Lemma 7.6] we introduce the strong topology τ_{s} on $\mathcal{H}$ as the initial topology which is generated by the family of maps

$$\Big\{\, \mathcal{H} \ni [u,N] \mapsto N \in \overline{\mathbb{N}}, \quad \mathcal{H} \ni [u,N] \mapsto \|u\|_{H_N} \in [0,\infty), \\ \mathcal{H} \ni [u,N] \mapsto (u, \Phi_N \varphi)_{H_N} \in \mathbb{R} \quad \Big|\, \varphi \in \mathcal{C} \,\Big\}.$$

We should hint that $\overline{\mathbb{N}}$ is topologised in the sense of the one-point compactification of $\mathbb{N}$. That means, a sequence $(N_k)_{k\in\mathbb{N}}$ in $\overline{\mathbb{N}}$ converges, if and only if, it is either constant for sufficiently large k, or it diverges definitely to infinity. The latter requires that for each $M \in \mathbb{N}$ there exists $k_0 \in \mathbb{N}$ such that $N_k \geq M$ for $k \geq k_0$. Obviously, regarding the asymptotic Hilbert space H, the trace topology of τ_{s} coincides with the usual topology of strong convergence on H, which is generated by the metric $d(u,v) = \|u - v\|_H$, $u, v \in H$. The reader who is familiar with the work of Kuwae and Shioya realizes that τ_{s} is not exactly the same topology on $\mathcal{H}$ as the one given by [28, Definition 2.4]. However, the emerging notion of convergence for a sequence $([u_k, N_k])_{k\in\mathbb{N}}$ in $\mathcal{H}$ is identical whenever the limit, say $[u^*, N^*]$, is a member of the asymptotic Hilbert space, i.e. $N^* = \infty$. Looking ahead, the Theorem of Mosco-Kuwae-Shioya and all this text's analysis in upcoming chapters only use the notion of convergence for exactly such sequences, whose set of accumulation points is wholly contained in the asymptotic Hilbert space H. Therefore, the definition of τ_{s} serves our purpose.

We now address the subject of the weak topology on $\mathcal{H}$. For $m \in \mathbb{N}$ we set

$$\mathcal{K}_m := \big\{ [u,N] \,\big|\, N \in \overline{\mathbb{N}},\, u \in H_N,\, \|u_N\|_{H_N} \leq m \big\} \tag{2.1.2}$$

and consider $\mathcal{K}_m$ equipped with the initial topology which is generated by the family of maps

$$\Big\{\, \mathcal{K}_m \ni [u,N] \mapsto N \in \overline{\mathbb{N}}, \quad \mathcal{K}_m \ni [u,N] \mapsto (u, \Phi_N \varphi)_{H_N} \in \mathbb{R} \quad \Big|\, \varphi \in \mathcal{C} \,\Big\}.$$

Motivated by [33, Lemma 2.13], we want to call a sequence $([u_k, N_k])_{k\in\mathbb{N}}$ in $\mathcal{H}$ weakly convergent if and only if, for suitable $m \in \mathbb{N}$, the sequence is contained and convergent in $\mathcal{K}_m$. So, we define τ_{w} as the final topology on $\mathcal{H}$ generated by the family of inclusion maps from $\mathcal{K}_m$ into $\mathcal{H}$ over the index $m \in \mathbb{N}$. As in the strong case above, the trace topology of τ_{w} regarding the asymptotic Hilbert space H coincides with the usual topology of weak convergence on H. Again, τ_{w} defines the same notion of convergence as its counterpart in [28, Definition 2.5] w.r.t. all the sequences $([u_k, N_k])_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} N_k = \infty$ (in $\overline{\mathbb{N}}$).

It is instructive to put the emphasis on sections when analysing the convergence in $\mathcal{H}$. The term *section* shall in this text describe an element $(u_N)_{N\in\overline{\mathbb{N}}} \in \mathcal{H}^{\overline{\mathbb{N}}}$ such that $u_N \in H_N$ for $N \in \overline{\mathbb{N}}$. We refer to u_∞ as the asymptotic element of the section. If $N \mapsto u_N$ is continuous as a map from $\overline{\mathbb{N}}$ into $(\mathcal{H}, \tau_{\mathrm{s}})$ or $(\mathcal{H}, \tau_{\mathrm{w}})$, then the section is called strongly, respectively weakly, continuous. We now explain, why it is sufficient to focus on sections in our context, underlining the relevance of next lemma. Clearly, if $([u_k, N_k])_{k\in\mathbb{N}}$ is a sequence in $\mathcal{H}$, converging in one of the topologies, then either N_k is a constant number in $\mathbb{N}$ for k large enough, or $\lim_{k\to\infty} N_k = \infty$ in $\overline{\mathbb{N}}$. The former case is uninteresting for our purpose as $(u_k)_k$ would essentially be a sequence in one Hilbert space, other than the asymptotic Hilbert space. In the latter case we may consider the disjoint union

$$\mathcal{H}_{\mathrm{alt}} := H \sqcup \Big(\bigsqcup_{k\in\mathbb{N}} H_{N_k} \Big) \quad \text{together with } (\Phi_{N_k})_{k\in\mathbb{N}},\ \Phi_\infty, \tag{2.1.3}$$

potentially regarding several copies of one and the same Hilbert space. Now, if u_∞ is an element of H, then the convergence of $([u_k, N_k])_{k\in\mathbb{N}}$ towards u_∞ in $\mathcal{H}$ for $k\to\infty$ is equivalent to the continuity of the section $(u_{N_k})_{k\in\overline{\mathbb{N}}}$ in $\mathcal{H}_{\text{alt}}$, w.r.t. any one of the topologies.

Lemma 2.1.1. *(i) For every $u \in H$ there is a strongly continuous section which has u as its asymptotic element.*

(ii) A section $(u_N)_{N\in\overline{\mathbb{N}}}$ is strongly continuous if and only if

$$\lim_{N\to\infty}(u_N, v_N)_{H_N} = (u_\infty, v_\infty) \tag{2.1.4}$$

holds true for every weakly continuous section $(v_N)_{N\in\overline{\mathbb{N}}}$. Vice versa, $(u_N)_{N\in\overline{\mathbb{N}}}$ is weakly continuous if and only if (2.1.4) holds true for every strongly continuous section $(v_N)_{N\in\overline{\mathbb{N}}}$.

(iii) The norm $[u, N] \mapsto \|u\|_{H_N}$ is a lower semi-continuous map from $(\mathcal{H}, \tau_{\text{w}})$ into the non-negative real numbers.

(iv) The set $\mathcal{K}_m$ from (2.1.2) is sequentially compact w.r.t. τ_{w} for each $m\in\mathbb{N}$.

The reading of [31, Section 7] is very instructive and all of the properties from (i) to (iv) of the above lemma can be found there. We put the proof of Lemma 2.1.1 (i) here, as it gives a useful insight of how a strongly continuous section with asymptotic element u is constructed.

Proof of (i). Let $u \in H$ and $\varphi_1, \varphi_2, \dots \in \mathcal{C}$, $(\alpha_i)_{i\in\mathbb{N}} \in l^2$ be chosen such that

$$u = \lim_{K\to\infty}\sum_{i=1}^{K}\alpha_i\varphi_i \quad \text{strongly in } H. \tag{2.1.5}$$

For each $N_0\in\mathbb{N}$ let k_{N_0} denote the maximal choice of a natural number such that for $k\in\{1,\dots,k_{N_0}\}$ it holds

$$\sup_{\substack{x\in\mathbb{R}^k\\ |x|_{\text{euc}}\le 1}} \Big| \underbrace{\big[(\Phi_N\varphi_i, \Phi_N\varphi_j)_{H_N} - (\varphi_i,\varphi_j)_H\big]_{i,j=1}^k}_{=:A^{(N,k)}\in\mathbb{R}^{k\times k}} x\Big|_{\text{euc}} \le \frac{1}{k},$$

$$\text{for all } N\in\mathbb{N} \text{ with } N\ge N_0, \tag{2.1.6}$$

or $k_{N_0} := 1$ in case (2.1.6) is untrue for every $k\in\mathbb{N}$. Clearly, $(k_{N_0})_{N_0\in\mathbb{N}}$ is non-decreasing. Moreover, $(k_{N_0})_{N_0\in\mathbb{N}}$ is unbounded, since for arbitrarily fixed $k\in\mathbb{N}$ the asymptotic equation

$$\begin{aligned}
4\lim_{N\to\infty}(\Phi_N\varphi_i, \Phi_N\varphi_j)_{H_N} &= \lim_{N\to\infty}\big(\Phi_N(\varphi_i+\varphi_j), \Phi_N(\varphi_i+\varphi_j)\big)_{H_N}\\
&\quad - \lim_{N\to\infty}\big(\Phi_N(\varphi_i-\varphi_j), \Phi_N(\varphi_i-\varphi_j)\big)_{H_N}\\
&= \big((\varphi_i+\varphi_j),(\varphi_i+\varphi_j)\big)_H - \big((\varphi_i-\varphi_j),(\varphi_i-\varphi_j)\big)_H = 4(\varphi_i,\varphi_j)_H,
\end{aligned}$$

with $i,j\in\{1,\dots,k\}$, implies that (2.1.6) is fulfilled for all except for finitely many $N\in\mathbb{N}$. If $N_0, l_{N_0}\in\mathbb{N}$ with $l_{N_0}\le k_{N_0}$, then on the one hand

$$\Big|\sum_{i,j=1}^{l_{N_0}}\alpha_i\alpha_j(\Phi_{N_0}\varphi_i, \Phi_{N_0}\varphi_j)_{H_{N_0}} - \sum_{i,j=1}^{\infty}\alpha_i\alpha_j(\varphi_i,\varphi_j)_H\Big|$$

$$\leq \left|(\alpha_i)_{i=1}^{l_{N_0}}\right|_{\text{euc}} \left|A^{(N_0,l_{N_0})}(\alpha_i)_{i=1}^{l_{N_0}}\right|_{\text{euc}}$$
$$+ 2 \sum_{i=l_{N_0}+1}^{\infty} \sum_{j=1}^{\infty} \alpha_i\alpha_j(\varphi_i,\varphi_j)_H + \sum_{i,j=l_{N_0}+1}^{\infty} \alpha_i\alpha_j(\varphi_i,\varphi_j)_H$$
$$\leq \frac{1}{l_{N_0}} \sum_{i=1}^{\infty} \alpha_i^2 + 2\|u\|_H \Big\| \sum_{j=l_{N_0}+1}^{\infty} \alpha_j\varphi_j \Big\|_H + \Big\| \sum_{j=l_{N_0}+1}^{\infty} \alpha_j\varphi_j \Big\|_H^2 \tag{2.1.7}$$

and on the other hand

$$\Big| \sum_{i=1}^{l_{N_0}} \alpha_i(\Phi_{N_0}\varphi_i, \Phi_{N_0}\varphi_j)_{H_{N_0}} - \sum_{i=1}^{\infty} \alpha_i(\varphi_i,\varphi_j)_H \Big|$$
$$\leq \left|A^{(N_0,l_{N_0})}(\alpha_i)_{i=1}^{l_{N_0}}\right|_{\text{euc}} + \Big| \sum_{i=l_{N_0}+1}^{\infty} \alpha_i(\varphi_i,\varphi_j)_H \Big|$$
$$\leq \frac{1}{l_{N_0}} \Big(\sum_{i=1}^{\infty} \alpha_i^2 \Big)^{\frac{1}{2}} + \Big\| \sum_{i=l_{N_0}+1}^{\infty} \alpha_i\varphi_i \Big\|_H \|\varphi_j\|_H, \quad j \leq l_{N_0}. \tag{2.1.8}$$

Hence, for any non-decreasing, unbounded sequence $(l_{N_0})_{N_0\in\mathbb{N}}$ of natural numbers, with $l_{N_0} \leq k_{N_0}$, $N_0 \in \mathbb{N}$ it holds

$$\lim_{N_0\to\infty} \sum_{i,j=1}^{l_{N_0}} \alpha_i \Phi_{N_0}\varphi_i = \sum_{i=1}^{\infty} \alpha_i\varphi_i \quad \text{strongly in } \mathcal{H}, \tag{2.1.9}$$

because in the limit of N_0 to infinity, (2.1.7) provides the convergence of the norms and (2.1.8) shows the weak convergence. For example, the desired strongly continuous section with asymptotic element u has been constructed if we choose $\alpha_i \in \mathbb{R}$ and $\varphi_i \in \mathcal{C}$ for $i \in \mathbb{N}$ such that (2.1.5) is the representation of u w.r.t. an orthonormal basis in H. This concludes th proof of (i). □

As pointed out in the discussion preceding Lemma 2.1.1, our applications require to look at the convergence of a sequence in $\mathcal{H}$ only in such a case, where its limit is a member of the asymptotic Hilbert space H. We consider the family of such topologies τ on $\mathcal{H}$, for which the map $[u, N] \mapsto N$ is continuous from $(\mathcal{H}, \tau)$ into $\overline{\mathbb{N}}$. We call two topologies τ_1, τ_2 on $\mathcal{H}$, which are from that family, *asymptotically equivalent*, if for every $u \in H$ and every sequence $([u_k, N_k])_{k\in\mathbb{N}}$ in $\mathcal{H}$ with $\lim_{k\to\infty} N_k = \infty$ (in $\overline{\mathbb{N}}$) it holds

$$\lim_{k\to\infty} u_k = u \text{ in } (\mathcal{H}, \tau_1) \quad \Longleftrightarrow \quad \lim_{k\to\infty} u_k = u \text{ in } (\mathcal{H}, \tau_2).$$

In this text, we always use the term of equivalence regarding topologies on $\mathcal{H}$ in this sense, i.e. meaning the asymptotic equivalence. We state some observations regarding Lemma 2.1.1.

Remark 2.1.2. (i) Let $D \subseteq H$ be a dense linear subspace and $(u_N)_{N\in\overline{\mathbb{N}}}$ be a section in $\mathcal{H}$ with $\sup_{N\in\mathbb{N}} \|u_N\|_{H_N} < \infty$. A necessary and sufficient criterion for $(u_N)_{N\in\overline{\mathbb{N}}}$ to be weakly continuous can be derived from Lemma 2.1.1, (ii) and (iv): For every $v \in D$ there is a strongly continuous section $(v_N)_{N\in\overline{\mathbb{N}}}$ with $v_\infty = v$ and

$$\lim_{N\to\infty} (u_N, v_N)_{H_N} = (u_\infty, v_\infty).$$

(ii) It is natural to ask (Q): Would an asymptotically equivalent weak and strong topology on $\mathcal{H}$ have emerged, had the construction been initiated with a different choice $\Phi'_N : D \to H_N$ instead of Φ_N for $N \in \overline{\mathbb{N}}$? Of course, this question makes only sense if $(\Phi'_N)_{N\in\overline{\mathbb{N}}}$ meets the analogue of (2.1.1) for all elements φ from the dense linear subspace $D \subset H$. The answer to (Q) is affirmative if and only if

$$\lim_{N\to\infty} (\Phi'_N\varphi, \Phi_N\eta)_{H_N} = (\varphi, \eta), \quad \varphi \in D,\, \eta \in \mathcal{C}. \tag{2.1.10}$$

The necessity of (2.1.10) for an affirmative answer to (Q) is clear by Lemma 2.1.1 (ii). On the other hand, for the sufficiency of (2.1.10) it is enough to prove the equivalence of the corresponding notion of convergence for sections in $\mathcal{H}$. The reason for this is, for any sequence $(N_k)_{k\in\mathbb{N}}$ in $\overline{\mathbb{N}}$ with limit ∞ we can retreat to an alternative sequence of Hilbert spaces as in (2.1.3), w.r.t. which (2.1.10), of course, implies its analogue

$$\lim_{k\to\infty} (\Phi'_{N_k}\varphi, \Phi_{N_k}\eta)_{H_{N_k}} = (\varphi, \eta), \quad \varphi \in D,\, \eta \in \mathcal{C}.$$

So, let us discuss the convergence of sections in $\mathcal{H}$ assuming that (2.1.10) is satisfied. First, the strong convergence of the section $(\Phi'_N\varphi)_{N\in\overline{\mathbb{N}}}$ for $\varphi \in D$ w.r.t the notion induced by $(\Phi_N)_{N\in\overline{\mathbb{N}}}$ follows, and vice versa. Hence, concerning the notion of weak convergence, the answer to (Q) is affirmative in view of Remark 2.1.2 (i). Via the duality stated in Lemma 2.1.1 (ii), the affirmative answer to (Q) can also be given for the strong convergence.

(iii) As we learn from [31, Proposition 7.2] there are isometric isomorphisms $\hat{\Phi}_N : H \to H_N$ for $N \in \overline{\mathbb{N}}$ such that (2.1.10) holds with $D = H$ and $\Phi'_N = \hat{\Phi}_N$, $N \in \overline{\mathbb{N}}$.

We now explore a standard scheme of particular interest which brings this abstract concept to life. As a general notation in this text, if Ω is a set and $\mathcal{A}$ is a collection of pairwise disjoint, non-empty subsets of Ω, then for $B \subseteq \Omega$ we define

$$B^{\sim,\mathcal{A}} := \{\alpha \in \mathcal{A} \,|\, \alpha \cap B \neq \emptyset\}.$$

If $\mathcal{A}$ is clear from context, then we simply write $B^{\sim}$, or alternatively $\widetilde{B}$, instead of $B^{\sim,\mathcal{A}}$. For example, if m_1, m_2 are two measures on a measurable space $(X, \mathcal{F})$ and B is a set of measurable, real-valued functions on X, then $L^2(X, m_1) \cap \widetilde{A}$ denotes the subset of $L^2(X, m_1)$ comprising all classes which have some m_1-representative in A, while $L^2(X, m_2) \cap \widetilde{A}$ denotes the subset of $L^2(X, m_2)$ comprising all classes with m_2-representative in A. In the following, let E be a completely regular Hausdorff space, i.e. a Hausdorff topological space satisfying the Tychonoff separation axiom T3$\frac{1}{2}$. We further assume that each finite Borel measure on E is a Radon measure. For example, this condition holds for Suslin spaces. By definition, a Hausdorff space is called *Suslin space* if it is the image set of a continuous surjection whose preimage set is a Polish space. This useful hint about the regularity of Borel measures in Suslin spaces is shown in [41, Theorem 1.17] as 'Satz von P. A. Meyer', where it is also mentioned that the product topological space of two Suslin spaces is again a Suslin space. The latter is presumed for Section 3.2.2 and Section 4.1. The space of (bounded) continuous functions, respectively real-valued Borel measurable functions, on E is denoted by $C(E)$, $(C_b(E))$, respectively $\mathcal{B}(E)$, $(\mathcal{B}_b(E))$. Moreover, we write $A \in \mathcal{B}(E)$ if A is an element of the Borel σ-algebra, i.e. if $\mathbf{1}_A \in \mathcal{B}(E)$. There is a one-to-one correspondence which identifies a finite Borel measure μ on E and a positive linear form I on $C_b(E)$. A proof can be found in [41, Chapter 8]. A finite Borel measure μ on E is uniquely determined by the values of $I_\mu(f) := \int_E f \,\mathrm{d}\mu$, $f \in C_b(E)$. In particular, the set $\mathcal{C} := L^2(E,\mu) \cap \widetilde{C}_b(E)$ of μ-classes of Borel measurable real-valued functions with representatives in $C_b(E)$ form a dense linear subspace of $L^2(E,\mu)$, the Hilbert space of μ-classes of all square integrable elements in $\mathcal{B}(E)$. Indeed, if an element u is in the orthogonal complement of $\mathcal{C}$ in $L^2(E,\mu)$, then the equation

$$I_1(f) := \int_E f \max\{u,0\}\,\mathrm{d}\mu = -\int_E f \min\{u,0\}\,\mathrm{d}\mu =: I_2(f), \quad f \in C_b(E),$$

implies $u(x) = 0$ for μ-a.e. $x \in E$. As a notation throughout this text, we define

$$\operatorname{supp}[\mu] := \big\{ x \in E \,\big|\, \mu(U) > 0 \text{ for every open set } U \text{ with } x \in U \subseteq E \big\},$$

the topological support of μ.

Let μ and $\mathcal{C}$ be as above. In a situation in which $(\mu_N)_{N\in\mathbb{N}}$ is a sequence of finite Borel measures and μ its limit in the sense of weak measure convergence on E, i.e.

$$\lim_{N\to\infty} \int_E f\,\mathrm{d}\mu_N = \int_E f\,\mathrm{d}\mu, \quad f \in C_b(E), \tag{2.1.11}$$

we can understand $L^2(E,\mu_N)$, $N \in \mathbb{N}$, as a sequence of convergent Hilbert spaces with asymptotic space $L^2(E,\mu)$ if

$$\operatorname{supp}[\mu_N] \subseteq \operatorname{supp}[\mu], \quad N \in \mathbb{N}, \tag{2.1.12}$$

holds true regarding the topological support of the measures. We remark that (2.1.12) is equivalent to the condition,

$$\text{if } \mu_N(U) > 0, \quad \text{then} \quad \mu(U) > 0, \quad \text{for } U \subseteq E \text{ open}, \ N \in \mathbb{N}.$$

Given (2.1.11) and (2.1.12), unless stated otherwise, $\Phi_N^\sim$, $N \in \mathbb{N}$, shall in this text automatically be defined as the linear operator which maps an element $\varphi \in \mathcal{C}$ onto the μ_N-class of one of its representatives from $C_b(E)$. We remark that $\Phi_N^\sim$ is indeed well-defined as an operator from $\mathcal{C}$ into $L^2(E,\mu_N)$ for $N \in \mathbb{N}$, because of (2.1.12), and that $(\Phi_N^\sim)_N$ fulfils the asymptotic equation of (2.1.1) by (2.1.11). As this situation represents the canonic set-up for converging Hilbert spaces within this text, we will use all related terminology and notions, which are given above and subsequently in the course of Chapter 2, quite fluently in that context, without additional declarations. For example, if $\mathcal{E}^N$ is a symmetric closed form on a Hilbert space H_N for $N \in \overline{\mathbb{N}}$, then the statement that $(\mathcal{E}^N)_{N\in\mathbb{N}}$ converges in the sense of Mosco towards $\mathcal{E}^\infty$ (see Theorem 2.2.1 below) can neither be called true nor wrong in mathematical sense, before a particular choice of asymptotic isometries $(\Phi_N)_{N\in\overline{\mathbb{N}}}$ in (2.1.1) is specified (on which the validity of the claim depends of course). However, in the case $H_N = L^2(E,\mu_N)$, $H_\infty = L^2(E,\mu)$, with (2.1.11) and (2.1.12) fulfilled, we say that $(\mathcal{E}^N)_{N\in\mathbb{N}}$ converges in the sense of Mosco towards $\mathcal{E}^\infty$, presuming the automatic choice $\Phi_N := \Phi_N^\sim$, $N \in \mathbb{N}$.

Let μ and $(\mu_N)_{N\in\mathbb{N}}$ be as above, with (2.1.11) and (2.1.12) satisfied. For classes of non-negative functions, convergence in

$$\mathcal{H} := L^2(E,\mu) \sqcup \Big(\bigsqcup_{N\in\mathbb{N}} L^2(E,\mu_N) \Big)$$

can be verified via a simple comparison argument, stated in the next lemma. We recall the notion of a (continuous) section, from above. The lemma proves the continuity of a section under the condition that there exist a suitable continuous majorante and minorante.

Lemma 2.1.3. *Let* $(u_N)_{N\in\overline{\mathbb{N}}}$, $(v_N^{\mathrm{min},l})_{N\in\overline{\mathbb{N}}}$ *and* $(v_N^{\mathrm{maj},l})_{N\in\overline{\mathbb{N}}}$ *for* $l\in\mathbb{N}$, *be sections in* $\mathcal{H}$ *such that*

$$0 \le v_N^{\mathrm{min},l} \le u_N \le v_N^{\mathrm{maj},l}\ \mu_N\text{-a.e.},\quad N,l\in\mathbb{N}.$$

(i) If $(v_N^{\mathrm{min},l})_{N\in\overline{\mathbb{N}}}$, $(v_N^{\mathrm{maj},l})_{N\in\overline{\mathbb{N}}}$ *are weakly continuous and*

$$\lim_{l\to\infty} v^{\mathrm{min},l} = u_\infty,\quad \lim_{l\to\infty} v^{\mathrm{maj},l} = u_\infty \quad \text{weakly in } L^2(E,\mu),$$

then $(u_N)_{N\in\overline{\mathbb{N}}}$ *is also weakly continuous.*

(ii) The analogue statement holds w.r.t. τ_{s}, *i.e. if* $(v_N^{\mathrm{min},l})_{N\in\overline{\mathbb{N}}}$, $(v_N^{\mathrm{maj},l})_{N\in\overline{\mathbb{N}}}$ *are strongly continuous and*

$$\lim_{l\to\infty} v^{\mathrm{min},l} = u_\infty,\quad \lim_{l\to\infty} v^{\mathrm{maj},l} = u_\infty \quad \text{strongly in } L^2(E,\mu),$$

then $(u_N)_{N\in\overline{\mathbb{N}}}$ *is also strongly continuous.*

Proof. We start with the proof of (i). Let $(u_N)_{N\in\overline{\mathbb{N}}}$, $(v_N^{\mathrm{min},l})_{N\in\overline{\mathbb{N}}}$ and $(v_N^{\mathrm{maj},l})_{N\in\overline{\mathbb{N}}}$ be as in the assumptions for $l\in\mathbb{N}$. There exists $m\in\mathbb{N}$ such that $(u_N)_{N\in\mathbb{N}}$ is contained in $\mathcal{K}_m$, as in (2.1.2), since it is dominated by a weakly convergent sequence $(v_N^{\mathrm{maj},l})_{N\in\mathbb{N}}$. By virtue of Lemma 2.1.1 (iv) it suffices to show that all weak accumulation points of $(u_N)_{N\in\mathbb{N}}$ coincide with u_∞. Let $u^*\in L^2(E,\mu)$ be chosen such that a subsequence of $(u_N)_{N\in\mathbb{N}}$ converges weakly to u^*. Since after retreating to the corresponding subsequence of Hilbert spaces and re-defining $\mathcal{H}$ accordingly the assumptions of this Lemma would still hold, we may w.l.o.g. assume that $u^*\in L^2(E,\mu)$ is the weak limit of the sequence $(u_N)_{N\in\mathbb{N}}$.

Let f be a non-negative, bounded and continuous function on E and $l\in\mathbb{N}$. The inequality

$$0 \le \int_E v_N^{\mathrm{min},l} f \,\mathrm{d}\mu_N \le \int_E u_N f\,\mathrm{d}\mu_N \le \int_E v_N^{\mathrm{maj},l} f\,\mathrm{d}\mu_N,\quad N\in\mathbb{N},$$

yields asymptotically

$$0 \le \int_E v_\infty^{\mathrm{min},l} f\,\mathrm{d}\mu \le \int_E u^* f\,\mathrm{d}\mu \le \int_E v_\infty^{\mathrm{maj},l} f\,\mathrm{d}\mu$$

by considering the limit for N to infinity. Now passing to the limit of l to infinity, we obtain

$$\int_E u_\infty f\,\mathrm{d}\mu = \int_E u^* f\,\mathrm{d}\mu, \tag{2.1.13}$$

at first for every non-negative, bounded and continuous function f. Of course, (2.1.13) immediately generalizes to every $f\in C_b(E)$ proving $u_\infty(x)=u^*(x)$ holds for μ-a.e. $x\in E$, as desired.

Under the assumptions formulated in (ii) we look at the inequalities

$$0 \le \int_E \left|v_N^{\mathrm{min},l}\right|^2 \mathrm{d}\mu_N \le \int_E |u_N|^2\,\mathrm{d}\mu_N \le \int_E \left|v_N^{\mathrm{maj},l}\right|^2 \mathrm{d}\mu_N,\quad N\in\mathbb{N},$$

for $l\in\mathbb{N}$ and by passing to the limit of N to infinity observe that

$$\int_E \left|v^{\mathrm{min},l}\right|^2 \mathrm{d}\mu \le \liminf_{N\to\infty} \int_E u_N^2\,\mathrm{d}\mu_N$$

as well as

$$\limsup_{N\to\infty} \int_E u_N^2 \, \mathrm{d}\mu_N \le \int_E \left|v^{\mathrm{maj},l}\right|^2 \mathrm{d}\mu.$$

Finally, considering the limit of l to infinity we obtain

$$\limsup_{N\to\infty} \int_E u_N^2 \, \mathrm{d}\mu_N \le \int_E u_\infty^2 \, \mathrm{d}\mu \le \liminf_{N\to\infty} \int_E u_N^2 \, \mathrm{d}\mu_N.$$

So, taking into account what has already been shown, the assumptions in (ii) imply that each subsequence of $(u_N)_{N\in\mathbb{N}}$ admits another (sub-)subsequence whose strong limit is u_∞. This concludes the proof. □

2.2 Convergence of closed symmetric forms

2.2.1 The theorem of Mosco-Kuwae-Shioya

Our reason for considering sequences of converging Hilbert spaces is to build a framework in which we analyse the approximation of semigroups of operators. In the context of problems motivated by statistical mechanics it is often not suitable to accommodate the approximation process in one fixed Hilbert space. The setting introduced in the preceding section provides enough flexibility. Kuwae and Shioya ([28]) define a topological space in which each point corresponds to a spectral structure on a variable Hilbert space. The corresponding term of convergence is referred to as *Mosco convergence* and named after a notion introduced in [23] by Umberto Mosco. The difference of Mosco's version is that his article fixes a Hilbert space on which all of the considered spectral structures operate. The central idea behind the Theorem of Mosco-Kuwae-Shioya, however, is thoroughly conveyed in the Mosco's original work and then nicely adapted and further developed in [28]. Correctly, we want to think of a spectral structure Σ in this context as a 6-tuple. It consists of

- a real, separable Hilbert space $(H, (\cdot,\cdot)_H)$,
- a non-positive definite self-adjoint operator $(L, \mathcal{D}(L))$ on H,
- a closed symmetric form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$,
- a spectral measure $(E_\lambda)_{\lambda\le 0}$ with support on $(-\infty, 0]$,
- a strongly continuous contraction semigroup $(T_t)_{t\ge 0}$,
- a strongly continuous contraction resolvent $(G_\alpha)_{\alpha>0}$.

The latter five entries of the tuple, generator, form, spectral measure, semigroup and resolvent, must be mutually associated with each other in the sense of [38, Section 1]. The form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is densely defined on H and all involved operators T_t, $t \ge 0$, and G_α, $\alpha > 0$, are symmetric operators on H. The spectral measure, $(E_\lambda)_{\lambda\le 0}$ in slightly informal style, actually represents an assignment E_A, which sends a set $A \in \mathcal{B}((-\infty, 0])$ to an orthogonal projection on H. Our notation in that matter orientates itself according to [39, Chapter 7], where all the properties of the spectral measure relevant in this text, in particular in the proof of Lemma 2.2.4, can be found. Given $u \in H$, we write $\int_{(-\infty,0]} f(\lambda)\, \mathrm{d}(E_\lambda u, u)_H$ to denote the integral of a function $f \in \mathcal{B}((-\infty, 0])$ w.r.t. the finite measure $(E_A u, u)_H$, $A \in \mathcal{B}((-\infty, 0])$, in case this integral exists. Analogously, $\int_{(-\infty,0]} f(\lambda)\,\mathrm{d}(E_\lambda u,v)_H$ for $u,v\in H$ denotes the integral of a function $f\in\mathcal{B}((-\infty,0])$ w.r.t. the signed measure $(E(A)u,v)_H$, $A\in\mathcal{B}((-\infty,0])$, in case this integral exists. The formal integral $\int_{(-\infty,0]} f(\lambda)\,\mathrm{d}E_\lambda$ always yields a self-adjoint operator on H, given $f\in\mathcal{B}((-\infty,0])$. Its domain comprises exactly those $u\in H$ for which $f(\lambda)$ is square integrable w.r.t. $\mathrm{d}(E_\lambda u,u)_H$. We have

$$L=\int_{(-\infty,0]}\lambda\,\mathrm{d}E_\lambda \quad \text{with} \quad \mathcal{D}(L)=\Big\{u\in H\,\Big|\,\int_{(-\infty,0]}\lambda^2\,\mathrm{d}(E_\lambda u,u)_H<\infty\Big\} \tag{2.2.1}$$

and

$$\mathcal{E}(u,v)=\int_{(-\infty,0]}|\lambda|\,\mathrm{d}(E_\lambda u,v)_H \quad \text{with} \quad \mathcal{D}(\mathcal{E})=\Big\{u\in H\,\Big|\,\int_{(-\infty,0]}|\lambda|\,\mathrm{d}(E_\lambda u,u)_H<\infty\Big\}. \tag{2.2.2}$$

Moreover,

$$T_t=\int_{(-\infty,0]}e^{\lambda t}\,\mathrm{d}E_\lambda,\quad t\ge 0 \tag{2.2.3}$$

and

$$G_\alpha=\int_{(-\infty,0]}\frac{1}{\alpha-\lambda}\,\mathrm{d}E_\lambda,\quad \alpha>0. \tag{2.2.4}$$

Starting either with a non-positive, self-adjoint operator L, a densely defined closed symmetric form $\mathcal{E}$, a strongly continuous contraction semigroup $(T_t)_{t\ge0}$, or a strongly continuous contraction resolvent $(G_\alpha)_{\alpha>0}$, each single one of the equations, (2.2.1) through (2.2.4), determines $(E_\lambda)_{\lambda\le0}$ uniquely and vice versa. Hence, the spectral structure Σ is already defined unambiguously by any of its components beyond the first. The term Mosco convergence most commonly refers to a sequence of closed symmetric forms, but has an equivalent formulation for each single one of the objects of Σ, i.e. generator, spectral measure, semigroup and resolvent. Each formulation fully characterizes the convergence of spectral structures already by its own. This is the statement of the Theorem of Mosco-Kuwae-Shioya.

We assume that we are given a sequence of spectral structures $(\Sigma_N)_{N\in\mathbb{N}}$ of the specified type, $\Sigma_N=(H_N,L_N,\mathcal{E}^N,(E^N_\lambda)_{\lambda\le0},(T^N_t)_{t\ge0},(G^N_\alpha)_{\alpha>0})$ for $N\in\mathbb{N}$, and let $\Sigma=(H,L,\mathcal{E},(E_\lambda)_{\lambda\le0},(T_t)_{t\ge0},(G_\alpha)_{\alpha>0})$ be another spectral structure. If a dense linear subspace $\mathcal{C}$ of H together with linear operators Φ_N mapping $\mathcal{C}$ into H_N for $N\in\mathbb{N}$ are chosen to the effect that the asymptotic equations (2.1.1) are satisfied, then we say that $(\mathcal{E}^N)_{N\in\mathbb{N}}$ converges to $\mathcal{E}$ in the sense of Mosco, given the validity of one of the equivalent statements in Theorem 2.2.1. We state it as it is written in [28, Theorem 2.4] and invite the reader to get informed about the proof in the original. The topological notions refer to the strong and weak topologies, τ_s and τ_w, on the disjoint union

$$\mathcal{H}:=H\sqcup\Big(\bigsqcup_{N\in\mathbb{N}}H_N\Big).$$

The validity of the equivalent statements in Theorem 2.2.1 depends only on the equivalence class of τ_s and τ_w respectively. The reader should recall Remark 2.1.4 (ii) for that matter. The space of continuous real-valued functions on $(-\infty,0]$ vanishing at infinity is denoted by $C_0((-\infty,0])$. That is by definition, for each $f\in C_0((-\infty,0])$ and $\varepsilon>0$ there exists a number $k\in\mathbb{N}$ with $f((-\infty,k))\subseteq(-\varepsilon,\varepsilon)$.

Theorem 2.2.1 (Mosco, Kuwae, Shioya)**.** *The following are equivalent.*

(i) *There exists* $\alpha > 0$ *such that* $\lim_{N\to\infty} G_\alpha^N u_N = G_\alpha u_\infty$ *holds strongly in* $\mathcal{H}$ *for every strongly continuous section* $(u_N)_{N\in\overline{\mathbb{N}}}$.

(ii) *There exists* $t > 0$ *such that* $\lim_{N\to\infty} T_t^N u_N = T_t u_\infty$ *holds strongly in* $\mathcal{H}$ *for every strongly continuous section* $(u_N)_{N\in\overline{\mathbb{N}}}$.

(iii) *For every strongly continuous section* $(u_N)_{N\in\overline{\mathbb{N}}}$ *and* $f \in C_0((-\infty,0])$ *the convergence*

$$\lim_{N\to\infty}\Big(\int_{(-\infty,0]} f(\lambda)\,\mathrm{d}E_\lambda^N\Big)u_N = \Big(\int_{(-\infty,0]} f(\lambda)\,\mathrm{d}E_\lambda\Big)u_\infty$$

holds strongly in $\mathcal{H}$.

(iv) (M1) *For every weakly continuous section* $(u_N)_{N\in\overline{\mathbb{N}}}$ *it holds*

$$\mathcal{E}(u_\infty,u_\infty) \le \liminf_{N\to\infty} \mathcal{E}^N(u_N,u_N).$$

(M2) *For every* $u \in \mathcal{D}(\mathcal{E})$ *there is* $u_N \in \mathcal{D}(\mathcal{E}^N)$, $N \in \mathbb{N}$, *such that* $\lim_{N\to\infty} u_N = u$ *holds strongly in* $\mathcal{H}$ *and*

$$\lim_{N\to\infty} \mathcal{E}^N(u_N,u_N) = \mathcal{E}(u,u).$$

The inequality in (M1) of Theorem 2.2.1 (iv) has to be read in the sense that in case $\#N$ with $u_N \in \mathcal{D}(\mathcal{E}^N)$ is infinite and accounts for a finite right hand side, then $u_\infty \in \mathcal{D}(\mathcal{E})$ and the stated inequality holds true.

Remark 2.2.2. (i) The notion of convergence for a sequence of bounded linear operators on variable Hilbert spaces, which lies behind Theorem 2.2.1 (i), (ii) and (iii) is a natural generalization of the strong operator topology. What is more,

$$\lim_{N\to\infty} (T_t^N \Phi_N\varphi, \Phi_N\varphi)_{H_N} = (T_t\varphi,\varphi)_H, \quad \varphi \in \mathcal{C},\, t \ge 0, \tag{2.2.5}$$

implies the strong convergence of $(T_t^N\Phi_N\varphi)_{N\in\mathbb{N}}$ towards $T_t\varphi$ for $\varphi \in \mathcal{C}$ and $t \ge 0$ due to symmetry and the semigroup property of the involved operators. Then, by Lemma 2.1.1 (ii), we get the weak convergence of $(T_t^N u_N)_{N\in\mathbb{N}}$ towards $T_t u_\infty$ for every weakly continuous section $(u_N)_{N\in\overline{\mathbb{N}}}$ and $t \ge 0$, as a consequence of (2.2.5) together with the symmetry and the contraction property of T_t^N for $N \in \mathbb{N}$. Finally, again by symmetry and Lemma 2.1.1 (ii), we conclude that (2.2.5) is actually equivalent to the statement in Theorem 2.2.1 (ii).

(ii) The reason why it suffices to show the convergence in Theorem 2.2.1 (i) and (ii) for just one particular index $\alpha > 0$, or $t > 0$ respectively, in order to get the convergence of the whole family of operators, is in fact the extended Stone-Weierstrass Theorem. We consider $C_0((-\infty,0])$ equipped with the supremum norm. Since, for $N \in \mathbb{N}$, the map $f \mapsto \int_{(-\infty,0]} f(\lambda)\,\mathrm{d}E_\lambda^N$ from $C_0((-\infty,0]$ into the algebra of bounded operators on H_N is multiplicative and continuous (w.r.t. the operator norm of bounded linear operators on H_N), it is easy to see that the family

$$\mathcal{A} := \Big\{ f \in C_0((-\infty,0]) \,\Big|\, \lim_{N\to\infty}\Big(\int_{(-\infty,0]} f(\lambda)\,\mathrm{d}E_\lambda^N\Big)u_N = \Big(\int_{(-\infty,0]} f(\lambda)\,\mathrm{d}E_\lambda\Big)u \text{ strongly for every strongly continuous section } (u_N)_{N\in\overline{\mathbb{N}}}\Big\}$$

is a closed subalgebra of $C_0((-\infty,0])$. Therefore, if $\mathcal{A}$ contains a point-separating, nowhere vanishing function on $(-\infty,0]$, such as $\lambda \mapsto e^{t\lambda}$ with $t>0$, or $\lambda \mapsto \frac{1}{\alpha-\lambda}$ with $\alpha>0$, then already it holds $C_0((-\infty,0]) \subseteq \mathcal{A}$.

To prove property (M1) of the statement in Theorem 2.2.1 (iv) it suffices to consider a weakly continuous section $(u_N)_{N\in\overline{\mathbb{N}}}$ on $\mathcal{H}$ with

$$\sup_{N\in\mathbb{N}} \mathcal{E}^N(u_N,u_N) < \infty,$$

after possibly dropping to a suitable subsequence. It turns out, that there is a further significant simplification of that. We are allowed to assume not only $\sup_{N\in\mathbb{N}} \mathcal{E}^N(u_N,u_N)<\infty$, but also the stronger condition

$$u_N = G_\alpha^N \Phi_N \varphi, \quad N\in\mathbb{N}, \tag{2.2.6}$$

for some arbitrarily fixed $\varphi \in \mathcal{C}$. Let us quickly depict the advantage we gain by that remark through an example. We consider the situation described in the second part of Section 2.1, where $(\mu_N)_{N\in\mathbb{N}}$ is a sequence of weakly convergent finite Borel measures on a $T3\frac{1}{2}$ Suslin Space E with limit μ and $\operatorname{supp}[\mu_N] \subseteq \operatorname{supp}[\mu]$, $N\in\mathbb{N}$. We assume that we have additional information about some property which G_α^N meets uniformly in $N\in\mathbb{N}$: For example, we say that $(G_\alpha^N)_{N\in\mathbb{N}}$ admits a uniform bound w.r.t. the operator norm on L^∞, defined respectively for $N\in\mathbb{N}$ for the restriction of G_α^N from $L^\infty(E,\mu_N)$ into $L^\infty(E,\mu_N)$ if this is possible. In this case, the verification of property (M1) in Theorem 2.2.1 (iv) only requires to look at a weakly continuous section $(u_N)_{N\in\overline{\mathbb{N}}}$ such that $\sup_{N\in\mathbb{N}} \mathcal{E}^N(u_N,u_N)<\infty$ as well as

$$\sup_{N\in\mathbb{N}} \|u_N\|_{L^\infty(E,\mu_N)} < \infty. \tag{2.2.7}$$

An analogue argument can likewise be derived from any other uniform property the sequence $(G_\alpha^N)_{N\in\mathbb{N}}$ might have, and make it much easier to prove (M1) in some applications. Also, the reader shall be hinted at the meaning of Remark 2.1.2 (ii) in this context, which states that one can possibly consider different choices for $\mathcal{C}$ to make the simplification of (2.2.6) in (M1) even more useful. The proof for the next lemma, which focusses on that issue, is very simple and apparent when carefully studying [28] or [23]. However, to the best knowledge of this text's author, it is not stated explicitly in the literature yet. That is why we want to put the short proof here.

Lemma 2.2.3. *We assume that there exists $\alpha\in(0,\infty)$ such that the following holds true for every $\varphi\in\mathcal{C}$ and every weak accumulation point u of $(G_\alpha^N\Phi_N\varphi)_{N\in\mathbb{N}}$ in $\mathcal{H}$: If $(G_\alpha^{N_k}\Phi_{N_k}\varphi)_{k\in\mathbb{N}}$ is a subsequence converging weakly to u, then $u\in\mathcal{D}(\mathcal{E})$ with*

$$\mathcal{E}(u,u) \le \liminf_{k\to\infty} \mathcal{E}^{N_k}\big(G_\alpha^{N_k}\Phi_{N_k}\varphi, G_\alpha^{N_k}\Phi_{N_k}\varphi\big).$$

Under this condition, all the equivalent claims from (i) to (iv) in Theorem 2.2.1 are already implied by property (M2) of statement (iv) alone.

Proof. The claim of this lemma is shown by verifying property (i) of Theorem 2.2.1 under the stated circumstances. First, we hint at a fact which is purely a consequence of (M2) from Theorem 2.2.1 (iv). In the Hilbert space $(\mathcal{D}(\mathcal{E}),\mathcal{E}_\alpha)$, where $\mathcal{E}_\alpha(\cdot,\cdot) := \mathcal{E}(\cdot,\cdot)+(\cdot,\cdot)_H$ and $\alpha\in(0,\infty)$ is as in the assumptions of this lemma, we pick an orthogonal basis $v^{(1)}, v^{(2)}, \dots$. Furthermore, let $v_N^{(i)} \in \mathcal{D}(\mathcal{E}^N)$ for $i, N \in \mathbb{N}$ be chosen to the effect that for each i it holds

$$\lim_{N\to\infty} \mathcal{E}^N\big(v_N^{(i)}, v_N^{(i)}\big) = \mathcal{E}\big(v^{(i)}, v^{(i)}\big) \quad \text{and} \quad \lim_{N\to\infty} v_N^{(i)} = v^{(i)} \text{ strongly in } \mathcal{H}.$$

By setting $\Phi'_N v^{(i)} := v_N^{(i)}$, $i \in \mathbb{N}$, we define a linear operator from the space of finite linear combinations $V :=\text{span}(\{v^{(i)} \,|\, i \in \mathbb{N}\})$ into $\mathcal{D}(\mathcal{E}^N)$ for $N \in \mathbb{N}$. The arguments of this proof use the corresponding terms of a strong and weak topology on the disjoint union

$$\mathcal{H}^{\mathcal{E},\alpha} := (\mathcal{D}(\mathcal{E}), \mathcal{E}_\alpha) \sqcup \Big(\bigsqcup_{N\in\mathbb{N}} \big(\mathcal{D}(\mathcal{E}^N), \mathcal{E}_\alpha^N\big)\Big), \quad \alpha > 0.$$

The rest of this proof is committed to the verification of the following claim.

$$\lim_{N\to\infty} G_\alpha^N u_N = G_\alpha u_\infty \text{ holds weakly in } \mathcal{H} \\ \text{for every weakly continuous section } (u_N)_{N\in\overline{\mathbb{N}}}. \tag{2.2.8}$$

Obviously, (2.2.8) is equivalent to the statement of Theorem 2.2.1 (i) due to symmetry of the involved operators and the duality relation between the strong and weak convergence as formulated in Lemma 2.1.1 (ii). In the next step of this proof we use the assumptions of this lemma to derive a convergence result in $\mathcal{H}^{\mathcal{E},\alpha}$, for fixed $\varphi \in \mathcal{C}$. Remarking that $\|G_\alpha^N \Phi_N \varphi\|_{H_N} \le \alpha^{-1} \|\Phi_N \varphi\|_{H_N}$ for $N \in \mathbb{N}$, we can use the sequential compactness of the weak topology on bounded subsets of $\mathcal{H}$, as stated in Lemma 2.1.1 (iv), to find a suitable subsequence such that both, (2.2.9) and (2.2.10) below, are fulfilled. We find a subsequence such that

$$\begin{aligned}\lim_{k\to\infty} \mathcal{E}_\alpha^{N_k}\big(G_\alpha^{N_k}\Phi_{N_k}\varphi, G_\alpha^{N_k}\Phi_{N_k}\varphi\big) &= \limsup_{N\to\infty} \mathcal{E}_\alpha^N\big(G_\alpha^N\Phi_N\varphi, G_\alpha^N\Phi_N\varphi\big) \\ &=: c \in \big[0, \alpha^{-1}\|\varphi\|_H^2\big]\end{aligned} \tag{2.2.9}$$

as well as

$$\text{there is a limit } u \in H \text{ of } (G_\alpha^{N_k}\Phi_{N_k}\varphi)_{k\in\mathbb{N}} \text{ w.r.t. the weak topology in } \mathcal{H}. \tag{2.2.10}$$

Then, in case $c > 0$, we estimate with the assumptions of this lemma

$$\begin{aligned}\sqrt{c} &= \lim_{k\to\infty} \frac{\big(G_\alpha^{N_k}\Phi_{N_k}\varphi, \Phi_{N_k}\varphi\big)_{H_{N_k}}}{\mathcal{E}_\alpha^{N_k}\big(G_\alpha^{N_k}\Phi_{N_k}\varphi, G_\alpha^{N_k}\Phi_{N_k}\varphi\big)^{1/2}} \\ &\le \frac{(u,\varphi)_H}{\mathcal{E}_\alpha(u,u)^{1/2}} = \frac{\mathcal{E}_\alpha(u, G_\alpha\varphi)}{\mathcal{E}_\alpha(u,u)^{1/2}} \le \mathcal{E}_\alpha(G_\alpha\varphi, G_\alpha\varphi)^{1/2}.\end{aligned}$$

Hence, we have $c \le \mathcal{E}_\alpha(G_\alpha\varphi, G_\alpha\varphi)$. We remark that $\lim_{N\to\infty} \Phi'_N v = v$ strongly in $\mathcal{H}$ for $v \in V$ is clear by construction of $(\Phi'_N)_{N\in\mathbb{N}}$. Lemma 2.1.1 (ii) is used repeatedly below. The strong convergence $(G_\alpha^N\Phi_N\varphi)_{N\in\mathbb{N}}$ towards $G_\alpha\varphi$ in $\mathcal{H}^{\mathcal{E},\alpha}$ now follows, because

$$\begin{aligned}\lim_{N\to\infty} \mathcal{E}_\alpha^N\big(G_\alpha^N\Phi_N\varphi, \Phi'_N v\big) &= \lim_{N\to\infty} (\Phi_N\varphi, \Phi'_N v)_{H_N} \\ &= (\varphi, v)_H = \mathcal{E}_\alpha(G_\alpha\varphi, v), \quad v \in V,\end{aligned}$$

tells us the weak convergence and then

$$c \le \mathcal{E}_\alpha(G_\alpha\varphi, G_\alpha\varphi) \le \liminf_{N\to\infty} \mathcal{E}_\alpha^N\big(G_\alpha^N\Phi_N\varphi, G_\alpha^N\Phi_N\varphi\big),$$

where the second inequality holds by Lemma 2.1.1 (iii), proves the convergence in norm. For the final step of this proof we fix a weakly continuous section $(u_N)_{N\in\overline{\mathbb{N}}}$ in $\mathcal{H}$ and set out to prove (2.2.8). Clearly, $\lim_{N\to\infty} G_\alpha^N u_N = G_\alpha u_\infty$ holds weakly in $\mathcal{H}^{\mathcal{E},\alpha}$, because

$$\sup_{N\in\mathbb{N}} \mathcal{E}_\alpha^N\left(G_\alpha^N u_N, G_\alpha^N u_N\right) \le \frac{1}{\alpha} \sup_{N\in\mathbb{N}} \|u_N\|_{H_N}^2 < \infty$$

and also

$$\begin{aligned}\lim_{N\to\infty} \mathcal{E}_\alpha^N\left(G_\alpha^N u_N, \Phi'_N v\right) &= \lim_{N\to\infty} (u_N, \Phi'_N v)_{H_N} \\ &= (u_\infty, v)_H = \mathcal{E}_\alpha(G_\alpha u_\infty, v), \quad v \in V.\end{aligned}$$

Using the strong convergence of $(G_\alpha^N \Phi_N \varphi)_{N\in\mathbb{N}}$ towards $G_\alpha\varphi$ in $\mathcal{H}^{\mathcal{E},\alpha}$ for every $\varphi \in \mathcal{C}$, as shown above, the claim of (2.2.8) follows from

$$\begin{aligned}\lim_{N\to\infty} \left(G_\alpha^N u_N, \Phi_N\varphi\right)_{H_N} &= \lim_{N\to\infty} \mathcal{E}_\alpha^N\left(G_\alpha^N u_N, G_\alpha^N \Phi_N \varphi\right) \\ &= \mathcal{E}_\alpha(G_\alpha u_\infty, G_\alpha\varphi) = (G_\alpha u_\infty, \varphi)_H, \quad \varphi \in \mathcal{C},\end{aligned}$$

and the estimate

$$\sup_{N\in\mathbb{N}} \left\|G_\alpha^N u_N\right\|_{H_N} \le \frac{1}{\alpha} \sup_{N\in\mathbb{N}} \|u_N\|_{H_N} < \infty.$$

This concludes the proof. □

2.2.2 Compatible classes

Before we continue our discussion about the convergence of closed symmetric forms we need to state a general preliminary about a closed symmetric and densely defined form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on a real, separable Hilbert space $(H, (\cdot,\cdot))$. For such a pair, we set $\mathcal{E}_\alpha(u,v) := \mathcal{E}(u,v) + \alpha(u,v)$ for $u, v \in \mathcal{D}(\mathcal{E})$ and $\alpha > 0$. The fact that $\mathcal{E}$ is closed means that $\mathcal{D}(\mathcal{E})$ with the inner product $\mathcal{E}_\alpha(\cdot,\cdot)$ is again a real, separable Hilbert space. The lemma gives a criterion for the identification $u = w$ regarding two elements u and w in H, one of which, say w, is assumed to be a member of $\mathcal{D}(\mathcal{E})$. The criterion reads as follows. If $\mathcal{D}(\mathcal{E}) \ni v \mapsto \varphi_v \in \mathcal{D}(\mathcal{E})$ is a map whose image set is bounded w.r.t. the norm $\|\cdot\|_H$ of H and which additionally fulfils the dual orthogonality relation

$$\mathcal{E}_1(w - \varphi_v, v) = 0 \quad \text{as well as} \quad (u - \varphi_v, v) = 0 \quad \text{for } v \in \mathcal{D}(\mathcal{E}) \tag{2.2.11}$$

in the Hilbert space $(\mathcal{D}(\mathcal{E}), \mathcal{E}_1)$, respectively H, then $u = w$ must be true.

Lemma 2.2.4. *Let $u \in H$ and $w \in \mathcal{D}(\mathcal{E})$. If, there exists a map $\mathcal{D}(\mathcal{E}) \ni v \mapsto \varphi_v \in \mathcal{D}(\mathcal{E})$ satisfying* (2.2.11) *and moreover $\sup_{v\in\mathcal{D}(\mathcal{E})} \|\varphi_v\|_H < \infty$, then $u = w$.*

Proof. If the generator $(L, \mathcal{D}(L))$ of $\mathcal{E}$ on H has pure point spectrum, then the claim is immediate. Let $(\lambda_i)_{i\in\mathbb{N}} \subset (-\infty, 0]$ be the eigenvalues of L with corresponding eigenvectors $(v_i)_{i\in\mathbb{N}}$. We have

$$\begin{aligned}(u, v_i) = (\varphi_{v_i}, v_i) &= \frac{1}{1-\lambda_i}(\varphi_{v_i}, (1-L)v_i) \\ &= \frac{1}{1-\lambda_i}\mathcal{E}_1(\varphi_{v_i}, v_i) = \frac{1}{1-\lambda_i}\mathcal{E}_1(w, v_i) \\ &= \frac{1}{1-\lambda_i}(w, (1-L)v_i) = (w, v_i), \quad i \in \mathbb{N},\end{aligned}$$

and since the linear span of $(v_i)_{i\in\mathbb{N}}$ is dense in H, we conclude $u = w$ as desired. The general case, where $(L, \mathcal{D}(L))$ is a non-positive, self-adjoint operator on H, needs a more detailed treatment, although the lines above already convey the right idea. Let $(E_\lambda)_{\lambda\in(-\infty,0]}$ denote the spectral measure of L. The role of the eigenvector in the previous argumentation must be taken over by an element from the image set of $E_{(\lambda_2,\lambda_1]} = \int_{(\lambda_2,\lambda_1]} \mathrm{d}E_\lambda$ for a bounded interval $(\lambda_2, \lambda_1] \subset (-\infty, 0]$. We start with the estimate

$$\begin{aligned}
&\Big|\mathcal{E}_1(v_1, E_{(\lambda_2,\lambda_1]}v_2) - (1-\lambda_1)(v_1, E_{(\lambda_2,\lambda_1]}v_2)\Big| \\
&= \Big|\Big(v_1, \Big[\int_{(-\infty,0]} -\lambda + \lambda_1 \,\mathrm{d}E_\lambda\Big] E_{(\lambda_2,\lambda_1]}v_2\Big)\Big| \\
&= \Big|\Big(v_1, \Big[\int_{(-\infty,0]} -\lambda + \lambda_1 \,\mathrm{d}E_\lambda\Big] E^2_{(\lambda_2,\lambda_1]}v_2\Big)\Big| \\
&= \Big|\Big(v_1, \Big[\int_{(\lambda_2,\lambda_1]} -\lambda + \lambda_1 \,\mathrm{d}E_\lambda\Big] E_{(\lambda_2,\lambda_1]}v_2\Big)\Big| \\
&\le (-\lambda_2 + \lambda_1)\|v_1\|_H \|E_{(\lambda_2,\lambda_1]}v_2\|_H, \quad v_1, v_2 \in \mathcal{D}(\mathcal{E}).
\end{aligned} \tag{2.2.12}$$

The goal is to prove $(w, v) = (u, v)$ for $v \in \mathcal{D}(\mathcal{E})$. It suffices to show $(w, E_{(-k,0]}v) = (u, E_{(-k,0]}v)$ for all $v \in \mathcal{D}(\mathcal{E})$ and $k \in \mathbb{N}$. With fixed $v \in \mathcal{D}(\mathcal{E})$ and $k \in \mathbb{N}$ we estimate

$$\begin{aligned}
&\big|(w, E_{(-k,0]}v) - (u, E_{(-k,0]}v)\big| \\
&\le \limsup_{n\to\infty} \Big|(w, E_{(-k,0]}v) - \sum_{i=1}^n \Big(\frac{1}{1+\frac{(i-1)k}{n}}\Big)\mathcal{E}_1\big(w, E_{(-\frac{ik}{n}, -\frac{(i-1)k}{n}]}v\big)\Big| \\
&\quad + \limsup_{n\to\infty} \Big|\sum_{i=1}^n \Big(\frac{1}{1+\frac{(i-1)k}{n}}\Big)\mathcal{E}_1\big(w, E_{(-\frac{ik}{n}, -\frac{(i-1)k}{n}]}v\big) - (u, E_{(-k,0]}v)\Big|.
\end{aligned} \tag{2.2.13}$$

It suffices to show that this upper bound equals zero for each $k \in \mathbb{N}$. The two summands of the right-hand side of (2.2.13) are treated individually. Concerning the first summand, utilizing (2.2.12) we obtain

$$\begin{aligned}
&\Big|(w, E_{(-k,0]}v) - \sum_{i=1}^n \Big(\frac{1}{1+\frac{(i-1)k}{n}}\Big)\mathcal{E}_1\big(w, E_{(-\frac{ik}{n}, -\frac{(i-1)k}{n}]}v\big)\Big| \\
&\le \sum_{i=1}^n \Big|(w, E_{(-\frac{ik}{n}, -\frac{(i-1)k}{n}]}v) - \Big(\frac{1}{1+\frac{(i-1)k}{n}}\Big)\mathcal{E}_1\big(w, E_{(-\frac{ik}{n}, -\frac{(i-1)k}{n}]}v\big)\Big|_H \\
&\le \sum_{i=1}^n \Big|\Big(1 + \frac{(i-1)k}{n}\Big)(w, E_{(-\frac{ik}{n}, -\frac{(i-1)k}{n}]}v) - \mathcal{E}_1\big(w, E_{(-\frac{ik}{n}, -\frac{(i-1)k}{n}]}v\big)\Big| \\
&\le \frac{k}{n}\|w\|_H \sum_{i=1}^n \big\|E_{(-\frac{ik}{n}, -\frac{(i-1)k}{n}]}v\big\|_H \\
&\le \frac{k}{\sqrt{n}}\|w\|_H \Big(\sum_{i=1}^n \big\|E_{(-\frac{ik}{n}, -\frac{(i-1)k}{n}]}v\big\|_H^2\Big)^{\frac{1}{2}} = \frac{k}{\sqrt{n}}\|w\|_H \big\|E_{(-k,0]}v\big\|_H
\end{aligned}$$

for $n \in \mathbb{N}$. For the second summand of the right-hand side of (2.2.13) we use the assumptions of this lemma. The calculation is similar as in the first case. Let $n \in \mathbb{N}$. We set $M := \sup_{\tilde{v}\in\mathcal{D}(\mathcal{E})} \|\varphi_{\tilde{v}}\|_H$ and choose $\varphi_i \in \mathcal{D}(\mathcal{E})$ for $i = 1, \ldots, n$ such that

$$\mathcal{E}_1\big(w, E_{(-\frac{ik}{n}, -\frac{(i-1)k}{n}]}v\big) = \mathcal{E}_1\big(\varphi_i, E_{(-\frac{ik}{n}, -\frac{(i-1)k}{n}]}v\big), \tag{2.2.14}$$

$$\big(\varphi_i, E_{(-\frac{ik}{n},-\frac{(i-1)k}{n}]}v\big) = \big(u, E_{(-\frac{ik}{n},-\frac{(i-1)k}{n}]}v\big). \tag{2.2.15}$$

Using (2.2.14),(2.2.15), and again (2.2.12) it holds

$$\begin{aligned}
&\Big|(u, E_{(-k,0]}v) - \sum_{i=1}^{n}\Big(\frac{1}{1+\frac{(i-1)k}{n}}\Big)\mathcal{E}_1\big(w, E_{(-\frac{ik}{n},-\frac{(i-1)k}{n}]}v\big)\Big| \\
&\le \sum_{i=1}^{n}\Big|(u, E_{(-\frac{ik}{n},-\frac{(i-1)k}{n}]}v) - \Big(\frac{1}{1+\frac{(i-1)k}{n}}\Big)\mathcal{E}_1\big(\varphi_i, E_{(-\frac{ik}{n},-\frac{(i-1)k}{n}]}v\big)\Big| \\
&\le \sum_{i=1}^{n}\Big|\Big(1+\frac{(i-1)k}{n}\Big)(\varphi_i, E_{(-\frac{ik}{n},-\frac{(i-1)k}{n}]}v) - \mathcal{E}_1\big(\varphi_i, E_{(-\frac{ik}{n},-\frac{(i-1)k}{n}]}v\big)\Big| \\
&\le \frac{k}{n}\sum_{i=1}^{n}\|\varphi_i\|_H \big\|E_{(-\frac{ik}{n},-\frac{(i-1)k}{n}]}v\big\|_H \\
&\le \frac{Mk}{\sqrt{n}}\Big(\sum_{i=1}^{n}\big\|E_{(-\frac{ik}{n},-\frac{(i-1)k}{n}]}v\big\|_H^2\Big)^{\frac{1}{2}} = \frac{Mk}{\sqrt{n}}\big\|E_{(-k,0]}v\big\|_H.
\end{aligned}$$

Hence, the right-hand side of (2.2.13) equals zero and the proof is complete. □

The setting under the condition of which we derive the analysis of the rest of this section, presumes once more a sequence of converging Hilbert spaces $(H_N, (\cdot,\cdot)_{H_N})$, $N \in \mathbb{N}$, with asymptotic space $(H, (\cdot,\cdot)_H)$, together with a family of closed symmetric forms, namely $(\mathcal{E}^N, \mathcal{D}(\mathcal{E}^N))$, defined densely on H_N, for $N \in \mathbb{N}$ and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, defined densely on H. We have the corresponding weak and strong topology from Section 2.1 on the disjoint union $\mathcal{H} := H \sqcup (\bigsqcup_{N\in\mathbb{N}} H_N)$. Additionally, we assume that we are given a dense linear subspace D of the Hilbert space $(\mathcal{D}(\mathcal{E}), \mathcal{E}_1)$ and linear maps Φ'_N from D into $\mathcal{D}(\mathcal{E}^N)$ for $N \in \mathbb{N}$ such that

$$\lim_{N\to\infty} \Phi'_N v = v \text{ strongly in } \mathcal{H}, \qquad \lim_{N\to\infty} \mathcal{E}^N(\Phi'_N v, \Phi'_N v) = \mathcal{E}(v,v), \quad v \in D. \tag{2.2.16}$$

Remark 2.2.5. (i) The additional assumption of (2.2.16) can always be retrieved from a situation, in which the family of forms fulfil property (M2) in Theorem 2.2.1 (iv), by defining Φ'_N, $N \in \mathbb{N}$, as in the proof of Lemma 2.2.3.

(ii) On the other hand, let us assume (2.2.16) is given. Regarding (2.2.16), Remark 2.1.2 (ii) tells us that we may take Φ'_N, $N \in \mathbb{N}$, as the asymptotic isometries in (2.1.1) and obtain an equivalent strong topology, as well as an equivalent weak topology on $\mathcal{H}$. So, simply let

$$\Phi_N := \Phi'_N, \quad \mathcal{D}(\Phi_N) = D, \quad N \in \mathbb{N},$$

from here on. Moreover, this family of maps induce a strong and a weak topology on the disjoint union of Hilbert spaces

$$\mathcal{H}^{\mathcal{E},\alpha} := (\mathcal{D}(\mathcal{E}), \mathcal{E}_\alpha) \sqcup \Big(\bigsqcup_{N\in\mathbb{N}} (\mathcal{D}(\mathcal{E}^N), \mathcal{E}^N_\alpha)\Big)$$

as well, for each $\alpha > 0$. Let $\varphi_1, \varphi_2, \dots \in D$ form an orthonormal basis of $(\mathcal{D}(\mathcal{E}), \mathcal{E}_\alpha)$, fixing $\alpha > 0$, and further l_{N_0}, for $N_0 \in \mathbb{N}$, be the maximal choice of a natural number such that for $k \in \{1, \dots, l_{N_0}\}$ it holds

$$\sup_{x\in\mathbb{R}^k} \big|\big[(\Phi_N\varphi_i,\Phi_N\varphi_j)_{H_N} - (\varphi_i,\varphi_j)_H\big]_{i,j=1}^k x\big|_{\mathrm{euc}} \le \frac{1}{k},$$
$$\text{and} \quad \sup_{x\in\mathbb{R}^k} \big|\big[\mathcal{E}_\alpha^N(\Phi_N\varphi_i,\Phi_N\varphi_j) - \delta_{ij}\big]_{i,j=1}^k x\big|_{\mathrm{euc}} \le \frac{1}{k},$$
$$\text{for all } N\in\mathbb{N} \text{ with } N\ge N_0.$$

The argumentation displayed in the proof of Lemma 2.1.1 (i) yields both the asymptotic equations,

$$\lim_{N_0\to\infty} \sum_{i=1}^{l_{N_0}} \mathcal{E}_\alpha(u,\varphi_i)\Phi_{N_0}\varphi_i = u \quad \text{strongly in } \mathcal{H}^{\mathcal{E},\alpha} \text{ and in } \mathcal{H} \text{ for } u\in\mathcal{D}(\mathcal{E}). \tag{2.2.17}$$

In particular, property (M2) in Theorem 2.2.1 (iv) holds true. There is another consequence of the dual convergence in (2.2.17). If $(u_N)_{N\in\overline{\mathbb{N}}}$ is a strongly continuous section in $\mathcal{H}^{\mathcal{E},\alpha}$, then e.g. with Lemma 2.1.1 (ii), we can conclude that $u_N - \sum_{i=1}^{l_N} \mathcal{E}_\alpha(u_\infty,\varphi_i)\Phi_N\varphi_i$ converges to zero strongly in $\mathcal{H}^{\mathcal{E},\alpha}$ as N tends to infinity. Hence, in particular

$$\limsup_{N\to\infty} \Big\| u_N - \sum_{i=1}^{l_N} \mathcal{E}_\alpha(u_\infty,\varphi_i)\Phi_N\varphi_i \Big\|_{H_N} = 0,$$

proving that $(u_N)_{N\in\overline{\mathbb{N}}}$ is a strongly continuous in $\mathcal{H}$ as well.

We shift the attention to property (M1) in Theorem 2.2.1 (iv), assuming (2.2.16) for the rest of this section.

Remark 2.2.6. (i) Mosco convergence of $(\mathcal{E}^N)_{N\in\mathbb{N}}$ towards $\mathcal{E}$ can be concluded from:

There exists $\alpha>0$ such that
$$\lim_{N\to\infty} \big(\Phi_N v, G_\alpha^N \Phi_N v\big)_{H_N} = (v, G_\alpha v)_H, \quad v\in D. \tag{2.2.18}$$

Indeed, (2.2.18) implies

$$\lim_{N\to\infty} \mathcal{E}_\alpha^N\big(G_\alpha^N\Phi_N v, G_\alpha^N\Phi_N v\big) = \mathcal{E}_\alpha(G_\alpha v, G_\alpha v), \quad v\in D.$$

Together with

$$\begin{aligned}\lim_{N\to\infty} \mathcal{E}_\alpha^N\big(G_\alpha^N\Phi_N v, \Phi_N w\big) &= \lim_{N\to\infty} (\Phi_N v, \Phi_N w)_{H_N} \\ &= (v,w)_H = \mathcal{E}_\alpha(G_\alpha v, w), \quad v,w\in D,\end{aligned} \tag{2.2.19}$$

we have the strong convergence of $(G_\alpha^N\Phi_N v)_{N\in\mathbb{N}}$ towards $G_\alpha v$ in $\mathcal{H}^{\mathcal{E},\alpha}$. Consequently, statement (i) of Theorem 2.2.1 is equivalent to (2.2.18).

(ii) Of course, the weak convergence of $(G_\alpha^N\Phi_N v)_{N\in\mathbb{N}}$ towards $G_\alpha v$ in $\mathcal{H}^{\mathcal{E},\alpha}$ for $v\in D$ and $\alpha>0$ is always true because of (2.2.19). If we fix $\alpha>0$ and assume that, every weakly continuous section in $\mathcal{H}^{\mathcal{E},\alpha}$ is also weakly continuous in $\mathcal{H}$, then automatically (2.2.18) is fulfilled. This is the statement of [33, Proposition 2.32].

(iii) We now assume (2.2.18). If $(u_k)_{k\in\mathbb{N}}$ is a sequence in $\mathcal{H}^{\mathcal{E},\alpha}$ and $u \in \mathcal{D}(\mathcal{E})$ such that

$$\lim_{k\to\infty} u_k = u \text{ holds weakly in } \mathcal{H}^{\mathcal{E},\alpha},$$

then the same asymptotic statement is shown to be true in $\mathcal{H}$, by verifying that for an arbitrary subsequence there is a (sub-)subsequence $(u_{k_l})_{l\in\mathbb{N}}$ such that

$$\lim_{l\to\infty} u_{k_l} = u \text{ holds weakly in } \mathcal{H}. \tag{2.2.20}$$

For sequences contained in $\mathcal{D}(\mathcal{E})$ this is obviously true. Otherwise, we may w.l.o.g. assume that $u_{k_l} \in H_{N_l}$ for $l \in \mathbb{N}$, where $(N_l)_{l\in\mathbb{N}}$ is a strictly increasing sequence in $\mathbb{N}$. In that case, by the first item of this remark we have

$$\begin{aligned}\lim_{l\to\infty} \left(u_{k_l}, \Phi_{N_l} v\right)_{H_{N_l}} &= \lim_{l\to\infty} \mathcal{E}_\alpha^{N_l}\left(u_{k_l}, G_\alpha^{N_l}\Phi_{N_l} v\right) \\ &= \mathcal{E}_\alpha(u, G_\alpha v) = (u,v)_H, \quad v \in D,\end{aligned}$$

proving (2.2.20).

For a set $A \subseteq \mathcal{H}^{\mathcal{E},1} \setminus \mathcal{D}(\mathcal{E})$ we consider the following property:

$$\begin{aligned}&\text{If } u_k \in A \text{ for } k \in \mathbb{N},\ u \in \mathcal{D}(\mathcal{E}) \text{ and } \lim_{k\to\infty} u_k = u \text{ holds weakly in } \mathcal{H}^{\mathcal{E},1},\\ &\text{then } \lim_{k\to\infty} u_k = u \text{ in the weak sense is also true for } \mathcal{H}.\end{aligned} \tag{2.2.21}$$

Remark 2.2.6 tells us that the Mosco convergence of $(\mathcal{E}^N)_{N\in\mathbb{N}}$ towards $\mathcal{E}$ is equivalent to the validity of (2.2.21) for the choice $A = \mathcal{H}^{\mathcal{E},1} \setminus \mathcal{D}(\mathcal{E})$. If I is an arbitrary index set, $A_i \subseteq \mathcal{H}^{\mathcal{E},1} \setminus \mathcal{D}(\mathcal{E})$ a set with property (2.2.21) for $i \in \mathbb{N}$, and moreover $A_i \cap \mathcal{D}(\mathcal{E}^N) \neq \emptyset$ for $N \in \mathbb{N}$, $i \in I$, then we call the union of product sets

$$\mathbf{C} := \bigcup_{i\in I} \prod_{N\in\mathbb{N}} \left(\mathcal{D}(\mathcal{E}^N) \cap A_i\right)$$

a *compatible class* in $\mathcal{H}^{\mathcal{E},1}$. Due to the weak sequential compactness of norm bounded sets in $\mathcal{H}^{\mathcal{E},1}$ as stated in Lemma 2.1.1 (iv), and the fact that weak convergence in $\mathcal{H}^{\mathcal{E},1}$ combined with weak convergence in $\mathcal{H}$ readily implies the weak convergence in $\mathcal{H}^{\mathcal{E},\alpha}$ for any $\alpha > 0$, we end up with the same notion of a compatible class if, in (2.2.21), the space $\mathcal{H}^{\mathcal{E},1}$ is replaced with $\mathcal{H}^{\mathcal{E},\alpha}$, for arbitrary $\alpha > 0$. We briefly explain why the notion of a compatible class and its discussion can be of practical use for problems related to Mosco convergence. The goal is obviously to show that $\prod_{N\in\mathbb{N}} \mathcal{D}(\mathcal{E}^N)$ is a compatible class. To achieve this it can be helpful to analyse other types of compatible classes at first. The generic example here is a sequence of converging subspaces $(V_N)_{N\in\mathbb{N}}$ such that V_N is a finite-dimensional subspace of H_N for $N \in \mathbb{N}$ and $\sup_{N\in\mathbb{N}} \dim(V_N) < \infty$. Let us consider for a moment the case, where all H_N, $N \in \mathbb{N}$, are finite-dimensional with a constant dimension $k \in \mathbb{N}$, i.e. we say $H_N = \mathbb{R}^k$, $N \in \mathbb{N}$, and we deal with a family of non-negative semidefinite symmetric matrices $(S^N)_{N\in\mathbb{N}} \subseteq \mathbb{R}^{k\times k}$, $S \in \mathbb{R}^{k\times k}$, with

$$\lim_{N\to\infty} S_{ij}^N = S_{ij}, \quad i,j = 1,\dots,k.$$

For $N \in \mathbb{N}$ there are orthonormal matrices $Q_N \in \mathbb{R}^{k\times k}$ and non-negative, real eigenvalues $\lambda_1^{(N)},\dots,\lambda_k^{(N)}$ with $Q_N^* S^N Q_N = \operatorname{diag}(\lambda_1^{(N)},\dots,\lambda_k^{(N)})$. Analogously, $Q^*SQ = \operatorname{diag}(\lambda_1,\dots,\lambda_k)$ for an orthogonal matrix $Q \in \mathbb{R}^{k\times k}$ and non-negative, real numbers

$\lambda_1, \dots, \lambda_k$. If $(x_N)_{N\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}^k$, then the following chain of implications holds true (from (2.2.22), to (2.2.23), to (2.2.24), to (2.2.25)).

$$\lim_{N\to\infty} x_N^{\mathrm{T}} S^N y = x^{\mathrm{T}} S y, \quad y \in \mathbb{R}^k. \tag{2.2.22}$$

$$\begin{aligned}\lim_{N\to\infty} (Q_N^* x_N)^{\mathrm{T}} \mathrm{diag}(\lambda_1^{(N)}, \dots, \lambda_k^{(N)}) z \\ = (Q^* x)^{\mathrm{T}} \mathrm{diag}(\lambda_1, \dots, \lambda_k) z, \quad z \in \mathbb{R}^k.\end{aligned} \tag{2.2.23}$$

$$\begin{aligned}\lim_{N\to\infty} (Q_N^* x_N)^{\mathrm{T}} \mathrm{diag}(\lambda_1^{(N)}, \dots, \lambda_k^{(N)}) (Q_N x_N) \\ = (Q^* x)^{\mathrm{T}} \mathrm{diag}(\lambda_1, \dots, \lambda_k) Q x.\end{aligned} \tag{2.2.24}$$

$$\lim_{N\to\infty} x_N^{\mathrm{T}} S^N x = x^{\mathrm{T}} S x. \tag{2.2.25}$$

Based on this observation in the Euclidean space, we can formulate a remark for our more general setting.

Remark 2.2.7. Let $(v_N^{(i)})_{N\in\overline{\mathbb{N}}}$ be a strongly continuous section in $\mathcal{H}^{\mathcal{E},1}$ for $i = 1, \dots, m$. If

$$u_N \in \mathrm{span}(\{v_N^{(1)}, \dots, v_N^{(m)}\}),\ N \in \mathbb{N}, \quad \text{and} \quad \sup_{N\in\mathbb{N}} \mathcal{E}_1(u_N, u_N) < \infty,$$

then we can choose a bounded sequence $(x_N)_{N\in\mathbb{N}}$ in $\mathbb{R}^m$ such that

$$u_N = (x_N)_1 v_N^{(1)} + \dots + (x_N)_m v_N^{(m)}, \quad N \in \mathbb{N}.$$

Furthermore, given the weak convergence of $(u_N)_{N\in\mathbb{N}}$ in $\mathcal{H}^{\mathcal{E},1}$ towards an element $u \in \mathcal{D}(\mathcal{E})$,

$$\mathcal{E}_1(u, w) = \lim_{N\to\infty} \mathcal{E}_1^N(u_N, w) \le \sum_{i=1}^m \limsup_{N\to\infty} |(x_N)_i| \left|\mathcal{E}^N\left(v_N^{(i)}, w\right)\right| = 0$$

holds true for all $w \in \mathcal{D}(\mathcal{E})$ with $\mathcal{E}_1(w, v_\infty^{(i)}) = 0$, $i = 1, \dots, m$. So, $u = x_1 v_\infty^{(1)} + \dots + x_m v_\infty^{(m)}$ for suitable $x \in \mathbb{R}^m$. Defining $S_{ij}^N = \mathcal{E}_1(v_N^{(i)}, v_N^{(j)})$ and $S_{ij} = \mathcal{E}_1(v_\infty^{(i)}, v_\infty^{(j)})$ for $i, j = 1, \dots, m$ we have (2.2.22) (with $k = m$). By (2.2.25), we conclude that $(u_N)_{N\in\mathbb{N}}$ converges strongly towards u in $\mathcal{H}^{\mathcal{E},1}$, hence it does so in $\mathcal{H}$. In particular,

$$\prod_{N\in\mathbb{N}} \mathrm{span}\big(\{v_N^{(1)}, \dots, v_N^{(m)}\}\big) \text{ is a compatible class in } \mathcal{H}^{\mathcal{E},1}.$$

How can this simple observation be exploited regarding the task of showing Mosco convergence of $(\mathcal{E}^N)_{N\in\mathbb{N}}$ towards $\mathcal{E}$? The idea is to approximate each element u_N in a weakly continuous section $(u_N)_{N\in\overline{\mathbb{N}}}$ in $\mathcal{H}^{\mathcal{E},1}$ through an element in some k-dimensional subspace $V_N^{(k)}$ of $\mathcal{D}(\mathcal{E}^N)$, respectively for $N \in \mathbb{N}$. If this can be done for every k in a suitable way, requiring a uniform approximation quality of the class

$$\mathbf{C} := \bigcup_{k\in\mathbb{N}} \prod_{N\in\mathbb{N}} V_N^{(k)},$$

then it is possible to show that $\prod_{N\in\mathbb{N}} \mathcal{D}(\mathcal{E}^N)$ is a compatible class, as desired, if $\mathbf{C}$ is. For that purpose it suffices to prove

$$\inf_{(w_N)_N} \limsup_{N\to\infty} \left(\left|\mathcal{E}^N(u_N - w_N, \Phi_N v)\right| + \left|(u_N - w_N, \Phi_N v)_{H_N}\right|\right) = 0$$

for every $v \in D$, where the infimum is taken over all elements

$$(w_N)_N \in \bigcup_{k\in\mathbb{N}} \prod_{N\in\mathbb{N}} \{u \in V_N^{(k)} | \mathcal{E}_1^N(u,u) \le R\}$$

with some fixed, but arbitrarily chosen $R \in (0,\infty)$ (independent from v). This is the statement of the next proposition. It provides the main tool from this section on abstract theory which is used to deal with property (M1) in the applications of upcoming chapters. For $R \in (0,\infty)$ we define

$$\mathbf{B}_R^{\mathcal{E}} := \prod_{N\in\mathbb{N}} \{u \in \mathcal{D}(\mathcal{E}^N) | \mathcal{E}_1^N(u,u) \le R\}.$$

Proposition 2.2.8. *We assume that* (2.2.16) *holds. Let* $\mathbf{C}$ *be a compatible class in* $\mathcal{H}^{\mathcal{E},1}$ *and* $A \subseteq \mathcal{H}^{\mathcal{E},1} \setminus \mathcal{D}(\mathcal{E})$ *with* $A \cap \mathcal{D}(\mathcal{E}^N) \neq \emptyset$ *for* $N \in \mathbb{N}$. *If, for any*

$$(u_N)_N \in \prod_{N\in\mathbb{N}} \mathcal{D}(\mathcal{E}^N) \cap A := \mathbf{C}'$$

there exists a constant $R \in (0,\infty)$ *such that*

for all $v \in D$:

$$\inf_{(w_N)_N \in \mathbf{C}\cap\mathbf{B}_R^{\mathcal{E}}} \limsup_{N\to\infty} \left(\left|\mathcal{E}^N(u_N - w_N, \Phi_N v)\right| + \left|(u_N - w_N, \Phi_N v)_{H_N}\right| \right) = 0,$$

then also $\mathbf{C}'$ *is a compatible class.*

Proof. We have to verify that A satisfies (2.2.21). To this end, let $(u'_k)_{k\in\mathbb{N}}$ be a sequence in A and $u' \in \mathcal{D}(\mathcal{E})$ be its weak limit in $\mathcal{H}^{\mathcal{E},1}$. We have to show that u' is also the weak limit of $(u'_k)_{k\in\mathbb{N}}$ in $\mathcal{H}$. It suffices to show that for every subsequence there is a (sub-)subsequence whose weak limit in $\mathcal{H}$ is u'. We choose $N_k \in \mathbb{N}$ for each $k \in \mathbb{N}$ such that $u'_k \in \mathcal{D}(\mathcal{E}^{N_k})$. After possibly dropping to suitable (sub-)subsequence we may assume that $(N_k)_{k\in\mathbb{N}}$ is strictly increasing and moreover that there exists a weak limit u^* of $(u'_k)_{k\in\mathbb{N}}$ in $\mathcal{H}$. We have to show $u^* = u'$. Now, we want to apply the assumption of this proposition. To do so, we take an arbitrary element $u''_N \in \mathcal{D}(\mathcal{E}^N) \cap A$ for $N \in \mathbb{N}$ and define

$$u_l := \begin{cases} u'_k & \text{if } l = N_k \text{ for some } k \in \mathbb{N}, \\ u''_l, & \text{if } l \notin \{N_k \,|\, k \in \mathbb{N}\} \end{cases}$$

for $l \in \mathbb{N}$. Let v be an element of $\mathcal{D}(\mathcal{E})$. We choose a strongly convergent approximation for v in the Hilbert space $(\mathcal{D}(\mathcal{E}), \mathcal{E}_1)$, say $(v_m)_{m\in\mathbb{N}}$, consisting of elements in D. By assumption, there exists a constant $R \in (0,\infty)$, and for each $m \in \mathbb{N}$ an element $(w_N^{(m)})_N \in \mathbf{C} \cap \mathbf{B}_R^{\mathcal{E}}$ with

$$\limsup_{N\to\infty} \left(\left|\mathcal{E}^N\big(u_N - w_N^{(m)}, \Phi_N v_m\big)\right| + \left|\big(u_N - w_N^{(m)}, \Phi_N v_m\big)_{H_N}\right| \right) \le \frac{1}{m}. \qquad (2.2.26)$$

Noting that u^* and u' are weak accumulation points of $(u_N)_{N\in\mathbb{N}}$ in $\mathcal{H}$, respectively $\mathcal{H}^{\mathcal{E},1}$, and repeatedly using the compactness result of Lemma 2.1.1 (iv), we can find a strictly increasing sequence of natural numbers $(L_k)_{k\in\mathbb{N}}$ and a sequence $(\varphi_m)_{m\in\mathbb{N}} \subset \mathcal{D}(\mathcal{E})$ via a diagonal procedure, in such a way, that in the limit of k to infinity it holds

$$u_{L_k} \overset{k}{\rightharpoonup} u^* \qquad \text{weakly in } \mathcal{H},$$

$$
\begin{aligned}
u_{L_k} &\overset{k}{\rightharpoonup} u' && \text{weakly in } \mathcal{H}^{\mathcal{E},1}, \\
w_{L_k}^{(m)} &\overset{k}{\rightharpoonup} \varphi_m && \text{weakly in } \mathcal{H}^{\mathcal{E},1}, \quad m \in \mathbb{N}.
\end{aligned}
$$

By virtue of Lemma 2.1.1 (iii) it holds

$$
\mathcal{E}_1(\varphi_m, \varphi_m) \le \liminf_{k\to\infty} \mathcal{E}_1^{L_k}\big(w_{L_k}^{(m)}, w_{L_k}^{(m)}\big) \le R, \quad m \in \mathbb{N}. \tag{2.2.27}
$$

For every $\varphi \in \mathcal{D}(\mathcal{E})$ we can write

$$
\begin{aligned}
|(u^* - \varphi, v)_H| \le{}& \big|(u^*, v)_H - (u^*, v_m)_H\big| && (2.2.28)\\
&+ \big|(u^*, v_m)_H - \big(u_{L_k}, \Phi_{L_k} v_m\big)_{H_{L_k}}\big| && (2.2.29)\\
&+ \big|\big(u_{L_k}, \Phi_{L_k} v_m\big)_{H_{L_k}} - \big(w_{L_k}^{(m)}, \Phi_{L_k} v_m\big)_{H_{L_k}}\big| && (2.2.30)\\
&+ \big|\big(w_{L_k}^{(m)}, \Phi_{L_k} v_m\big)_{H_{L_k}} - (\varphi_m, v_m)_H\big| && (2.2.31)\\
&+ \big|(\varphi_m, v_m)_H - (\varphi, v)_H\big| && (2.2.32)
\end{aligned}
$$

as well as

$$
\begin{aligned}
|\mathcal{E}_1(u' - \varphi, v)| \le{}& \big|\mathcal{E}_1(u', v) - \mathcal{E}_1(u', v_m)\big| && (2.2.33)\\
&+ \big|\mathcal{E}_1(u', v_m) - \mathcal{E}_1^{L_k}\big(u_{L_k}, \Phi_{L_k} v_m\big)\big| && (2.2.34)\\
&+ \big|\mathcal{E}_1^{L_k}\big(u_{L_k}, \Phi_{L_k} v_m\big) - \mathcal{E}_1^{L_k}\big(w_{L_k}^{(m)}, \Phi_{L_k} v_m\big)\big| && (2.2.35)\\
&+ \big|\mathcal{E}_1^{L_k}\big(w_{L_k}^{(m)}, \Phi_{L_k} v_m\big) - \mathcal{E}_1(\varphi_m, v_m)\big| && (2.2.36)\\
&+ \big|\mathcal{E}_1(\varphi_m, v_m) - \mathcal{E}_1(\varphi, v)\big|. && (2.2.37)
\end{aligned}
$$

In particular, the above inequalities are true if φ is an accumulation point of $(\varphi_m)_{m\in\mathbb{N}}$ in the Hilbert space $(\mathcal{D}(\mathcal{E}), \mathcal{E}_1)$, say $\lim_{n\to\infty} \varphi_{m_n} = \varphi$ holds for a suitable subsequence $(\varphi_{m_n})_{n\in\mathbb{N}}$. Due to (2.2.27), such an accumulation point φ and the according subsequence exist. Let $\varepsilon > 0$. We can first choose $m \in \{m_n \,|\, n \in \mathbb{N}\}$ such that $m \ge 6/\varepsilon$ and the summands of (2.2.28) and (2.2.32), respectively (2.2.33) and (2.2.37), on the right-hand side of the above inequalities take a value smaller equal $\varepsilon/5$ each. Then, depending on the value of m, we can choose $k \in \mathbb{N}$ such that the summands of (2.2.29), (2.2.30) and (2.2.31), respectively (2.2.34), (2.2.35) and (2.2.36), take value smaller equal $\varepsilon/5$ each. Here, in view of (2.2.30) and (2.2.35), the condition $m \ge 6/\varepsilon$ together with (2.2.26) has been used. Since the choice for $\varepsilon > 0$ and for $v \in \mathcal{D}(\mathcal{E})$ are arbitrary, the desired identity $u' = u^*$ follows from Lemma 2.2.4. This concludes the proof. □

Theorem 2.2.9. *Under the assumption of* (2.2.16), *$\mathcal{E}$ is the Mosco limit of $(\mathcal{E}^N)_{N\in\mathbb{N}}$ on $\mathcal{H}$, if and only if, there exists $\alpha > 0$ and a compatible class $\mathbf{C}$ with the following property: For any $u \in D$ there exists a constant $R \in (0,\infty)$ such that*

$$
\inf_{(w_N)_N} \limsup_{N\to\infty} \Big(\big|\mathcal{E}^N\big(G_\alpha^N \Phi_N u - w_N, \Phi_N v\big)\big| + \big|\big(G_\alpha^N \Phi_N u - w_N, \Phi_N v\big)_{H_N}\big| \Big) = 0
$$

holds for every $v \in D$. The infimum is taken over all $(w_N)_N \in \mathbf{C} \cap \mathbf{B}_R^{\mathcal{E}}$.

Proof. Let $\alpha > 0$ be as in the assumptions and $u \in D$. The assumption of Proposition 2.2.8 is satisfied for the set $A := \{G_\alpha^N \Phi_N u \,\big|\, N \in \mathbb{N}\}$. Therefore, the property of (2.2.21) is true for A. In (2.2.21), the space $\mathcal{H}^{\mathcal{E},1}$ may be replaced by $\mathcal{H}^{\mathcal{E},\alpha}$ without compromising its validity. The weak convergence of $(G_\alpha^N \Phi_N u)_{N\in\mathbb{N}}$ towards $G_\alpha u$ in $\mathcal{H}^{\mathcal{E},\alpha}$, which is evident from

$$\begin{aligned}\lim_{N\to\infty} \mathcal{E}_\alpha^N\big(G_\alpha^N \Phi_N u, \Phi_N v\big) &= \lim_{N\to\infty} (\Phi_N u, \Phi_N v)_{H_N} \\ &= (v, w)_H = \mathcal{E}_\alpha(G_\alpha u, v), \quad v \in D,\end{aligned}$$

now implies

$$\lim_{N\to\infty} \big(\Phi_N u, G_\alpha^N \Phi_N u\big)_{H_N} = (u, G_\alpha u)_H.$$

Since the latter is true for every $u \in D$, the claim of the theroem follows from Remark 2.2.6 (i). □

Chapter 3 Standard gradient forms on $\mathbb{R}^d$

3.1 The method of Finite Elements

3.1.1 The tent function and Finite Elements

The goal is to analyse the function $\chi^{(d)}$ on the d-dimensional Euclidean space which is the generalization of the unit tent function $\chi^{(1)}$ supported on $[-1,1]$, i.e.

$$\chi^{(1)}(x) := \big(\min(\{1-x, 1+x\})\big)_+, \quad x \in \mathbb{R}.$$

Here and in the following, $f_+(x) := \max(\{f(x), 0\})$, $x \in \mathbb{R}^d$, denotes the positive part of a real-valued function f on $\mathbb{R}^d$, $d \in \mathbb{N}$. It is not immediately clear how a generalization of that function for the cases $d \geq 2$ should look like. The result of our considerations displayed below suggests

$$\chi^{(d)}(x) := \Big(\min_{i,j\in\{1,\dots,d\}} \big(\{1 + x_i - x_j, 1 + x_i, 1 - x_i\}\big)\Big)_+, \quad x \in \mathbb{R}^d, \tag{3.1.1}$$

which we take as a definition. The reasoning leading there starts with the idea that we have to construct a piecewise affine linear, continuous function and first looks for these domains on which $\chi^{(d)}$ restricts to an affine linear function. The restriction of $\chi^{(d)}$ onto such a domain can then simply be conceived through means of a suitable linear interpolation method. We are thus looking for a triangulation of the d-dimensional Euclidean space. The *Coxeter-Freudenthal-Kuhn triangulation* particularly fits into this purpose. The close connection between the function $\chi^{(d)}$ from (3.1.1) and the Coxeter-Freudenthal-Kuhn triangulation is explained below and the relevant properties of $\chi^{(d)}$ are discussed in Lemma 3.1.3 and Theorem 3.1.6.

Let $d \in \mathbb{N}$, $d \geq 2$, and $\mathbf{e}_i$ denote the i-th unit vector of $\mathbb{R}^d$. We consider the image of the convex set

$$\mathcal{C} := \{x \in \mathbb{R}^d \,|\, 0 \leq x_d \leq \cdots \leq x_1 \leq 1\}$$

under an arbitrary permutation of coordinates. For a given element σ from the symmetric group $\mathcal{S}_d$ over $\{1,\dots,d\}$ we denote by A_σ the orthogonal transformation on $\mathbb{R}^d$ such that $A_\sigma \mathbf{e}_i := \mathbf{e}_{\sigma(i)}$ for $i = 1,\dots,d$. Obviously, we have

$$[0,1]^d = \bigcup_{\sigma\in\mathcal{S}_d} A_\sigma(\mathcal{C}).$$

This concept of a triangulation traces back to [1, 2, 3] and therefrom gets its name. The reader can find information on the historical background of the Coxeter-Freudenthal-Kuhn triangulation, along with a descriptive introduction, further explanations and more recent applications, in [21, Chapter 2]. Any of the sets $A_\sigma(\mathcal{C})$ admit an interpretation as the convex hulls of the traces of paths in $[0,1]^d$ with a specific property. We consider exactly such paths which start in the origin, end in the point $\mathbf{e} = \mathbf{e}_1 + \cdots + \mathbf{e}_d$, and only walk along the edges of the unit d-cube in positive direction. To make this idea precise, let $\mathcal{T}$ be the set comprising all tuples

$$T = (T(0), \dots, T(d)) \in \prod_{i=0}^{d} \{0,1\}^d$$

of length $d+1$ for which

$$T(0) = 0, \quad T(d) = \mathbf{e}, \tag{3.1.2}$$

and moreover,

$$|T(i) - T(i-1)|_{\text{euc}} = 1 \quad \text{for } i = 1, \dots, d. \tag{3.1.3}$$

Then, for $T \in \mathcal{T}$ we denote by

$$\mathcal{C}_T := \Big\{ \sum_{i=0}^{d} \lambda_i T(i) \,\Big|\, \lambda_0, \dots, \lambda_d \in [0,1], \sum_{i=0}^{d} \lambda_i = 1 \Big\} \tag{3.1.4}$$

the set of all convex linear combinations of the components of T.

Remark 3.1.1. (i) The set $\mathcal{T}$ can be characterized by its one-to-one correspondence with the symmetric group $\mathcal{S}_d$. Let $T \in \mathcal{T}$. Due to (3.1.3), the components of $T(i)$ differ from the components of $T(i-1)$ for exactly one index $j \in \{1, \dots, d\}$, for given $i = 1, \dots, d$. Moreover, from (3.1.2) and (3.1.3), we deduce that the sum of the coordinates

$$s(T(i)) := (T(i))_1 + \dots + (T(i))_d$$

is increased by one compared to the value of $s(T(i-1))$, for each $i = 1, \dots, d$. Therefore, defining $\sigma_T(i) := j$, where j is the index of the component in which $T(i)$ and $T(i-1)$ differ, we obtain

$$\mathbf{e}_{\sigma_T(i)} = T(i) - T(i-1), \quad i = 1, \dots, d. \tag{3.1.5}$$

Taking the sum of (3.1.5) up to a certain index we obtain

$$T(i) = \sum_{k=1}^{i} \mathbf{e}_{\sigma_T(k)}, \quad i = 1, \dots, d. \tag{3.1.6}$$

The map $i \mapsto \sigma_T(i)$ is a permutation on $\{1, \dots, d\}$, indeed. The surjectivity follows from (3.1.6) and the fact that $(T(0))_j = 0$ while $(T(d))_j = 1$ for each $j \in \{1, \dots, d\}$. Now, $\sigma_T \in \mathcal{S}_d$ since σ_T maps the set $\{1, \dots, d\}$ into itself.

(ii) Vice versa, given a permutation $\sigma \in \mathcal{S}_d$, we can find the corresponding element $T \in \mathcal{T}$ with $\sigma_T = \sigma$ by setting $T(0) := 0$ and $T(i) := \sum_{k=1}^{i} \mathbf{e}_\sigma(k)$ for $i = 1, \dots, d$.

(iii) Let $T \in \mathcal{T}$. The k-th component of $T(i)$ writes

$$(T(i))_k = \mathbf{1}_{\sigma_T(\{1,\dots,i\})}(k) \quad \text{for } k, i \in \{1, \dots, d\}$$

due to (3.1.6). In particular,

$$0 \le x_{\sigma_T(d)} \le x_{\sigma_T(d-1)} \le \dots \le x_{\sigma_T(1)} \le 1 \tag{3.1.7}$$

for $x \in \{T(0), \dots, T(d)\}$. Now, the representation

$$\mathcal{C}_T = \big\{ x \in \mathbb{R}^d \,\big|\, 0 \le x_{\sigma_T(d)} \le x_{\sigma_T(d-1)} \le \dots \le x_{\sigma_T(1)} \le 1 \big\} \tag{3.1.8}$$

is shown as follows. The inclusion '$\subseteq$' holds, because the condition (3.1.7) is stable under taking convex combinations of points $x \in \mathbb{R}^d$ having that property. On the other hand, for $x \in \mathbb{R}^d$ it holds

$$x = x_{\sigma_T(d)} T(d) + (x_{\sigma_T(d-1)} - x_{\sigma_T(d)}) T(d-1)$$

$$
\begin{aligned}
&+ (x_{\sigma_T(d-2)} - x_{\sigma_T(d-1)})T(d-2)\\
&+ \dots\\
&+ (x_{\sigma_T(1)} - x_{\sigma_T(2)})T(1) + (1 - x_{\sigma_T(1)})T(0)
\end{aligned}
$$

because of (3.1.5). So, if x satisfies (3.1.7), then $x \in \mathcal{C}_T$. This proves the inclusion '$\supseteq$'.

For a $(d+1)$-tuple of the form

$$
\begin{aligned}
T &= \big(T(0), \dots, T(d)\big)\\
&= \big(\mathbf{k} + T_0(0), \mathbf{k} + T_0(1), \dots, \mathbf{k} + T_0(d)\big) =: \mathbf{k} + T_0
\end{aligned}
$$

for some $\mathbf{k} \in \mathbb{Z}^d$ and $T_0 \in \mathcal{T}$ we set $\mathcal{C}_T := \{\mathbf{k} + x \,|\, x \in \mathcal{C}_{T_0}\}$ and $\sigma_T := \sigma_{T_0}$. Of course, $\mathbf{k}$ and T_0 are unique in the above representation of T, so there is no ambiguity in the definitions of $\mathcal{C}_T$ and σ_T. Moreover, for each $i \in \{0, \dots, d\}$ we define an affine linear function from $\mathbb{R}^d$ into $\mathbb{R}$ by

$$
h_T^i(x) := \begin{cases} 1 - x_{\sigma_T(1)} + \mathbf{k}_{\sigma_T(1)} & \text{if } i = 0,\\ x_{\sigma_T(i)} - \mathbf{k}_{\sigma_T(i)} - x_{\sigma_T(i+1)} + \mathbf{k}_{\sigma_T(i+1)} & \text{if } i \in \{1, \dots, d-1\},\\ x_{\sigma_T(d)} - \mathbf{k}_{\sigma_T(d)} & \text{if } i = d. \end{cases}
$$

Remark 3.1.2. Let $T = \mathbf{k} + T_0$ for some $\mathbf{k} \in \mathbb{Z}^d$ and $T_0 \in \mathcal{T}$.

(i) Let $i \in \{0, \dots, d\}$. h_T^i is the unique affine linear function from $\mathbb{R}^d$ into $\mathbb{R}$ which interpolates the $(d+1)$-sized sample $(T(j), \delta_{ij})_{j=0,\dots,d}$, i.e. for which

$$
(T(j), \delta_{ij}) \in \big\{(x, h_T^i(x)) \in \mathbb{R}^d \times \mathbb{R} \,\big|\, x \in \mathbb{R}^d\big\}, \quad j = 0, \dots, d.
$$

In particular,

$$
\sum_{i=0}^{d} h_T^i(x) = 1, \quad x \in \mathbb{R}^d. \tag{3.1.9}
$$

(ii) $\nabla h_T^i = \mathbf{1}_{\{1,\dots,d\}}(i)\mathbf{e}_{\sigma_T(i)} - \mathbf{1}_{\{0,\dots,d-1\}}(i)\mathbf{e}_{\sigma_T(i+1)}$ for $i \in \{0, \dots, d\}$.

Lemma 3.1.3. *(i) Let $d \in \mathbb{N}$ with $d \geq 2$. For $x \in \mathbb{R}^d$ it holds $\chi^{(d)}(x) = 0$ unless there exists $T \in \mathcal{T}$ and $i \in \{0, \dots, d\}$ such that $x \in \mathcal{C}_{-T(i)+T}$. In that case, it holds $\chi^{(d)}(x) = h^i_{-T(i)+T}(x) \geq 0$.*

(ii) $\chi^{(d)}$ is a Lipschitz continuous function, supported on $[-1,1]^d$, with values in $[0,1]$ and $\displaystyle\int_{\mathbb{R}^d} \chi^{(d)}(x)\,\mathrm{d}x = 1$ for $d \in \mathbb{N}$.

Proof. (i) Let $T' = \mathbf{k} + T$ for some $T \in \mathcal{T}$ and $\mathbf{k} \in \mathbb{Z}^d$ with $N_{T'} := \{T'(0), \dots, T'(d)\}$. At first, we prove the claim

$$
\begin{aligned}
0 \in N_{T'} \quad \text{if and only if} \quad &|x_i - x_j| \leq 1 \quad \forall i,j \in \{1, \dots, d\},\, x \in N_{T'},\\
&\text{and } |x_i| \leq 1 \quad \forall i \in \{1, \dots, d\},\, x \in N_{T'}.
\end{aligned} \tag{3.1.10}
$$

If $T'(i_0) = 0$ for some $i_0 \in \{0, \dots, d\}$, then we can compute the coordinates of all other nodes in $N_{T'}$ via (3.1.6). We have $0 = T'(i_0) = \mathbf{k} + T(i_0)$ and hence

$$
T'(i) = \mathbf{k} + T(i) = \mathbf{k} + T(i_0) - \sum_{k=i+1}^{i_0} \mathbf{e}_{\sigma_T(k)} = -\sum_{k=i+1}^{i_0} \mathbf{e}_{\sigma_T(k)} \quad \text{if } 0 \leq i < i_0,
$$

while

$$T'(i) = \mathbf{k} + T(i) = \mathbf{k} + T(i_0) + \sum_{k=i_0+1}^{i} \mathbf{e}_{\sigma_T(k)} = \sum_{k=i_0+1}^{i} \mathbf{e}_{\sigma_T(k)} \quad \text{if } i_0 < i \leq d.$$

For the coordinates of a node $T'(i) \in \mathbb{N}_{T'}$ this means

$$\{(T'(i))_j \,|j = 1, \dots d\} \subseteq \{-1, 0\}, \quad \text{if } i \leq i_0,$$

while

$$\{(T'(i))_j \,|j = 1, \dots d\} \subseteq \{0, 1\}, \quad \text{if } i \geq i_0.$$

In particular, the inequalities on right-hand side of (3.1.10) are fulfilled. To prove the other direction of the stated equivalence (3.1.10), we now assume that the inequalities on right-hand side hold true. We choose the index $i_0 \in \{0, \dots, d\}$ such that the node $T'(i_0)$ minimizes the modulus of the sum of the coordinates $|s(x)|$ with $s(x) := |x_1 + \dots + x_d|$, $x \in N_{T'}$, i.e. $|s(T'(i_0))| \leq |s(x)|$ for all $x \in N_{T'}$. There are the three possible cases of either $s(T'(i_0)) = 0$, or $s(T'(i_0)) \geq 1$, or else $s(T'(i_0)) \leq -1$. The case of $s(T'(i_0)) \geq 1$ leads to a contradiction by distinguishing again between two subcases. The first subcase to consider is $s(T'(i_0)) \geq 1$ and $i_0 = 0$. Then, we have

$$s(T'(d)) = s(T'(0)) + d \geq 1 + d$$

which contradicts $|(T'(d))_i| \leq 1$ for all $i = 1, \dots, d$. In the second subcase, where $s(T'(i_0)) \geq 1$ and $i_0 > 0$, we can argue via (3.1.5) that

$$s(T'(i_0 - 1)) = s(\mathbf{k}) + s(T(i_0 - 1)) = s(\mathbf{k}) + s(T(i_0)) - 1 = s(T'(i_0)) - 1$$

contradicts the minimality of $|s(T'(i_0))|$. Similarly, the case of $s(T'(i_0)) \leq -1$ leads to a contradiction by distinguishing between two subcases. If $s(T'(i_0)) \leq -1$ and $i_0 = d$, then

$$s(T'(0)) = s(T'(d)) - d \leq -1 - d$$

in contradiction to $|(T'(0))_i| \leq 1$ for all $i = 1, \dots, d$. If $s(T'(i_0)) \leq -1$ and $i_0 < d$, then again by (3.1.5) we obtain

$$s(T'(i_0 + 1)) = s(\mathbf{k}) + s(T(i_0 + 1)) = s(\mathbf{k}) + s(T(i_0)) + 1 = s(T'(i_0)) + 1,$$

which contradicts the minimality of $|s(T'(i_0))|$. Therefore, we must have $s(T'(i_0)) = 0$. In view of $|(T'(i_0))_i - (T'(i_0))_j| \leq 1$ for $i, j \in \{1, \dots, d\}$, this implies $T'(i_0) = 0$ and the proof of (3.1.10) is complete.

Now, we are prepared to approach the proof of statement (i). We start by showing that $\chi^{(d)}$ vanishes on $\mathcal{C}_{T'}$ if $0 \notin N_{T'}$. So, we assume that the equivalent statements of (3.1.10) do not hold. Since $\max_{i=1,\dots,d} |x_i - y_i| \leq 1$ for $x, y \in \mathbb{N}_{T'}$, there are three possible cases in view of (3.1.10). Either

$$\exists i, j \in \{1, \dots, d\} : \quad \forall x \in N_{T'} : \quad 1 + x_i - x_j \leq 0, \tag{3.1.11}$$

or

$$\exists i \in \{1, \dots, d\} : \quad \forall x \in N_{T'} : \quad 1 + x_i \leq 0, \tag{3.1.12}$$

or else

$$\exists i \in \{1, \dots, d\} : \quad \forall x \in N_{T'} : \quad 1 - x_i \leq 0. \tag{3.1.13}$$

For $x \in \mathcal{C}_{T'}$ there are $\lambda_0, \dots, \lambda_d \in [0,1]$ with $\sum_{i=0}^d \lambda_i = 1$ such that

$$x = \mathbf{k} + \sum_{k=0}^{d} \lambda_i T(i) = \sum_{k=0}^{d} \lambda_i T'(i), \tag{3.1.14}$$

by the definition of $\mathcal{C}_T$, (3.1.4). Since the inequalities of (3.1.11), (3.1.12) and (3.1.13) are stable under taking convex linear combinations, they generalize to all points $x \in \mathcal{C}_{T'}$, respectively. Hence $\chi^{(d)}(x) = 0$ for $x \in \mathcal{C}_{T'}$ by construction.

Now, we look at the other case where the equivalent statements of (3.1.10) are true and say $0 = T'(i_0)$ for some $i_0 \in \{0, \dots, d\}$. In the following, the equality $h^{i_0}_{-T(i_0)+T}(x) = \chi^{(d)}(x)$ for $x \in \mathcal{C}_{T'}$ is verified by showing both, '$\leq$' and '$\geq$', one after another. For two affine linear functions f and g from $\mathbb{R}^d$ into $\mathbb{R}$ with $g(0) = f(0)$ it holds

$$f(x) \geq g(x) \text{ for } x \in N_{T'} \quad \Longrightarrow \quad f(x) \geq g(x) \text{ for } x \in \mathcal{C}_{T'} \tag{3.1.15}$$

using the linear combination of (3.1.14) for general $x \in \mathcal{C}_{T'}$. Because of (3.1.10) and (3.1.15), the functions $\alpha_{ij}(x) := 1 + x_i - x_j$, $\beta_i(x) := 1 + x_i$, $\gamma_i(x) = 1 - x_i$, $x \in \mathbb{R}^d$, with indices $i, j \in \{1, \dots, d\}$, belong to the family

$$A_{T'} := \left\{ f : \mathbb{R}^d \to \mathbb{R} \,\middle|\, f \text{ is a affine linear, } f(\mathcal{C}_{T'}) \subset [0, \infty),\ f(0) = 1 \right\}$$

comprising all affine linear functions, which are non-negative on $\mathcal{C}_{T'}$ and take the value 1 at the origin. We have $0 = T'(i_0) = \mathbf{k} + T(i_0)$ and hence $T' = -T(i_0) + T$. By Remark 3.1.2 (i) we obtain $h^{i_0}_{-T(i_0)+T}(x) = \delta_0(x)$ for $x \in N_{T'}$. Through (3.1.15) we can conclude

$$h^{i_0}_{-T(i_0)+T}(x) \leq f(x) \quad \forall x \in \mathcal{C}_{T'},\ f \in A_{T'},$$

and so,

$$h^{i_0}_{-T(i_0)+T}(x) \leq \min_{i,j \in \{1,\dots,d\}} \left(\{\alpha_{i,j}(x), \beta_i(x), \gamma_i(x)\} \right) \leq \chi^{(d)}(x), \quad x \in \mathcal{C}_{T'}.$$

On the other hand, we can calculate

$$h^{i_0}_{-T(i_0)+T}(x) = \begin{cases} 1 - x_{\sigma_T(1)} & \text{if } i_0 = 0, \\ x_{\sigma_T(i_0)} + 1 - x_{\sigma_T(i_0+1)} & \text{if } i_0 \in \{1, \dots, d-1\}, \\ x_{\sigma_T(d)} + 1 & \text{if } i_0 = d, \end{cases}$$

for $x \in \mathbb{R}^d$, using $\mathbf{k} = 0$ if $i_0 = 0$, or else $\mathbf{k} = -\sum_{k=1}^{i_0} \mathbf{e}_{\sigma_T(k)}$ by (3.1.6). This implies $h^{i_0}_{-T(i_0)+T}(x) \geq \chi^{(d)}(x)$ for $x \in \mathcal{C}_{T'}$ by construction of $\chi^{(d)}$. The proof of part (i) of this lemma is complete.

(ii) We move to the second part of the proof. Since the claim for $d = 1$ is obvious from the definition of $\chi^{(1)}$, we only deal with the case $d \geq 2$. The class of Lipschitz continuous functions is stable under taking the minimum or maximum of two elements. So, the Lipschitz continuity of $\chi^{(d)}$ and also $\chi^{(d)}(x) \subseteq [0,1]$, $x \in \mathbb{R}^d$, is clear by construction, while

$$\operatorname{supp}[\chi^{(d)}] \subseteq \bigcup_{T \in \mathcal{T}, i=0,\dots,d} \mathcal{C}_{-T(i)+T} \subseteq [-1,1]^d$$

follows from part (i). The equality $\int_{\mathbb{R}^d} \chi(x)\,\mathrm{d}x = 1$ is left to be shown. The following calculation is based on the observation

$$\bigcup_{\substack{T'=\mathbf{k}+T\\ \mathbf{k}\in\mathbb{Z}^d, T\in\mathcal{T}}} \mathcal{C}_{T'} \setminus \operatorname{int}(\mathcal{C}_{T'}) \quad \text{has Lebesgue measure zero,} \quad \mathbb{R}^d = \bigcup_{\substack{T'=\mathbf{k}+T\\ \mathbf{k}\in\mathbb{Z}^d, T\in\mathcal{T}}} \mathcal{C}_{T'},$$

the fact that $h^i_{\mathbf{k}+T}(x) = h^i_T(x-\mathbf{k})$ for $x \in \mathbb{R}^d$, $T \in \mathcal{T}$, $\mathbf{k} \in \mathbb{Z}^d$, $i \in \{0,\dots,d\}$, as well as the statement of (i), the shift invariance of the Lebesgue measure and (3.1.9). Indeed,

$$\begin{aligned}
\int_{\mathbb{R}^d} \chi^{(d)}(x)\,\mathrm{d}x &= \sum_{\substack{T'=\mathbf{k}+T\\ \mathbf{k}\in\mathbb{Z}^d, T\in\mathcal{T}}} \int_{\operatorname{int}(\mathcal{C}_{T'})} \chi^{(d)}(x)\,\mathrm{d}x \\
&= \sum_{\substack{T'=-T(i)+T\\ T\in\mathcal{T}, i\in\{0,\dots,d\}}} \int_{\operatorname{int}(\mathcal{C}_{T'})} h^i_{-T(i)+T}(x)\,\mathrm{d}x \\
&= \sum_{\substack{T'=-T(i)+T\\ T\in\mathcal{T}, i\in\{0,\dots,d\}}} \int_{\operatorname{int}(\mathcal{C}_{T'})} h^i_T\big(x+T(i)\big)\,\mathrm{d}x \\
&= \sum_{T\in\mathcal{T}, i\in\{0,\dots,d\}} \int_{\operatorname{int}(\mathcal{C}_T)} h^i_T(y)\,\mathrm{d}y = \sum_{T\in\mathcal{T}} \int_{\operatorname{int}(\mathcal{C}_T)} \mathrm{d}y = \int_{[0,1]^d} 1\,\mathrm{d}y = 1.
\end{aligned}$$

This concludes the proof. □

We analyse the tent function $\chi^{(d)}$ in more detail and fix $d \in \mathbb{N}$, $d \geq 2$. It can be useful to have an explicit representation for $\chi^{(d)}$ as given by (3.1.21) below. The family of indicator functions of the sets $\mathcal{C}_T$, $T \in \mathcal{T}$, do not sum up to the indicator function of the unit cube $[0,1]^d$, because these sets have intersection at the boundary. We would like to have a partition of the semi-open cube

$$[0,1)^d = \bigsqcup_{T\in\mathcal{T}} \mathcal{C}^{\mathrm{dis}}_T, \tag{3.1.16}$$

where $\mathcal{C}^{\mathrm{dis}}_T$ is contained in $\mathcal{C}_T$ for $T \in \mathcal{T}$ and the family $\{\mathcal{C}^{\mathrm{dis}}_T \,|\, T \in \mathcal{T}\}$ are pairwise disjoint. The symbol $<_{\sigma,i}$ for $i \in \{2,\dots,d\}$ and $\sigma \in \mathcal{S}_d$ shall denote the relation on $\mathbb{R}$ which coincides with '$<$' in case $\sigma(i-1) < \sigma(i)$ and with '$\leq$' in case $\sigma(i-1) > \sigma(i)$. For $T \in \mathcal{T}$ we define

$$\mathcal{C}^{\mathrm{dis}}_T := \big\{x \in \mathbb{R}^d \,\big|\, 0 \leq x_{\sigma_T(d)} <_{\sigma_T,d} x_{\sigma_T(d-1)} <_{\sigma_T,d-1} \cdots <_{\sigma_T,2} x_{\sigma_T(1)} < 1\big\}. \tag{3.1.17}$$

Obviously, the topological closure of $\mathcal{C}^{\mathrm{dis}}_T$ coincides with $\mathcal{C}_T$ for $T \in \mathcal{T}$. In the next remark we argue why the definition of (3.1.17) matches (3.1.16) and state some immediate consequences of (3.1.16). We set $\mathcal{C}^{\mathrm{dis}}_{\mathbf{k}+T} := \{\mathbf{k}+x \,|\, x \in \mathcal{C}^{\mathrm{dis}}_T\}$ and moreover $\eta_{\mathbf{k}+T} := \mathbf{1}_{\mathcal{C}^{\mathrm{dis}}_{\mathbf{k}+T}}$ for $T \in \mathcal{T}$ and $\mathbf{k} \in \mathbb{Z}^d$.

Remark 3.1.4. (i) Let $x \in [0,1)^d$. The lexicographic ordering of the set of tuples $A_x := \{(x_i, i) \,|\, i = 1,\dots,d\}$ defines a permutation $\sigma_x \in \mathcal{S}_d$ by setting

$$\sigma_x(j) = i :\Longleftrightarrow (x_i, i) \text{ is the } j\text{-th element in the lexicographic ordering of the set } A_x \text{ in decreasing order.}$$

It holds

$$\sigma_T = \sigma_x \quad \text{if and only if} \quad x \in \mathcal{C}_T^{\text{dis}}, \tag{3.1.18}$$

as we convince ourselves now. We note that (3.1.18) implies (3.1.16), since $T \mapsto \sigma_T$ is a one-to-one map between $\mathcal{T}$ and $\mathcal{S}_d$. First, if $T \in \mathcal{T}$ and x is an element of $\mathcal{C}_T^{\text{dis}}$ as in (3.1.17), then clearly

$$x_{\sigma_T(1)} \geq x_{\sigma_T(2)} \geq \cdots \geq x_{\sigma_T(d)}.$$

Moreover, by definition of $<_{\sigma_T,i}$, the strict inequality $x_{\sigma_T(i-1)} > x_{\sigma_T(i)}$ must hold for all $i = 2, \dots, d$ for which $\sigma_T(i-1) < \sigma_T(i)$. In other words,

$$(x_{\sigma_T(1)}, \sigma_T(1)) >_{\text{lex.}} (x_{\sigma_T(2)}, \sigma_T(2)) >_{\text{lex.}} \cdots >_{\text{lex.}} (x_{\sigma_T(d)}, \sigma_T(d)) \tag{3.1.19}$$

is the decreasing lexicographical ordering of the elements of A_x. Hence, $\sigma_T = \sigma_x$. Vice versa, if $T \in \mathcal{T}$ such that $\sigma_T = \sigma_x$, then (3.1.19) holds true and so $x_{\sigma_T(i)} <_{\sigma_T,i} x_{\sigma_T(i-1)}$ for $i = 2, \dots, d$ by definition of $<_{\sigma_T,i}$. Therefore, $x \in \mathcal{C}_T^{\text{dis}}$. We have seen, $x \in \mathcal{C}_T^{\text{dis}}$ if and only if $\sigma_T = \sigma_x$ and so (3.1.16) holds indeed.

(ii) As a consequence of (3.1.16) it holds

$$\begin{aligned}\sum_{T \in \mathcal{T}, \mathbf{k} \in \mathbb{Z}^d} \eta_{\mathbf{k}+T}(x) &= \sum_{T \in \mathcal{T}, \mathbf{k} \in \mathbb{Z}^d} \eta_T(x - \mathbf{k}) \\ &= \sum_{\mathbf{k} \in \mathbb{Z}^d} \mathbf{1}_{[0,1)^d}(x - \mathbf{k}) = 1, \quad x \in \mathbb{R}^d.\end{aligned}$$

(iii) Due to (ii) we have

$$\chi^{(d)}(x) = \sum_{T \in \mathcal{T}, \mathbf{k} \in \mathbb{Z}^d} \eta_{\mathbf{k}+T}(x) \chi(x), \quad x \in \mathbb{R}^d. \tag{3.1.20}$$

In view of $\text{supp}[\eta_{\mathbf{k}+T}] = \mathcal{C}_{\mathbf{k}+T}$, we derive from Lemma 3.1.3 (i) that $\eta_{\mathbf{k}+T}(x)\chi(x) = 0$ whenever $\mathbf{k} \notin \{T(0), \dots, T(d)\}$ for fixed $T \in \mathcal{T}$ and $x \in \mathbb{R}^d$. So, (3.1.20) transforms into

$$\chi^{(d)}(x) = \sum_{T \in \mathcal{T}, i=0,\dots,d} \eta_{-T(i)+T}(x) h^i_{-T(i)+T}(x), \quad x \in \mathbb{R}^d. \tag{3.1.21}$$

Let $d \in \mathbb{N}$ be fixed. We set $\mathcal{T}_+ := \{\mathbf{k} + T \,|\, T \in \mathcal{T}, \mathbf{k} \in \mathbb{Z}^d\}$ in the case of $d \geq 2$ and $\mathcal{T}_+ := \{(k, k+1) \,|\, k \in \mathbb{Z}\}$ in the case of $d = 1$. To minimize the notational effort, given a tuple $T = (T(0), \dots, T(d))$ from $\mathcal{T}_+$, we denote the set of nodes $\{T(0), \dots, T(d)\}$ again by T. For an assignment of weights $w : T \to \mathbb{R}$ we define

$$\nabla_T w := \sum_{\substack{\{x,y\} \subseteq T \\ |x-y|_{\text{euc}}=1}} \big(w(x) - w(y)\big)(x - y) \in \mathbb{R}^d.$$

Remark 3.1.5. Let $T \in \mathcal{T}_+$ and w a real-valued function on T. The vector $\nabla_T w$ is the gradient of the unique affine linear function h from $\mathbb{R}^d$ to $\mathbb{R}$ interpolating the $(d+1)$-sized sample $(x, w(x))_{x \in T}$. If $d = 1$, then

$$h(x) = w(T(0))\big(T(1) - x\big) + w(T(1))\big(x - T(0)\big), \quad x \in \mathbb{R},$$

with

$$h' = w(T(1)) - w(T(0)) = \nabla_T w.$$

In case $d \geq 2$ the analogue statement is derived as follows. Due to Remark 3.1.2 (i) the interpolating function h has the representation

$$h(x) = \sum_{i=0}^{d} w(T(i))h_T^i(x), \quad x \in \mathbb{R}^d.$$

Furthermore, by Remark 3.1.2 (ii), it holds

$$\begin{aligned}\nabla h = \sum_{i=0}^{d} w(T(i))\,\nabla h_T^i &= \sum_{k=1}^{d} \big(w(T(k)) - w(T(k-1))\big)\mathbf{e}_{\sigma_T(k)} \\ &= \sum_{k=1}^{d} \big(w(T(k)) - w(T(k-1))\big)\big(T(k) - T(k-1)\big) = \nabla_T w. \quad (3.1.22)\end{aligned}$$

The notational hint at the dimension of the preimage set regarding the tent function $\chi^{(d)}$ can be dropped without ambiguity because it is always clear from the context. So, we prefer to write χ instead of $\chi^{(d)}$ from here on. As discussed above, χ is Lipschitz continuous and therefore weakly differentiable on $\mathbb{R}^d$. Regarding the next theorem we remark that the sum in (3.1.23) below is locally finite, which assures that the assignment yields a locally Lipschitz continuous function. For the case $d = 1$ we complement the notation by setting $\eta_T := \mathbf{1}_{[T(0),T(1))}$ for $T \in \mathcal{T}_+$.

Theorem 3.1.6. *Let $d \in \mathbb{N}$.*

(i) For the weak gradient of the tent function χ it holds

$$(\nabla\chi, \mathbf{e}_i)_{\mathrm{euc}}(x) = \sum_{\substack{T\in\mathcal{T}_+ \\ \{-\mathbf{e}_i,0\}\subseteq T}} \eta_T(x) - \sum_{\substack{T\in\mathcal{T}_+ \\ \{0,\mathbf{e}_i\}\subseteq T}} \eta_T(x) \ \mathrm{d}x\text{-a.e.}$$

(ii) Given weights $w = (w_\mathbf{k})_{\mathbf{k}\in\mathbb{Z}^d}$ the weak gradient ∇_x w.r.t. the variable x of the function

$$\Lambda_\chi w(x) := \sum_{\mathbf{k}\in\mathbb{Z}^d} w_\mathbf{k}\chi(x - \mathbf{k}), \quad x \in \mathbb{R}^d, \tag{3.1.23}$$

is the piecewise constant $\mathbb{R}^d$-valued function class with

$$\nabla_x(\Lambda_\chi w)(x) = \sum_{T\in\mathcal{T}_+} (\nabla_T w|_T)\eta_T(x) \ \mathrm{d}x\text{-a.e.}$$

Proof. (i) For the case $d = 1$ the statement of (i) simplifies to the equation

$$\chi'(x) = \mathbf{1}_{[-1,0)}(x) - \mathbf{1}_{[0,1)}(x) = \eta_{T_1}(x) - \eta_{T_2}(x), \quad x \in \mathbb{R},$$

with $T_1 := (-1, 0)$ and $T_2 := (0, 1)$. As to the case $d \geq 2$, we have

$$\chi(x) = \sum_{i=0}^{d} \sum_{\substack{T\in\mathcal{T}_+ \\ T(i)=0}} \eta_T(x)h_T^i(x), \quad x \in \mathbb{R}^d,$$

by virtue of (3.1.21). If x is a point in the interior of $\mathcal{C}_T$ for some $T \in \mathcal{T}_+$, then there are two possible cases depending on whether $0 \in T$, or $0 \notin T$. In the latter case χ vanishes in a neighbourhood of x. In the former case, with say $0 = T(j)$ for some $j \in \{0, \ldots, d\}$, the tent function χ equals the affine linear function h_T^j in a neighbourhood of x and by Remark 3.1.2 (ii) in turn yields three subcases in which

$$(\nabla\chi, \mathbf{e}_i)_{\text{euc}}(x) = (\nabla h_T^j, \mathbf{e}_i)_{\text{euc}} = \begin{cases} 1 & \text{if } j > 0 \text{ and } \mathbf{e}_{\sigma_T(j)} = \mathbf{e}_i, \\ -1 & \text{if } j < d \text{ and } \mathbf{e}_{\sigma_T(j+1)} = \mathbf{e}_i, \\ 0 & \text{else,} \end{cases}$$
$$= \begin{cases} 1 & \text{if } j > 0 \text{ and } T(j-1) = -\mathbf{e}_i, \\ -1 & \text{if } j < d \text{ and } T(j+1) = \mathbf{e}_i, \\ 0 & \text{else.} \end{cases}$$

For the last representation we used (3.1.5). We note that for general $T \in \mathcal{T}_+$ the condition,

$$\text{there exists } j \in 1, \ldots, d \text{ with } T(j) = 0 \text{ and } T(j-1) = -\mathbf{e}_i,$$

is equivalent to $\{-\mathbf{e}_i, 0\} \subseteq T$, while the condition,

$$\text{there exists } j \in 0, \ldots, d-1 \text{ with } T(j) = 0 \text{ and } T(j+1) = \mathbf{e}_i,$$

is equivalent to $\{0, \mathbf{e}_i\} \subseteq T$. Since

$$dx\big(\mathcal{C}_T \setminus \operatorname{int}(\mathcal{C}_T)\big) = 0, \quad \mathbf{1}_{\operatorname{int}(\mathcal{C}_T)} \leq \eta_T(\cdot) \leq \mathbf{1}_{\mathcal{C}_T}, \quad T \in \mathcal{T}_+,$$

while $\sum_{T \in \mathcal{T}_+} \eta_T = \mathbf{1}_{\mathbb{R}^d}$ we conclude

$$(\nabla\chi, \mathbf{e}_i)_{\text{euc}}(x) = \sum_{\substack{T \in \mathcal{T}_+ \\ \{-\mathbf{e}_i, 0\} \subseteq T}} \eta_T(x) - \sum_{\substack{T \in \mathcal{T}_+ \\ \{0, \mathbf{e}_i\} \subseteq T}} \eta_T(x) \ dx\text{-a.e.}$$

as desired.

(ii) Using the result of (i) we compute

$$\begin{aligned} \nabla_x(\Lambda_\chi w)(x) &= \sum_{i=1}^d \sum_{\mathbf{k} \in \mathbb{Z}^d} w_{\mathbf{k}} \Big(\sum_{\substack{T \in \mathcal{T}_+ \\ \{-\mathbf{e}_i, 0\} \subseteq T}} \eta_T(x - \mathbf{k}) - \sum_{\substack{T \in \mathcal{T}_+ \\ \{0, \mathbf{e}_i\} \subseteq T}} \eta_T(x - \mathbf{k}) \Big) \mathbf{e}_i \\ &= \sum_{i=1}^d \sum_{\mathbf{k} \in \mathbb{Z}^d} w_{\mathbf{k}} \Big(\sum_{\substack{T \in \mathcal{T}_+ \\ \{\mathbf{k} - \mathbf{e}_i, \mathbf{k}\} \subseteq T}} \eta_T(x) - \sum_{\substack{T \in \mathcal{T}_+ \\ \{\mathbf{k}, \mathbf{k} + \mathbf{e}_i\} \subseteq T}} \eta_T(x) \Big) \mathbf{e}_i \\ &= \sum_{i=1}^d \sum_{T \in \mathcal{T}_+} \sum_{\substack{\mathbf{k} \in \mathbb{Z}^d \\ \{\mathbf{k}, \mathbf{k} + \mathbf{e}_i\} \subseteq T}} \eta_T(x)(w_{\mathbf{k} + \mathbf{e}_i} - w_{\mathbf{k}}) \mathbf{e}_i \\ &= \sum_{T \in \mathcal{T}_+} \eta_T(x)\, \nabla_T w|_T, \quad x \in \mathbb{R}^d. \end{aligned}$$

This concludes the proof. □

Remark 3.1.7. In the case $d = 1$ we have

$$\sum_{k \in \mathbb{Z}} \chi(x - k) = \sum_{k \in \mathbb{Z}} \mathbf{1}_{[-1,0)}(x - k)(1 + x - k) + \mathbf{1}_{[0,1)}(x - k)(1 - x + k)$$

$$
\begin{aligned}
&= \sum_{k\in\mathbb{Z}} \mathbf{1}_{[-1,0)}(x-k)(1+x-k) + \mathbf{1}_{[0,1)}(x-k+1)(-x+k) \\
&= \sum_{k\in\mathbb{Z}} \mathbf{1}_{[k-1,k)}(x)(1+x-k) - \mathbf{1}_{[k-1,k)}(x)(-x+k) \\
&= \sum_{k\in\mathbb{Z}} \mathbf{1}_{[k-1,k)}(x) = 1, \quad x \in \mathbb{R}.
\end{aligned}
$$

For $d \in \mathbb{N}$ with $d \geq 2$ analogous steps can be carried out by means of (3.1.21), (3.1.9) and Remark 3.1.4 (ii). Indeed, we have

$$
\begin{aligned}
\sum_{\mathbf{k}\in\mathbb{Z}^d} \chi(x-\mathbf{k}) &= \sum_{\mathbf{k}\in\mathbb{Z}^d} \sum_{i=0}^{d} \sum_{\substack{T\in\mathcal{T}_+\\ T(i)=0}} \eta_T(x-\mathbf{k}) h_T^i(x-\mathbf{k}) \\
&= \sum_{\mathbf{k}\in\mathbb{Z}^d} \sum_{i=0}^{d} \sum_{\substack{T\in\mathcal{T}_+\\ T(i)=\mathbf{k}}} \eta_T(x) h_T^i(x) = \sum_{T\in\mathcal{T}_+} \sum_{i=0}^{d} \eta_T(x) h_T^i(x) \\
&= \sum_{T\in\mathcal{T}_+} \sum_{i=0}^{d} \eta_T(x) h_T^i(x) = \sum_{T\in\mathcal{T}_+} \eta_T(x) = 1, \quad x \in \mathbb{R}^d.
\end{aligned}
$$

3.1.2 Mosco convergence of gradient forms

This section is concerned with Mosco convergence of standard gradient-type Dirichlet forms with varying reference measure, which exactly fits into the framework of Mosco-Kuwae-Shioya. The set-up is a similar one as in [31]. One of the main results of this section, Theorem 3.1.12, imposes no further condition on the limiting Dirichlet form, other than Hamza's condition for closability. In particular, in contrast to [31, Theorems 1.1 and 1.3], no assumptions on the uniqueness of the domain have to be made. Afterwards in Theorem 3.1.13, to provide a version of the man result, which uses milder, 'localized' conditions, we also assume the uniqueness of the limiting form domain in the sense made precise below. The principal tool in this section is the next proposition. It is the link between the previous discussion on Finite Elements and the topic of Mosco convergence. We introduce some notation.

For a real-valued function f on $\mathbb{R}^d$ we set $f_r^{\mathbf{k}}(x) := f(\frac{x}{r} - \mathbf{k})$, $x \in \mathbb{R}^d$, $\mathbf{k} \in \mathbb{Z}^d$, and moreover,

$$
\omega(f, \delta, K) := \sup_{\substack{x\in K, y\in\mathbb{R}^d\\ |x-y|_{\mathrm{euc}}\leq\delta}} \{|f(x) - f(y)|\}, \quad \delta \in (0,\infty),\, K \subseteq \mathbb{R}^d,
$$

as a value in the extended non-negative real line $\mathbb{R}_0^+ \cup \{\infty\}$. The following notions used in Proposition 3.1.8 below refer to the measure space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mathrm{d}x)$. We set $0\cdot\infty = 0$. For a measurable, non-negative function f on $\mathbb{R}^d$ which takes values in $\mathbb{R}_0^+ \cup \{\infty\}$, we define

$$
\|f\|_{\mathcal{L}^1(A)} := \int_A f(x)\,\mathrm{d}x \in \mathbb{R}_0^+ \cup \{\infty\}, \quad A \in \mathcal{B}(\mathbb{R}^d),
$$

and

$$
\|f\|_{\mathcal{L}^\infty(A)} := \operatorname*{ess\,sup}_{x\in A} f(x) \in \mathbb{R}_0^+ \cup \{\infty\}, \quad A \in \mathcal{B}(\mathbb{R}^d).
$$

Let f, g be bounded, measurable functions on $\mathbb{R}^d$ such that there exists $R \in (0,\infty)$ with $f(x) = 0$, $g(x) = 0$ for $|x|_{\text{euc}} > R$. For a locally integrable function h on $\mathbb{R}^d$ we set

$$I_r^{f,g} h(x) := r^{-d} \sum_{\mathbf{k}\in\mathbb{Z}^d} \int_{\mathbb{R}^d} h(y) g_r^{\mathbf{k}}(y)\,\mathrm{d}y\, f_r^{\mathbf{k}}(x), \quad x \in \mathbb{R}^d.$$

For a locally $\mathrm{d}x$-integrable function f on $\mathbb{R}^d$ we set

$$f_{\text{ave},r}(x) := (2r)^{-d} \int_{[-r,r]^d} f(x+y)\,\mathrm{d}y, \quad x \in \mathbb{R}^d.$$

Beyond that, in Proposition 3.1.8, the functions $\xi(x) := \mathbf{1}_{[0,1)^d}(x)$, $x \in \mathbb{R}^d$,

$$d_i^+ \chi(x) := \sum_{\substack{T\in\mathcal{T}_+ \\ \{-\mathbf{e}_i,0\}\subseteq T}} \eta_T(x), \quad x \in \mathbb{R}^d,\ i = 1,\ldots,d,$$

and

$$S_i\xi(x) := \int_0^1 \xi(x - t\,\mathbf{e}_i)\,\mathrm{d}t, \quad x \in \mathbb{R}^d,\ i = 1,\ldots,d,$$

play a significant role. For a Lipschitz continuous function u on $\mathbb{R}^d$ we denote by $\partial_i u$ its weak i-th partial derivative, for $i = 1,\ldots d$, as an element in the space $L^\infty(\mathbb{R}^d)$. Then, for any integrable function ρ on $\mathbb{R}^d$, the class $(\partial_i u)^2\rho$ is an element in the space $L^1(\mathbb{R}^d)$.

Proposition 3.1.8. *Assume that we are given a non-negative, integrable function ρ on $\mathbb{R}^d$ and a Lipschitz continuous function u with $-1 \le u(x) \le 1$, $x \in \mathbb{R}^d$. The inequalities (i) to (iii) hold true for every $r \in (0,\infty)$, every $g \in C_b^1(\mathbb{R}^d)$ and every measurable function κ on $\mathbb{R}^d$ with values in $[0,1]$.*

(i) $$\Big|\int_{\mathbb{R}^d} (u - I_r^{\chi,\xi} u) g\kappa\rho\,\mathrm{d}x\Big| \le \|g\|_\infty \big\|\kappa\rho - I_r^{\xi,\chi}(\kappa\rho)\big\|_{\mathcal{L}^1(\mathbb{R}^d)} + \omega(g, 2r\sqrt{d}, \operatorname{supp}[\kappa\rho]).$$

(ii) $$\forall i = 1,\ldots,d: \quad \Big|\int_{\mathbb{R}^d} \partial_i(u - I_r^{\chi,\xi} u)(\partial_i g)\kappa\rho\,\mathrm{d}x\Big| \le \Big\{\|\partial_i g\|_\infty \big\|1 + 6^d\rho^{-1}(\kappa\rho)_{\text{ave},3r}\big\|^{\frac12}_{\mathcal{L}^\infty(\mathbb{R}^d)} \big\|\kappa\rho - I_r^{(S_i\xi),(d_i^+\chi)}(\kappa\rho)\big\|^{\frac12}_{\mathcal{L}^1(\mathbb{R}^d)} + 6^d\omega(\partial_i g, 4r\sqrt{d}, \operatorname{supp}[\kappa\rho])\big\|\rho^{-1}(\kappa\rho)_{\text{ave},3r}\big\|_{\mathcal{L}^\infty(\mathbb{R}^d)}\Big\} \big\|(\partial_i u)^2\rho\big\|^{\frac12}_{L^1(\mathbb{R}^d)}.$$

(iii) $$\int_{\mathbb{R}^d} |\nabla I_r^{\chi,\xi} u|^2_{\text{euc}}\kappa\rho\,\mathrm{d}x \le 6^d \big\|\rho^{-1}(\kappa\rho)_{\text{ave},3r}\big\|_{\mathcal{L}^\infty(\mathbb{R}^d)} \sum_{i=1}^d \big\|(\partial_i u)^2\rho\big\|_{L^1(\mathbb{R}^d)}.$$

Proof. Let ρ and u be as above and further $r \in (0,\infty)$, $g \in C_b^1(\mathbb{R}^d)$ and $\kappa(x) \in [0,1]$, $x \in \mathbb{R}^d$, be as in the assumptions and fixed throughout this proof. We set $\tilde\rho := \kappa\rho$ and start with an abstract estimate, which is used in the proof of (i) and (ii). Let $R \in (0,\infty)$ and φ, ψ be two bounded, measurable functions from $\mathbb{R}^d$ into $[0,\infty)$ such that $\varphi(x) = 0$, $\psi(x) = 0$ for $|x|_{\text{euc}} > R$. For $h \in C_b(\mathbb{R}^d)$ the inequalities

$$\Big|h(x)\tilde\rho(x) - I_r^{\varphi,\psi}(h\tilde\rho)(x)\Big|$$

$$
\begin{aligned}
&= \Big| h(x)\tilde{\rho}(x) - I_r^{\varphi,\psi}\big(h(x)\tilde{\rho}(\cdot) - h(x)\tilde{\rho}(\cdot) + h(\cdot)\tilde{\rho}(\cdot)\big)(x) \Big| \\
&\le \big| h(x)\tilde{\rho}(x) - h(x) I_r^{\varphi,\psi}\tilde{\rho}(x) \big| \\
&\quad + r^{-d} \sum_{\mathbf{k}\in\mathbb{Z}^d} \int_{\{y\colon |y/r-\mathbf{k}|_{\mathrm{euc}} \le R\}} |h(x) - h(y)| \tilde{\rho}(y) \psi_r^{\mathbf{k}}(y)\,\mathrm{d}y\, \varphi_r^{\mathbf{k}}(x) \\
&\le |h(x)| \big| \tilde{\rho}(x) - I_r^{\varphi,\psi}\tilde{\rho}(x) \big| + \omega(h, 2Rr\,\mathrm{supp}[\tilde{\rho}]) I_r^{\varphi,\psi}(\tilde{\rho})(x) \qquad (3.1.24)
\end{aligned}
$$

are valid for each point $x \in \mathbb{R}^d$. The last estimate holds because $\varphi_r^{\mathbf{k}}(x) = 0$ for $|x/r - \mathbf{k}|_{\mathrm{euc}} > R$. In the following, the proofs of (i) to (iii) are addressed one after another. To shorten the mathematical notation within this proof, we simply write

$$
(f,h)_2 := \int_{\mathbb{R}^d} f(x) h(x)\,\mathrm{d}x
$$

during its course, whenever f and h are (classes of) functions for which the integral on the right-hand side is standardly defined. The verification of (i) starts with an estimate for $I_r^{\xi,\chi}\tilde{\rho}$ in $\mathcal{L}^1(\mathbb{R}^d)$. We apply Tonelli's theorem twice to exchange the infinite sum with the integral and obtain

$$
\begin{aligned}
\int_{\mathbb{R}^d} I_r^{\xi,\chi}|f|\,\mathrm{d}x &= r^{-d} \sum_{\mathbf{k}\in\mathbb{Z}^d} \big(\chi_r^{\mathbf{k}}, |f|\big)_2 \int_{\mathbb{R}^d} \xi_r^{\mathbf{k}}(x)\,\mathrm{d}x \\
&= \Big(\sum_{\mathbf{k}\in\mathbb{Z}^d} \chi_r^{\mathbf{k}}, |f| \Big)_2 \int_{\mathbb{R}^d} \xi(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} |f|\,\mathrm{d}x
\end{aligned}
$$

for an integrable function f on $\mathbb{R}^d$, under use of Remark 3.1.7 and the transformation formula. In particular,

$$
\int_{\mathbb{R}^d} I_r^{\xi,\chi}\tilde{\rho}\,\mathrm{d}x \le 1. \qquad (3.1.25)
$$

Again, by Remark 3.1.7 and the transformation formula we conclude $I_r^{\chi,\xi}|u|(x) \le |u(x)| \le 1$, $x \in \mathbb{R}^d$, and hence $\int_{\mathbb{R}^d} (I_r^{\chi,\xi}|u|)|g|\tilde{\rho}\,\mathrm{d}x < \infty$. With the theorem of Fubini-Tonelli it follows

$$
\begin{aligned}
&\int_{\mathbb{R}^d} (u - I_r^{\chi,\xi}u) g\tilde{\rho}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} ug\tilde{\rho}\,\mathrm{d}x - r^{-d} \int_{\mathbb{R}^d} \sum_{\mathbf{k}\in\mathbb{Z}^d} \big(u, \xi_r^{\mathbf{k}}\big)_2 \chi_r^{\mathbf{k}}(y) g(y) \tilde{\rho}(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^d} ug\tilde{\rho}\,\mathrm{d}x - r^{-d} \int_{\mathbb{R}^d} u(x) \sum_{\mathbf{k}\in\mathbb{Z}^d} \big(\chi_r^{\mathbf{k}}, g\tilde{\rho}\big)_2 \xi_r^{\mathbf{k}}(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} u(x) \big(g(x)\tilde{\rho}(x) - I_r^{\xi,\chi}(g\tilde{\rho})(x) \big)\,\mathrm{d}x. \qquad (3.1.26)
\end{aligned}
$$

To find the desired upper bound of this integral as claimed in (i), we make use of (3.1.24) and also (3.1.25) to end up with

$$
\begin{aligned}
\Big| \int_{\mathbb{R}^d} (u - I_r^{\chi,\xi}u) g\tilde{\rho}\,\mathrm{d}x \Big| &\le \int_{\mathbb{R}^d} |g| \big| \tilde{\rho} - I_r^{\xi,\chi}(\tilde{\rho}) \big| + \omega(g, 2Rr, \mathrm{supp}[\tilde{\rho}]) I_r^{\xi,\chi}(\tilde{\rho})\,\mathrm{d}x \\
&\le \|g\|_\infty \big\| \tilde{\rho} - I_r^{\xi,\chi}\tilde{\rho} \big\|_{\mathcal{L}^1(\mathbb{R}^d)} + \omega(g, 2Rr, \mathrm{supp}[\tilde{\rho}])
\end{aligned}
$$

if we set $R := \sqrt{d}$. With that choice of R we ensure that $[-1,1]^d$ is contained in the centred Euclidean d-ball with radius R and hence $\chi(x) = 0$, $\xi(x) = 0$ for $|x|_{\mathrm{euc}} > R$. This concludes the proof of (i).

We turn to the proof of (ii) and fix $i \in \{1, \dots, d\}$. We start with some preliminary calculations, the first of which is an application of Theorem 3.1.6 (i). With

$$\sum_{\substack{T \in \mathcal{T}_+ \\ \{0, \mathbf{e}_i\} \subseteq T}} \eta_T(x) = \sum_{\substack{T \in \mathcal{T}_+ \\ \{-\mathbf{e}_i, 0\} \subseteq T}} \eta_T(x - \mathbf{e}_i), \quad x \in \mathbb{R}^d,$$

we conclude

$$\begin{aligned}
\big(\nabla \chi_r^{\mathbf{k}}, \mathbf{e}_i\big)_{\mathrm{euc}}(x) &= \frac{1}{r}(\nabla \chi, \mathbf{e}_i)_{\mathrm{euc}}\big(\frac{x}{r} - \mathbf{k}\big) \\
&= \frac{1}{r}\Big(d_i^+ \chi\big(\frac{x}{r} - \mathbf{k}\big) - d_i^+ \chi\big(\frac{x}{r} - \mathbf{k} - \mathbf{e}_i\big)\Big) \\
&= \frac{1}{r}\Big((d_i^+ \chi)_r^{\mathbf{k}}(x) - (d_i^+ \chi)_r^{\mathbf{k}+\mathbf{e}_i}(x)\Big) \; \mathrm{d}x\text{-a.e.}, \quad \mathbf{k} \in \mathbb{Z}^d.
\end{aligned} \tag{3.1.27}$$

The second is a consequence of the translation invariance of the Lebesgue measure and the fundamental theorem of calculus in the Sobolev space $H^{1,\infty}((-r,0))$. Since $H^{1,\infty}((-r,0))$ contains the Lipschitz continuous functions on $[-r,0]$, we can write

$$\begin{aligned}
&\int_{\mathbb{R}^d} u(x)\big(\xi_r^{\mathbf{k}}(x) - \xi_r^{\mathbf{k}-\mathbf{e}_i}(x)\big) \,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} u(x)\big(\mathbf{1}_{[0,r)}(x - r\mathbf{k}) - \mathbf{1}_{[0,r)}(x + r\mathbf{e}_i - r\mathbf{k})\big) \,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} \big(u(x) - u(x - r\mathbf{e}_i)\big)\mathbf{1}_{[0,r)}(x - r\mathbf{k}) \,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} \Big(\int_{-r}^0 \partial_i u(x + t\mathbf{e}_i) \,\mathrm{d}t\Big)\mathbf{1}_{[0,r)}(x - r\mathbf{k}) \,\mathrm{d}x, \quad \mathbf{k} \in \mathbb{Z}^d.
\end{aligned} \tag{3.1.28}$$

Then, considering Fubini's theorem, we continue the above calculation with suitable transformations of integrals to obtain

$$\begin{aligned}
&= \int_{-r}^0 \int_{\mathbb{R}^d} \partial_i u(x + t\mathbf{e}_i)\mathbf{1}_{[0,r)}(x - r\mathbf{k}) \,\mathrm{d}x \,\mathrm{d}t \\
&= \int_{-r}^0 \int_{\mathbb{R}^d} \partial_i u(x)\mathbf{1}_{[0,r)}(x - r\mathbf{k} - t\mathbf{e}_i) \,\mathrm{d}x \,\mathrm{d}t \\
&= \int_{\mathbb{R}^d} \partial_i u(x)\Big(r\int_{-1}^0 \mathbf{1}_{[0,r)}(x - r\mathbf{k} - rs\mathbf{e}_i) \,\mathrm{d}s\Big) \,\mathrm{d}x \\
&= r\int_{\mathbb{R}^d} \partial_i u(x)\Big(\int_{-1}^0 \mathbf{1}_{[0,1)}\big(\frac{x}{r} - \mathbf{k} - s\mathbf{e}_i\big) \,\mathrm{d}s\Big) \,\mathrm{d}x \\
&= r\int_{\mathbb{R}^d} \partial_i u(x)(S_i\xi)_r^{\mathbf{k}}(x) \,\mathrm{d}x, \quad \mathbf{k} \in \mathbb{Z}^d.
\end{aligned} \tag{3.1.29}$$

With (3.1.27), (3.1.28) and (3.1.29) we arrive at

$$\begin{aligned}
\partial_i(I_r^{\chi,\xi} u)(x) &= r^{-d-1} \sum_{\mathbf{k} \in \mathbb{Z}^d} \big(u, \xi_r^{\mathbf{k}}\big)_2\big((d_i^+ \chi)_r^{\mathbf{k}}(x) - (d_i^+ \chi)_r^{\mathbf{k}+\mathbf{e}_i}(x)\big) \\
&= r^{-d-1} \sum_{\mathbf{k} \in \mathbb{Z}^d} \big(u, \xi_r^{\mathbf{k}} - \xi_r^{\mathbf{k}-\mathbf{e}_i}\big)_2 (d_i^+ \chi)_r^{\mathbf{k}}(x)
\end{aligned}$$

$$= I_r^{(d_i^+\chi),(S_i\xi)}(\partial_i u)(x)\ \mathrm{d}x\text{-a.e.} \tag{3.1.30}$$

In a similar way as in (3.1.26) and by taking into account (3.1.30) we find a representation of the relevant integral in claim (ii). Before doing so, we remark that

$$\sum_{k\in\mathbb{Z}^d}(d_i^+\chi)_r^{\mathbf{k}}(x) = \sum_{\mathbf{k}\in\mathbb{Z}^d}\ \sum_{\substack{T\in\mathcal{T}_+\\ \{\mathbf{k}-\mathbf{e}_i,\mathbf{k}\}\subseteq T}} \eta_T\Big(\frac{x}{r}\Big) = 1, \quad x\in\mathbb{R}^d, \tag{3.1.31}$$

and estimate

$$\begin{aligned} I_r^{(d_i^+\chi),(S_i\xi)}|f|(x) &= r^{-d}\sum_{\mathbf{k}\in\mathbb{Z}^d}\big((S_i\xi)_r^{\mathbf{k}},|f|\big)_2(d_i^+\chi)_r^{\mathbf{k}}(x)\\ &\le \|f\|_{L^\infty(\mathbb{R}^d)}\|S_i\xi\|_{\mathcal{L}^1(\mathbb{R}^d)}\sum_{k\in\mathbb{Z}^d}(d_i^+\chi)_r^{\mathbf{k}}(x)\\ &= \|f\|_{L^\infty(\mathbb{R}^d)}, \quad x\in\mathbb{R}^d,\ f\in L^\infty(\mathbb{R}^d), \end{aligned}$$

to argue that

$$\int_{\mathbb{R}^d}\big(I_r^{(d_i^+\chi),(S_i\xi)}|\partial_i u|\big)|\partial_i g|\tilde\rho\,\mathrm{d}x < \infty.$$

With (3.1.30) and the theorem of Fubini-Tonelli it follows

$$\begin{aligned} &\int_{\mathbb{R}^d}\partial_i(u - I_r^{\chi,\xi}u)(\partial_i g)\tilde\rho\,\mathrm{d}x\\ &= \int_{\mathbb{R}^d}(\partial_i u)(\partial_i g)\tilde\rho\,\mathrm{d}x - r^{-d}\int_{\mathbb{R}^d}\partial_i u(x)\sum_{\mathbf{k}\in\mathbb{Z}^d}\big((d_i^+\chi)_r^{\mathbf{k}},(\partial_i g)\tilde\rho\big)_2(S_i\xi)_r^{\mathbf{k}}(x)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^d}\partial_i u\big((\partial_i g)\tilde\rho - I_r^{(S_i\xi),(d_i^+\chi)}((\partial_i g)\tilde\rho)\big)\,\mathrm{d}x. \end{aligned} \tag{3.1.32}$$

Similarly as in the proof of (i) above, (3.1.24) suggests how to find the claimed upper bound for the modulus of the integral in (3.1.32). Before we apply that strategy, we discuss two more estimates, which allow us to derive the upper bound for the claimed inequality as stated in (ii). We remark that

$$(d_i^+\chi)_r^{\mathbf{k}}(x) \le \mathbf{1}_{K(\mathbf{k},r)}(x), \quad (S_i\xi)_r^{\mathbf{k}}(x) \le \mathbf{1}_{K'(\mathbf{k},r)}(x), \quad x\in\mathbb{R}^d,$$

with

$$K(\mathbf{k},r) := \big\{x\in\mathbb{R}^d \,\big|\, r\mathbf{k}_i - r \le x_i \le r\mathbf{k}_i \text{ and } r\mathbf{k}_j - r \le x_j \le r\mathbf{k}_j + r,\ j\ne i\big\}$$

and

$$K'(\mathbf{k},r) := \big\{x\in\mathbb{R}^d \,\big|\, r\mathbf{k}_i \le x_i \le r\mathbf{k}_i + 2r \text{ and } r\mathbf{k}_j \le x_j \le r\mathbf{k}_j + r,\ j\ne i\big\}$$

for $\mathbf{k}\in\mathbb{Z}^d$. These observations together with $\sum_{\mathbf{k}\in\mathbb{Z}^d}(S_i\xi)_r^{\mathbf{k}}(x) = 1$ yield the first estimate

$$\begin{aligned} I_r^{(S_i\xi),(d_i^+\chi)}\tilde\rho(x) &= r^{-d}\sum_{\mathbf{k}\in\mathbb{Z}^d}\int_{\mathbb{R}^d}(d_i^+\chi)_r^{\mathbf{k}}(y)\tilde\rho(y)\,\mathrm{d}y(S_i\xi)_r^{\mathbf{k}}(x)\\ &\le r^{-d}\sum_{\substack{\mathbf{k}\in\mathbb{Z}^d:\\ x\in K'(\mathbf{k},r)}}\int_{\{y\in K(\mathbf{k},r)\}}\tilde\rho(y)\,\mathrm{d}y(S_i\xi)_r^{\mathbf{k}}(x) \end{aligned}$$

$$
\begin{aligned}
&\le r^{-d} \sum_{\mathbf{k}\in\mathbb{Z}^d} \int_{[-3r,3r]^d} \tilde\rho(x+y)\,\mathrm{d}y (S_i\xi)_r^{\mathbf{k}}(x) \\
&\le 6^d \tilde\rho_{\mathrm{ave},3r}(x) \sum_{\mathbf{k}\in\mathbb{Z}^d} (S_i\xi)_r^{\mathbf{k}}(x) = 6^d \tilde\rho_{\mathrm{ave},3r}(x), \quad x\in\mathbb{R}^d. \qquad (3.1.33)
\end{aligned}
$$

Moreover, (3.1.33) further results in a second estimate,

$$
\begin{aligned}
&\rho^{-1}\big(\tilde\rho - I_r^{(S_i\xi),(d_i^+\chi)}\tilde\rho\big)^2 \\
&= \rho^{-1}\Big(\tilde\rho\big(\tilde\rho - I_r^{(S_i\xi),(d_i^+\chi)}\tilde\rho\big) + \big(I_r^{(S_i\xi),(d_i^+\chi)}\tilde\rho\big)\big(I_r^{(S_i\xi),(d_i^+\chi)}\tilde\rho - \tilde\rho\big)\Big) \\
&\le \big|\tilde\rho - I_r^{(S_i\xi),(d_i^+\chi)}\tilde\rho\big| + 6^d\rho^{-1}\tilde\rho_{\mathrm{ave},3r}\big|I_r^{(S_i\xi),(d_i^+\chi)}\tilde\rho - \tilde\rho\big|. \qquad (3.1.34)
\end{aligned}
$$

Now, we apply (3.1.32), (3.1.24) and then Cauchy's inequality together with (3.1.33) and (3.1.34) to infer

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^d} \partial_i(u - I_r^{\chi,\xi}u)(\partial_i g)\tilde\rho\,\mathrm{d}x\Big| \\
&\le \int_{\mathbb{R}^d} |\partial_i u|\Big(|\partial_i g|\big|\tilde\rho - I_r^{(S_i\xi),(d_i^+\chi)}(\tilde\rho)\big| + \omega(\partial_i g, 4r\sqrt{d}, \mathrm{supp}[\tilde\rho]) I_r^{(S_i\xi),(d_i^+\chi)}(\tilde\rho)\Big)\,\mathrm{d}x \\
&\le \|(\partial_i u)^2\rho\|_{L^1(\mathbb{R}^d)}^{\frac12}\Big\{\|\partial_i g\|_\infty \big\|(1+6^d\rho^{-1}\tilde\rho_{\mathrm{ave},3r})\big(I_r^{(S_i\xi),(d_i^+\chi)}\tilde\rho - \tilde\rho\big)\big\|_{\mathcal{L}^1(\mathbb{R}^d)}^{\frac12} \\
&\qquad + 6^d\omega(g, 4r\sqrt{d}, \mathrm{supp}[\tilde\rho])\|\rho^{-1}\tilde\rho_{\mathrm{ave},3r}\big\|_{\mathcal{L}^\infty(\mathbb{R}^d)}\Big\}.
\end{aligned}
$$

This concludes the proof of (ii).

We approach the missing proof of (iii). Again, we fix $i\in\{1,\dots,d\}$. By (3.1.31) it holds

$$
\sum_{\mathbf{k}\in\mathbb{Z}^d}\int_{\mathbb{R}^d}(S_i\xi)_r^{\mathbf{k}}(y)\,\mathrm{d}y(d_i^+\chi)_r^{\mathbf{k}}(x) = \sum_{\mathbf{k}\in\mathbb{Z}^d}(d_i^+\chi)_r^{\mathbf{k}}(x) = 1, \quad x\in\mathbb{R}^d.
$$

Hence, $I_r^{(d_i^+\chi),(S_i\xi)}\mathbf{1}_A(x)$, $A\in\mathcal{B}(\mathbb{R}^d)$, defines a probability measure for each $x\in\mathbb{R}^d$. With (3.1.30), Jensen's inequality, the theorem of Fubini-Tonelli and (3.1.33) we conclude

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\big(\partial_i(I_r^{\chi,\xi}u)(x)\big)^2\tilde\rho(x)\,\mathrm{d}x = \int_{\mathbb{R}^d}\big(I_r^{(d_i^+\chi),(S_i\xi)}(\partial_i u)(x)\big)^2\tilde\rho(x)\,\mathrm{d}x \\
&\le \int_{\mathbb{R}^d} I_r^{(d_i^+\chi),(S_i\xi)}\big((\partial_i u)^2\big)(x)\tilde\rho(x)\,\mathrm{d}x = \int_{\mathbb{R}^d}(\partial_i u(x))^2 I_r^{(S_i\xi),(d_i^+\chi)}\tilde\rho(x)\,\mathrm{d}x \\
&\le 6^d\big\|\rho^{-1}\tilde\rho_{\mathrm{ave},3r}\big\|_{\mathcal{L}^\infty(\mathbb{R}^d)}\big\|(\partial_i u)^2\rho\big\|_{L^1(\mathbb{R}^d)}.
\end{aligned}
$$

The statement of (iii) follows by summing up over $i=1,\dots,d$. This concludes the proof. □

We give a useful estimate related to Proposition 3.1.8.

Remark 3.1.9. Let φ, ψ be two bounded, measurable functions from $\mathbb{R}^d$ into $[0,1]$ such that $\varphi(x)=0$, $\psi(x)=0$ for $x\in\mathbb{R}^d\setminus[-2,2]^d$,

$$
\int_{\mathbb{R}^d}\psi(x)\,\mathrm{d}x = 1 \quad \text{and} \quad \sum_{\mathbf{k}\in\mathbb{Z}^d}\varphi(x-\mathbf{k}) = 1, \quad x\in\mathbb{R}^d.
$$

Then, for a locally integrable function f on $\mathbb{R}^d$ and $r \in (0, \infty)$ it holds

$$\begin{aligned}\left|f(x) - I_r^{\varphi,\psi} f(x)\right| &\le \sum_{\mathbf{k}\in\mathbb{Z}^d} \left|f(x) - r^{-d}\int_{\mathbb{R}^d} f(y)\psi_r^{\mathbf{k}}(y)\,\mathrm{d}y\right|\varphi_r^{\mathbf{k}}(x) \\ &\le r^{-d}\sum_{\mathbf{k}\in\mathbb{Z}^d}\left|\int_{\mathbb{R}^d}(f(x)-f(y))\psi_r^{\mathbf{k}}(y)\,\mathrm{d}y\right|\varphi_r^{\mathbf{k}}(x) \\ &\le r^{-d}\sum_{\mathbf{k}\in\mathbb{Z}^d}\int_{\substack{\{y\in\mathbb{R}^d\,|\\ x-y\in[-4r,4r]^d\}}} |f(x)-f(y)|\,\mathrm{d}y\varphi_r^{\mathbf{k}}(x) \\ &\le 8^d(|f(x)-f(\cdot)|)_{\mathrm{ave},4r}(x), \quad x\in\mathbb{R}^d.\end{aligned}$$

In particular,

$$\left|f(x) - I_r^{\chi,\xi} f(x)\right| \le 8^d(|f(x)-f(\cdot)|)_{\mathrm{ave},4r}(x), \quad x\in\mathbb{R}^d,$$

and for every $i = 1, \ldots, d$ it holds

$$\left|f(x) - I_r^{(S_i\xi),(d_i^+\chi)} f(x)\right| \le 8^d(|f(x)-f(\cdot)|)_{\mathrm{ave},4r}(x), \quad x\in\mathbb{R}^d.$$

The starting point for our discussion about symmetric closed forms of gradient-type and their convergence is a density function ρ on $\mathbb{R}^d$ of a certain type specified below. Its name traces back to [5] and it is famous for providing the closability for such gradient-type forms as are considered below. In the context of a closable symmetric bilinear form on a Hilbert space, we use the term 'closure' or 'smallest closed extension' as they are defined in [38, Chapter 1] or [20, Chapter I]. This text focusses on convergence results for such symmetric forms, whose domain is a dense subspace of the respective Hilbert space.

Condition 3.1.10 (Hamza's condition)**.** Let ρ be a non-negative, measurable function on $\mathbb{R}^d$ and

$$R(\rho) := \left\{x_0 \in \mathbb{R}^d \,\middle|\, \text{there exists } U \subseteq \mathbb{R}^d \text{ open: } x_0 \in U,\, \rho^{-1}|_U \in \mathcal{L}^1(U, \mathrm{d}x)\right\}.$$

If $\rho(x) = 0$, $\mathrm{d}x$-a.e. on $\mathbb{R}^d \setminus R(\varrho)$, then ρ is said to meet Hamza's condition.

Basically, Hamza's condition yields continuity of the restriction map $u|_K$, $u \in L^2(\mathbb{R}^d, \rho\,\mathrm{d}x)$, seen as a map into $L^1(\mathbb{R}^d, \mathrm{d}x)$, for every compact set K contained in $R(\rho)$. This results in a natural continuous embedding

$$L^2(\mathbb{R}^d, \rho\,\mathrm{d}x) \hookrightarrow L^1_{\mathrm{loc}}(R(\rho), \mathrm{d}x). \tag{3.1.35}$$

Under Hamza's condition, the partial derivative ∂_i is a well-defined map from $L^2(\mathbb{R}^d, \rho\,\mathrm{d}x) \cap \widetilde{C}^1(\mathbb{R}^d)$ into $L^2(\mathbb{R}^d, \rho\,\mathrm{d}x) \cap \widetilde{C}(\mathbb{R}^d)$ for $i = 1, \ldots, d$. Hence, there is a gradient operating on classes of functions, ∇u for $u \in L^2(\mathbb{R}^d, \rho\,\mathrm{d}x) \cap \widetilde{C}^1(\mathbb{R}^d)$. The assignment

$$\mathcal{E}^\rho(u, v) := \int_{\mathbb{R}^d} (\nabla u, \nabla v)^2_{\mathrm{euc}}\,\rho\,\mathrm{d}x, \quad u, v \in L^2(\mathbb{R}^d, \rho\,\mathrm{d}x) \cap \widetilde{C}^1_b(\mathbb{R}^d), \tag{3.1.36}$$

yields a densely defined, symmetric bilinear form on $L^2(\mathbb{R}^d, \rho\,\mathrm{d}x)$, which is closable. We denote the domain of its closure on $L^2(\mathbb{R}^d, \rho\,\mathrm{d}x)$ by $\mathcal{D}(\mathcal{E}^\rho)$. The action of $\mathcal{E}^\rho$ extends onto $\mathcal{D}(\mathcal{E}^\rho)$ and $(\mathcal{E}^\rho, \mathcal{D}(\mathcal{E}^\rho))$ defines a Dirichlet form on $L^2(\mathbb{R}^d, \rho\,\mathrm{d}x)$. The latter means that

$$(0 \vee u) \wedge 1 =: v \in \mathcal{D}(\mathcal{E}^\rho), \quad \mathcal{E}^\rho(v,v) \le \mathcal{E}^\rho(u,u), \quad \text{for every } u \in \mathcal{D}(\mathcal{E}^\rho). \tag{3.1.37}$$

In the line above, the operations $\vee$ and $\wedge$ denote the maximum, respectively the minimum, taken in the sense of $\rho\,\mathrm{d}x$-classes of measurable functions. We say that the unit contraction operates on $(\mathcal{E}^\rho, \mathcal{D}(\mathcal{E}^\rho))$ if (3.1.37) holds. Since convergence in $L^2(\mathbb{R}^d, \rho\,\mathrm{d}x)$ implies the convergence in $L^1_{\mathrm{loc}}(R(\rho), \mathrm{d}x)$, we have a natural embedding

$$\big(\mathcal{D}(\mathcal{E}^\rho), \mathcal{E}_1^{\frac{1}{2}}(\cdot,\cdot)\big) \hookrightarrow H^{1,1}_{\mathrm{loc}}(R(\rho), \mathrm{d}x) \tag{3.1.38}$$

and we have a closed extension of $(\mathcal{E}^\rho, \mathcal{D}(\mathcal{E}^\rho))$ through the assignment

$$\mathcal{E}^\rho(u,v) = \int_{\mathbb{R}^d} (\nabla u, \nabla v)^2_{\mathrm{euc}}\, \rho\,\mathrm{d}x, \quad u, v \in \mathcal{D}_{\max}(\mathcal{E}^\rho),$$

with

$$\mathcal{D}_{\max}(\mathcal{E}^\rho) := \Big\{ u \in L^2(\mathbb{R}^d, \rho\,\mathrm{d}x) \cap H^{1,1}_{\mathrm{loc}}(R(\rho), \mathrm{d}x) \;\Big|\; \int_{\mathbb{R}^d} |\nabla u|^2_{\mathrm{euc}}\, \rho\,\mathrm{d}x < \infty \Big\}.$$

The integrals above, of course, have to be read in the sense of an a.e.-defined integrand w.r.t. the corresponding measure $\rho\,\mathrm{d}x$, and we write '$u \in H^{1,1}_{\mathrm{loc}}(R(\rho), \mathrm{d}x)$' for $u \in L^2(\mathbb{R}^d, \rho\,\mathrm{d}x)$, if the image of u under the embedding (3.1.35) is a member of $H^{1,1}_{\mathrm{loc}}(R(\rho), \mathrm{d}x)$.

Remark 3.1.11. (i) Let ρ be $\mathrm{d}x$-integrable over $\mathbb{R}^d$. For every $f \in \mathrm{Lip}_b(\mathbb{R}^d)$, the space of bounded, Lipschitz continuous function on $\mathbb{R}^d$, there exists a sequence $(f_n)_{N\in\mathbb{N}} \in C_b^\infty(\mathbb{R}^d)$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for $x \in \mathbb{R}^d$, while

$$\sup_{n\in\mathbb{N}} \|f_n\|_\infty \le \|f\|_\infty \quad \text{and} \quad \sup_{n\in\mathbb{N}} \|\partial_i f_n\|_\infty \le L_f, \quad i = 1, \dots, d.$$

L_f denotes the Lipschitz constant of f on $\mathbb{R}^d$. For example, the sequence $(f_n)_{n\in\mathbb{N}}$ can easily be constructed, using standard mollifying techniques, as the convolution $f_n := f * \varphi_n$, where $(\varphi_n)_{n\in\mathbb{N}}$ is an approximate identity. The application [20, Lemma 2.12 of Chapter I] in combination of Lebesgue's dominated convergence (to show convergence in $L^2(\mathbb{R}^d, \rho\,\mathrm{d}x)$) is a standard argument with two helpful implications. For one thing, $\mathcal{D}(\mathcal{E}^\rho)$ is the minimal domain of a closed symmetric form extending the action of $\mathcal{E}^\rho$ on the pre-domain $L^2(\mathbb{R}^d, \rho\,\mathrm{d}x) \cap \widetilde{C_b^\infty}(\mathbb{R}^d)$. Secondly, the class $L^2(\mathbb{R}^d, \rho\,\mathrm{d}x) \cap \widetilde{\mathrm{Lip}_b}(\mathbb{R}^d)$ is contained in $\mathcal{D}(\mathcal{E}^\rho)$.

(ii) In the case $d = 1$, some authors prefer the alternative description of $\mathcal{D}_{\max}(\mathcal{E}^\rho)$ written as

$$\Big\{ u \in L^2(\mathbb{R}, \rho\,\mathrm{d}x) \;\Big|\; \text{there is an absolutely continuous function } f \text{ on } R(\rho) \text{ with } f = u,\ \mathrm{d}x\text{-a.e. on } R(\rho),\ \int_{R(\rho)} (f')^2 \rho\,\mathrm{d}x < \infty \Big\}.$$

Both representations are equal, as one for example can check with [11, Section 1.1.2].

The formulation of the next theorem uses the term of a 1-capacity associated with $(\mathcal{E}^\rho, \mathcal{D}(\mathcal{E}^\rho))$. It refers to the definition from [38, Chapter 2], where it is shown, alongside other potential theoretic properties of Dirichlet forms, that the assignment

$$\mathrm{Cap}_1(U) := \inf\left(\left\{\mathcal{E}_1^\rho(u,u) \,\middle|\, u \in \mathcal{D}(\mathcal{E}^\rho),\ u \geq 1 \text{ holds } \rho\,\mathrm{d}x\text{-a.e. on } U\right\}\right)$$

for an open set $U \subseteq \mathbb{R}^d$ (with $\inf(\emptyset) := \infty$), and

$$\mathrm{Cap}_1(A) := \inf_{\substack{U \subseteq \mathbb{R}^d \text{ open} \\ A \subseteq U}} \mathrm{Cap}_1(U)$$

for an arbitrary subset A of $\mathbb{R}^d$, yields a Choquet capacity on $\mathbb{R}^d$.

Now, let ρ and ρ_N, for each $N \in \mathbb{N}$, be probability densities on $\mathbb{R}^d$ which fulfil Hamza's condition 3.1.10. We assume the weak convergence of measures on $\mathbb{R}^d$, regarding the sequence $(\rho_N\,\mathrm{d}x)_{N\in\mathbb{N}}$ with limit $\rho\,\mathrm{d}x$, as well as the inclusion $\mathrm{supp}[\rho_N\,\mathrm{d}x] \subseteq \mathrm{supp}[\rho\,\mathrm{d}x]$, $N \in \mathbb{N}$, concerning their topological support. We use the conventions introduced in the end of Section 2.1.

Theorem 3.1.12. *If there is a sequence $(\Omega_m)_{m\in\mathbb{N}}$ of relatively compact, open sets in $\mathbb{R}^d$ with $\mathrm{cl}(\Omega_m) \subset \Omega_{m+1}$, $m \in \mathbb{N}$, such that the 1-capacity associated with $(\mathcal{E}^\rho, \mathcal{D}(\mathcal{E}^\rho))$ of the set $\mathbb{R}^d \setminus \Omega_m$ converges to zero as m tends to infinity and moreover*

$$\forall m \in \mathbb{N}: \quad \limsup_{k\to\infty} \sup_{N\in\mathbb{N}} \left\| \rho_N^{-1} (\rho_N)_{\mathrm{ave},\frac{1}{k}} \right\|_{\mathcal{L}^\infty(\Omega_m)} < \infty \tag{3.1.39}$$

$$\textit{as well as} \quad \lim_{k\to 0} \limsup_{N\to\infty} \int_{\Omega_m} \left(|\rho_N(x) - \rho_N(\cdot)|\right)_{\mathrm{ave},\frac{1}{k}}(x)\,\mathrm{d}x = 0, \tag{3.1.40}$$

then $(\mathcal{E}^\rho, \mathcal{D}(\mathcal{E}^\rho))$ is the Mosco limit of the sequence $(\mathcal{E}^{\rho_N}, \mathcal{D}(\mathcal{E}^{\rho_N}))_{N\in\mathbb{N}}$.

Proof. We start with the proof of property (M1). Let $m \in \mathbb{N}$ be fixed in the first part of this proof. There exists a continuously differentiable function κ_m on $\mathbb{R}^d$ with

$$\mathbf{1}_{\mathrm{cl}(\Omega_m)}(x) \leq \kappa_m(x) \leq \mathbf{1}_{\Omega_{m+1}}(x), \quad x \in \mathbb{R}^d.$$

The densities $\kappa_m\rho$ and $\kappa_m\rho_N$, for $N \in \mathbb{N}$, fulfil the Hamza condition 3.1.10 and moreover $\kappa_m\rho\,\mathrm{d}x$ is the limit of the sequence $(\kappa_m\rho_N\,\mathrm{d}x)_{N\in\mathbb{N}}$ in the sense of weak measure convergence on $\mathbb{R}^d$. Referring to the structure of the disjoint union

$$\mathcal{H}_m := L^2(\mathbb{R}^d, \kappa_m\rho\,\mathrm{d}x) \sqcup \Big(\bigsqcup_{N\in\mathbb{N}} L^2(\mathbb{R}^d, \kappa_m\rho_N\,\mathrm{d}x) \Big)$$

we have the strong convergence of $|\nabla\Phi_N^{\sim} u|_{\mathrm{euc}} = \Phi_N^{\sim}|\nabla u|_{\mathrm{euc}}$ towards $|\nabla u|_{\mathrm{euc}}$ as N tends to infinity, for each $u \in L^2(\mathbb{R}^d, \kappa_m\rho\,\mathrm{d}x) \cap \widetilde{C_b^1}(\mathbb{R}^d)$. This conjures up a situation as in (2.2.16) of Section 2.2.2, where in particular, $(\mathcal{D}(\mathcal{E}^{\kappa_m\rho_N}), \mathcal{E}_\alpha^{\kappa_m,\rho_N})$, $N \in \mathbb{N}$, form a sequence of converging Hilbert spaces on their own right with asymptotic space $(\mathcal{D}(\mathcal{E}^{\kappa_m\rho}), \mathcal{E}_\alpha^{\kappa_m\rho})$ for $\alpha \in (0,\infty)$. We set

$$\mathcal{H}_m^{\mathcal{E},\alpha} := (\mathcal{D}(\mathcal{E}^{\kappa_m\rho}), \mathcal{E}_\alpha^{\kappa_m\rho}) \sqcup \Big(\bigsqcup_{N\in\mathbb{N}} (\mathcal{D}(\mathcal{E}^{\kappa_m\rho_N}), \mathcal{E}_\alpha^{\kappa_m\rho_N}) \Big), \quad \alpha \in (0,\infty).$$

We discuss certain compatible classes in $\mathcal{H}_m^{\mathcal{E},1}$. The first observation concerns the assignment $\Lambda_\chi : w \mapsto \Lambda_\chi w$ as in Theorem 3.1.6, which maps a lattice weight $w \in \mathbb{R}^{\mathbb{Z}^d}$ into the space of locally bounded, locally Lipschitz continuous functions on $\mathbb{R}^d$. We set $\Lambda_{\chi,r}w(x) := \Lambda_\chi w(x/r)$ for $x \in \mathbb{R}^d$, $w \in \mathbb{R}^{\mathbb{Z}^d}$, $r \in (0,\infty)$. Furthermore, we define a finite-dimensional subspace of the bounded Lipschitz continuous functions on $\mathbb{R}^d$ by

$$V_{m,r} := \Lambda_{\chi,r}\big(\{w \in \mathbb{R}^{\mathbb{Z}^d} \,\big|\, w_{\mathbf{k}} = 0 \text{ for } \mathbf{k} \in \mathbb{Z}^d,\ r\mathbf{k} \notin \Omega_{m+2}\}\big), \quad r \in (0,\infty). \tag{3.1.41}$$

We fix $r \in (0,\infty)$ and an element $u \in \mathcal{D}(\mathcal{E}^{\kappa_m\rho}) \cap \widetilde{V}_{m,r}$, say

$$u(x) = \sum_{\mathbf{k}\in\mathbb{Z}^d} w_{\mathbf{k}}\chi\big(\frac{x}{r} - \mathbf{k}\big) \quad \kappa_m\rho\,\mathrm{d}x\text{-a.e.}$$

for some $w \in \mathbb{R}^{\mathbb{Z}^d}$, $w_{\mathbf{k}} = 0$ if $r\mathbf{k} \notin \Omega_{m+2}$. The weak gradient of u is calculated with Theorem 3.1.6 and the chain rule. Since η_T is the indicator function of a set which has negligible boundary w.r.t. the Lebesgue measure for $T \in \mathcal{T}_+$, we conclude

$$\begin{aligned}
\mathcal{E}^{\kappa_m\rho}(u,u) &= \frac{1}{r^2}\sum_{T\in\mathcal{T}_+} \big|\,\nabla_T w|_T\big|^2_{\mathrm{euc}} \int_{\mathbb{R}^d} \eta_T\big(\frac{x}{r}\big)\rho(x)\kappa_m(x)\,\mathrm{d}x \\
&= \lim_{N\to\infty} \frac{1}{r^2}\sum_{T\in\mathcal{T}_+} \big|\,\nabla_T w|_T\big|^2_{\mathrm{euc}} \int_{\mathbb{R}^d} \eta_T\big(\frac{x}{r}\big)\rho_N(x)\kappa_m(x)\,\mathrm{d}x \\
&= \lim_{N\to\infty} \mathcal{E}^{\kappa_m\rho_N}(\Phi_N^{\sim}u, \Phi_N^{\sim}u).
\end{aligned} \tag{3.1.42}$$

by virtue of the Portmanteau Theorem. Due to (3.1.42) and the fact that $V_{m,r}$ is a finite-dimensional vector space for fixed $r \in (0,\infty)$ we find ourselves in the situation of Remark 2.2.7. So,

$$\mathbf{C}_m^0 := \bigcup_{r\in(0,\infty)} \prod_{N\in\mathbb{N}} \mathcal{D}(\mathcal{E}^{\kappa_m\rho_N}) \cap \widetilde{V}_{m,r}$$

is a compatible class in $\mathcal{H}_m^{\mathcal{E},1}$. The next step in the discussion of compatible classes requires some preliminary observations. First,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \big(|\kappa_m(x)\rho_N(x) - \kappa_m(\cdot)\rho_N(\cdot)|\big)_{\mathrm{ave,r}}(x)\,\mathrm{d}x \\
&\le \int_{\mathbb{R}^d} \big(\kappa_m(x)|\rho_N(x) - \rho_N(\cdot)|\big)_{\mathrm{ave,r}}(x)\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^d} \big(|\kappa_m(x) - \kappa_m(\cdot)|\rho_N(\cdot)\big)_{\mathrm{ave,r}}(x)\,\mathrm{d}x \\
&\le \int_{\Omega_{m+1}} \big(|\rho_N(x) - \rho_N(\cdot)|\big)_{\mathrm{ave,r}}(x)\,\mathrm{d}x + \omega(\kappa_m, r\sqrt{d}, \mathrm{cl}(\Omega_{m+1}))
\end{aligned} \tag{3.1.43}$$

for $r \in (0,\infty)$ and $N \in \mathbb{N}$. The next observation refers to the case where $r > 0$ is small enough such that

$$\big\{x \in \mathbb{R}^d \,\big|\, x - y \in [-r,r]^d \text{ for some } y \in \mathrm{cl}(\Omega_{m+1})\big\} \subseteq \Omega_{m+2}.$$

If so, then for $N \in \mathbb{N}$ it holds

$$\begin{aligned}
\big\|\rho_N^{-1}(\kappa_m\rho_N)_{\mathrm{ave,r}}\big\|_{\mathcal{L}^\infty(\mathbb{R}^d)} &\le \big\|\rho_N^{-1}(\mathbf{1}_{\Omega_{m+1}}\rho_N)_{\mathrm{ave,r}}\big\|_{\mathcal{L}^\infty(\mathbb{R}^d)} \\
&\le \big\|\rho_N^{-1}(\rho_N)_{\mathrm{ave,r}}\big\|_{\mathcal{L}^\infty(\Omega_{m+2})}
\end{aligned} \tag{3.1.44}$$

and moreover

$$\mathcal{D}(\mathcal{E}^{\kappa_m\rho_N}) \cap \widetilde{V}_{m,r} = \mathcal{D}(\mathcal{E}^{\kappa_m\rho_N}) \cap \widetilde{\Lambda_{\chi,r}}(\mathbb{R}^{\mathbb{Z}^d}).$$

Let

$$F_N := \Big\{ f \in \mathrm{Lip}(\mathbb{R}^d) \,\Big|\, |f(x)| \le 1,\, x \in \mathbb{R}^d, \text{ and } \int_{\mathbb{R}^d} |\nabla f|^2_{\mathrm{euc}}\, \rho_N \,\mathrm{d}x \le 2 \Big\}, \quad N \in \mathbb{N},$$

and further

$$\mathbf{C}_m^{\mathrm{Lip}} := \prod_{N\in\mathbb{N}} \mathcal{D}(\mathcal{E}^{\kappa_m \rho_N}) \cap \widetilde{F_N}.$$

We want to apply Proposition 2.2.8 to show that $\mathbf{C}_m^{\mathrm{Lip}}$ is a compatible class in $\mathcal{H}_m^{\mathcal{E},1}$. To this end, let $(u_N)_{N\in\mathbb{N}} \in \mathbf{C}_m^{\mathrm{Lip}}$ and $\tilde{u}_N$ be a representative of u_N in F_N for $N \in \mathbb{N}$. We denote the $\kappa_m\rho_N \,\mathrm{d}x$-class of $I_r^{\chi,\xi}\tilde{u}_N$ in by $u_N^{(r)}$ for $N \in \mathbb{N}$ and $r \in (0,\infty)$. For each $v \in L^2(\mathbb{R}^d, \kappa_m \rho \,\mathrm{d}x) \cap \widetilde{C_b^1}(\mathbb{R}^d)$ there exists a zero sequence $(r_k)_{k\in\mathbb{N}}$ of positive numbers such that

$$\limsup_{N\to\infty} \big|\mathcal{E}^{\kappa_m\rho_N}\big(u_N - u_N^{(r_k)}, \widetilde{\Phi_N v}\big)\big| + \Big| \int_{\mathbb{R}^d} \big(u_N - u_N^{(r_k)}\big)(\widetilde{\Phi_N v})\kappa_m \rho_N \,\mathrm{d}x \Big|$$

converges to zero for $k \to \infty$, due to (3.1.43), (3.1.44), the assumptions of this theorem, together with Proposition 3.1.8 (i) and (ii) and Remark 3.1.9. Moreover, by (3.1.44) and Proposition 3.1.8 (iii) we may additionally assume

$$\sup_{k\in\mathbb{N}} \sup_{N\in\mathbb{N}} \mathcal{E}_1^{\kappa_m\rho_N}\big(u_N^{(r_k)}, u_N^{(r_k)}\big) < \infty.$$

Hence, the assumptions of Proposition 2.2.8 are fulfilled by the choices $V := L^2(\mathbb{R}^d, \kappa_m \rho \,\mathrm{d}x) \cap \widetilde{C_b^1}(\mathbb{R}^d)$, $\mathbf{C} := \mathbf{C}_m^0$ and

$$A := \bigsqcup_{N\in\mathbb{N}} \mathcal{D}(\mathcal{E}^{\kappa_m\rho_N}) \cap \widetilde{F_N}$$

proving that $\mathbf{C}_m^{\mathrm{Lip}}$ is indeed a compatible class in $\mathcal{H}_m^{\mathcal{E},1}$. We look for yet another compatible class in $\mathcal{H}_m^{\mathcal{E},1}$ which is large enough to conclude the proof of (M1). Let $N \in \mathbb{N}$. Each element $u \in \mathcal{D}(\mathcal{E}^{\rho_N})$ can be approximated in the Hilbert space $(\mathcal{D}(\mathcal{E}^{\rho_N}), \mathcal{E}_1^{\rho_N})$ by a sequence in $L^2(\rho_N \,\mathrm{d}x) \cap \widetilde{C_b^1}(\mathbb{R}^d)$ by construction. Since $(\mathcal{E}^{\rho_N}, \mathcal{D}(\mathcal{E}^{\rho_N}))$ is a Dirichlet form, each element $u \in \mathcal{D}(\mathcal{E}^{\rho_N})$ with $|u(x)| \le 1$, $\rho_N \,\mathrm{d}x$-a.e., can be approximated weakly in the Hilbert space $(\mathcal{D}(\mathcal{E}^{\rho_N}), \mathcal{E}_1^{\rho_N})$ by a sequence in

$$\big\{ -1 \vee v \wedge 1 \,\big|\, v \in L^2(\rho_N \,\mathrm{d}x) \cap \widetilde{C_b^1}(\mathbb{R}^d) \big\}$$

applying [20, Lemma 2.12 of Chapter I]. In particular, the fact that $\mathbf{C}_m^{\mathrm{Lip}}$ is a compatible class implies that also

$$\mathbf{C}_m := \prod_{N\in\mathbb{N}} \Big\{ u \in \mathcal{D}(\mathcal{E}^{\kappa_m\rho_N}) \,\Big|\, \text{there exists } v \in \mathcal{D}(\mathcal{E}^{\rho_N}) \text{ such that } \mathcal{E}^{\rho_N}(v,v) \le 1,\ |v(x)| \le 1 \text{ holds } \rho_N \,\mathrm{d}x\text{-a.e., and } u(x) = v(x) \text{ holds } \kappa_m\rho_N \,\mathrm{d}x\text{-a.e.} \Big\}$$

is a compatible class in $\mathcal{H}_m^{\mathcal{E},1}$, again by Proposition 2.2.8. The class of $\mathbf{C}_m$ serves our purpose. Closing our discussion on compatible classes we turn towards the actual verification of (M1).

The strategy is to make use of Lemma 2.2.3. To check that the condition specified in Lemma 2.2.3 is satisfied with $\mathcal{C} := L^2(\mathbb{R}^d, \rho\, \mathrm{d}x) \cap \widetilde{C_b}(E)$, it suffices to show the following claim: We fix $\varphi \in \mathcal{C}$ and set $u_N := G_1^N \Phi_N^{\sim} \varphi$ for $N \in \mathbb{N}$. Possibly after rescaling φ with the factor $1/\|\varphi\|_\infty$, the sub-Markovianity of the resolvent operator G_1^N associated with $\mathcal{E}^{\rho_N}$ yields

$$(u_N)_{N\in\mathbb{N}} \in \prod_{N\in\mathbb{N}} \big\{ v \in \mathcal{D}(\mathcal{E}^{\rho_N}) \,\big|\, |v(x)| \le 1,\ \rho_N\, \mathrm{d}x\text{-a.e., and } \mathcal{E}^{\rho_N}(v,v) \le 1 \big\}.$$

We claim that any weakly converging subsequence $(u_{N_k})_{k\in\mathbb{N}}$ with limit u^*, referring to the topological structure of

$$\mathcal{H} := L^2(\mathbb{R}^d, \rho\, \mathrm{d}x) \sqcup \Big(\bigsqcup_{N\in\mathbb{N}} L^2(\mathbb{R}^d, \rho_N\, \mathrm{d}x) \Big)$$

now, fulfils

$$u^* \in \mathcal{D}(\mathcal{E}^\rho) \quad \text{with} \quad \mathcal{E}^\rho(u^*, u^*) \le \liminf_{k\to\infty} \mathcal{E}^{\rho_{N_k}}(u_{N_k}, u_{N_k}). \tag{3.1.45}$$

Let $m \in \mathbb{N}$. In the following, we denote the class in $L^2(\mathbb{R}^d, \kappa_m \rho_N)$ which coincides with u_N in $\kappa_m \rho_N\, \mathrm{d}x$-a.e.-sense again by u_N. Analogously, we denote the element in $L^2(\mathbb{R}^d, \kappa_m \rho)$ which coincides with u^* in $\kappa_m \rho\, \mathrm{d}x$-a.e.-sense again by u^*. If a subsequence $(u_{N_k})_{k\in\mathbb{N}}$ is given as above, then $(u_{N_k})_{k\in\mathbb{N}}$ converges weakly towards u^* in $\mathcal{H}_m$ for every $m \in \mathbb{N}$. Moreover, $(u_N)_{N\in\mathbb{N}}$ is a member of $\mathbf{C}_m$. Since every sequence in

$$\prod_{N\in\mathbb{N}} \big\{ v \in \mathcal{D}(\mathcal{E}^{\kappa_m \rho_N}) \,\big|\, \mathcal{E}_1^{\kappa_m \rho_N}(v,v) \le 2 \big\}$$

has a weakly convergent subsequence in $\mathcal{H}_m^{\mathcal{E},1}$ and $\mathbf{C}_m$ is a compatible class in $\mathcal{H}_m^{\mathcal{E},1}$, for every $m \in \mathbb{N}$, we can achieve the following, by repeatedly dropping to a suitable subsequence: For every subsequence of $(u_N)_{N\in\mathbb{N}}$ converging weakly to u^* there exists a (sub-)subsequence such that

$$u_{N_k} \overset{k\to\infty}{\rightharpoonup} u^* \text{ in } \mathcal{H} \quad \text{and} \quad u_{N_k} \overset{k\to\infty}{\rightharpoonup} u^* \text{ in } \mathcal{H}_m^{\mathcal{E},1}, \quad m \in \mathbb{N}. \tag{3.1.46}$$

This includes the statement $u^* \in \mathcal{D}(\mathcal{E}^{\kappa_m \rho})$ for each $m \in \mathbb{N}$. As a consequence of (3.1.46) and the weak convergence of $(u_{N_k})_{k\in\mathbb{N}}$ towards u^* in $\mathcal{H}_m$ we obtain the weak convergence of $(u_{N_k})_{k\in\mathbb{N}}$ towards u^* in $\mathcal{H}_m^{\mathcal{E},\alpha}$ for every $\alpha > 0$ and $m \in \mathbb{N}$. In particular,

$$\begin{aligned} \mathcal{E}_\alpha^{\kappa_m \rho}(u^*, u^*) &\le \liminf_{k\to\infty} \mathcal{E}_\alpha^{\kappa_m \rho_{N_k}}\big(u_{N_k}, u_{N_k}\big) \\ &\le \liminf_{k\to\infty} \mathcal{E}_\alpha^{\rho_{N_k}}(u_{N_k}, u_{N_k}), \quad \alpha > 0,\ m \in \mathbb{N}. \end{aligned} \tag{3.1.47}$$

Next, we settle an issue with the domains. Let $m \in \mathbb{N}$. From $u^* \in \mathcal{D}(\mathcal{E}^{\kappa_{m+1}\rho})$ we can infer $\kappa_m u^* \in \mathcal{D}(\mathcal{E}^\rho)$ as follows. Indeed, if $(\varphi_l)_{l\in\mathbb{N}}$ is a sequence in $C_b^1(\mathbb{R}^d)$ such that

$$\lim_{l\to\infty} \int_{\mathbb{R}^d} \big|\varphi_l - u^*\big|^2 \kappa_{m+1} \rho\, \mathrm{d}x = 0, \quad \lim_{l_0\to\infty} \sup_{l\ge l_0} \int_{\mathbb{R}^d} \big|\, \nabla(\varphi_l - \varphi_{l_0})\big|^2 \kappa_{m+1} \rho\, \mathrm{d}x = 0,$$

then, with $\kappa_{m+1}(x) = 1$, $x \in \mathrm{supp}[\kappa_m]$, and the chain rule, it follows

$$\lim_{l\to\infty} \int_{\mathbb{R}^d} \big|\kappa_m \varphi_l - \kappa_m u^*\big|^2 \rho\, \mathrm{d}x = 0 \quad \text{and}$$

$$\lim_{l_0\to\infty}\sup_{l\geq l_0}\int_{\mathbb{R}^d}\big|\nabla(\kappa_m\varphi_l-\kappa_m\varphi_{l_0})\big|^2\rho\,\mathrm{d}x=0,$$

proving $\kappa_m u^*\in\mathcal{D}(\mathcal{E}^\rho)$. Since u^* coincides with $\kappa_m u^*$ on Ω_m, and by assumption $(\mathrm{cl}(\Omega_m))_{m\in\mathbb{N}}$ is a nest w.r.t. the Dirichlet form $(\mathcal{E}^\rho,\mathcal{D}(\mathcal{E}^\rho))$, the fact that $u^*\in\mathcal{D}(\mathcal{E}^\rho)$ follows from the uniqueness of Silverstein's extension for $(\mathcal{E}^\rho,\mathcal{D}(\mathcal{E}^\rho))$, as provided by [26, Theorem 6.2] (not exclusively). By letting α tend to zero and then m tend to infinity in (3.1.47), we obtain the inequality of (3.1.45) as desired. Only (M2) is left to show.

The verification of (M2) is analogous to the beginning of this proof, where the convergence of the Hilbert spaces $(\mathcal{D}(\mathcal{E}^{\kappa_m\rho_N}),\mathcal{E}_1^{\kappa_m\rho_N})$, $N\in\mathbb{N}$, has been considered. Now, referring to the structure of $\mathcal{H}$ and defining $(\Phi_N^\sim)_{N\in\mathbb{N}}$ accordingly as in the end of Section 2.1, we have the strong convergence of $|\nabla\Phi_N^\sim u|_{\mathrm{euc}}=\Phi_N^\sim|\nabla u|_{\mathrm{euc}}$ towards $|\nabla u|_{\mathrm{euc}}$ as N tends to infinity for each $u\in L^2(\rho\,\mathrm{d}x)\cap\widetilde{C_b^1}(\mathbb{R}^d)$. Thus, we have a situation, where the condition of (2.2.16) from Section 2.2.2 holds regarding the forms $(\mathcal{E}^{\rho_N},\mathcal{D}(\mathcal{E}^{\rho_N}))$ on $L^2(\mathbb{R}^d,\rho_N\,\mathrm{d}x)$, $N\in\mathbb{N}$, together with $(\mathcal{E}^\rho,\mathcal{D}(\mathcal{E}^\rho))$ on $L^2(\mathbb{R}^d,\rho\,\mathrm{d}x)$. Remark 2.2.5 then provides property (M2). This concludes the proof. □

A localization of the conditions in Theorem 3.1.12 would be desirable, as a plain example can explain. We consider the Hamza density $\rho=\rho_N=\mathbf{1}_{(0,1)^d}$, $N\in\mathbb{N}$, on $\mathbb{R}^d$. The corresponding form $(\mathcal{E}^\rho,\mathcal{D}(\mathcal{E}^\rho))$ defined as above for that choice of ρ is the standard energy form of the Sobolev space with $\mathcal{D}(\mathcal{E}^\rho)=H^{1,2}((0,1)^d)$. Even in that simple case the conditions of Theorem 3.1.12 are violated, because the 1-capacity of the boundary A of $(0,1)^d$ is strictly positive. In other words, $H_0^{1,2}((0,1)^d)\neq H^{1,2}((0,1)^d)$. It is therefore not possible to choose Ω_m in the assumptions of Theorem 3.1.12 such that $\Omega_m\subset\mathbb{R}^d\setminus A$ holds for all $m\in\mathbb{N}$. On the other hand, if $x\in A$ and $x\in\Omega$ for some open set $\Omega\subseteq\mathbb{R}^d$, then obviously $\|\rho^{-1}(\rho)_{\mathrm{ave},r}\|_{\mathcal{L}^\infty(\Omega)}=\infty$ for every $r>0$ due to $\rho_{\mathrm{ave},r}(x)>0$. In such situations, one way out can be $\mathcal{D}_{\max}(\mathcal{E}^\rho)=\mathcal{D}(\mathcal{E}^\rho)$, as certainly holds true in the given example, because this means the property '$u\in\mathcal{D}(\mathcal{E}^\rho)$' can be checked locally. The vast part of the proof of Theorem 3.1.12, including the discussion on compatible classes, works independently from the assumption that the 1-capacities of $(\mathbb{R}^d\setminus\Omega_m)_{m\in\mathbb{N}}$ converge to zero. All steps to obtain the estimate of (3.1.47) are valid under the condition that $(\Omega_m)_{m\in\mathbb{N}}$ is a sequence of relatively compact, open sets in $\mathbb{R}^d$, such that for $m\in\mathbb{N}$: $\mathrm{cl}(\Omega_m)\subset\Omega_{m+1}$, (3.1.39) and (3.1.40) hold, while κ_m is continuous with

$$\mathbf{1}_{\mathrm{cl}(\Omega_m)}(x)\leq\kappa_m(x)\leq\mathbf{1}_{\Omega_{m+1}}(x),\quad x\in\mathbb{R}^d.$$

Theorem 3.1.13. *If*

$$\mathcal{D}(\mathcal{E}^\rho)=\mathcal{D}_{\max}(\mathcal{E}^\rho)$$

and moreover for all $x_0\in R_\rho$ there exists an open neighbourhood Ω of x_0, such that

$$\lim_{k\to\infty}\sup_{N\to\infty}\big\|\rho_N^{-1}(\rho_N)_{\mathrm{ave},\frac{1}{k}}\big\|_{\mathcal{L}^\infty(\Omega)}<\infty \tag{3.1.48}$$

$$\text{and}\quad\lim_{k\to\infty}\limsup_{N\to\infty}\int_\Omega\big(|\rho_N(x)-\rho_N(\cdot)|\big)_{\mathrm{ave},\frac{1}{k}}(x)\,\mathrm{d}x=0, \tag{3.1.49}$$

then $(\mathcal{E}^\rho,\mathcal{D}(\mathcal{E}^\rho))$ is the Mosco limit of the sequence $(\mathcal{E}^{\rho_N},\mathcal{D}(\mathcal{E}^{\rho_N}))$, $N\in\mathbb{N}$.

Proof. Defining

$$\Omega_m:=\Big\{x\in R(\rho)\cap(-m,m)^d\,\Big|\,y\in R(\rho)\text{ holds, if }y\in\mathbb{R}^d,|y-x|_{\mathrm{euc}}<\frac{1}{m}\Big\}$$

for $m \in \mathbb{N}$ yields a sequence of relatively compact, open sets in $\mathbb{R}^d$, such that $\mathrm{cl}(\Omega_m) \subset \Omega_{m+1} \subset R(\rho)$, $m \in \mathbb{N}$, and

$$R(\rho) = \bigcup_{m\in\mathbb{N}} \Omega_m.$$

For fixed $m \in \mathbb{N}$ there is a finite open cover $\{U_i \,|\, i = 1, \dots, l_m\}$, with $l_m \in \mathbb{N}$, for the compact set $\mathrm{cl}(\Omega_m)$, such that (3.1.48) and (3.1.49) from above hold with $\Omega = U_i$ for $i = 1, \dots, l_m$. Since l_m depends neither on k, nor on N, obviously

$$\text{for all } m \in \mathbb{N} \quad \text{it holds} \quad \lim_{k\to\infty} \sup_{N\to\infty} \big\| \rho_N^{-1} (\rho_N)_{\mathrm{ave},\frac{1}{k}} \big\|_{\mathcal{L}^\infty(\Omega_m)} < \infty$$
$$\text{and} \quad \lim_{k\to\infty} \limsup_{N\to\infty} \int_{\Omega_m} \big(|\rho_N(x) - \rho_N(\cdot)| \big)_{\mathrm{ave},\frac{1}{k}}(x)\,\mathrm{d}x = 0.$$

Addressing property (M1) now, we proceed exactly as in the proof of Theorem 3.1.12 until we reach to (3.1.47). For $m \in \mathbb{N}$, κ_m denotes a continuous function with

$$\mathbf{1}_{\mathrm{cl}(\Omega_m)}(x) \le \kappa_m(x) \le \mathbf{1}_{\Omega_{m+1}}(x), \quad x \in \mathbb{R}^d.$$

If we can deduce

$$u^* \in \mathcal{D}(\mathcal{E}^\rho) \quad \text{with} \quad \mathcal{E}^\rho(u^*, u^*) \le \liminf_{k\to\infty} \mathcal{E}^{\rho_{N_k}}(u_{N_k}, u_{N_k}) \tag{3.1.50}$$

from the fact that u^* is the weak limit in $\mathcal{H}$ of $(u_{N_k})_{k\in\mathbb{N}}$, where $u_{N_k} \in \mathcal{D}(\mathcal{E}^{\rho_{N_k}})$, while assuming $u^* \in \mathcal{D}(\mathcal{E}^{\kappa_m\rho})$ with

$$\begin{aligned} \mathcal{E}_\alpha^{\kappa_m\rho}(u^*, u^*) &\le \liminf_{k\to\infty} \mathcal{E}_\alpha^{\kappa_m\rho_{N_k}}\big(u_{N_k}, u_{N_k}\big) \\ &\le \liminf_{k\to\infty} \mathcal{E}_\alpha^{\rho_{N_k}}(u_{N_k}, u_{N_k}), \quad \alpha > 0,\, m \in \mathbb{N}, \end{aligned} \tag{3.1.51}$$

then (M1) is shown. This task has been dealt with in the proof of Theorem 3.1.12, as well. Here, however, there is an even faster argument to show this claim, due to the additional condition on the form domains. The conclusion $u^* \in H^{1,1}_{\mathrm{loc}}(R(\rho), \mathrm{d}x)$ is immediate from $u^* \in \mathcal{D}(\mathcal{E}^{\kappa_m\rho})$, $m \in \mathbb{N}$, because each compact set K with $K \subset R(\rho)$ is contained in Ω_{m_K} for a suitable choice $m_K \in \mathbb{N}$ and hence $K \subset R(\kappa_{m_K}\rho)$. Furthermore, by monotone convergence we have

$$\int_{R(\rho)} |\nabla u^*|^2_{\mathrm{euc}}\, \rho \,\mathrm{d}x = \sup_{m\in\mathbb{N}} \int_{R(\rho)} |\nabla u^*|^2_{\mathrm{euc}}\, \kappa_m \rho \,\mathrm{d}x = \sup_{m\in\mathbb{N}} \mathcal{E}^{\kappa_m\rho}(u^*, u^*).$$

Letting α tend to zero in (3.1.51) we conclude the proof of (M1). The verification of property (M2) runs completely analogous as in the proof of Theorem 3.1.12, as the sets $(\Omega_m)_{m\in\mathbb{N}}$ play no role there. This concludes the proof. □

3.2 Modifications

3.2.1 Perturbation with densities

There are cases, in which (3.1.39), (3.1.40), (3.1.48) and (3.1.49) from the conditions of Theorem 3.1.12 and Theorem 3.1.13 are particularly easy to verify. One may think of a sequence of probability densities $(\rho_N)_{N\in\mathbb{N}}$ constituting an equi-continuous family of strictly positive functions on $\mathbb{R}^d$, for example. In other cases, the situation may be more obscure. The motivation behind the subsequent analysis can be explained as follows. If one finds a factorization $\rho_N = \tilde{\rho}_N \rho_N^\circ$ for each $N \in \mathbb{N}$ in such a way, that the conditions of Theorem 3.1.12 and Theorem 3.1.13 are known to hold regarding the sequence $(\rho_N^\circ)_{N\in\mathbb{N}}$, then ρ_N can be thought of as a perturbation of ρ_N° with distorting density $\tilde{\rho}_N$. Sometimes, the quantity, by which we can decide whether $(\tilde{\rho}_N)_{N\in\mathbb{N}}$ is admissible as a sequence of perturbing densities, becomes surprisingly simple, if the functions $\tilde{\rho}_N$ belong to a particular class. This is the case in Lemma 3.2.1 below. It considers the class of real-valued functions on $\mathbb{R}^d$ which are monotone on each line, parallel to the coordinate axis. The only other property, the sequence of perturbing densities has to meet, is a uniform bound of the supremum norm, whose local existence even suffices regarding (3.1.39), (3.1.40), (3.1.48) and (3.1.49) of Theorem 3.1.12 and Theorem 3.1.13. Our interest in the class of functions with the named monotonicity property is invoked by a generalization of a problem for $d = 1$. We ask the following. Let $(\rho_N^\circ)_{N\in\mathbb{N}}$ be a sequence of probability densities, which meet the above mentioned conditions (maybe even $\rho_N^\circ = 1$, $N \in \mathbb{N}$), and $\tilde{\rho}_N = \exp(-f_N)$, where f_N is a function of bounded variation on $\mathbb{R}$, for $N \in \mathbb{N}$. What conditions for $(f_N)_{N\in\mathbb{N}}$ make a perturbation result for $(\tilde{\rho}_N \rho_N^\circ)_{N\in\mathbb{N}}$ available? Writing $f_N = f_N^+ - f_N^-$ as the difference of two bounded, monotone increasing functions we get $\tilde{\rho}_N = \exp(-f_N^+)\exp(f_N^-)$. Hence, we may assume the monotonicity when answering that question. As the supremum norm of f_N^+, respectively f_N^-, from the decomposition above are bounded by the total variation of f_N, the essential condition on $(f_N)_{N\in\mathbb{N}}$ states that the total variation norms of that sequence are bounded. The merit of a perturbation result can also be seen from the other point of view. If we successfully validated the conditions of Theorem 3.1.12 and Theorem 3.1.13 for the densities $(\rho_N)_{N\in\mathbb{N}}$ we are interested in, then we get the statements of those theorems for the whole family $\{(\tilde{\rho}_N \rho_N)_N\}$ of all certified perturbations 'for free'.

Below, $\mathrm{dist}_{\max}(A, B)$ denotes the distance of two subsets $A, B \subseteq \mathbb{R}^d$ w.r.t. the maximum norm $|x|_{\max} := \max_{i=1,\dots,d} |x_i|$, $x \in \mathbb{R}^d$, i.e.

$$\mathrm{dist}_{\max}(A, B) := \inf \big\{ |x - y|_{\max} \,\big|\, x \in A,\, y \in B \big\}.$$

Lemma 3.2.1. *Let f be a non-negative, $\mathrm{d}x$-integrable function on $\mathbb{R}^d$, $\Omega \in \mathcal{B}(\mathbb{R}^d)$ with non-empty interior and $r \in (0, \infty)$ such that*

$$\delta := \int_\Omega (|f(x) - f(\cdot)|)_{\mathrm{ave},2r}(x)\,\mathrm{d}x < \infty.$$

If a measurable, non-negative and bounded function g on Ω satisfies one of the monotonicity properties

$$g(x_1, \dots, x_d) \le g(y_1, \dots, y_d) \quad \text{for} \quad x, y \in \Omega : x_1 \le y_1, \dots, x_d \le y_d,$$

or

$$g(x_1, \dots, x_d) \ge g(y_1, \dots, y_d) \quad \text{for} \quad x, y \in \Omega : x_1 \le y_1, \dots, x_d \le y_d,$$

then

$$\int_{\Omega'} \big(\big|(gf)(x) - (gf)(\cdot)\big|\big)_{\mathrm{ave},r}(x)\,\mathrm{d}x \le \big(1 + 2^{d+1}\big)\delta \sup_{x\in\Omega} g(x)$$

for every measurable set $\Omega' \subseteq \Omega$ with $\mathrm{dist}_{\max}(\Omega', \mathbb{R}^d \setminus \Omega) > 3r$.

Proof. We focus on the assumption

$$g(x_1, \dots, x_d) \le g(y_1, \dots, y_d) \quad \text{for} \quad x, y \in \Omega : x_1 \le y_1, \dots, x_d \le y_d,$$

because the other case works analogous. Let $r \in (0,\infty)$ and sets $\Omega' \subseteq \Omega \subset \mathbb{R}^d$ be fixed as above with $\mathrm{dist}_{\max}(\Omega', \mathbb{R}^d \setminus \Omega) > 3r$. Using the estimate

$$|g(x)f(x) - g(y)f(y)| \leq g(x)|f(x) - f(y)| + f(y)|g(x) - g(y)|, \quad x, y \in \Omega.$$

we obtain

$$\begin{aligned}
&\int_{\Omega'} \big(\big|(gf)(x) - (gf)(\cdot)\big|\big)_{\mathrm{ave},r}(x)\,\mathrm{d}x \\
&\quad\leq \int_{\Omega'} g(x)\big(\big|f(x) - f(\cdot)\big|\big)_{\mathrm{ave},r}(x)\,\mathrm{d}x + \int_{\Omega'} \big(f(\cdot)\big|g(x) - g(\cdot)\big|\big)_{\mathrm{ave},r}(x)\,\mathrm{d}x. \qquad (3.2.1)
\end{aligned}$$

We define

$$\Omega'_r := \big\{x \in \mathbb{R}^d \,\big|\, \mathrm{dist}_{\max}(x, \Omega') < r\big\}.$$

and $\tilde{f} := \mathbf{1}_{\Omega'_r} f$. To bound the second summand on the right-hand side of (3.2.1) we estimate

$$\begin{aligned}
&\int_{\Omega'}\int_{[-r,r]^d} f(x+y)|g(x) - g(x+y)|\,\mathrm{d}y\,\mathrm{d}x \\
&\leq \int_{\Omega'} (g(x + r\mathbf{e}) - g(x - r\mathbf{e})) \int_{[-r,r]^d} f(x+y)\,\mathrm{d}y\,\mathrm{d}x \\
&\leq \int_{\mathbb{R}^d} (g(x + r\mathbf{e}) - g(x - r\mathbf{e})) \int_{[-r,r]^d} \tilde{f}(x+y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} g(x) \int_{[-r,r]^d} \tilde{f}(x - r\mathbf{e} + y) - \tilde{f}(x + r\mathbf{e} + y)\,\mathrm{d}y\,\mathrm{d}x \\
&\leq \int_{\mathbb{R}^d} g(x) \int_{[-r,r]^d} \big|\tilde{f}(x - r\mathbf{e} + y) - \tilde{f}(x)\big| \\
&\qquad\qquad + \big|\tilde{f}(x) - \tilde{f}(x + r\mathbf{e} + y)\big|\,\mathrm{d}y\,\mathrm{d}x. \qquad (3.2.2)
\end{aligned}$$

We observe that by $\mathrm{dist}_{\max}(\Omega'_r, \mathbb{R} \setminus \Omega)\} \geq 2r$ it holds

$$\int_{[-2r,0]^d} \big|\tilde{f}(x+y) - \tilde{f}(x)\big|\,\mathrm{d}y = \int_{[0,2r]^d} \big|\tilde{f}(x+y) - \tilde{f}(x)\big|\,\mathrm{d}y = 0,$$

for $x \in \mathbb{R}^d \setminus \Omega$. To bound the right-hand side of (3.2.2) by the value $2\sup_{z\in\Omega} g(z)$ we calculate

$$\begin{aligned}
&\int_{\mathbb{R}^d} g(x) \int_{[-r,r]^d} \big|\tilde{f}(x - r\mathbf{e} + y) - \tilde{f}(x)\big|\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} g(x) \int_{[-2r,0]^d} \big|\tilde{f}(x+y) - \tilde{f}(x)\big|\,\mathrm{d}y\,\mathrm{d}x \\
&\leq \int_{\Omega} g(x) \int_{[-2r,0]^d} \big|f(x+y) - f(x)\big|\,\mathrm{d}y\,\mathrm{d}x \\
&\leq \sup_{z\in\Omega} g(z) \int_{\Omega}\int_{[-2r,2r]^d} \big|f(x+y) - f(x)\big|\,\mathrm{d}y\,\mathrm{d}x
\end{aligned}$$

and analogously

$$\int_{\mathbb{R}^d} g(x) \int_{[-r,r]^d} \big|\tilde{f}(x) - \tilde{f}(x + r\mathbf{e} + y)\big|\,\mathrm{d}y\,\mathrm{d}x$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^d} g(x) \int_{[0,2r]^d} \big|\tilde{f}(x+y) - \tilde{f}(x)\big| \,\mathrm{d}y \,\mathrm{d}x \\
&\le \int_{\Omega} g(x) \int_{[0,2r]^d} \big|f(x+y) - f(x)\big| \,\mathrm{d}y \,\mathrm{d}x \\
&\le \sup_{z\in\Omega} g(z) \int_{\Omega} \int_{[-2r,2r]^d} \big|f(x+y) - f(x)\big| \,\mathrm{d}y \,\mathrm{d}x.
\end{aligned}
$$

Now, the claim of the Lemma follows from (3.2.1) and (3.2.2), since

$$
\int_{\Omega} \int_{[-2r,2r]^d} \big|f(x+y) - f(x)\big| \,\mathrm{d}y \,\mathrm{d}x \le 2^d \delta.
$$

□

The question we have ignored so far in our discussion on perturbations concerns the weak measure convergence. Whether or not the weak convergence of measures $\rho_N^\circ \,\mathrm{d}x$ towards $\rho^\circ \,\mathrm{d}x$ for N to infinity implies the same property for a perturbation $\tilde{\rho}_N \rho_N^\circ \,\mathrm{d}x$, $N \in \mathbb{N}$, in relation to $\tilde{\rho}\rho^\circ \,\mathrm{d}x$, is unclear - even more so, if the perturbing densities do not represent continuous functions. One strategy to obtain the desired convergence emerges from Lemma 2.1.3, by re-interpreting the problem as a question of weak convergence within the frame of converging Hilbert space $L^2(\rho_N^\circ)$, $N \in \mathbb{N}$, with asymptotic space $L^2(\rho^\circ)$. For that matter, Lemma 3.2.3 below can provide the required majorante and minorante for a sequence of monotone functions. We consider the case $d = 1$. For two bounded, monotone increasing functions f, g on the real line let

$$
\begin{aligned}
d^*(f,g) := \inf \Big\{ \delta > 0 \,\Big|\, \forall x \in \mathbb{R} : \quad & f(x+\delta) + \delta \ge g(x) \\
& \wedge\, g(x+\delta) + \delta \ge f(x) \Big\}.
\end{aligned}
$$

Remark 3.2.2. We have

$$
f(x+\delta) + \delta \ge g(x) \ge f(x-\delta) - \delta \quad \text{for all } x \in \mathbb{R} \text{ and } \delta > d^*(f,g)
$$

for two bounded, monotone increasing functions f, g on $\mathbb{R}$ and hence

$$
\big[f(x-\delta), f(x+\delta)\big] \cap \big[g(x) - \delta, g(x) + \delta\big] \ne \emptyset, \quad x \in \mathbb{R},\, \delta > d^*(f,g).
$$

Lemma 3.2.3. *Let f and f_n, $n \in \mathbb{N}$, be bounded, monotone increasing functions on $\mathbb{R}$ with $\lim_{n\to\infty} d^*(f, f_n) = 0$.*

(i) For $\varepsilon > 0$ there exist bounded, continuous functions, $\tilde{f}^{\min,\varepsilon}$ and $\tilde{f}_n^{\min,\varepsilon}$ for $n \in \mathbb{N}$, such that

$$
\begin{aligned}
&\inf_{y\in\mathbb{R}} f_n(y) \le \tilde{f}_n^{\min,\varepsilon}(x) \le f_n(x) \quad \textit{for } x \in \mathbb{R},\, n \in \mathbb{N}, \\
&\big|\tilde{f}^{\min,\varepsilon}(x) - f(x)\big| \le f(x) - f(x-\varepsilon) \quad \textit{for } x \in \mathbb{R}
\end{aligned}
$$

and

$$
\lim_{n\to\infty} \big\|\tilde{f}_n^{\min,\varepsilon} - \tilde{f}^{\min,\varepsilon}\big\|_\infty = 0.
$$

(ii) Similarly, for $\varepsilon > 0$ there exist bounded, continuous functions, $\tilde{f}^{\mathrm{maj},\varepsilon}$ and $\tilde{f}_n^{\mathrm{maj},\varepsilon}$ for $n \in \mathbb{N}$, such that

$$
f_n(x) \le \tilde{f}_n^{\mathrm{maj},\varepsilon}(x) \le \sup_{y\in\mathbb{R}} f_n(y) \quad \textit{for } x \in \mathbb{R},\, n \in \mathbb{N},
$$

$$\left|\tilde{f}^{\mathrm{maj},\varepsilon}(x) - f(x)\right| \le f(x+\varepsilon) - f(x) \quad \textit{for } x \in \mathbb{R}$$

and

$$\lim_{n\to\infty} \left\|\tilde{f}_n^{\mathrm{maj},\varepsilon} - \tilde{f}^{\mathrm{maj},\varepsilon}\right\|_\infty = 0.$$

Proof. Items (i) and (ii) are treated simultaneously. Let $\varepsilon > 0$ be fixed throughout this proof and $\tilde{\varepsilon} := \varepsilon/3$. At first, candidates for the functions $\tilde{f}^{\mathrm{min},\varepsilon}$, $\tilde{f}^{\mathrm{maj},\varepsilon}$, $\tilde{f}_n^{\mathrm{min},\varepsilon}$ and $\tilde{f}_n^{\mathrm{maj},\varepsilon}$ are defined. Then, it is easy to see and is shown by elementary calculations that the candidates possess the stated properties indeed. Let $(\varphi_k)_{k\in\mathbb{Z}}$ be a partition of unity on subordinate to the cover $(I_k)_{k\in\mathbb{Z}}$ with $I_k := (\tilde{\varepsilon}k - \tilde{\varepsilon}, \tilde{\varepsilon}k + \tilde{\varepsilon})$ and $\varphi_k \in C_{\mathrm{comp}}(I_k)$ for $k \in \mathbb{Z}$. There exists $N \in \mathbb{N}$ such that $\delta_n := 2d^*(f, f_n) < \tilde{\varepsilon}$ for all $n \in \mathbb{N}$, $n \ge N$. We choose

$$y_n(k) \in \big[f_n(\tilde{\varepsilon}k - \delta_n), f_n(\tilde{\varepsilon}k + \delta_n)\big] \cap \big[f(\tilde{\varepsilon}k) - \delta_n, f(\tilde{\varepsilon}k) + \delta_n\big], \quad k \in \mathbb{Z},\, n \ge N,$$

and define

$$\tilde{f}^{\mathrm{min},\varepsilon}(x) := \sum_{k\in\mathbb{Z}} f(\tilde{\varepsilon}k)\varphi_{k+2}(x), \quad \tilde{f}^{\mathrm{maj},\varepsilon}(x) := \sum_{k\in\mathbb{Z}} f(\tilde{\varepsilon}k)\varphi_{k-2}(x), \quad x \in \mathbb{R},$$

as well as

$$\begin{aligned} \tilde{f}_n^{\mathrm{min},\varepsilon}(x) &:= \sum_{k\in\mathbb{Z}} y_n(k)\varphi_{k+2}(x), \\ \tilde{f}_n^{\mathrm{maj},\varepsilon}(x) &:= \sum_{k\in\mathbb{Z}} y_n(k)\varphi_{k-2}(x), \quad x \in \mathbb{R},\, n \in \mathbb{N},\, n \ge N. \end{aligned}$$

Although the statement of this lemma bears relevancy only for the asymptotic of $n \to \infty$, we want to define

$$\begin{aligned} \tilde{f}_n^{\mathrm{min},\varepsilon}(x) &:= \sum_{k\in\mathbb{Z}} f_n(\tilde{\varepsilon}k)\varphi_{k+2}(x), \\ \tilde{f}_n^{\mathrm{maj},\varepsilon}(x) &:= \sum_{k\in\mathbb{Z}} f_n(\tilde{\varepsilon}k)\varphi_{k-2}(x), \quad x \in \mathbb{R},\, n \in \mathbb{N},\, n < N, \end{aligned}$$

for the sake of completeness. The rest of the proof is a verification of the desired properties by simple estimates, which shall be displayed here only for the case of (i), as (ii) is analogous. Let $n \in \mathbb{N}$. We have $\inf_{y\in\mathbb{R}} f_n(y) \le \tilde{f}_n^{\mathrm{min},\varepsilon}(x)$, $x \in \mathbb{R}$ by construction. Moreover, if $x \in \mathbb{R}$ and $k \in \mathbb{Z}$ such that $x \in [\tilde{\varepsilon}k, \tilde{\varepsilon}k + \tilde{\varepsilon})$, then $\varphi_k(x) + \varphi_{k+1}(x) = 1$ and monotonicity yields

$$\begin{aligned} f_n^{\mathrm{min},\varepsilon}(x) &\le \sum_{l\in\mathbb{Z}} f_n(\tilde{\varepsilon}l + \tilde{\varepsilon})\varphi_{l+2}(x) \\ &= f_n\big(\tilde{\varepsilon}(k-2) + \tilde{\varepsilon}\big)\varphi_k(x) + f_n\big(\tilde{\varepsilon}(k-1) + \tilde{\varepsilon}\big)\varphi_{k+1}(x) \\ &\le f_n(x)\varphi_k(x) + f_n(x)\varphi_{k+1}(x) = f_n(x). \end{aligned}$$

Next, again with $x \in \mathbb{R}$ and $k \in \mathbb{Z}$ such that $x \in [\tilde{\varepsilon}k, \tilde{\varepsilon}k + \tilde{\varepsilon})$, it holds

$$\begin{aligned} \left|\tilde{f}^{\mathrm{min},\varepsilon}(x) - f(x)\right| &\le \sum_{l\in\mathbb{Z}} |f(\tilde{\varepsilon}l) - f(x)|\varphi_{l+2}(x) \\ &= |f(\tilde{\varepsilon}(k-2)) - f(x)|\varphi_k(x) + |f(\tilde{\varepsilon}(k-1)) - f(x)|\varphi_{k+1}(x) \\ &\le \big(f(x) - f(x - 3\tilde{\varepsilon})\big)\big(\varphi_k(x) + \varphi_{k+1}(x)\big) \end{aligned}$$

$$= f(x) - f(x-\varepsilon).$$

This yields the second of the stated properties in (i). Finally, the last property follows from the estimate

$$\begin{aligned} \left| \tilde{f}_n^{\min,\varepsilon}(x) - \tilde{f}^{\min,\varepsilon}(x) \right| &\le \sum_{k\in\mathbb{Z}} |y_n(k) - f(\tilde{\varepsilon}k)| \varphi_{k+2}(x) \\ &\le \delta_n \sum_{k\in\mathbb{Z}} \varphi_{k+2}(x) = 2d^*(f, f_n), \quad x \in \mathbb{R}. \end{aligned}$$

This concludes the proof □

3.2.2 Mosco convergence of superposed forms

The key for the disintegration method, which derives a result on Mosco convergence of gradient forms in infinite dimension and is the topic of Section 4, lies in the superposition of one-component forms on the real line. Here, a bit more general, we use the analysis of compatible classes from Section 2.2.2 to derive a convergence theorem for superposed standard gradient forms on $\mathbb{R}^d$. To this end, we consider a sequence of mixing measures $(\nu_N)_{N\in\mathbb{N}}$, Borel probability measures on a completely regular Hausdorff space E. Furthermore, we assume that each finite Borel measure on E is a Radon measure, as would be true for a Suslin space. For $N \in \mathbb{N}$ respectively, the gradient forms which are superposed have reference measures $m_N(z, A)$, $A \in \mathcal{B}(\mathbb{R}^d)$, depending on a variable $z \in E$. Hence, we define $m_N(z, \cdot)$, $z \in E$, as Markov probability kernel from $(E, \mathcal{B}(E))$ to $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, that is

$$\begin{aligned} m_N(z, \cdot) &\text{ is a Borel probability measure on } \mathbb{R}^d \text{ for each } z \in E, \\ m_N(\cdot, A) &\text{ is a measurable map from } E \text{ to } [0,1] \text{ for each } A \in \mathcal{B}(\mathbb{R}^d). \end{aligned}$$

The idea of the proof of Theorem 3.2.9 is a similar one as developed for Theorem 3.1.12. Therefore, we need to transport the essential ingredients provided by Proposition 3.1.8 into the current setting. This is done in Proposition 3.2.4 below. Dropping the index N, let ν and $m(z, A)$, $A \in \mathcal{B}(\mathbb{R}^d)$, $z \in E$, generically be a Borel probability on E, respectively a Markov probability kernel from $(E, \mathcal{B}(E))$ to $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. We define the semi-direct product of ν and m as the probability measure,

$$(\nu \times m)(B) := \int_{E\times\mathbb{R}^d} \mathbf{1}_B(z, x) m(z, \mathrm{d}x) \, \mathrm{d}\nu(z), \quad B \in \mathcal{B}(E \times \mathbb{R}^d).$$

In the following, we assume that $m(z, \cdot)$ is absolutely continuous w.r.t. the Lebesgue measure $\mathrm{d}x$ for ν-a.e. $z \in E$ and that $\rho(z, x)$, $(z, x) \in E \times \mathbb{R}^d$, is a non-negative, measurable function on $E \times \mathbb{R}$ with $\rho(z, \cdot) = \frac{\mu(z,\cdot)}{\mathrm{d}x}$, ν-a.e. $z \in E$.

Proposition 3.2.4. *Let $u \in \mathcal{B}_b(E) \otimes \mathrm{Lip}_b(\mathbb{R}^d)$ with $-1 \le u(z, x) \le 1$ for $(z, x) \in E \times \mathbb{R}^d$ such that*

$$\int_E \int_{\mathbb{R}^d} |\nabla u(z, \cdot)|^2_{\mathrm{euc}}(x) m(z, \mathrm{d}x) \, \mathrm{d}\nu(x) < \infty.$$

The inequalities (i) to (iii) hold true for every $r \in (0, \infty)$, every $g \in \mathcal{B}_b(E) \otimes C_b^1(\mathbb{R}^d)$ and every measurable function κ on $E \times \mathbb{R}^d$ with values in $[0, 1]$. Let

$$C_r := 6^d \operatorname*{ess\,sup}_{(z,x)\in E\times\mathbb{R}^d} \rho^{-1}(z, x) \big((\kappa\rho(z, \cdot))_{\mathrm{ave},3r}(x) \big) \quad \textit{w.r.t. } \nu \times \mathrm{d}x \quad \textit{and}$$

$$c_r := 8^d \int_{E\times\mathbb{R}^d} \big|(\kappa\rho)(z,x) - (\kappa\rho(z,\cdot))_{\mathrm{ave},4r}(x)\big| \,\mathrm{d}x\,\mathrm{d}\nu(z).$$

(i) $$\Big|\int_{E\times\mathbb{R}^d} \big(u(z,x) - I_r^{\chi,\xi}u(z,\cdot)(x)\big)(g\kappa)(z,x)\,\mathrm{d}(\nu\times m)(z,x)\Big|$$
$$\le c_r\|g\|_\infty + \sup_{z\in E}\omega\big(g(z,\cdot), 2r\sqrt{d}, \mathrm{supp}[(\kappa\rho)(z,\cdot)]\big).$$

(ii) $$\Big|\int_{E\times\mathbb{R}^d} \big(\nabla u(z,\cdot) - \nabla I_r^{\chi,\xi}u(z,\cdot), \nabla g(z,\cdot)\big)_{\mathrm{euc}}(x)\kappa(z,x)\,\mathrm{d}(\nu\times m)(z,x)\Big|$$
$$\le \Big(\sqrt{c_r(1+C_r)}\beta_g + C_r\gamma_g(r)\Big)$$
$$\times\Big(\int_E\int_{\mathbb{R}^d} |\nabla u(z,\cdot)|^2_{\mathrm{euc}}(x)\,\mathrm{d}(\nu\times m)(z,x)\Big)^{\frac12},$$
where $\beta_g\in(0,\infty)$ is a constant and $\gamma_g(\cdot)$ is a function on $(0,\infty)$, which only depends on g, and $\lim_{r'\to 0}\gamma_g(r') = 0$.

(iii) $$\int_{E\times\mathbb{R}^d} |\nabla I_r^{\chi,\xi}u(z,\cdot)|^2_{\mathrm{euc}}(x)(\kappa\rho)(z,x)\,\mathrm{d}(\nu\times m)(z,x)$$
$$\le C_r\int_E\int_{\mathbb{R}^d} |\nabla u(z,\cdot)|^2_{\mathrm{euc}}(x)\,\mathrm{d}(\nu\times m)(z,x).$$

Proof. For fixed $z\in E$ the function $u(z,\cdot)$ is a Lipschitz continuous function on $\mathbb{R}^d$ with values in $[-1,1]$. The strategy is the same for each of the estimates. We apply Proposition 3.2.4 in combination with Remark 3.1.9 onto the inner integral over $\mathbb{R}$. In case of (i), this reads

$$\Big|\int_{E\times\mathbb{R}^d} \big(u(z,x) - I_r^{\chi,\xi}u(z,\cdot)(x)\big)(g\kappa)(z,x)\,\mathrm{d}(\nu\times m)(z,x)\Big|$$
$$\le \int_E\Big|\int_{\mathbb{R}^d} \big(u(z,x) - I_r^{\chi,\xi}u(z,\cdot)(x)\big)(g\kappa)(z,x)\rho(z,x)\,\mathrm{d}x\Big|\,\mathrm{d}\nu(z)$$
$$\le \int_E 8^d\int_{\mathbb{R}^d} \big|(\kappa\rho)(z,x) - (\kappa\rho(z,\cdot))_{\mathrm{ave},4r}(x)\big|\,\mathrm{d}x\,\mathrm{d}\nu(z)$$
$$+\sup_{z\in E}\omega\big(g(z,\cdot), 2r\sqrt{d}, \mathrm{supp}[(\kappa\rho)(z,\cdot)]\big)$$

and the claim is shown. We address the case of (ii). For $i=1,\dots,d$ we have by the same method

$$\Big|\int_{E\times\mathbb{R}^d} \partial_i\big(u(z,\cdot) - I_r^{\chi,\xi}u(z,\cdot)\big)\partial_i g(x)\kappa(z,x)\,\mathrm{d}(\nu\times m)(z,x)\Big|$$
$$\le \int_E\Big|\int_{\mathbb{R}^d} \partial_i\big(u(z,\cdot) - I_r^{\chi,\xi}u(z,\cdot)\big)\partial_i g(x)\kappa(z,x)\rho(x)\,\mathrm{d}x\Big|\,\mathrm{d}\nu(z)$$
$$\le \sup_{(z,x)\in E\times\mathbb{R}^d} |\partial_i g(z,x)|\sqrt{1+C_r}$$
$$\times\int_E\Big(8^d\int_{\mathbb{R}^d} \big|(\kappa\rho)(z,x) - (\kappa\rho(z,\cdot))_{\mathrm{ave},4r}(x)\big|\,\mathrm{d}x\Big)^{\frac12}$$
$$\times\int_{\mathbb{R}^d} |\partial_i u(z,\cdot)(x)|^2\rho(z,x)\,\mathrm{d}x\Big)^{\frac12}\,\mathrm{d}\nu(z)$$

$$+ C_r \sup_{z\in E} \omega(|\partial_i g(z,\cdot)|, 4r\sqrt{d}, \operatorname{supp}[(\kappa\rho)(z,\cdot)])$$
$$\times \int_{\mathbb{R}^d} |\partial_i u(z,\cdot)(x)|^2 \rho(z,x)\,\mathrm{d}x\Big)^{\frac{1}{2}} \,\mathrm{d}\nu(z)$$

On the first summand of the right-hand side we apply the Hölder inequality and on the second summand we apply Jensen's inequality, both on the outer integrals w.r.t. $\mathrm{d}\nu$. After this, the right-hand side admits the upper bound

$$\Big(\sup_{(z,x)\in E\times\mathbb{R}^d} |\partial_i g(z,x)| \sqrt{c_r(1+C_r)}$$
$$+ C_r \sup_{z\in E} \omega(|\partial_i g(z,\cdot)|, 4r\sqrt{d}, \operatorname{supp}[(\kappa\rho)(z,\cdot)])\Big)$$
$$\times \Big(\int_{E\times\mathbb{R}^d} |\partial_i u(z,\cdot)(x)|^2 m(z,\mathrm{d}x)\,\mathrm{d}\nu(z)\Big)^{\frac{1}{2}}$$

The claim of (ii) now follows from the inequality

$$\sum_{i=1}^d a_i\sqrt{b_i} \le d\big(\max_{i=1,\dots,d} |a_i|\big)\Big(\sum_{i=1}^d b_i\Big)^{\frac{1}{2}}, \quad a_i, b_i \in (0,\infty),$$

by summing up over i. The prove of (iii) follows with the same strategy as (i). □

To be able to define a symmetric closed form on $L^2(E,\nu\times m)$ as the superposition of standard gradient forms on $\mathbb{R}^d$, the Hamza condition is required to hold in ν-a.e. sense for the disintegrating densities. We impose the following condition.

Condition 3.2.5. The density $\rho(z,\cdot)$ meets Condition 3.1.10 for ν-a.e. $z\in E$.

Hence, for ν-a.e. $z\in E$ the gradient form $\mathcal{E}^{\rho(z,\cdot)}$ is defined on $L^2(\mathbb{R}^d, m(z,\cdot))$. We recall the natural embedding (3.1.38) from Section 3.1.2 to understand in which sense the next two definitions should be read. Let

$$\begin{aligned}\mathcal{D}(\mathcal{E}) := \Big\{ u \in L^2(E\times\mathbb{R}^d, \nu\times m) \,\Big|\, & u(z,\cdot)\in\mathcal{D}\big(\mathcal{E}^{\rho(z,\cdot)}\big),\ \nu\text{-a.e. } z\in E,\\ & \nabla u(z,\cdot)(x) = V(z,x),\ (\nu\times m)\text{-a.e. } (z,x)\in E\times\mathbb{R}^d,\\ & \text{for some element } V\in L^2\big(E\times\mathbb{R}^d,\mathbb{R}^d,\nu\times m\big)\Big\}\end{aligned}$$

and

$$\mathcal{E}(u,v) := \int_{E\times\mathbb{R}^d} \big(\nabla u(z,\cdot)(x), \nabla v(z,\cdot)(x)\big)_{\text{euc}}\,\mathrm{d}m(x)\,\mathrm{d}\nu(z), \quad u,v\in\mathcal{D}(\mathcal{E}).$$

We commit ourselves to the approximation of $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ in the sense of Mosco now. Before we start, some preliminaries and notational matters are handled. For a set M and functions $f: E\to M$, $g:\mathbb{R}^d\to M$, we define $(f\times g)(z,x) := \big(f(z),g(x)\big)\in M\times M$ for $(z,x)\in E\times\mathbb{R}^d$. We now assume additionally that E is a locally convex topological vector space. The topological dual of E is denoted by E'. The linear space

$$\mathcal{F}C_b := \big\{ F(l_1,\dots,l_m) \,\big|\, m\in\mathbb{N},\ F\in C_b(\mathbb{R}^m),\ l_1,\dots,l_m\in E'\big\} \tag{3.2.3}$$

induces the set $L^2(E,\nu)\cap\widetilde{\mathcal{F}C_b}$, a dense subspace of $L^2(E,\nu)$, which plays a distinguished role in the subsequent discussion. For a proof of the density of this subspace, see [17, Remark 3.1] and the references therein. The statement also holds with $\mathcal{F}C_b^\infty$, instead of $\mathcal{F}C_b$. The latter is defined by re-writing the right-hand side of (3.2.3), while replacing $C_b(\mathbb{R}^m)$ by $\in C_b^\infty(\mathbb{R}^m)$. For a linear space V of functions from E to $\mathbb{R}$ and a linear space W of functions from $\mathbb{R}^d$ to $\mathbb{R}$ we define the algebraic product $V \otimes W$ as the linear span of the set $\{f(z)\cdot g(x) \,|\, f \in V,\, g \in W\}$ within the vector space of real-valued functions on $E \times \mathbb{R}$.

Remark 3.2.6. Let V be a linear space of functions from E to $\mathbb{R}$. If $\widetilde{V}$ is dense in $L^2(E,\nu)$, then the algebraic product $V \otimes C_b^\infty(\mathbb{R}^d)$ induces a dense linear subspace $L^2(E\times\mathbb{R},\nu\times m) \cap (V \otimes C_b^\infty(\mathbb{R}^d))^\sim$ of $L^2(E\times\mathbb{R},\nu\times m)$, which is even dense in $(\mathcal{D}(\mathcal{E}),\mathcal{E}_1)$. Indeed, let $\mathcal{A}$ be a countable subset of $C_b^\infty(\mathbb{R}^d)$, whose linear span is dense in $C_b^1(\mathbb{R}^d)$ w.r.t. the topology of local uniform convergence of functions and their first-order derivatives. A sequence $(\varphi_n)_{n\in\mathbb{N}}$ converges to φ in that topology if and only if

$$\lim_{n\to\infty} \sup_{x\in K} \big(|\varphi_n - \varphi| + |\,\nabla\varphi_n - \nabla\varphi|_{\mathrm{euc}}\big) = 0 \qquad \text{holds for every compact set } K \subset \mathbb{R}^d.$$

Then, from

$$\int_E f(z) \int_{\mathbb{R}^d} u(z,x)\varphi(x) + \big(\nabla u(z,\cdot)(x), \nabla\varphi(x)\big)_{\mathrm{euc}}\, \mathrm{d}m(z,\mathrm{d}x)\, \mathrm{d}\nu(z) = 0 \qquad \text{for all } f \in V \text{ and } \varphi \in \mathcal{A}$$

for given $u \in \mathcal{D}(\mathcal{E})$, we can conclude

$$\int_{\mathbb{R}^d} u(z,x)\varphi(x) + \big(\nabla u(z,\cdot)(x), \nabla\varphi(x)\big)_{\mathrm{euc}}\, \mathrm{d}m(z,\mathrm{d}x) = 0, \quad \nu\text{-a.e. } z \in E,$$

for each $\varphi \in \mathcal{A}$, by density of $L^2(E,\nu) \cap \widetilde{V}$. However, since $\mathcal{A}$ is countable, even

$$\int_{\mathbb{R}^d} u(z,x)\varphi(x) + \big(\nabla u(z,\cdot)(x), \nabla\varphi(x)\big)_{\mathrm{euc}}\, \mathrm{d}m(z,\mathrm{d}x) = 0 \quad \text{for all } \varphi \in \mathcal{A}$$

holds for ν-a.e. $z \in E$. The latter proves $u(z,\cdot) = 0$, ν-a.e. $z \in E$, since for any generic Hamza probability density $\rho_0(x)$, $x \in \mathbb{R}^d$, the space $(\mathrm{span}(\mathcal{A}))^\sim$ is dense in $(\mathcal{E}^{\rho_0}, \mathcal{D}(\mathcal{E}^{\rho_0}))$. In particular, $\mathcal{F}C_b \otimes C_b^\infty(\mathbb{R}^d)$ and $\mathcal{F}C_b^\infty \otimes C_b^\infty(\mathbb{R}^d)$ are dense subspaces of $(\mathcal{D}(\mathcal{E}),\mathcal{E}_1)$.

Let $m_N(z,\cdot)$, $z \in E$, be a Markov kernel and ν_N a mixing measure on E, for every $N \in \mathbb{N}$ as in the beginning of this section. We assume that the analogue of Condition 3.2.5 is satisfied for every $N \in \mathbb{N}$. Let $\mathcal{E}^N$ be defined analogously as the form $\mathcal{E}$ above with m, ν, replaced by m_N, ν_N for $N \in \mathbb{N}$. The frame within which we discuss Mosco convergence here is slightly different from the standard one described at the end of Section 2.1. We are again interested in the question whether $\mathcal{E}$ is the limit of $(\mathcal{E}^N)_{N\in\mathbb{N}}$ in the sense of Mosco and hence want to see $L^2(E\times\mathbb{R}^d,\nu\times m)$ as the asymptotic Hilbert space for $L^2(E\times\mathbb{R}^d,\nu_N\times m_N)$, $N \in \mathbb{N}$. However, the main application presented in Section 4 below, which motivates this discourse about the superposition of forms, benefits from a more general approach, where we do not ask for the condition

$$\mathrm{supp}[\nu_N \times m_N] \subseteq \mathrm{supp}[\nu \times m], \quad N \in \mathbb{N}, \tag{3.2.4}$$

as would be the standard assumption regarding the topological support of the measures. Instead, we fix a continuous linear operator J from E into E, such that the topological support of ν is contained in the set of fix points of J. We assume

$$\operatorname{supp}[\nu] \subseteq \{z \in E \,|\, Jz = z\} \quad \text{together with}$$
$$\operatorname{supp}[(\nu_N \times m_N) \circ (J \times \mathrm{id}_{\mathbb{R}^d})^{-1}] \subseteq \operatorname{supp}[\nu \times m], \quad N \in \mathbb{N}. \tag{3.2.5}$$

Obviously, (3.2.5) is less restrictive than (3.2.4). On top of that, let $\nu \times m$ be the limit of $(\nu_N \times m_N)_{N\in\mathbb{N}}$ in the sense that

$$\lim_{N\to\infty} \int_E f(z) \int_{\mathbb{R}^d} \varphi(x) m_N(z, \mathrm{d}x)\, \mathrm{d}\nu_N = \int_E f(z) \int_{\mathbb{R}^d} \varphi(x) m(z, \mathrm{d}x)\, \mathrm{d}\nu \qquad \text{for } f \in \mathcal{F}C_b,\ \varphi \in C_b(\mathbb{R}^d). \tag{3.2.6}$$

Because of (3.2.5), it is legal to define Φ_N for $N \in \mathbb{N}$ as the map which sends an element u from the linear subspace

$$\mathcal{C} := L^2(E \times \mathbb{R}^d, \nu \times m) \cap \big(\mathcal{F}C_b \otimes C_b(\mathbb{R}^d)\big)^{\sim},$$

onto the class in $L^2(E \times \mathbb{R}^d, \nu_N \times m_N)$ of the function $f(Jz, x)$, $(z, x) \in E \times \mathbb{R}^d$, for one of its bounded, continuous representatives f with $u(z, x) = f(z, x)$ for $(\nu \times m)$-a.e. $(z, x) \in E \times \mathbb{R}^d$ and $f \in C_b(E \times \mathbb{R}^d)$. If $u(z, x) = f(z)\varphi(x)$ and $v(z, x) = g(z)\eta(x)$, in $(\nu\times m)$-a.e. sense for $(z, x) \in E\times\mathbb{R}^d$ respectively, with $f, g \in \mathcal{F}C_b$ and $\varphi, \eta \in C_b(\mathbb{R}^d)$, then

$$\begin{aligned}
&\lim_{N\to\infty} \int_{E\times\mathbb{R}^d} (\Phi_N u)(\Phi_N v)\, \mathrm{d}(\nu_N \times m_N) \\
&= \lim_{N\to\infty} \int_E f(Jz) g(Jz) \int_{\mathbb{R}^d} \varphi(x)\eta(x) m_N(z, \mathrm{d}x)\, \mathrm{d}\nu_N \\
&= \int_E f(Jz) g(Jz) \int_{\mathbb{R}^d} \varphi(x)\eta(x) m(z, \mathrm{d}x)\, \mathrm{d}\nu \\
&= \int_E f(z) g(z) \int_{\mathbb{R}^d} \varphi(x)\eta(x) m(z, \mathrm{d}x)\, \mathrm{d}\nu \\
&= \int_{E\times\mathbb{R}^d} uv\, \mathrm{d}(\nu \times m),
\end{aligned} \tag{3.2.7}$$

is implied by (3.2.6). Via linearity (3.2.7) generalizes to $u, v \in \mathcal{C}$. We interpret $L^2(E \times \mathbb{R}^d, \nu_N \times m_N)$, $N \in \mathbb{N}$, as a sequence of converging Hilbert spaces with asymptotic space $L^2(E\times\mathbb{R}^d, \nu\times m)$ now, as suggested by the asymptotic isometries $(\Phi_N)_{N\in\mathbb{N}}$. On their disjoint union

$$\mathcal{H} := \Big(\bigsqcup_{N\in\mathbb{N}} L^2(E \times \mathbb{R}^d, \nu_N \times m_N)\Big) \sqcup L^2(E \times \mathbb{R}^d, \nu \times m)$$

we obtain the corresponding notions of the strong and the weak topology.

Remark 3.2.7. (i) Given $f \in \mathcal{F}C_b$ and $\varphi \in C_b(\mathbb{R}^d)$, let u_N for $N \in \mathbb{N}$ denote the class in $L^2(E\times\mathbb{R}^d, \nu_N\times m_N)$ of the function $f(z)\varphi(x)$, $(z, x) \in E\times\mathbb{R}^d$ and u be defined analogously. Due to

$$\lim_{N\to\infty} \int_E f(z)(g\circ J)(z) \int_{\mathbb{R}^d} \varphi(x)\eta(x) m_N(z, \mathrm{d}x)\, \mathrm{d}\nu_N$$

$$= \int_E f(z)g(z) \int_{\mathbb{R}^d} \varphi(x)\eta(x)m(z,\mathrm{d}x)\,\mathrm{d}\nu, \quad g \in \mathcal{F}C_b,\ \eta \in C_b(\mathbb{R}^d),$$

and

$$\lim_{N\to\infty} \int_E f(z)^2 \int_{\mathbb{R}^d} \varphi(x)^2 m_N(z,\mathrm{d}x)\,\mathrm{d}\nu_N = \int_E f(z)^2 \int_{\mathbb{R}^d} \varphi(x)^2 m(z,\mathrm{d}x)\,\mathrm{d}\nu,$$

both of which are implied by (3.2.6), $\lim_{N\to\infty} u_N = u$ holds strongly in $\mathcal{H}$.

(ii) Any other valid choice of a continuous linear operator $J' : E \to E$, instead of J, w.r.t. which the analogue of (3.2.5) is still satisfied, would result in the equivalent topological notions on $\mathcal{H}$ in the sense of Remark 2.1.2 (ii), because

$$\lim_{N\to\infty} \int_E (f\circ J')(z)(g\circ J)(z) \int_{\mathbb{R}^d} \varphi(x)\eta(x)m_N(z,\mathrm{d}x)\,\mathrm{d}\nu_N$$
$$= \int_E f(z)g(z) \int_{\mathbb{R}^d} \varphi(x)\eta(x)m(z,\mathrm{d}x)\,\mathrm{d}\nu, \quad f,g \in \mathcal{F}C_b,\ \varphi,\eta \in C_b(\mathbb{R}^d),$$

follows from (3.2.6).

(iii) From the assumption,

$$\text{if } (\nu_N \times m_N)\circ(J\times \mathrm{id}_{\mathbb{R}^d})^{-1}(V) > 0, \text{ then}$$
$$(\nu \times m)\circ(J\times \mathrm{id}_{\mathbb{R}^d})^{-1}(V) > 0, \quad \text{for } V \subseteq E\times\mathbb{R}^d \text{ open}, N\in\mathbb{N},$$

which is the statement of (3.2.5), and continuity of the projection onto the first coordinate $E\times\mathbb{R}^d \to E$, it follows

$$\text{if } \nu_N\circ J^{-1}(U) > 0, \quad \text{then} \quad \nu\circ J^{-1}(U) > 0, \quad \text{for } U\subseteq E \text{ open}, N\in\mathbb{N},$$

and hence $\mathrm{supp}[\nu_N\circ J^{-1}] \subseteq \mathrm{supp}[\nu]$. Analogous to the previous procedure, the assumption of (3.2.6), now with the choice $\varphi = \mathbf{1}_{\mathbb{R}^d}$, produces a sequence of converging Hilbert spaces. With $N\in\mathbb{N}$, the map Φ_N^{pr}, which sends an element u from the linear subspace $L^2(E,\nu_N)\cap\widetilde{\mathcal{F}C_b}$ onto the class in $L^2(E,\nu_N)$ of the function $f(Jz)$, $z\in E$, for one of its bounded, continuous representatives f with $u(z) = f(z)$ for ν-a.e. $z\in E$, allows to interpret the spaces $L^2(E,\nu_N)$, $N\in\mathbb{N}$, as a sequence of converging Hilbert spaces with asymptotic space $L^2(E,\nu)$. We set

$$\mathcal{H}^{\mathrm{pr}} := \Big(\bigsqcup_{N\in\mathbb{N}} L^2(E,\nu_N)\Big) \sqcup L^2(E,\nu).$$

That settles the issues concerning the convergence of Hilbert spaces in this context. For the sake of Mosco convergence, however, an additional assumption has to be made. The equation

$$\lim_{N\to\infty} \int_E \Big|\int_{\mathbb{R}^d} \varphi(x)m_N(z,\mathrm{d}x)\Big|^2 \mathrm{d}\nu_N(z) = \int_E \Big|\int_{\mathbb{R}^d} \varphi(x)m(z,\mathrm{d}x)\Big|^2 \mathrm{d}\nu(z), \quad \varphi\in C_b(\mathbb{R}^d) \quad (3.2.8)$$

is required. What (3.2.8) actually means is the strong convergence in $\mathcal{H}^{\mathrm{pr}}$ of the ν_N-class of the function $\int_{\mathbb{R}^d}\varphi(x)m_N(z,\mathrm{d}x)$, $z\in E$, towards the ν-class of $\int_{\mathbb{R}^d}\varphi(x)m(z,\mathrm{d}x)$, $z\in E$, for N to infinity and given $\varphi\in C_b(\mathbb{R}^d)$. Indeed, the strong convergence in $\mathcal{H}^{\mathrm{pr}}$ for any sequence of this kind has to be named as an additional assumption, since it is a prerequisite for the proof of Theorem 3.2.9, while the basic condition of (3.2.6) only yields its weak convergence. So, in (3.2.8), the missing feature, i.e. the convergence of the respective norms, is demanded. At this point, it is useful to highlight another consequence of the strong convergence in $\mathcal{H}^{\mathrm{pr}}$ we just discussed. It holds

$$\lim_{N\to\infty} \int_E f_N(z) g(Jz) \int_{\mathbb{R}^d} \varphi(x) m_N(z,\mathrm{d}x)\,\mathrm{d}\nu_N(z) \\ = \int_E f(z) g(z) \int_{\mathbb{R}^d} \varphi(x) m(z,\mathrm{d}x)\,\mathrm{d}\nu(z), \quad \varphi \in C_b(\mathbb{R}^d), g \in \mathcal{F}C_b, \quad (3.2.9)$$

whenever $f(z)$ and $f_N(z)$, $z \in E$, are representatives of elements $u \in L^2(E,\nu)$, respectively $u_N \in L^2(E,\nu_N)$ for $N \in \mathbb{N}$, such that u is the weak limit of $(u_N)_{N\in\mathbb{N}}$ in $\mathcal{H}^{\mathrm{pr}}$. On the other hand, (3.2.9) means nothing different than the weak convergence in $\mathcal{H}$ of the sequence $(\tilde{u}_N)_{N\in\mathbb{N}}$, where $\tilde{u}_N \in L^2(E\times\mathbb{R}^d, \nu_N \times m_N)$ is the class with $u_N(z,x) = f_N(z)$, $(\nu_N \times m_N)$-a.e. $(z,x) \in E\times\mathbb{R}^d$. The weak limit of $(\tilde{u}_N)_{N\in\mathbb{N}}$ in $\mathcal{H}$ is the class $u \in L^2(E\times\mathbb{R}^d, \nu\times m)$, which is analogously related to the function f. In this sense, a weakly continuous section in $\mathcal{H}^{\mathrm{pr}}$ can also be interpreted as a weakly convergent section in $\mathcal{H}$, as a consequence of (3.2.8).

The proof of Theorem 3.2.9 below utilizes a version of (3.2.6), displayed by the following remark, which is a consequence of the Portmanteau theorem.

Remark 3.2.8. For a non-negative function $f \in \mathcal{F}C_b$ the weak convergence of the measures

$$\int_E m_N(z,A) f(z)\,\mathrm{d}\nu_N(z), \quad A \in \mathcal{B}(\mathbb{R}^d),$$

towards

$$\int_E m(z,A) f(z)\,\mathrm{d}\nu(z), \quad A \in \mathcal{B}(\mathbb{R}^d),$$

as N tends to infinity holds true by (3.2.6). Writing

$$F(l_1,\dots,l_m) = \max\{F,0\}\circ(l_1,\dots,l_m) - \max\{-F,0\}\circ(l_1,\dots,l_m)$$

for $m \in \mathbb{N}$, $F \in C_b(\mathbb{R}^m)$ and $l_1,\dots,l_m \in E'$, we state a consequence of the Portmanteau theorem. If the set of discontinuities

$$U_\varphi := \mathbb{R}^d \setminus \left\{ x \in \mathbb{R}^d \,\middle|\, \varphi \text{ is continuous at } x \right\}$$

for a given function $\varphi \in \mathcal{B}_b(\mathbb{R}^d)$ is negligible w.r.t. the Lebesgue measure, then

$$\lim_{N\to\infty} \int_E f(z) \int_{\mathbb{R}^d} \varphi(x)\,\mathrm{d}m_N(z,\mathrm{d}x)\,\mathrm{d}\nu_N \\ = \int_E f(z) \int_{\mathbb{R}^d} \varphi(x)\,\mathrm{d}m(z,\mathrm{d}x)\,\mathrm{d}\nu, \quad f \in \mathcal{F}C_b.$$

Let $(\mathcal{E}^N, \mathcal{D}(\mathcal{E}^N))$ be defined analogously as $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ with μ_N, ρ_N and ν_N replacing μ, ρ and ν, respectively.

Theorem 3.2.9. *Let (3.2.5), (3.2.6) and (3.2.8) hold true. We understand $L^2(E\times\mathbb{R}^d, \nu_N \times m_N)$, $N \in \mathbb{N}$, as a sequence of converging Hilbert spaces with asymptotic space $L^2(E\times\mathbb{R}^d, \nu\times m)$ in the way explained above.*

If, for every $m \in \mathbb{N}$ it holds

$$\limsup_{k\to\infty} \sup_{N\in\mathbb{N}} \operatorname*{ess\,sup}_{(z,x)\in E\times[-m,m]^d} \rho_N^{-1}(z,x)\cdot\left(\rho_N(z,\cdot)_{\mathrm{ave},\frac{1}{k}}(x)\right) < \infty$$

(the essential supremum above is taken w.r.t. the product measure $\nu_N \times \mathrm{d}x$ for $N \in \mathbb{N}$ respectively) as well as

$$\lim_{k\to\infty} \limsup_{N\to\infty} \int_{E\times[-m,m]^d} \big|\rho_N(z,x) - \big(\rho_N(z,\cdot)_{\mathrm{ave},\frac{1}{k}}(x)\big)\big| \,\mathrm{d}x \,\mathrm{d}\nu_N(z) = 0,$$

then $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is the Mosco limit of the sequence $(\mathcal{E}^N, \mathcal{D}(\mathcal{E}^N))_{N\in\mathbb{N}}$.

Proof. We start with the proof property of (M1). We set $\Omega_m := (-m,m)^d$ and let $m \in \mathbb{N}$ be fixed in the first part of the proof. To a large extent, this proof pursues the same strategy as the one of Theorem 3.1.12. We choose a continuously differentiable function κ_m on $\mathbb{R}^d$ with

$$\mathbf{1}_{\mathrm{cl}(\Omega_m)}(x) \le \kappa_m(x) \le \mathbf{1}_{\Omega_{m+1}}(x), \quad x \in \mathbb{R}^d \quad \text{and} \quad \sup_{m\in\mathbb{N}} \sup_{x\in\mathbb{R}^d} |\nabla\kappa_m(x)|_{\mathrm{euc}} < C$$

for a constant C independent of m. The densities $\kappa_m(x)\rho_N(z,x)$, $N \in \mathbb{N}$, and $\kappa_m(x)\rho(z,x)$, $(z,x) \in E \times \mathbb{R}^d$ meet the superposed Hamza condition 3.2.5. Since κ_m is bounded, continuous we can define the topological notions on

$$\mathcal{H}_m := L^2\big(E \times \mathbb{R}^d, \rho(\nu \times \kappa_m \,\mathrm{d}x)\big) \sqcup \Big(\bigsqcup_{N\in\mathbb{N}} L^2\big(E \times \mathbb{R}^d, \rho_N(\nu_N \times \kappa_m \,\mathrm{d}x)\big)\Big)$$

analogously as we did for $\mathcal{H}$, via (3.2.6). Let $(\Psi_N)_{N\in\mathbb{N}}$ denote the asymptotic isometries, defined from $L^2(\rho(\nu \times \kappa_m \,\mathrm{d}x)) \cap (\mathcal{F}C_b \otimes C_b(\mathbb{R}^d))^\sim$ into $L^2(\kappa_m\rho_N(\nu_N \times \kappa_m \,\mathrm{d}x))$ respectively for $N \in \mathbb{N}$, with the analogous property of (3.2.7). With the asymptotic equation

$$\lim_{N\to\infty} \int_E \mathcal{E}^{\kappa_m\rho_N(z,\cdot)}\big(\Psi_N u(z,\cdot), \Psi_N u(z,\cdot)\big) \,\mathrm{d}\nu_N(z)$$
$$= \int_E \mathcal{E}^{\kappa_m\rho(z,\cdot)}\big(u(z,\cdot), u(z,\cdot)\big) \,\mathrm{d}\nu(z), \quad u \in (\mathcal{F}C_b \otimes C_b^1(\mathbb{R}^d))^\sim$$

we initialize another sequence of converging Hilbert spaces, $(\mathcal{D}(\tilde{\mathcal{E}}^{N,m}), \tilde{\mathcal{E}}_1^{N,m})$, $N \in \mathbb{N}$, with asymptotic space $(\mathcal{D}(\tilde{\mathcal{E}}^m), \tilde{\mathcal{E}}_1^m)$. The sequence of forms $(\tilde{\mathcal{E}}^{N,m})_{N\in\mathbb{N}}$ and their domains are defined analogously as the sequence $(\mathcal{E}^N)_{N\in\mathbb{N}}$, with $\kappa_m(x)\rho_N(z,x)$ replacing ρ_N for $N \in \mathbb{N}$. In the same way, $(\tilde{\mathcal{E}}^m, \mathcal{D}(\tilde{\mathcal{E}}^m))$ is defined like $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ with $\kappa_m(x)\rho(z,x)$ taking the role of $\rho(z,x)$. We set

$$\mathcal{H}_m^{\mathcal{E},1} := \big(\mathcal{D}(\tilde{\mathcal{E}}^{N,m}), \tilde{\mathcal{E}}_1^{N,m}\big) \sqcup \Big(\bigsqcup_{N\in\mathbb{N}} \big(\mathcal{D}(\tilde{\mathcal{E}}^{N,m}), \tilde{\mathcal{E}}_1^{N,m}\big)\Big).$$

In the subsequent discussion on compatible classes in $\mathcal{H}_m^{\mathcal{E},1}$ we adopt some notation from the proof of Theorem 3.1.12. Let $\Lambda_\chi : w \mapsto \Lambda_\chi w$ be as in Theorem 3.1.6, mapping a lattice weight $w \in \mathbb{R}^{\mathbb{Z}^d}$ into the space of locally bounded, locally Lipschitz continuous functions on $\mathbb{R}^d$. For $r \in (0,\infty)$ we set $\Lambda_{\chi,r}w(x) := \Lambda_\chi w(x/r)$ for $x \in \mathbb{R}^d$, $w \in \mathbb{R}^{\mathbb{Z}^d}$, and moreover

$$V_{m,r} := \Lambda_{\chi,r}\big(\{w \in \mathbb{R}^{\mathbb{Z}^d} \,\big|\, w_{\mathbf{k}} = 0 \text{ for } \mathbf{k} \in \mathbb{Z}^d,\, r\mathbf{k} \notin [-m-2, m+2]\}\big),$$

similarly as in (3.1.41) from Section 3.1.2. Here, $\mathcal{B}_{b,\le 1}(E)$ denotes the space of real-valued, measurable functions on E whose absolute value is smaller equal 1 on E. By choice of $V_{m,r}$, we have

$$\mathcal{D}(\tilde{\mathcal{E}}^{N,m}) \cap (\mathcal{B}_{b,\le 1}(E) \otimes V_{m,r})^\sim = \mathcal{D}(\tilde{\mathcal{E}}^{N,m}) \cap (\mathcal{B}_{b,\le 1}(E) \otimes \Lambda_{\chi,r})^\sim.$$

First, we claim that

$$\mathbf{C}_m^0 := \bigcup_{r\in(0,\infty)} \prod_{N\in\mathbb{N}} \mathcal{D}(\tilde{\mathcal{E}}^{N,m}) \cap (\mathcal{B}_{b,\le 1}(E) \otimes V_{m,r})^{\sim}$$

is a compatible class in $\mathcal{H}_m^{\mathcal{E},1}$. At this point, the argumentation is a bit more involved than its analogue in the proof of Theorem 3.1.12. It is for this part of the proof, the assumption of (3.2.8) has been made. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{H}_m^{\mathcal{E},1}\backslash\mathcal{D}(\tilde{E}^m)$ which converges weakly to an element $u \in \mathcal{D}(\tilde{\mathcal{E}}^m)$. What we have to show (according to the definition of a compatible class) is the following: If, for every $k \in \mathbb{N}$, the class of u_k (to whatever Hilbert space H_k of the sequence, which constitutes $\mathcal{H}_m$, this may refer) has a representative in $\mathcal{B}_{b,\le 1}(E)\otimes V_{m,r}$, then u is the weak limit of $(u_k)_{k\in\mathbb{N}}$ also in $\mathcal{H}$. This is our commitment in the subsequent part.

Let $\varphi_1,\ldots,\varphi_L \in C_b(\mathbb{R}^d)$ be a numbering of all functions from $V_{m,r}$. L depends only on d, m, and r. Then, there are $f_1^{(k)},\ldots f_L^{(k)} \in \mathcal{B}_{b,\le 1}(E)$ such that

$$u_k(z,x) = f_1^{(k)}(z)\varphi_1(x) + \cdots + f_1^{(k)}(z)\varphi_L(x), \quad \text{in a.e.-sense of } H_k. \tag{3.2.10}$$

Next, we use Lemma 2.1.1 (iv) multiple times to obtain convergent (sub-)subsequences in a certain way. We apply Lemma 2.1.1 (iv) w.r.t. the classes of $(f_i^{(k)})_{k\in\mathbb{N}}$ within $\mathcal{H}^{\text{pr}}$ for $i = 1,\ldots,L$. The following statement is based on the observation that all conditions, which have been imposed on the sequences $(m_N)_{N\in\mathbb{N}}$ and $(\nu_N)_{N\in\mathbb{N}}$, would hold for any subsequences of those, as well. Due to Lemma 2.1.1 (iv), in order to prove that $\mathbf{C}_m^0$ is a compatible class in $\mathcal{H}_m^{\mathcal{E},1}$ it suffices to show:

If $u_N \in \mathcal{D}(\tilde{\mathcal{E}}^{N,m}) \cap (\mathcal{B}_{b,\le 1}(E)\otimes V_{m,r})^{\sim}$, $u \in \mathcal{D}(\tilde{\mathcal{E}}^m)$ and $q_1,\ldots,q_L \in L^2(E,\nu)$ such that $\lim_{N\to\infty} u_N = u$ holds weakly in $\mathcal{H}_m^{\mathcal{E},1}$,

$$\lim_{N\to\infty}\int_E f_i^{(N)}(z)g(z)\,\mathrm{d}\nu_N(z) = \int_E q_i(z)g(z)\,\mathrm{d}\nu(z), \quad g \in \mathcal{F}C_b,$$

for $i = 1,\ldots,L$, where $\rho_N(\nu_N \times \kappa_m\,\mathrm{d}x)$-a.e.:

$$u_N(z,x) = f_1^{(N)}(z)\varphi_1(x) + \cdots + f_L^{(N)}(z)\varphi_L(x),\ N \in \mathbb{N},$$

then $\lim_{N\to\infty} u_N = u$ holds weakly in $\mathcal{H}_m$. (3.2.11)

Lemma 2.1.1 (ii) is used two times now, for the pairing of a strongly with a weakly continuous section in $\mathcal{H}_m$, on each occasion. First, in the lines following (3.2.8), we discussed how, as a consequence of (3.2.8), the weak convergence of $\mathcal{H}^{\text{pr}}$ can be re-interpreted as a weak convergence on the 'larger' frame with both coordinates, z and x, i.e. as weak convergence on $\mathcal{H}_m$, in this case. That is why, for $g \in \mathcal{F}C_b$ and $\eta \in C_b(\mathbb{R}^d)$ it holds

$$\lim_{N\to\infty}\int_{E\times\mathbb{R}^d}\Big(\sum_{i=1}^L f_i^{(N)}(z)\varphi_i(x)\Big)g(z)\eta(x)\kappa_m(x)\,\mathrm{d}m_N(x)\,\mathrm{d}\nu_N(z)$$
$$= \int_{E\times\mathbb{R}^d}\Big(\sum_{i=1}^L q_i(z)\varphi_i(x)\Big)g(z)\eta(x)\kappa_m(x)\,\mathrm{d}m(x)\,\mathrm{d}\nu(z). \tag{3.2.12}$$

In other words, we know that $(u_N)_{N\in\mathbb{N}}$ does in fact converge weakly in $\mathcal{H}_m$ and its weak limit u^* can be identified from the right-hand side of the equation above by

$u^*(z,x) = \sum_{i=1}^L q_i(z)\varphi_i(x)$, $(\nu \times \kappa_m m)$-a.e. $(z,x) \in E \times \mathbb{R}^d$. Secondly, by virtue of Remark 3.2.8 combined with Theorem 3.1.6, we also know

$$\lim_{N\to\infty} \int_{E\times\mathbb{R}^d} \Big(\sum_{i=1}^L f_i^{(N)}(z) \big(\nabla\varphi_i(x), \nabla\eta(x) \big)_{\text{euc}} \Big) g(z)\kappa_m(x)\, \mathrm{d}m_N(x)\, \mathrm{d}\nu_N(z)$$

$$= \int_{E\times\mathbb{R}^d} \Big(\sum_{i=1}^L q_i(z) \big(\nabla\varphi_i(x), \nabla\eta(x) \big)_{\text{euc}} \Big) g(z)\kappa_m(x)\, \mathrm{d}m(x)\, \mathrm{d}\nu(z) \quad (3.2.13)$$

for $g \in \mathcal{F}C_b$ and $\eta \in C_b^1(\mathbb{R}^d)$. Now, combining (3.2.12) and (3.2.13), the designated weak limit u of $(u_N)_{N\in\mathbb{N}}$ in $\mathcal{H}_m^{\mathcal{E},1}$ reads $\sum_{i=1}^L q_i(z)\varphi_i(x)$, $(\nu\times\kappa_m m)$-a.e. $(z,x) \in E\times\mathbb{R}^d$. Consequently, $u^* = u$ and (3.2.11) is proven.

From here on, we keep to a shorter argumentation, because we have passed the point in which this proof deviates decisively from the proof of Theorem 3.1.12. We set

$$\mathbf{C}_m^{\mathrm{L}} := \prod_{N\in\mathbb{N}} \Big\{ u \in \mathcal{D}(\tilde{\mathcal{E}}^{N,m}) \cap (\mathcal{B}_b(E) \otimes \mathrm{Lip}_b(\mathbb{R}^d))^\sim \,\Big|\, \text{there is a representative } \tilde{u} \in \mathcal{B}_b(E) \otimes \mathrm{Lip}_b(\mathbb{R}^d) \text{ with } |\tilde{u}(z,x)| \le 1,\ z\in E,\ x \in \mathbb{R}^d, \text{ and } \int_{E\times\mathbb{R}^d} |\nabla_x \tilde{u}|_{\text{euc}}^2 \,\mathrm{d}(\nu_N \times m_N) \le 2 \Big\}.$$

With Proposition 3.2.4 it can be shown that $\mathbf{C}_m^{\mathrm{L}}$ defines a compatible class, using the fact that $\mathbf{C}_m^0$ is a compatible class together with Proposition 2.2.8. This is done analogously as in Theorem 3.1.12. We consider $(u_N)_N \in \mathbf{C}_m^{\mathrm{L}}$ and apply Proposition 3.2.4 w.r.t. each index $N \in \mathbb{N}$ for the respective measures m_N, ν_N and a representative of u_N, to check the assumption of Proposition 2.2.8. The decisive observation at this point is, that the upper bounds found in (i) and (ii) of Proposition 3.2.4 converge to zero as r tends to zero, independently of N, because $\tilde{\mathcal{E}}^{N,m}(u_N,u_N) \le 2$ for $N \in \mathbb{N}$.

The compatible class which is large enough to conclude the proof of (M1) writes

$$\mathbf{C}_m := \prod_{N\in\mathbb{N}} \Big\{ u \in \mathcal{D}(\tilde{\mathcal{E}}^{N,m}) \,\Big|\, \text{there exists } v \in \mathcal{D}(\mathcal{E}^N) \text{ such that } \mathcal{E}^N(v,v) \le 1,\ |v(z,x)| \le 1 \text{ holds } \mathrm{d}(\nu_N \times m_N)\text{-a.e., and } u(z,x) = v(z,x) \text{ holds } \mathrm{d}(\nu_N \times \kappa_m m_N)\text{-a.e.} \Big\}$$

The fact that this is indeed a compatible class in $\mathcal{H}_m^{\mathcal{E},1}$ can be concluded from Proposition 2.2.8 easily. We just use the analogous arguments as in Theorem 3.1.12. To check the assumptions Proposition 2.2.8, we use the density of $(\mathcal{B}_b(E) \otimes C_b^1(\mathbb{R}^d))^\sim$ in $(\mathcal{D}(\mathcal{E}^N), \mathcal{E}_1^N)$ for $N \in \mathbb{N}$ and then fact that the unit contraction operates on $(\mathcal{E}, \mathcal{D}(\mathcal{E}^N))$.

The last part again works analogously to the proof of Theorem 3.1.12. For simplicity, we make no notational indication when changing from a class in $L^2(E\times\mathbb{R}^d, \nu_N \times m_N)$ to the class it induces in $L^2(E\times\mathbb{R}^d, \nu_N \times \kappa_m m_N)$. Analogously, we proceed with $\in L^2(E\times\mathbb{R}^d, \nu\times m)$ and $L^2(E\times\mathbb{R}^d, \nu\times\kappa_m m)$. We exploit the simplification regarding (M1), which Lemma 2.2.3 brings for Dirichlet forms. So, we fix an element

$$(u_N)_{N\in\mathbb{N}} \in \prod_{N\in\mathbb{N}} \Big\{ v \in \mathcal{D}(\mathcal{E}^N) \,\Big|\, |v(z,x)| \le 1,\ \mathrm{d}(\nu\times m)\text{-a.e.} \Big\}.$$

We have to show that any weakly converging subsequence $(u_{N_k})_{k\in\mathbb{N}}$ with limit u^*, referring to the topological structure of

$$\mathcal{H} := L^2(E\times\mathbb{R}^d, \nu\times m) \sqcup \Big(\bigsqcup_{N\in\mathbb{N}} L^2(E\times\mathbb{R}^d, \nu_N\times m_N)\Big)$$

now, fulfils

$$u^* \in \mathcal{D}(\mathcal{E}) \quad \text{with} \quad \mathcal{E}^N(u^*,u^*) \le \liminf_{k\to\infty} \mathcal{E}^{N_k}(u_{N_k},u_{N_k}). \tag{3.2.14}$$

We remark that in such an instance $(u_{N_k})_{k\in\mathbb{N}}$ converges weakly towards u^* in $\mathcal{H}_m$ for every $m\in\mathbb{N}$. Moreover, for $m\in\mathbb{N}$ the sequence $(u_N)_{N\in\mathbb{N}}$ is a member of $\mathbf{C}_m$ and every sequence in

$$\prod_{N\in\mathbb{N}} \Big\{u\in\mathcal{D}(\tilde{\mathcal{E}}^{N,m}) \,\Big|\, \tilde{\mathcal{E}}_1^{N,m}(u,u)\le 2\Big\}$$

has a weakly convergent subsequence in $\mathcal{H}_m^{\mathcal{E},1}$. Hence, we can achieve the following for every accumulation point u^* of $(u_N)_{N\in\mathbb{N}}$ w.r.t. the weak topology of $\mathcal{H}$, by repeatedly dropping to a suitable subsequence: For every subsequence of $(u_N)_{N\in\mathbb{N}}$ converging weakly to u^* there exists a (sub-)subsequence such that

$$u_{N_k} \overset{k\to\infty}{\rightharpoonup} u^* \text{ in } \mathcal{H} \quad \text{and} \quad u_{N_k} \overset{k\to\infty}{\rightharpoonup} u^* \text{ in } \mathcal{H}_m^{\mathcal{E},1}, \quad m\in\mathbb{N}. \tag{3.2.15}$$

This includes the statement that $u^*\in\mathcal{D}(\tilde{\mathcal{E}}^m)$ for each $m\in\mathbb{N}$. As a consequence of (3.2.15) and the weak convergence of $(u_{N_k})_{k\in\mathbb{N}}$ towards u^* in $\mathcal{H}_m$ we obtain the weak convergence of $(u_{N_k})_{k\in\mathbb{N}}$ towards u^* in $H_m^{\mathcal{E},\alpha}$ for every $\alpha>0$ and $m\in\mathbb{N}$. In particular,

$$\tilde{\mathcal{E}}_\alpha^m(u^*,u^*) \le \liminf_{N\to\infty} \tilde{\mathcal{E}}_\alpha^{N_k,m}(u_{N_k},u_{N_k}) nonumber \tag{3.2.16}$$

$$\le \liminf_{k\to\infty} \mathcal{E}_\alpha^{N_k}(u_{N_k},u_{N_k}), \quad \alpha>0,\, m\in\mathbb{N}. \tag{3.2.17}$$

An issue with the domains must be settled. Let $m\in\mathbb{N}$. We know that that $\kappa_m(x)u^*(z,x)$, $z\in E$, $x\in\mathbb{R}^d$ is a member of $\mathcal{D}(\mathcal{E})$, which follows from $u\in\mathcal{D}(\tilde{\mathcal{E}}^{m+1})$. The fact, that the right-hand side of the estimate is independent of m, together with $\sup_m \|\,|\nabla\kappa_m|_{\mathrm{euc}}\|_\infty<\infty$, implies

$$\sup_{m\in\mathbb{N}} \mathcal{E}(\kappa_m u^*, \kappa_m u^*) < \infty.$$

Of course, the latter yields $u^*\in\mathcal{D}(\mathcal{E})$. By letting $\alpha\to 0$ and then $m\to\infty$ we obtain the inequality of (3.2.14) as desired. This concludes the proof of (M1).

The verification of (M2) is simple. Referring to the structure of $\mathcal{H}$ and defining $(\Phi_N)_{N\in\mathbb{N}}$ accordingly, we have the strong convergence of $|\nabla_x\Phi_N u|_{\mathrm{euc}} = \Phi_N|\nabla_x u|_{\mathrm{euc}}$ towards $|\nabla_x u|_{\mathrm{euc}}$ as N tends to infinity for each $u\in\mathcal{D}(\mathcal{E})\cap\cap(\mathcal{F}C_b\otimes C_b^1(\mathbb{R}^d))^\sim$. In particular, $(\mathcal{D}(\mathcal{E}^N),\mathcal{E}_1^N)$, $N\in\mathbb{N}$, form a sequence of converging Hilbert spaces on their own right with limit $(\mathcal{D}(\mathcal{E}),\mathcal{E}_1)$. This implies (M2) and concludes the proof. □

Chapter 4 The disintegration method

4.1 Classical Dirichlet forms on topological vector spaces

In this chapter we elaborate the method of disintegration to study the approximation of a classical gradient form on a locally convex topological vector space E, which is further assumed to be a (Hausdorff) Suslin space, in terms of Mosco convergence. A densely included Hilbert space $(H, \langle \cdot, \cdot \rangle_H) \subseteq E$ shall be fixed to serve as a tangential space, with the inclusion map being continuous from H to E. Identifying H with its dual, we have

$$E' \subseteq H' = H \subseteq E \tag{4.1.1}$$

and all inclusions in the line above are dense. The gradient of a cylindrical smooth function

$$f \in \mathcal{F}C_b^\infty := \big\{ F(l_1, \dots, l_m) \,\big|\, m \in \mathbb{N},\, F \in C_b^\infty(\mathbb{R}^m),\, l_1, \dots, l_m \in E' \big\}$$

at a point $z \in E$ is defined to be the unique element $\nabla f(z) \in H$ which represents the bounded linear functional

$$\langle \nabla f(z), h \rangle_H := \lim_{s \to 0} \frac{f(z + sh) - f(z)}{s}, \quad h \in H. \tag{4.1.2}$$

Let μ be a Borel probability measure on E. Throughout Chapter 4, we use the notation and convention introduced at the end of Section 2.1. The classic theory of Dirichlet forms in infinite dimension gives the answer to the question of closability of the gradient form

$$\mathcal{E}(u, v) := \int_E \langle \nabla u, \nabla v \rangle_H \, \mathrm{d}\mu, \quad u, v \in L^2(E, \mu) \cap \widetilde{\mathcal{F}C_b^\infty}, \tag{4.1.3}$$

on $L^2(E, \mu)$ and the preliminary question whether the action of the gradient can be defined for μ-classes in the first place, i.e.

$$\nabla f = \nabla g \; \mu\text{-a.e.} \quad \text{for } g, f \in \mathcal{F}C_b^\infty \text{ with } f = g \; \mu\text{-a.e.}, \tag{4.1.4}$$

by disintegrating μ and decomposing $\mathcal{E}$, along a suitable class of one-dimensional subspaces of E. Thanks to this method, it is effectively enough, even for the infinite dimensional setting, to have a closability condition for a classic energy form on the real line, such as Condition 3.1.10 provides for the form defined in (3.1.36) of Section 3.1.2 with $d = 1$. In what follows, we pursue a similar strategy, making the results of Chapter 3 applicable in order to derive a statement on Mosco convergence, related to a sequence of weakly converging probability measures $(\mu_N)_{N \in \mathbb{N}}$ on E, whose limit is μ. At first, we recall the standard criterion for (4.1.4) and the closability of the form in (4.1.3). The right-hand side of (4.1.2) exits even for arguments in E. Given $k \in E \setminus \{0\}$, let

$$\frac{\partial f}{\partial k}(z) := \lim_{s \to 0} \frac{f(z + sk) - f(z)}{s}$$

denote the Gâteaux-type derivative of a function $f \in \mathcal{F}C_b^\infty$ at a point $z \in E$ in the direction of k. An element $k \in E \setminus \{0\}$ is called μ-admissible, if for one thing, the 'directional' analogue of (4.1.4) is fulfilled, i.e. if

$$\frac{\partial f}{\partial k} = \frac{\partial g}{\partial k} \; \mu\text{-a.e.} \quad \text{for } g, f \in \mathcal{F}C_b^\infty \text{ with } f = g \; \mu\text{-a.e.},$$

and for the other, the component form

$$\mathcal{E}_k(u,v) := \int_E \frac{\partial u}{\partial k}\frac{\partial v}{\partial k}\,\mathrm{d}\mu, \quad u,v \in L^2(E,\mu)\cap \widetilde{\mathcal{F}C_b^\infty}, \tag{4.1.5}$$

is closable on $L^2(E,\mu)$. For μ-admissible $k \in E\setminus\{0\}$ we denote the smallest closed extension of (4.1.5) on $L^2(E,\mu)$ by $(\mathcal{E}_k, \mathcal{D}(\mathcal{E}_k))$. All the statements, we briefly recall in the next paragraph, are proven in [17, Section 3]. In the article we just cited, the presentation of the matter is self-contained, while references to earlier works are given, including [6, 7, 8, 9, 12] to name some relevant ones in this context.

If there is an orthonormal basis K_0 in H, which consists of μ-admissible elements, then (4.1.4) holds and the form in (4.1.3) is closable on $L^2(E,\mu)$. Denoting its closure by $(\mathcal{E},\mathcal{D}(\mathcal{E}))$, we obtain a Dirichlet form. This means that $\mathcal{D}(\mathcal{E})$ is a dense linear subspace of $L^2(E,\mu)$ and the unit contraction operates on $(\mathcal{E},\mathcal{D}(\mathcal{E}))$. For fixed $k\in E\setminus\{0\}$ and $l \in E'$ such that $l(k)=1$ the vector space E decomposes into the direct sum

$$\pi_k(E)\oplus \operatorname{span}(\{k\}), \quad \text{where} \quad \pi_k(z) := z - l(z)k, \quad z\in E.$$

The assignment $(\pi_k(z), l(z))$, $z\in E$, yields a vector space isomorphism from E into $\pi_k(E)\times\mathbb{R}$, with inverse mapping $z+sk$, $z\in\pi_k(E)$, $s\in\mathbb{R}$. In the following, we always regard the image measure of μ under the map π_k as a probability measure on $(E,\mathcal{B}(E))$ with support contained in $\pi_k(E)$, and make the choice of l clear by context. There is family of probability measures $m(z,\cdot)$, $z\in E$, on $\mathcal{B}(\mathbb{R})$ such that

$$\begin{aligned}\int_E f\,\mathrm{d}\mu &= \int_{E\times\mathbb{R}} f(z+sk)\,\mathrm{d}\big(\mu\circ(\pi_k,l)^{-1}\big)(z,s)\\ &= \int_E\int_{\mathbb{R}} f(z+sk)m(z,\mathrm{d}s)\,\mathrm{d}\nu(z) \quad (\text{with } \nu := \mu\circ\pi_k^{-1}),\end{aligned} \tag{4.1.6}$$

holds for $f\in\mathcal{B}(E)$ non-negative, or $f\in\mathcal{B}_b(E)$. The above equation determines $m(z,\cdot)$, $z\in E$, uniquely in a.e.-sense w.r.t. $\mathrm{d}\nu(z)$. The element k is μ-admissible, if and only if, the image $\mu\circ(\pi_k,l)^{-1}$ of μ under the map (π_k,l) satisfies Condition 3.2.5 with $d=1$. The latter requires that for ν-a.e. $z\in E$ the Radon-Nikodym derivative $\rho(z,s) = \frac{\mathrm{d}m(z,\cdot)}{\mathrm{d}s}(s)$, $s\in\mathbb{R}$, exists and meets Condition 3.1.10. As this characterization of μ-admissibility implies, the validity of Condition 3.2.5 regarding the image of μ under (π_k,l) is true either for all elements $l\in E'$ with $l'(k)=1$, or for none of them. We fix a μ-admissible element $k\in E\setminus\{0\}$ together with $l\in E'$ with $l'(k)=1$ and set $\nu := \mu\circ\pi_k^{-1}$. The isometric isomorphism, which naturally arises from the disintegration given in (4.1.6), between $L^2(E,\mu)$ and the direct integral $\int^{\oplus} L^2(\mathbb{R},\rho(z,\cdot)\,\mathrm{d}s)\,\mathrm{d}\nu(z)$ in the sense of [4, Chapter 2.1], accounts for the definitions of (4.1.7) and (4.1.8) below. We also recall

$$L^2\big(\mathbb{R}^d,\rho(z,\cdot)\,\mathrm{d}s\big)\hookrightarrow L^1_{\mathrm{loc}}\big(R(\rho(z,\cdot)),\mathrm{d}s\big)$$

if $z\in E$ such that $\rho(z,\cdot)$ fulfils Condition 3.1.10. The partial derivative $u\mapsto\frac{\partial u}{\partial k}$ extends to a closed linear operator on $L^2(E,\mu)$ with domain

$$\begin{aligned}\mathcal{D}_{\max}(\mathcal{E}_k) := \Big\{u\in L^2(E,\mu)\,\Big|\, & u(z+\,\cdot\,k)\in H^{1,1}_{\mathrm{loc}}\big(R(\rho(z,\cdot))\big),\ \nu\text{-a.e. } z\in E,\\ & u(z+\,\cdot\,k)'(s) = v(z,s),\ \mu\circ(\pi_k,l)^{-1}\text{-a.e. } (z,s)\in E\times\mathbb{R}\\ & \text{for some element } v\in L^2\big(E\times\mathbb{R},\mu\circ(\pi_k,l)^{-1}\big)\Big\},\end{aligned} \tag{4.1.7}$$

via the assignment

$$\frac{\partial u}{\partial k}(z+sk) := u(z+\,\cdot\,k)'(s), \quad \mu\circ(\pi_k,l)^{-1}\text{-a.e.}, \quad (z,s)\in E\times\mathbb{R}. \tag{4.1.8}$$

The component form of (4.1.5) extends to a Dirichlet form $(\mathcal{E}_k, \mathcal{D}_{\max}(\mathcal{E}_k))$ on $L^2(E,\mu)$. The action of $\frac{\partial}{\partial k}$ on $\mathcal{D}_{\max}(\mathcal{E}_k)$ is referred to as the μ-stochastic partial derivative in the direction of k. To introduce the concept of a total μ-stochastic derivative, let a dense linear subspace K of H be given such that each element $k\in K\setminus\{0\}$ is μ-admissible with $l\in E'$, $l(k)=1$, fixed and let

$$\mathcal{D}^+_K(\mathcal{E}) := \Big\{u\in \bigcap_{k\in K\setminus\{0\}} \mathcal{D}_{\max}(\mathcal{E}_k)\,\Big|\, \text{there exists } \nabla u\in L^2(E,H,\mu): \\ \langle\nabla u, k\rangle_H = \frac{\partial u}{\partial k}\ \mu\text{-a.e.}, \quad k\in K\setminus\{0\}\Big\}.$$

The gradient in the definition above coincides with the gradient on $L^2(E,\mu)\cap\widetilde{\mathcal{F}C_b^\infty}$ given through (4.1.2) and (4.1.4). The form $\mathcal{E}$ of (4.1.3) extends to a Dirichlet form with domain $\mathcal{D}^+_K(\mathcal{E})$. If $K_0\subset K$ and K_0 is an orthonormal basis in H, then it holds

$$\mathcal{E}(u,v) = \sum_{k\in K_0}\int_E \frac{\partial u}{\partial k}\frac{\partial v}{\partial k}\,\mathrm{d}\mu, \quad u,v\in\mathcal{D}^+_K(\mathcal{E}).$$

A larger domain in $L^2(E,\mu)$ onto which the bilinear form on the right-hand side of the equation above can naturally be extended reads

$$\mathcal{D}^{\max}_{K_0}(\textstyle\sum_k \mathcal{E}_k) := \Big\{u\in\bigcap_{k\in K_0}\mathcal{D}_{\max}(\mathcal{E}_k)\,\Big|\,\sum_{k\in K_0}\mathcal{E}_k(u,u)<\infty\Big\}.$$

The sequence

$$(\mathcal{E},\mathcal{D}(\mathcal{E}))\subseteq(\mathcal{E},\mathcal{D}^+_K(\mathcal{E}))\subseteq\big(\mathcal{E},\mathcal{D}^{\max}_{K_0}(\textstyle\sum_k\mathcal{E}_k)\big) \tag{4.1.9}$$

describes Dirichlet forms on $L^2(E,\mu)$ extending each other.

The question under which circumstances the domains of all forms in (4.1.9) do actually coincide is relevant in view of Theorem 4.1.3 below, as it would be sufficient for (4.1.19). The reader should have in mind that from an application point of view one is most likely interested in the minimal form $(\mathcal{E},\mathcal{D}(\mathcal{E}))$, where only the tangential space H is fixed from a given context (and not the subspace K or the basis K_0). The arguments derived in this section to find approximations for $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ in the sense of Mosco, however, do require a suitable representation of the minimal gradient form as an infinite sum of one-component forms and alongside a corresponding characterization of $\mathcal{D}(\mathcal{E})$. In every concrete example, it is thus a preliminary task for the applicability of the abstract convergence result of this section to find a suitable orthonormal basis K_0 of the given Hilbert space H in order to guarantee a decomposition of the minimal gradient form. The concept of Markov uniqueness is closely related. It gives a necessary and sufficient condition for all form domains in (4.1.9) to coincide, if every $k\in K\setminus\{0\}$ is a so-called *well-μ-admissible* element, under the additional assumption that, referring to (4.1.1), either K equals E', or that the strong dual topology on E' is metrizable with K being densely included in E'. Being a stronger notion than μ-admissibility, the well-μ-admissibility of k by definition requires $\rho(z,\cdot)$ to be the representative of an element in $H^{1,1}_{\mathrm{loc}}(\mathbb{R})$ for ν-a.e. $z\in E$ and moreover

$$\big((\ln\circ\rho)(z,\cdot)'\big)_{z\in E}\in\int^{\oplus} L^2(\mathbb{R},\rho(z,\cdot)\,\mathrm{d}s)\,\mathrm{d}\nu(z).$$

It forces the space $L^2(E,\mu) \cap \widetilde{\mathcal{F}C_b^\infty}$ to be contained in the generator domain of $(\mathcal{E}_k, \mathcal{D}_{\max}(\mathcal{E}_k))$. Under the named assumptions, equality of the form domains in (4.1.9) is equivalent to the uniqueness of all Markovian self-adjoint extensions on $L^2(E,\mu)$ of the operator L with domain $L^2(E,\mu) \cap \widetilde{\mathcal{F}C_b^\infty}$, characterized via

$$\int_E (-Lu)v \, \mathrm{d}\mu = \mathcal{E}(u,v), \quad u \in L^2(E,\mu) \cap \widetilde{\mathcal{F}C_b^\infty},\, v \in \mathcal{D}(\mathcal{E}).$$

This is the statement of [22, Theorem 1.9]. If we merely assume that K is a dense subspace of E' (without the metrizability of E') consisting of μ-well-admissible elements, then by virtue of [16, Theorem 3.1] it holds $\mathcal{D}_K^+(\mathcal{E}) = \mathcal{D}_{K_0}^{\max}(\sum_k \mathcal{E}_k)$ and the Markov uniqueness of $(L, L^2(E,\mu) \cap \widetilde{\mathcal{F}C_b^\infty})$ is still a sufficient criterion for $\mathcal{D}(\mathcal{E}) = \mathcal{D}_K^+(\mathcal{E})$.

Our survey on the approximation of gradient forms on E puts the focus on component forms of the type in (4.1.5) at first. We assume we are given a sequence of weakly convergent probability measures $(\mu_N)_{N\in\mathbb{N}}$ on $(E, \mathcal{B}(E))$ whose limit is μ, together with a convergent sequence $(k_N)_{N\in\mathbb{N}}$ in E with limit k, presuming the μ_N-admissibility of $k_N \neq 0$ for $N \in \mathbb{N}$, the μ-admissibility of $k \neq 0$, and the inclusion

$$\operatorname{supp}[\mu_N] \subseteq \operatorname{supp}[\mu], \quad N \in \mathbb{N}, \tag{4.1.10}$$

regarding the topological supports of the measures. This constitutes the basic setting in which it makes sense to ask whether the corresponding component forms converge in the sense of Mosco. For $(\mu_N)_N$ and $(k_N)_N$ chosen like this, the results of Section 3.2.2 provide us with a sufficient criterion for the Mosco convergence of $(\mathcal{E}^{k_N}, \mathcal{D}(\mathcal{E}_{k_N}))_N$ towards $(\mathcal{E}_k, \mathcal{D}(\mathcal{E}_k))$, where $(\mathcal{E}_k, \mathcal{D}(\mathcal{E}_k))$ is as above and $(\mathcal{E}^{k_N}, \mathcal{D}(\mathcal{E}_{k_N}))$ analogously denotes the smallest closed extension of

$$\mathcal{E}_{k_N}(u,v) := \int_E \frac{\partial u}{\partial k_N} \frac{\partial v}{\partial k_N} \, \mathrm{d}\mu_N, \quad u,v \in L^2(E,\mu_N) \cap \widetilde{\mathcal{F}C_b^\infty},$$

on $L^2(E,\mu_N)$ for $N \in \mathbb{N}$.

Proposition 4.1.1. *We assume that $(\mu_N)_{N\in\mathbb{N}}$ is a tight sequence of probability measures. $(\mathcal{E}^{k_N}, \mathcal{D}(\mathcal{E}_{k_N}))_{N\in\mathbb{N}}$ Mosco converges to $(\mathcal{E}_k, \mathcal{D}(\mathcal{E}_k))$ if the following condition is imposed on the sequence $(k_N)_N$ (and its limit k):*

We can choose $l \in E'$ with $l(k) = 1$ and $l_N \in E'$ with $l_N(k_N) = 1$ for $N \in \mathbb{N}$ such that $(\star)$ holds, where we set

$$\pi_k(z) := z - l(z)k, \quad \pi_{k_N}(z) : z - l_N(z)k_N, \quad z \in E,\, N \in \mathbb{N}.$$

$(\star)$ l is the limit of $(l_N)_{N\in\mathbb{N}}$ in E' w.r.t. the strong dual topology. The images $\mu_N \circ (\pi_{k_N}, l_N)^{-1}$, $N \in \mathbb{N}$, and $\mu \circ (\pi_k, l)^{-1}$, form a family of Borel probability measures on $E \times \mathbb{R}$, which, for the choice $J := \pi_k$, meet the assumptions of Theorem 3.2.9.

Reciting the assumptions of Theorem 3.2.9 in this context, we require for a family of disintegrating Hamza probabilities $m(z, \mathrm{d}s) = \rho(z,s)\,\mathrm{d}s$, respectively $m_N(z,\mathrm{d}s) = \rho_N(z,s)\,\mathrm{d}s$, $z \in E$, on $\mathcal{B}(\mathbb{R})$ to meet:

- $\displaystyle\int_E f \,\mathrm{d}\mu = \int_E \int_{\mathbb{R}} f(z+sk) m(z,\mathrm{d}s) \,\mathrm{d}(\mu \circ \pi_k^{-1})(z)$, *respectively*
  $\displaystyle\int_E f \,\mathrm{d}\mu_N = \int_E \int_{\mathbb{R}} f(z+sk_N) m_N(z,\mathrm{d}s) \,\mathrm{d}(\mu_N \circ \pi_{k_N}^{-1})(z)$, *$N \in \mathbb{N}$, for $f \in \mathcal{B}_b(E)$.*
- $\operatorname{supp}[\mu_N \circ \big((\pi_k \times \mathrm{id}_{\mathbb{R}}) \circ (\pi_{k_N}, l_N)\big)^{-1}] \subseteq \operatorname{supp}[\mu \circ (\pi_k, l)^{-1}], \quad N \in \mathbb{N}.$

- $\lim_{N\to\infty} \int_E (f\circ\pi_{k_N})\cdot(\varphi\circ l_N)\,\mathrm{d}\mu_N = \int_E (f\circ\pi_k)\cdot(\varphi\circ l)\,\mathrm{d}\mu,\ f\in\mathcal{F}C_b,\ \varphi\in C_b(\mathbb{R}).$

- $$\lim_{N\to\infty}\int_E \Big|\int_{\mathbb{R}}\varphi(s)\,\mathrm{d}m_N(z,\mathrm{d}s)\Big|^2\,\mathrm{d}(\mu_N\circ\pi_{k_N}^{-1})(z)$$
$$= \int_E \Big|\int_{\mathbb{R}}\varphi(s)\,\mathrm{d}m(z,\mathrm{d}s)\Big|^2\,\mathrm{d}(\mu\circ\pi_k^{-1})(z),\quad \varphi\in C_b(\mathbb{R}).$$

- $\limsup_{k\to\infty}\sup_{N\in\mathbb{N}}\operatorname*{ess\,sup}_{(z,s)\in E\times[-m,m]} \rho_N^{-1}(z,s)\cdot\big(\rho_N(z,\cdot)_{\mathrm{ave},\frac{1}{k}}(s)\big)<\infty$ *for* $m\in\mathbb{N}$, *where the essential supremum above is taken w.r.t. the product measure* $(\mu_N\circ\pi_{k_N}^{-1})\times\mathrm{d}s$.

- $\lim_{k\to\infty}\limsup_{N\to\infty}\int_E\int_{[-m,m]}\big|\rho_N(z,s)-\big(\rho_N(z,\cdot)_{\mathrm{ave},\frac{1}{k}}(s)\big)\big|\,\mathrm{d}s\,\mathrm{d}(\mu_N\circ\pi_{k_N}^{-1})(z)=0.$

Proof. The subject of this proposition is a statement about Mosco convergence on

$$\mathcal{H} := L^2(E,\mu)\sqcup\Big(\bigsqcup_{N\in\mathbb{N}} L^2(E,\mu_N)\Big).$$

Due to (4.1.10), the maps $\Phi_N^{\sim}$, from $\mathcal{C} := L^2(E,\mu)\cap\widetilde{\mathcal{F}C_b}$ into $L^2(E,\mu_N)$ for $N\in\mathbb{N}$ respectively, can be defined as in the end of Section 2.1, providing a strong and a weak topology on $\mathcal{H}$. Before we explain step by step why the claim of this proposition follows from Theorem 3.2.9 and the compatibility equation

$$\lim_{N\to\infty}\int_E (g\circ\pi_k\circ\pi_{k_N})\cdot(\varphi\circ l_N)\cdot f\,\mathrm{d}\mu_N$$
$$= \int_E (g\circ\pi_k)\cdot(\varphi\circ l)\cdot f\,\mathrm{d}\mu,\quad f,g\in\mathcal{F}C_b,\ \varphi\in C_b(\mathbb{R}), \tag{4.1.11}$$

we give the short proof for (4.1.11). Obviously,

$$\lim_{N\to\infty}\tilde{l}\circ\pi_k\circ\pi_{k_N}=\tilde{l}\circ\pi_k\quad\text{strongly in } E'\quad\text{for } \tilde{l}\in E'. \tag{4.1.12}$$

If $h\in C_b(\mathbb{R}^m)$ for some $m\in\mathbb{N}$ and K is a compact subset in E, then

$$\Big|\int_E h\big(\tilde{l}_N^{(1)},\dots,\tilde{l}_N^{(m)}\big)\,\mathrm{d}\mu_N-\int_E h\big(\tilde{l}^{(1)},\dots,\tilde{l}^{(m)}\big)\,\mathrm{d}\mu\Big|$$
$$\le \sup_{z\in K}\big|h\big(\tilde{l}_N^{(1)}(z),\dots,\tilde{l}_N^{(m)}(z)\big)-h\big(\tilde{l}^{(1)}(z),\dots,\tilde{l}^{(m)}(z)\big)\big|$$
$$+\|h\|_\infty\sup_{N\in\mathbb{N}}\mu_N(E\setminus K),\quad \tilde{l}_N^{(i)},\tilde{l}^{(i)}\in E',\ i=1,\dots,m,\ N\in\mathbb{N}.$$

Hence, due to the tightness of $(\mu_N)_{N\in\mathbb{N}}$,

$$\lim_{N\to\infty}\tilde{l}_N^{(i)}=\tilde{l}^{(i)}\quad\text{strongly in } E',\quad \tilde{l}_N^{(i)},\tilde{l}^{(i)}\in E',\ i=1,\dots,m,\ N\in\mathbb{N},$$
$$\text{implies}\quad \lim_{N\to\infty}\int_E h\big(\tilde{l}_N^{(1)},\dots,\tilde{l}_N^{(m)}\big)\,\mathrm{d}\mu_N=\int_E h\big(\tilde{l}^{(1)},\dots,\tilde{l}^{(m)}\big)\,\mathrm{d}\mu$$
$$\text{for every } m\in\mathbb{N} \text{ and } h\in C_b(\mathbb{R}^m). \tag{4.1.13}$$

This proves (4.1.11) in view of (4.1.12).

We address the main part of this proof. Theorem 3.2.9 settles the Mosco convergence of a sequence of closed symmetric forms within another frame of converging

Hilbert spaces. It can be applied to the spaces $H_N := L^2(E\times\mathbb{R}, \mu_N\circ(\pi_{k_N}, l_N)^{-1})$, $N\in\mathbb{N}$, with asymptotic space $H := L^2(E\times\mathbb{R}, \mu\circ(\pi_k, l)^{-1})$. The asymptotic isometries, denoted by $(\Psi_N)_{N\in\mathbb{N}}$ in this proof, which define the corresponding topological notions on

$$\widehat{\mathcal{H}} := H \sqcup \Big(\bigsqcup_{N\in\mathbb{N}} H_N\Big)$$

are based on the assumption of (3.2.5), i.e.

$$\operatorname{supp}[\mu_N\circ\big((\pi_k\times\mathrm{id}_{\mathbb{R}})\circ(\pi_{k_N}, l_N)\big)^{-1}] \subseteq \operatorname{supp}[\mu\circ(\pi_k, l)^{-1}], \quad N\in\mathbb{N}. \tag{4.1.14}$$

For $N\in\mathbb{N}$ the map Ψ_N is defined on the space $\widehat{\mathcal{C}} := H\cap\big(\mathcal{F}C_b\times C_b(\mathbb{R})\big)^{\sim}$ and sends an element $v\in\widehat{\mathcal{C}}$ onto the class in H_N of the function $f(\pi_k z, s)$, $(z,s)\in E\times\mathbb{R}$, if f is a representative for u in H. In the course of this proof, let $(T_t)_{t\geq 0}$ on H and $(T_t^N)_{t\geq 0}$ on H_N for $N\in\mathbb{N}$ be the semigroups for which the analogue of the statement in Theorem 2.2.1 (ii) is fulfilled as part of the assumptions of this proposition. For an element $u\in\mathcal{H}$, we define $\widehat{u}\in\widehat{\mathcal{H}}$ as the class which is determined for $(z,s)\in E\times\mathbb{R}$ by

$$\widehat{u}(z,s) := \begin{cases} u(z+sk_N),\ \mu_N\circ(\pi_{k_N}, l_N)^{-1}\text{-a.e. } (z,s), & \text{if } u\in L^2(E,\mu_N),\\ u(z+sk),\ \mu\circ(\pi_k, l)^{-1}\text{-a.e. } (z,s), & \text{if } u\in L^2(E,\mu).\end{cases}$$

The compatibility equation (4.1.11) translates into

$$\lim_{N\to\infty}\big(\Psi_N v, \widehat{\widetilde{\Phi_N} u}\big)_{H_N} = \big(v, \widehat{u}\big)_H, \quad v\in\widehat{\mathcal{C}},\ u\in\mathcal{C},$$

or equivalently

$$\lim_{N\to\infty}\big((\Psi_N v)\circ(\pi_{k_N}, l_N), \widetilde{\Phi_N} u\big)_{L^2(E,\mu_N)}$$
$$= \big(v\circ(\pi_k, l), u\big)_{L^2(E,\mu)}, \quad v\in\widehat{\mathcal{C}},\ u\in\mathcal{C}.$$

By virtue of Remark 2.1.2 (ii), given a sequence $(u_N)_{N\in\mathbb{N}}$ in $\mathcal{H}$ and an element $u\in L^2(E,\mu)$, this means that

$$\lim_{N\to\infty} u_N = u \text{ strongly in } \mathcal{H} \quad \text{if and only if} \quad \lim_{N\to\infty}\widehat{u_N} = \widehat{u} \text{ strongly in } \widehat{\mathcal{H}}.$$

We have

$$\lim_{N\to\infty}\widehat{\widetilde{\Phi_N} u} = \widehat{u} \quad \text{strongly in } \widehat{\mathcal{H}}, \quad u\in\mathcal{C},$$

and hence

$$\lim_{N\to\infty} T_t^N\widehat{\widetilde{\Phi_N} u} = T_t\widehat{u} \quad \text{strongly in } \widehat{\mathcal{H}}.$$

This, in turn, leads to

$$\lim_{N\to\infty}\big(T_t^N\widehat{\widetilde{\Phi_N} u}\big)\circ(\pi_{k_N}, l_N) = (T_t\widehat{u})\circ(\pi_k, l) \quad \text{strongly in } \mathcal{H}.$$

In view of Theorem 2.2.1 and Remark 2.2.2, we have just shown the Mosco convergence on $\mathcal{H}$ for a sequence of closed symmetric forms, the ones which belong to the semigroup of operators

$$S_t^N u := \big(T_t^N\widehat{u}\big)\circ(\pi_{k_N}, l_N), \quad u\in L^2(E,\mu_N),\ t\geq 0,$$

respectively for $N \in \mathbb{N}$. It converges in the sense of Mosco to the form on $L^2(E,\mu)$ whose semigroup is given by

$$S_t u := \big(T_t \widehat{u}\big) \circ (\pi_k, l), \quad u \in L^2(E,\mu),\, t \geq 0.$$

In the last part of this proof, we want to understand why $(S_t^N)_{t\geq 0}$ is the semigroup of $(\mathcal{E}_{k_N}, \mathcal{D}(\mathcal{E}_{k_N}))$ for $N \in \mathbb{N}$ and $(S_t)_{t\geq 0}$ is the semigroup of $(\mathcal{E}_k, \mathcal{D}(\mathcal{E}_k))$. Below, we call the symmetric bilinear form on $L^2(E,\mu)$, defined via

$$(u,v) \mapsto \tilde{\mathcal{E}}(\widehat{u}, \widehat{v}) \quad \text{with domain} \quad \big\{ w \in L^2(E,\mu) \,\big|\, \widehat{w} \in \mathcal{D}(\tilde{\mathcal{E}}) \big\},$$

the image form of $(\tilde{\mathcal{E}}, \mathcal{D}(\tilde{\mathcal{E}}))$ under the map $(z,s) \mapsto z + sk$, $(z,s) \in E \times \mathbb{R}$, for a given symmetric bilinear form $(\tilde{\mathcal{E}}, \mathcal{D}(\tilde{\mathcal{E}}))$ on H. The term 'form core' depicts a linear subspace which is dense in the Hilbert space $(\mathcal{D}(\tilde{\mathcal{E}}), \tilde{\mathcal{E}}_1)$.

Let $\rho(z,s) := \frac{\mathrm{d}m(z,\cdot)}{\mathrm{d}s}(s)$, $z \in E$, $s \in \mathbb{R}$, where $m(z,\cdot)$, $z \in E$ is a family of disintegration measures for $\mu \circ (\pi_k, l)^{-1}$ as in (4.1.6). By the choice of $(T_t)_{t\geq 0}$ and because $u \mapsto \widehat{u}$ defines an isomorphism from $L^2(E,\mu)$ into H with inverse map $v \mapsto v \circ (\pi_k, l)$. the form belonging to $(S_t)_{t\geq 0}$ is the image form of the one considered in Theorem 3.2.9 under the map $(z,s) \mapsto z + sk$, $(z,s) \in E \times \mathbb{R}$. That means, the form of $(S_t)_{t\geq 0}$ is the image form of the closed symmetric form on H which reads

$$(u,v) \mapsto \int_E \mathcal{E}^{\rho(z,\cdot)}(u(z,\cdot), v(z,\cdot))\, \mathrm{d}(\mu \circ \pi_k^{-1}) \tag{4.1.15}$$

for u, v in its domain

$$\Big\{ w \in L^2\big(E \times \mathbb{R}, \mu \circ (\pi_k, l)^{-1}\big) \,\Big|\, w(z,\cdot) \in \mathcal{D}\big(\mathcal{E}^{\rho(z,\cdot)}\big),\ \mu \circ \pi_k^{-1}\text{-a.e. } z \in E,$$
$$w(z,\cdot)'(s) = q(z,s),\ \mu \circ (\pi_k, l)^{-1}\text{-a.e. } (z,s) \in E \times \mathbb{R},$$
$$\text{for some element } q \in L^2\big(E \times \mathbb{R}, \mu \circ (\pi_k, l)^{-1}\big) \Big\}.$$

So, the form associated with $(S_t)_{t\geq 0}$ is an extension of $(\mathcal{E}_k, \mathcal{D}(\mathcal{E}_k))$. On the other hand, a form core of the closed symmetric form in (4.1.15) is given by the space $H \cap (\mathcal{F}C_b^\infty \otimes C_b^\infty(\mathbb{R}))^\sim$, as pointed out in Remark 3.2.6. Since the map

$$z \mapsto f(\pi_k(z))\varphi(l(z)), \quad z \in E,$$

again belongs to $\mathcal{F}C_b^\infty$ for every $f \in \mathcal{F}C_b^\infty$ and $\varphi \in C_b^\infty(\mathbb{R})$, the space $L^2(E,\mu) \cap \widetilde{\mathcal{F}C_b^\infty}$ is a core for the image form of (4.1.15) under $(z,s) \mapsto z + sk$, $(z,s) \in E \times \mathbb{R}$, i.e. the one associated to $(S_t)_{t\geq 0}$. This proves that $(\mathcal{E}_k, \mathcal{D}(\mathcal{E}_k))$ is associated to $(S_t)_{t\geq 0}$. With the very same argumentation one shows that $(S_t^N)_{t\geq 0}$ is the semigroup of $(\mathcal{E}_{k_N}, \mathcal{D}(\mathcal{E}_{k_N}))$ for $N \in \mathbb{N}$ and the proof of this proposition is complete. □

For the above proof to work, the sequence $(l_N)_{N\in\mathbb{N}}$ is required to converge to l uniformly on compact sets K, meaning

$$\sup_{N\in\mathbb{N}} \sup_{z\in K} |l_N(z)| < \infty \quad \text{and} \quad \lim_{N\to\infty} \sup_{z\in K} |l_N(z) - l(z)| = 0$$

for every compact set K contained in E. If the weak* topology on E' already delivers that, then it is enough to assume the weak* convergence of the sequence $(l_N)_{N\in\mathbb{N}}$ towards l. Certainly, this would be the case if, for example, E is a Banach space as in the subsequent Section 4.2. The availability of such a simplification, however, doesn't play a role in the rest of this text.

Remark 4.1.2. In the situation of Proposition 4.1.1 one part of the conditions of Theorem 3.2.9, namely (3.2.6) of Section 3.2.2, is automatically fulfilled under the general assumptions of the proposition. Indeed,

$$\lim_{N\to\infty}\int_E (f\circ\pi_{k_N})\cdot(\varphi\circ l_N)\,\mathrm{d}\mu_N=\int_E (f\circ\pi_k)\cdot(\varphi\circ l)\,\mathrm{d}\mu,\quad f\in\mathcal{F}C_b,\,\varphi\in C_b(\mathbb{R}),$$

follows from the same arguments displayed in the first part of the previous proof. Beyond the requirement of (4.1.14) concerning the topological support of the measures, what remains to be checked for Proposition 4.1.1 regarding the applicability of Theorem 3.2.9 are three conditions, formulated in terms of the disintegrating measures $m(z,\cdot)$, $z\in E$, which relate to $\mu\circ(\pi_k,l)^{-1}$ via (4.1.6), and $m_N(z,\cdot)$, $z\in E$, which relate to $\mu_N\circ(\pi_{k_N},l_N)^{-1}$ via the analogue of (4.1.6) for $N\in\mathbb{N}$. We set $\nu_N:=\mu_N\circ\pi_{k_N}^{-1}$, $N\in\mathbb{N}$, and $\nu:=\mu\circ\pi_k^{-1}$. The first condition corresponds to (3.2.8) of Section 3.2.2 and in this context reads

$$\lim_{N\to\infty}\int_E\Big|\int_{\mathbb{R}}\varphi(s)\,\mathrm{d}m_N(z,\mathrm{d}s)\Big|^2\mathrm{d}\nu_N(z)$$
$$=\int_E\Big|\int_{\mathbb{R}}\varphi(s)\,\mathrm{d}m(z,\mathrm{d}s)\Big|^2\mathrm{d}\nu(z),\quad\varphi\in C_b(\mathbb{R}).\tag{4.1.16}$$

The other two conditions are

$$\limsup_{k\to\infty}\sup_{N\in\mathbb{N}}\operatorname*{ess\,sup}_{(z,s)\in E\times[-m,m]}\rho_N^{-1}(z,s)\cdot\big(\rho_N(z,\cdot)_{\mathrm{ave},\frac{1}{k}}(s)\big)<\infty\tag{4.1.17}$$

(the essential supremum above is taken w.r.t. the product measure $\nu_N\times\mathrm{d}s$ for $N\in\mathbb{N}$ respectively), and

$$\lim_{k\to\infty}\limsup_{N\to\infty}\int_E\int_{[-m,m]}\big|\rho_N(z,s)-\big(\rho_N(z,\cdot)_{\mathrm{ave},\frac{1}{k}}(s)\big)\big|\,\mathrm{d}s\nu_N(z)=0,\tag{4.1.18}$$

for each $m\in\mathbb{N}$, where $\rho_N(z,\cdot)$ is a probability density on $\mathbb{R}^d$ with $\rho_N(z,s)\,\mathrm{d}s=m_N(z,\mathrm{d}s)$, ν_N-a.e. $z\in E$, $N\in\mathbb{N}$.

Now we are prepared to address the approximation problem for the minimal gradient form $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ on $L^2(E,\mu)$, assuming that the form in (4.1.3) is closable. A result on Mosco convergence is provided in Theorem 4.1.3 below, under a condition which requires the component-wise convergence of Dirichlet forms and the characterization of $\mathcal{D}(\mathcal{E})$ in terms of the component forms. Let $(\mu_N)_{N\in\mathbb{N}}$ be as above a tight sequence of probability measures, which weakly converges to μ. For each $N\in\mathbb{N}$ we fix a family K_0^N, which is either a countable or a finite collection of μ_N-admissible elements in $E\setminus\{0\}$, indexed by

$$k_N^i\quad\text{for}\quad i\in\mathcal{I}_N:=\big\{n\in\mathbb{N}\,\big|\,n\le|K_0^N|\big\},$$

such that the following three conditions are met: We have

$$\sum_{k\in K_0^N}l(k)^2<\infty\quad\text{for all}\quad l\in E'.$$

Secondly, every $i\in\mathbb{N}$ is contained in $\mathcal{I}_N$ for N large enough and

$$k^i\neq0,\quad k^i:=\lim_{N\to\infty}k_N^i,\quad i\in\mathbb{N},\quad\text{exists in E.}$$

Thirdly, $K_0 := \{k^i \,|\, i \in \mathbb{N}\}$ is a set of μ-admissible elements in H and forms an orthonormal basis in H.

By virtue of [17, Theorem 3.8], the form

$$\mathcal{E}^N(u,v) := \sum_{k \in K_0^N} \int_E \frac{\partial u}{\partial k}\frac{\partial v}{\partial k}\,\mathrm{d}\mu_N, \quad u,v \in L^2(E,\mu_N) \cap \widetilde{\mathcal{F}C_b^\infty},$$

is well-defined and has a smallest closed extension, a Dirichlet form $(\mathcal{E}^N, \mathcal{D}(\mathcal{E}^N))$, on $L^2(E,\mu_N)$ for $N \in \mathbb{N}$. For $k \in K_0$ we denote by $(\mathcal{E}_k, \mathcal{D}(\mathcal{E}_k))$ the closure of (4.1.5) on $L^2(E,\mu)$ as usual.

Theorem 4.1.3. *If the equality*

$$\mathcal{D}(\mathcal{E}) = \Big\{ u \in \bigcap_{k \in K_0} \mathcal{D}(\mathcal{E}_k) \,\Big|\, \sum_{k \in K_0} \mathcal{E}_k(u,u) < \infty \Big\} \tag{4.1.19}$$

holds true and the sequence $(k_N^i)_N$, with $N \in \mathbb{N}$ large enough such that $i \in \mathcal{I}_N$, meets the condition of Proposition 4.1.1 for each $i \in \mathbb{N}$, then $(\mathcal{E}^N, \mathcal{D}(\mathcal{E}^N))_{N\in\mathbb{N}}$ converges to $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ in the sense of Mosco.

Proof. We check the assumptions of Theorem 2.2.1 (iv), starting with property (M1). In the disjoint union of Hilbert spaces $L^2(E,\mu_N)$, $N \in \mathbb{N}$, and $L^2(E,\mu)$, let $(u_N)_{N\in\overline{\mathbb{N}}}$ be a weakly continuous section such that $\#\{N \in \mathbb{N} \,|\, u_N \in \mathcal{D}(\mathcal{E}^N)\}$ is infinite, providing $\liminf_{N\to\infty} \mathcal{E}^N(u_N,u_N) < \infty$. For the subsequent estimate we recall that every $i \in \mathbb{N}$ is contained in $\mathcal{I}_N$ for N large enough. Due to Proposition 4.1.1, Fatou's Lemma and the assumption on the form domain, we have $u_\infty \in \mathcal{D}(\mathcal{E})$ and

$$\begin{aligned}\mathcal{E}(u_\infty,u_\infty) &= \sum_{i\in\mathbb{N}} \mathcal{E}_{k^i}(u_\infty,u_\infty) \le \sum_{i\in\mathbb{N}} \liminf_{N\to\infty} \int_E \Big(\frac{\partial u_N}{\partial k_N^i}\Big)^2 \mathrm{d}\mu_N \\ &\le \liminf_{N\to\infty} \sum_{i\in\mathcal{I}_N} \int_E \Big(\frac{\partial u_N}{\partial k_N^i}\Big)^2 \mathrm{d}\mu_N = \liminf_{N\to\infty} \mathcal{E}^N(u_N,u_N).\end{aligned}$$

This concludes the proof of (M1). As to the remaining property (M2), we convince ourselves that (2.2.16) of Section 2.2.2 is established in our setting by default. Indeed, $L^2(E,\mu) \cap \widetilde{\mathcal{F}C_b^\infty}$ is a dense linear subspace of $(\mathcal{D}(\mathcal{E}), \mathcal{E}_1)$ and the usual choice of $(\Phi_N^\sim)_{N\in\mathbb{N}}$, as in the end of Section 2.1, delivers

$$\lim_{N\to\infty} \mathcal{E}^N(\Phi_N^\sim u, \Phi_N^\sim u) = \mathcal{E}(u,u), \quad u \in L^2(E,\mu) \cap \widetilde{\mathcal{F}C_b^\infty}$$

via the weak measure convergence of $(\mu_N)_{N\in\mathbb{N}}$ towards μ. (M1) follows from Remark 2.2.5 (ii). This concludes the proof. ∎

4.2 A perturbation result in infinite dimension

In the following $(\Omega, \mathcal{A}, m)$ is a finite measure space. In the preceding Section 4.1, we looked at a sequence $(\mu_N)_{N\in\mathbb{N}}$ of weakly converging probability measures on a locally convex space E and discussed the Mosco convergence of one-component forms and then gradient forms, having μ_N as a reference measure for $N \in \mathbb{N}$ respectively. Now, we are interested in a perturbation result for the case $E := L^p(\Omega, m)$, $1 \le p \le \infty$, which focusses on a family of densities

$$\varrho_N(z) = \exp\Big(\gamma \int_\Omega (f_N \circ z)(\omega)\,\mathrm{d}m(\omega)\Big), \qquad z \in E,\ N \in \mathbb{N}, \tag{4.2.1}$$

and

$$\varrho(z) = \exp\Big(\gamma \int_\Omega (f \circ z)(\omega)\,\mathrm{d}m(\omega)\Big), \qquad z \in E, \tag{4.2.2}$$

with $\gamma \in \{-1, 1\}$ and bounded, monotone increasing functions f_N and f on the real line. The task of verifying the assumptions of Proposition 4.1.1 w.r.t. the sequence of perturbed measures $(\varrho_N\mu_N)_{N\in\mathbb{N}}$ and the expected limit $\varrho\mu$, given admissible elements k_N, $N \in \mathbb{N}$, and k in $E \setminus \{0\}$, where k is the limit of $(k_N)_{N\in\mathbb{N}}$ in E, allows for an essential simplification in some relevant cases, thanks to the results of Section 3.2.1. It suffices to settle that problem for the unperturbed measures μ_N, $N \in \mathbb{N}$, and μ, if two conditions are satisfied. The first one reads $\lim_{N\to\infty} d^*(f, f_N) = 0$. Then, secondly, for every point $x \in \mathbb{R}$, at which f is discontinuous, we need to assume that

$$\text{for } \mu\text{-a.e. } z \in E \quad \text{the level set} \quad \{\omega \,|\, z(\omega) = x\} \quad \text{is } m\text{-negligible.} \tag{4.2.3}$$

Of course, choosing two different representatives $\tilde{z}_1$ and $\tilde{z}_2$ for $z \in E$, the sets $A_1 := \{\omega \,|\, \tilde{z}_1(\omega) = x\}$ and $A_2 := \{\omega \,|\, \tilde{z}_2(\omega) = x\}$ may define different elements in $\mathcal{A}$ for fixed $x \in \mathbb{R}$, with $m(A_1) = m(A_2)$ however. That is why the statement of (4.2.3) is unambiguous. Both of the mentioned conditions for $(f_N)_{N\in\mathbb{N}}$ and f are invariant (their status of validity doesn't change) if we multiply the functions f and f_N with positive constants c, respectively c_N for $N \in \mathbb{N}$, such that $\lim_{N\to\infty} c_N = c$. For example, normalization constants depending on N can be absorbed by properly rescaled choices for f_N, $N \in \mathbb{N}$, and f in that way. Lemma 4.2.2 below shows the weak measure convergence of $(\varrho_N\mu_N)_{N\in\mathbb{N}}$ towards $\varrho\mu$ under these conditions, while the subsequent Theorem 4.2.4 deals with the remaining assumptions of Proposition 4.1.1. A note on the measurability of the functions in (4.2.1) and (4.2.2) precedes the lemma.

Remark 4.2.1. (i) For $f \in C_b(\mathbb{R})$ the assignment of (4.2.2) is continuous in the variable z as a map from E into $\mathbb{R}$ due to Lebesgue's dominated convergence.

(ii) Let $-\infty < a < b < \infty$. Since the indicator function $\mathbf{1}_{[a,b)}$ can be written as the pointwise limit on $\mathbb{R}$ of a sequence of uniformly bounded, continuous functions, another application of Lebesgue's dominated convergence together with (i) proves that there is a sequence of continuous functions on E which converge pointwisely towards $\exp(\gamma \int_\Omega \mathbf{1}_{[a,b)} \circ z\,\mathrm{d}m)$, $z \in E$.

(iii) The family $\{[a,b) \,|\, -\infty < a < b < \infty\}$, together with $\emptyset$ and $\mathbb{R}$, form a π-system in $\mathbb{R}$, whose elements generate $\mathcal{B}(\mathbb{R})$. With the monotone class theorem, it follows that

$$\Big\{f : \mathbb{R} \to \mathbb{R} \,\Big|\, z \mapsto \exp\Big(\gamma \int_\Omega f \circ z\,\mathrm{d}m\Big) \text{ from } E \text{ to } \mathbb{R} \text{ is measurable}\Big\}$$

contains the space $\mathcal{B}_b(\mathbb{R})$.

(iv) The analogous argumentation of (i) to (iii) shows the measurability of the assignment

$$z \mapsto \exp\Big(\gamma \int_A (f \circ z)(\omega)\,\mathrm{d}m(\omega)\Big), \qquad z \in E,$$

for any element $A \in \mathcal{A}$.

As in the standard setting from the end of Section 2.1, we assume $\mathrm{supp}[\mu_N] \subseteq \mathrm{supp}[\mu]$ for $N \in \mathbb{N}$ below.

Lemma 4.2.2. *Let μ be the limit of probabilities $(\mu_N)_{N\in\mathbb{N}}$ in the sense of weak measure convergence on E. Further, let f and f_N, $N \in \mathbb{N}$, be bounded, monotone increasing functions on $\mathbb{R}$ such that (4.2.3) is satisfied for every point $x \in \mathbb{R}$, at which f is discontinuous, and also $\lim_{N\to\infty} d^*(f, f_N) = 0$. For $\gamma \in \{-1, 1\}$ and the densities from (4.2.1), (4.2.2) we can conclude the weak convergence of weighted measures $(\varrho_N\mu_N)_{N\in\mathbb{N}}$ towards $\varrho\mu$.*

Proof. Justifying the use of Lebesgue's dominated convergence in various cases below, we point out that $\lim_{N\to\infty} d^*(f, f_N) = 0$ necessitates the uniform boundedness of the sequence $(f_N)_{N\in\mathbb{N}}$. The set of discontinuities of f is a countable subset of $\mathbb{R}$, which we denote by U_f. Then, we have

$$\int_{\{\omega\in\Omega\,|\,z(\omega)\in U_f\}} \mathrm{d}m(\omega) \leq \sum_{x\in U_f} \int_{\{\omega\in\Omega\,|\,z(\omega)=x\}} \mathrm{d}m(\omega) = 0, \quad \mu\text{-a.e. } z \in E.$$

In the following we use Lemma 3.2.3 and adopt its notation. By Lebesgue's dominated convergence

$$\lim_{\varepsilon\to 0} \exp\Big(\gamma \int_\Omega \tilde{f}^{\mathrm{min},\varepsilon}(z(\omega))\,\mathrm{d}m(\omega)\Big) = \varrho(z) \quad \mu\text{-a.e. } z \in E,$$

and likewise

$$\lim_{\varepsilon\to 0} \exp\Big(\gamma \int_\Omega \tilde{f}^{\mathrm{maj},\varepsilon}(z(\omega))\,\mathrm{d}m(\omega)\Big) = \varrho(z) \quad \mu\text{-a.e. } z \in E.$$

A second use of Lebesgue's dominated convergence yields the asymptotic

$$\lim_{\varepsilon\to 0} \int_E \Big| e^{\gamma\int_\Omega \tilde{f}^{\mathrm{min},\varepsilon}\circ z\,\mathrm{d}m} - \varrho(z)\Big|^2 \mathrm{d}\mu(z) = 0, \tag{4.2.4}$$

respectively

$$\lim_{\varepsilon\to 0} \int_E \Big| e^{\gamma\int_\Omega \tilde{f}^{\mathrm{maj},\varepsilon}\circ z\,\mathrm{d}m} - \varrho(z)\Big|^2 \mathrm{d}\mu(z) = 0. \tag{4.2.5}$$

As the remaining part of this proof shows, the comparison criterion given in Lemma 2.1.3 is applicable here, choosing $e^{\gamma\int_\Omega \tilde{f}_N^{\mathrm{min},\varepsilon}\circ z\,\mathrm{d}m}$, $z \in E$, as minorante and $e^{\gamma\int_\Omega \tilde{f}_N^{\mathrm{maj},\varepsilon}\circ z\,\mathrm{d}m}$, $z \in E$, as majorante, for $\varrho_N(z)$, $z \in E$, $N \in \mathbb{N}$, in case $\gamma = 1$. In case $\gamma = -1$ the roles of the minorante and the majorante are just swapped. What Lemma 2.1.3 provides, after checking its conditions, is the weak convergence of $(\varrho_N)_{N\in\mathbb{N}}$ towards ϱ - more precisely, of the respective classes of these functions - within the frame of converging Hilbert spaces $L^2(E, \mu_N)$, $N \in \mathbb{N}$, with asymptotic space $L^2(E, \mu)$. This, of course, yields the weak measure convergence of $(\varrho_N\mu_N)_{N\in\mathbb{N}}$ towards $\varrho\mu$ as claimed. What remains to be shown is

$$\lim_{N\to\infty} \Big| \int_E g(z) e^{\gamma\int_\Omega \tilde{f}_N^{\mathrm{min},\varepsilon}\circ z\,\mathrm{d}m}\,\mathrm{d}\mu_N(z) - \int_E g(z) e^{\gamma\int_\Omega \tilde{f}^{\mathrm{min},\varepsilon}\circ z\,\mathrm{d}m}\,\mathrm{d}\mu(z)\Big| = 0 \tag{4.2.6}$$

as well as

$$\lim_{N\to\infty} \Big| \int_E g(z) e^{\gamma\int_\Omega \tilde{f}_N^{\mathrm{maj},\varepsilon}\circ z\,\mathrm{d}m}\,\mathrm{d}\mu_N(z) - \int_E g(z) e^{\gamma\int_\Omega \tilde{f}^{\mathrm{maj},\varepsilon}\circ z\,\mathrm{d}m}\,\mathrm{d}\mu(z)\Big| = 0 \tag{4.2.7}$$

for every $g \in C_b(E)$ and $\varepsilon \in (0,\infty)$. We use the estimate

$$\left|e^a - e^b\right| = \left|e^b(e^{a-b} - 1)\right| \le e^{|b|}\left(e^{|a-b|} - 1\right), \quad a, b \in \mathbb{R},$$

to deduce

$$\left|e^{\gamma \int_\Omega \varphi_1 \circ z \,\mathrm{d}m} - e^{\gamma \int_\Omega \varphi_2 \circ z \,\mathrm{d}m}\right| \le e^{\|\varphi_2\|_\infty}\left(e^{\|\varphi_1 - \varphi_2\|_\infty} - 1\right) \qquad \text{for } \varphi_1, \varphi_2 \in C_b(\mathbb{R}),\ z \in E.$$

With the triangular inequality, we can bound the modulus brackets within (4.2.6) from above by

$$\sup_{z\in E} |g(z)|\, e^{\|\tilde{f}^{\min,\varepsilon}\|_\infty}\left(e^{\|\tilde{f}_N^{\min,\varepsilon} - \tilde{f}^{\min,\varepsilon}\|_\infty} - 1\right) + \left|\int_E g(z) e^{\gamma \int_\Omega \tilde{f}^{\min,\varepsilon} \circ z \,\mathrm{d}m}\,\mathrm{d}\mu_N(z) - \int_E g(z) e^{\gamma \int_\Omega \tilde{f}^{\min,\varepsilon} \circ z \,\mathrm{d}m}\,\mathrm{d}\mu(z)\right|, \tag{4.2.8}$$

and analogously the modulus brackets within (4.2.7) from above by

$$\sup_{z\in E} |g(z)|\, e^{\|\tilde{f}^{\mathrm{maj},\varepsilon}\|_\infty}\left(e^{\|\tilde{f}_N^{\mathrm{maj},\varepsilon} - \tilde{f}^{\min,\varepsilon}\|_\infty} - 1\right) + \left|\int_E g(z) e^{\gamma \int_\Omega \tilde{f}^{\mathrm{maj},\varepsilon} \circ z \,\mathrm{d}m}\,\mathrm{d}\mu_N(z) - \int_E g(z) e^{\gamma \int_\Omega \tilde{f}^{\mathrm{maj},\varepsilon} \circ z \,\mathrm{d}m}\,\mathrm{d}\mu(z)\right|, \tag{4.2.9}$$

for $N \in \mathbb{N}$, $\varepsilon > 0$ and $g \in C_b(E)$ respectively. Since $e^{\gamma \int_\Omega \tilde{f}^{\min,\varepsilon} \circ z \,\mathrm{d}m}$ and $e^{\gamma \int_\Omega \tilde{f}^{\mathrm{maj},\varepsilon} \circ z \,\mathrm{d}m}$ are bounded, continuous functions in the variable $z \in E$, the value of (4.2.8) and that of (4.2.9) converge to zero as N tends to infinity, in view of Lemma 3.2.3 with $\varepsilon > 0$ and $g \in C_b(E)$ fixed. This concludes the proof. □

Remark 4.2.3. Lemma 4.2.2 bears even a stronger result than the weak measure convergence. As a second look reveals, even the strong convergence of $(\varrho_N)_{N\in\mathbb{N}}$ towards ϱ - more precisely, of the respective classes of these functions - within the frame of converging Hilbert spaces $L^2(E, \mu_N)$, $N \in \mathbb{N}$, with asymptotic space $L^2(E, \mu)$, holds true. In the proof, the weak convergence of that sequence is shown. Then, by a second application of Lemma 4.2.2, we also have

$$\lim_{N\to\infty} \int_E \varrho_N^2 \,\mathrm{d}\mu_N = \lim_{N\to\infty} \int_E \exp\left(\gamma \int_\Omega (2 f_N \circ z)(\omega)\,\mathrm{d}m(\omega)\right)\mathrm{d}\mu_N = \exp\left(\gamma \int_\Omega (2 f \circ z)(\omega)\,\mathrm{d}m(\omega)\right) = \int_E \varrho^2 \,\mathrm{d}\mu. \tag{4.2.10}$$

Let ϱ_N, μ_N for $N \in \mathbb{N}$ and ϱ, μ fulfil the assumptions of the preceding Lemma. Since all densities involved are bounded from below and above, the notions of μ- and $\varrho\mu$-admissibility, respectively μ_N- and $\varrho_N\mu_N$-admissibility for $N \in \mathbb{N}$, coincide. We fix a μ_N-admissible element $k_N \in E\backslash\{0\}$ for $N \in \mathbb{N}$ and a μ-admissible element $k \in E\backslash\{0\}$ such that $\lim_{N\to\infty} k_N = k$ in E. Also, the topological support of μ and $\varrho\mu$, respectively μ_N and $\varrho_N\mu_N$ for $N \in \mathbb{N}$, coincide of course.

Theorem 4.2.4. *If the sequence $(k_N)_{N\in\mathbb{N}}$ (with limit k) satisfies the assumption of Proposition 4.1.1 in relation to the unweighted measures μ_N, $N \in \mathbb{N}$, and μ, then it* *also does so in relation to the perturbed measures $\varrho_N\mu_N$, $N\in\mathbb{N}$, and $\varrho\mu$. In particular for this case, denoting by $(\mathcal{E}_{k_N},\mathcal{D}(\mathcal{E}_{k_N}))$ the smallest closed extension of*

$$\mathcal{E}_{k_N}(u,v):=\int_E \frac{\partial u}{\partial k_N}\frac{\partial v}{\partial k_N}\varrho_N\,\mathrm{d}\mu_N,\quad u,v\in L^2(E,\varrho_N\mu_N)\cap\widetilde{\mathcal{F}C_b^\infty},$$

on $L^2(E,\varrho_N\mu_N)$ for $N\in\mathbb{N}$ now, and likewise by $(\mathcal{E}_k,\mathcal{D}(\mathcal{E}_k))$ the smallest closed extension of

$$\mathcal{E}_k(u,v):=\int_E \frac{\partial u}{\partial k}\frac{\partial v}{\partial k}\varrho\,\mathrm{d}\mu,\quad u,v\in L^2(E,\varrho\mu)\cap\widetilde{\mathcal{F}C_b^\infty},$$

on $L^2(E,\varrho\mu)$, the forms $(\mathcal{E}_{k_N},\mathcal{D}(\mathcal{E}_{k_N}))_{N\in\mathbb{N}}$ converge to $(\mathcal{E}_k,\mathcal{D}(\mathcal{E}_k))$ in the sense of Mosco.

Proof. Let $(k_N)_N$ meet the assumption of Proposition 4.1.1 in relation to the unweighted measures μ_N, $N\in\mathbb{N}$, and μ. There exists a sequence $(l_N)_{N\in\mathbb{N}}$ and an element l in E', as in the formulation of Proposition 4.1.1, which validate this quality of $(k_N)_N$. We see in this proof that the same choices $(l_N)_N$ and l are validating in this sense for $(k_N)_N$ regarding the perturbed measures $\varrho_N\mu_N$, $N\in\mathbb{N}$, and $\varrho\mu$. As usual, $\pi_k(z):=z-l(z)k$, $\pi_{k_N}(z):=z-l_N(z)k_N$ for $z\in E$ and $N\in\mathbb{N}$. The weak measure convergence of $(\varrho_N\mu_N)_N$, stated in Lemma 4.2.2, yields the tightness of that sequence, because E is a Polish space. Let $\tilde m(z,\cdot)$, $z\in E$, a family of Borel probability measures on $\mathbb{R}$, be related to $\mu\circ(\pi_k,l)^{-1}$ via (4.1.6), and likewise another family $\tilde m_N(z,\cdot)$, $z\in E$, be related to $\mu_N\circ(\pi_{k_N},l_N)^{-1}$ in the same way. We prefer to write $\tilde m_N(z,\cdot)$, $z\in E$, here (rather than just $m_N(z,\cdot)$, $z\in E$) and analogously $\tilde m(z,\cdot)$, $z\in E$, to avoid any ambiguity, as $m(A)$, $A\in\mathcal{A}$, already denotes the measure to define the functions in (4.2.1) and (4.2.2). We set $\nu:=\mu\circ\pi_k^{-1}$ and $\nu_N:=\mu_N\circ\pi_{k_N}^{-1}$ for $N\in\mathbb{N}$. From the equation

$$\begin{aligned}\int_E f\varrho\,\mathrm{d}\mu &= \int_{E\times\mathbb{R}} f(z+sk)\varrho(z+sk)\,\mathrm{d}\big(\mu\circ(\pi_k,l)^{-1}\big)(z,s)\\ &= \int_E\int_{\mathbb{R}} f(z+sk)\varrho(z+sk)\tilde m(z,\mathrm{d}s)\,\mathrm{d}\nu(z),\quad f\in\mathcal{B}_b(E),\end{aligned}$$

and likewise

$$\begin{aligned}\int_E f\varrho_N\,\mathrm{d}\mu_N &= \int_{E\times\mathbb{R}} f(z+sk_N)\varrho_N(z+sk_N)\,\mathrm{d}\big(\mu_N\circ(\pi_{k_N},l_N)^{-1}\big)(z,s)\\ &= \int_E\int_{\mathbb{R}} f(z+sk_N)\varrho_N(z+sk_N)\tilde m_N(z,\mathrm{d}s)\,\mathrm{d}\nu_N(z),\quad f\in\mathcal{B}_b(E),\end{aligned}$$

for $N\in\mathbb{N}$, we can read the disintegrating densities for the perturbed measures. Remark 4.1.2 names the three conditions, (4.1.16), (4.1.17) and (4.1.18), which have to be shown for the perturbed case now, under the assumption that they hold true in the unperturbed case (i.e. they hold true for $\varrho_N=\varrho=\mathbf{1}_E$). Here, (4.1.17) needs no proof, because the functions $(\varrho_N)_{N\in\mathbb{N}}$ are bounded from below and above, uniformly in N. We have to show

$$\lim_{\alpha\to\infty}\limsup_{N\to\infty}\int_E\int_{[-M,M]}\big|F_{N,\alpha}(z,s)\big|\,\mathrm{d}s\nu_N(z)=0,\tag{4.2.11}$$

$$\begin{aligned}F_{N,\alpha}(z,s):=&\ \varrho_N(z+sk_N)\,\frac{\mathrm{d}\tilde m_N(z,\cdot)}{\mathrm{d}s}(s)\\ &-\Big(\varrho_N(z+\cdot\,k_N)\,\frac{\mathrm{d}\tilde m_N(z,\cdot)}{\mathrm{d}s}\Big)_{\mathrm{ave},\frac{1}{\alpha}}(s),\quad z\in E,\ s\in\mathbb{R},\ N,\alpha\in\mathbb{N},\end{aligned}$$

for each $M \in \mathbb{N}$, and also

$$\lim_{N\to\infty} \int_E \Big| \int_{\mathbb{R}} \varphi(s)\varrho_N(z+sk_N)\,\mathrm{d}\tilde{m}_N(z,\mathrm{d}s)\Big|^2 \mathrm{d}\nu_N(z) \\ = \int_E \Big| \int_{\mathbb{R}} \varphi(s)\varrho(z+sk)\,\mathrm{d}\tilde{m}(z,\mathrm{d}s)\Big|^2 \mathrm{d}\nu(z), \quad \varphi \in C_b(\mathbb{R}), \tag{4.2.12}$$

knowing that both of these equations are satisfied in case $\varrho = \varrho_N = \mathbf{1}_E$, $N \in \mathbb{N}$.

Let $N \in \mathbb{N}$. We choose two sets Ω_N^+, Ω_N^- from $\mathcal{A}$ with $\Omega = \Omega_N^+ \cup \Omega_N^-$, $k_N(\omega) \geq 0$, m-a.e. $\omega \in \Omega_N^+$, and $k_N(\omega) \leq 0$, m-a.e. $\omega \in \Omega_N^-$. Then, for $z \in E$ and $s \in \mathbb{R}$ it holds

$$\begin{aligned} &\varrho_N(z+sk_N) \\ &= \exp\Big(\gamma \int_{\Omega_N^+} f_N\big(z+sk_N(\omega)\big)\,\mathrm{d}m(\omega) + \gamma \int_{\Omega_N^-} f_N\big(z+sk_N(\omega)\big)\,\mathrm{d}m(\omega)\Big) \\ &= \exp\Big(\gamma \int_{\Omega_N^+} f_N\big(z+sk_N(\omega)\big)\,\mathrm{d}m(\omega)\Big) \exp\Big(\gamma \int_{\Omega_N^-} f_N\big(z+sk_N(\omega)\big)\,\mathrm{d}m(\omega)\Big). \end{aligned}$$

The right-hand side is the product of two strictly positive functions, one of them being monotone increasing and the other monotone decreasing in the variable s while z is fixed. Both functions are uniformly bounded for $(z,s) \in E \times \mathbb{R}$ by the value

$$C := \exp\big(m(\Omega) \sup_{N\in\mathbb{N}} \sup_{x\in\mathbb{R}} f_N(x)\big).$$

Let further $M \in \mathbb{N}$ now. A double application of Lemma 3.2.1 yields

$$\int_{[-M,M]} \big|F_{N,\alpha}(z,s)\big|\,\mathrm{d}s \\ \leq 5^2 C^2 \int_{[-M-1,M+1]} \Big| \frac{\mathrm{d}\tilde{m}_N(z,\cdot)}{\mathrm{d}s}(s) - \Big(\frac{\mathrm{d}\tilde{m}_N(z,\cdot)}{\mathrm{d}s}\Big)_{\mathrm{ave},\frac{4}{\alpha}}(s)\Big|\,\mathrm{d}s$$

uniformly in $z \in E$, for $\alpha \in \mathbb{N}$ large enough and $N \in \mathbb{N}$. This proves (4.2.11).

To prove the missing equation of (4.2.12) we define

$$\mathcal{H} := L^2(E\times\mathbb{R}, \mu \circ (\pi_k, l)^{-1}) \sqcup \Big(\bigsqcup_{N\in\mathbb{N}} L^2(E\times\mathbb{R}, \mu_N \circ (\pi_{k_N}, l_N)^{-1}\Big)$$

and

$$\mathcal{H}^{\mathrm{pr}} := L^2(E,\nu) \sqcup \Big(\bigsqcup_{N\in\mathbb{N}} L^2(E,\nu_N)\Big),$$

with their respective strong and weak topologies, in analogy to Section 3.2.2 for the choice $J := \pi_k$. By Lemma 2.1.1 (ii) it suffices to show

$$\lim_{N\to\infty} \int_E u_N(z) \int_{\mathbb{R}} \varphi(s)\varrho_N(z+sk_N)\,\mathrm{d}\tilde{m}_N(z,\mathrm{d}s)\,\mathrm{d}\nu_N(z) \\ = \int_E u_\infty(z) \int_{\mathbb{R}} \varphi(s)\varrho(z+sk)\,\mathrm{d}\tilde{m}(z,\mathrm{d}s)\,\mathrm{d}\nu(z) \tag{4.2.13}$$

for $\varphi \in C_b(\mathbb{R})$ and each weakly continuous section $(u_N)_{N\in\overline{\mathbb{N}}}$ in $\mathcal{H}^{\mathrm{pr}}$. Let φ and $(u_N)_{N\in\overline{\mathbb{N}}}$ with these properties be fixed in the following argumentation why (4.2.13), and hence (4.2.12), holds true indeed.

Some of the steps below are justified by the boundedness of squared norms,

$$\begin{aligned}\sup_{N\in\mathbb{N}}\int_{E\times\mathbb{R}}|u_N(z)|^2\,\mathrm{d}\big(\mu_N\circ(\pi_{k_N},l_N)^{-1}\big)&=\sup_{N\in\mathbb{N}}\int_E|u_N\circ\pi_{k_N}|^2\,\mathrm{d}\mu_N\\&=\sup_{N\in\mathbb{N}}\int_E|u_N|^2\,\mathrm{d}\nu_N<\infty,\end{aligned}$$

regarding a sequence in $\mathcal{H}$, a sequence in the disjoint union of $L^2(E,\mu_N)$, $L^2(E,\mu)$ over $N\in\mathbb{N}$, and a sequence in $\mathcal{H}^{\mathrm{pr}}$. As pointed pout in Section 3.2.2 subsequently to (3.2.9), we can understand $(u_N)_{N\in\overline{\mathbb{N}}}$ as a weakly continuous section in $\mathcal{H}$ as well because (4.2.12) holds for the choice $\varrho=\varrho_N=\mathbf{1}_E$ corresponding to (3.2.8) of Section 3.2.2 in the unperturbed case. We can rewrite (4.2.13) as

$$\lim_{N\to\infty}\int_E(u_N\circ\pi_{k_N})\cdot(\varphi\circ l_N)\cdot\varrho_N\,\mathrm{d}\mu_N = \int_E(u_\infty\circ\pi_k)\cdot(\varphi\circ l)\cdot\varrho\,\mathrm{d}\mu \quad (4.2.14)$$

Now, to show (4.2.14) in turn, it suffices to show

$$\lim_{N\to\infty}\int_E(u_N\circ\pi_{k_N})\cdot(\varphi\circ l_N)\cdot g\,\mathrm{d}\mu_N = \int_E(u_\infty\circ\pi_k)\cdot(\varphi\circ l)\cdot g\,\mathrm{d}\mu,\quad g\in\mathcal{F}C_b,\quad (4.2.15)$$

by virtue of Lemma 4.2.2 in combination with Remark 4.2.3, and again Lemma 2.1.1 (ii). However, (4.2.15) is equivalent to

$$\lim_{N\to\infty}\int_{E\times\mathbb{R}}u_N(z)\varphi(s)g(z+sk_N)\,\mathrm{d}\big(\mu_N\circ(\pi_{k_N},l_N)^{-1}\big)(z,s) = \int_{E\times\mathbb{R}}u_\infty(z)\varphi(s)g(z+sk)\,\mathrm{d}\big(\mu\circ(\pi_k,l)^{-1}\big)(z,s),\quad g\in\mathcal{F}C_b.\quad (4.2.16)$$

Due to the weak continuity of $(u_N)_{N\in\overline{\mathbb{N}}}$ in $\mathcal{H}$ we can argue as follows. Using Lemma 2.1.1 (ii) one more time, (4.2.16) is clear from

$$\begin{aligned}\lim_{N\to\infty}\int_E(\varphi\circ l_N)\cdot g\cdot h\,\mathrm{d}\mu_N&=\int_E(\varphi\circ l)\cdot g\cdot h\,\mathrm{d}\mu,\\\lim_{N\to\infty}\int_E(\varphi^2\circ l_N)\cdot g^2\,\mathrm{d}\mu_N&=\int_E(\varphi^2\circ l)\cdot g^2\,\mathrm{d}\mu,\quad g,h\in\mathcal{F}C_b.\end{aligned}$$

The latter holds true by (4.1.11) within the proof of Proposition 4.1.1. This concludes the proof of this theorem. □

Remark 4.2.5. Via the Jordan decomposition every function with globally bounded variation on the real line can be written as the difference of two bounded, monotone increasing functions. Since the integral is linear, the exponentials considered in (4.2.1) or (4.2.2) factorize. Hence, ba a double application of Lemma 4.2.2 and of Theorem 4.2.4, the perturbation theory of this section also applies to such instances, in which f_N for $N\in\mathbb{N}$ and f are functions of bounded variation, as well. The condition of Lemma 4.2.2 and of Theorem 4.2.4 then need to be verified for the respective monotone functions obtained in the Jordan decomposition. This strategy is explicitly applied in Chapter 5.

Chapter 5 Scaling limits of interface models

5.1 Tightness and height maps

Let $N \in \mathbb{N}$ and m_N be a probability measure on $(\mathbb{R}^{k_N}, \mathcal{B}(\mathbb{R}^{k_N}))$. The starting point for the subsequent discussion shall be an m_N-symmetric Markovian transition function $(p_t^N)_{t\geq 0}$ on $(\mathbb{R}^{k_N}, \mathcal{B}(\mathbb{R}^{k_N}))$ with the property

$$\lim_{t\downarrow 0} p_t^N u(x) = u(x)\ m_N\text{-a.e.} \quad x \in \mathbb{R}^{k_N}, \text{ for all } u \in \mathcal{B}_b(\mathbb{R}^{k_N}).$$

The terminology of an m_N-symmetric Markovian transition function is used here in the same sense as in [38, Chapter 1.4]. This means $p_t^N(x, \cdot\,)$ is a measure on $(\mathbb{R}^{k_N}, \mathcal{B}(\mathbb{R}^{k_N}))$ with $p_t^N(x, \mathbb{R}^{k_N}) \leq 1$ for each $x \in \mathbb{R}^{k_N}$ and moreover $p_t^N(\,\cdot\,, A) \in \mathcal{B}_b(\mathbb{R}^{k_N})$ for each $A \in \mathcal{B}(\mathbb{R}^{k_N})$ and $t \geq 0$. We set $p_t^N u(x) := \int_{\mathbb{R}^{k_N}} u(y) p_t(x, \mathrm{d}y)$ for $x \in \mathbb{R}^{k_N}$, $u \in \mathcal{B}_b(\mathbb{R}^{k_n})$ and $t \geq 0$. The condition of m_N-symmetry expresses the property

$$\int_{\mathbb{R}^{k_N}} u(x)(p_t v)(x)\,\mathrm{d}m_N(x) = \int_{\mathbb{R}^{k_N}} (p_t u)(x) v(x)\,\mathrm{d}m_N(x), \quad \text{for } u, v \in \mathcal{B}_b(\mathbb{R}^{k_N}).$$

As described in the beginning of [38, Chapter 4.2] in combination with [38, Lemma 1.4.3], there is a unique strongly continuous contraction semigroup $(T_t^N)_{t\geq 0}$ on $L^2(\mathbb{R}^{k_N}, m_N)$ of symmetric, sub-Markovian operators such that $T_t^N u = p_t^N u$ for $u \in \mathcal{B}_b(\mathbb{R}^{k_N})$, $t \geq 0$. Hence, there is also an associated densely defined and symmetric Dirichlet form $(\mathcal{E}^N, \mathcal{D}(\mathcal{E}^N))$ on $L^2(\mathbb{R}^{k_N}, m_N)$. We assume that $\mathcal{E}^N$ is strongly local and regular, as defined in [38, Chapter 1.1], and that $\mathcal{E}^N$ is conservative, i.e. $p_t^N \mathbf{1}_{\mathbb{R}^{k_N}} = \mathbf{1}_{\mathbb{R}^{k_N}}$ for $t \geq 0$. The proof of 5.1.1 essentially uses that $\mathcal{E}^N$ is required to admit a carré du champ $\Gamma_{\mathcal{E}^N}$ in the sense of [19, Definition 4.1.2 of Chapter I]. By definition, $\Gamma_{\mathcal{E}^N}$ is the unique symmetric bilinear form from $\mathcal{D}(\mathcal{E}^N) \times \mathcal{D}(\mathcal{E}^N)$ into $L^1(\mathbb{R}^{k_N}, m_N)$ such that

$$2\mathcal{E}^N(uv, u) - \mathcal{E}^N(v, u^2) = \int_{\mathbb{R}^{k_N}} v\Gamma_{\mathcal{E}^N}(u, u)\,\mathrm{d}m_N, \quad u, v \in \mathcal{D}(\mathcal{E}^N) \cap L^\infty(\mathbb{R}^{k_N}, m_N).$$

As follows from [38, Chapters 4 & 7], there is a unique probability measure P_N on $\Omega_N := C([0,\infty), \mathbb{R}^{k_N})$, endowed with the σ-algebra $\mathcal{C}_{\Omega_N}$ generated by the collection of cylinder sets $\{\omega \mid (\omega(t_1), \dots, \omega(t_m)) \in A\}$, $A \in \mathcal{B}((\mathbb{R}^{k_N})^m)$, $t_1, \dots, t_m \in [0,\infty)$, such that

$$\begin{aligned}&\int_{\Omega_N} F_1(\omega(t_1)) \cdot \dots \cdot F_m(\omega(t_m))\,\mathrm{d}P_N(\omega)\\ &\quad = \int_{\{(y_0,\dots,y_m)\in(\mathbb{R}^{k_N})^{m+1}\}} \prod_{i=0}^{m-1} F_{m-i}(y_{m-i}) p^N_{t_{m-i}-t_{m-i-1}}(y_{m-i-1}, \mathrm{d}y_{m-i})\,\mathrm{d}m_N(y_0)\end{aligned} \tag{5.1.1}$$

for $F_1, \dots, F_m \in C_b(\mathbb{R}^{k_N})$, $0 \leq t_1 \leq \dots \leq t_m < \infty$, and $m \in \mathbb{N}_{\geq 2}$. In the equation above, the term $\prod_{i=0}^{m-1} F_{m-i}(y_{m-i}) p^N_{t_{m-i}-t_{m-i-1}}(y_{m-i-1}, \mathrm{d}y_{m-i})$, for fixed $y_0 \in \mathbb{R}^{k_N}$, is just a formal expression for the weighted product measure

$$F_m(y_m) p^N_{t_m - t_{m-1}}(y_{m-1}, \mathrm{d}y_m) \cdots F_2(y_2) p^N_{t_2-t_1}(y_1, \mathrm{d}y_2) F_1(y_1) p^N_{t_1}(y_0, \mathrm{d}y_1)$$

on $(\mathbb{R}^{k_N})^m$ integrating the variables $y_1, \dots, y_m$.

As needed in the context of scaling limits of dynamical interface models we want to interpret the canonical process $X_t := \omega(t) \in \mathbb{R}^{k_N}$, $t \geq 0$, $\omega \in \Omega_N$ on the probability space $(\Omega_N, \mathcal{C}_{\Omega_N}, P_N)$, as a model for $(d+1)$-dimensional surface evolving in time for some $d \in \mathbb{N}$, i.e. we want to associate X_t with the graph of a real function h_t defined on a bounded domain $Y \subset \mathbb{R}^d$ for each $t \geq 0$. This involves to come up with a suitable scaling in space and time. At this point, we agree on some basic simplifications. For one thing we set $Y = [0,1]^d$, as we do not want to make the geometry of the domain Y a prominent theme in this article, and for the other we say $(h_t)_{t\geq 0}$ is supposed to emerge from the process $(X_t)_{t\geq 0}$ through a transformation of the type

$$h_t = \Lambda(X_{ct}), \tag{5.1.2}$$

where c is a positive constant and Λ is an injective linear map from $\mathbb{R}^{k_N}$ into $\mathcal{B}_b(Y)$. Both, the space and the time scaling are thus linear. A path space for the process $(h_t)_{t\geq 0}$ is given by the image space $\mathrm{Im}(\Lambda)$ of Λ, which is a k_N-dimensional vector space. In this sense, the technical dimension k_N corresponds to the number of degrees of freedom the observed dynamical interface has. Suppose we are given models $(\Omega_N, \mathcal{C}_{\Omega_N}, P_N)$, $N \in \mathbb{N}$, with $k_N \uparrow \infty$ as $N \to \infty$. Taking the scaling limit is the endeavour to investigate the asymptotic behaviour of the left-hand side of (5.1.2) as $N \to \infty$. The scope of this paper covers the convergence in distribution after setting up suitable function spaces on which to consider the weak convergence of measures. Of course, the assignment of (5.1.2) must depend on N, which shall be indicated by an index. We now define the according constant $c_N \in (0,\infty)$ and linear map Λ_N from $\mathbb{R}^{k_N}$ into $\mathcal{B}_b([0,1]^d)$ for each $N \in \mathbb{N}$. Λ_N is uniquely determined by its value $\xi_{N,i}$ on the scaled unit vector $N^{1-\frac{d}{2}}\mathbf{e}_i$ for $i = 1, \ldots, k_N$, respectively. In essence, the additional conditions, which we inflict on Λ_N now, are supposed to ensure that $\xi_{N,i}$ is a non-negative function and that the value of its integral does not depend on i. To understand the reasoning behind the following definition, the reader should be aware of the excess freedom, this framework provides w.r.t. spatial scaling. Let R_N be the multiplication by some positive number depending on $N \in \mathbb{N}$. We could start with differently scaled transition functions $(\tilde{p}_t^N)_t$ instead of $(p_t^N)_t$, where $\tilde{p}_t^N u(x) := \int_{\mathbb{R}^{k_N}} u(R_N y) p_t(\frac{x}{R_N}, \mathrm{d}y)$, $u \in \mathcal{B}_b(\mathbb{R}^{k_N})$, $x \in \mathbb{R}^{k_N}$. If p_t is m_N-symmetric, then $\tilde{p}_t$ is $m_N \circ R_N^{-1}$-symmetric for $t \geq 0$. Such a re-scaling results in a model $(\Omega_N, \mathcal{C}_{\Omega_N}, \tilde{P}_N)$ instead of $(\Omega_N, \mathcal{C}_{\Omega_N}, P_N)$ with $\tilde{P}_N = P_N \circ R_N^{-1}$, which we could consider just as well. Hence, there is enough freedom to define the actual value of the integral of $\xi_{N,i}$ as a convention. We assume

$$0 \leq \xi_{N,i}(\cdot) \leq 1 \quad \text{and} \quad \int_{[0,1]^d} \xi_{N,i}(z)\,\mathrm{d}z = N^{-d} \quad \text{for } i = 1, \ldots, k_N.$$

Thus,

$$\Lambda_N x := N^{\frac{d}{2}-1} \sum_{i=1}^{k_N} x_i \xi_{N,i} \quad \text{for } x \in \mathbb{R}^{k_N}. \tag{5.1.3}$$

The subsequent arguments require the existence of a uniform bound $n^* \in \mathbb{N}$ with

$$\sup_{N\in\mathbb{N}} \sup_{z\in[0,1]^d} \left| \left\{ i \in \mathbb{N} \,\middle|\, 1 \leq i \leq k_N,\ \xi_{N,i}(z) \neq 0 \right\} \right| \leq n^*. \tag{5.1.4}$$

The map which identifies the canonical process $X = (X_t)_{t\geq 0}$ on Ω_N with a dynamical interface $(h_t)_{t\geq 0}$ as in (5.1.2) is denoted by U_N. For the relevant processes considered in this chapter we always set $c_N := N^2$. So,

$$(U_N X)_t := \Lambda_N X_{N^2 t}, \quad t \geq 0. \tag{5.1.5}$$

U_N is a map from Ω_N into $\big(\mathcal{B}_b([0,1]^d)\big)^{[0,\infty)}$. The first problem which is addressed consists of finding a suitable path space such that the the image measures $(P_N \circ U_N)_{N\in\mathbb{N}}$ form a tight sequence of probability measures. The idea presented in Proposition 5.1.1 has become standard in the discourse of symmetric Markov processes.

From here on, we set $H := L^2((0,1)^d, \mathrm{d}z)$. The set-up of (5.1.6) is assumed for the rest of this chapter. Let H_0 be a Hilbert space which is continuously and densely embedded into H, the embedding denoted by $J : H_0 \hookrightarrow H$. We can interpret H as a subspace of H_0' through the Riesz isomorphism and the Gelfand triple

$$H_0 \subset H = H' \subset H_0' \tag{5.1.6}$$

with central space $(H, \langle\cdot,\cdot\rangle)$. Moreover we assume that J is a Hilbert-Schmidt operator and denote its Hilbert-Schmidt norm by $\|J\|_{\mathrm{HS}}$. It is then consistent in notation to denote the dual pairing of an element $u \in H_0$ with an element $v \in H_0'$ by $\langle u, v\rangle$. The respective norms of these Hilbert spaces are denoted by $|\cdot|_{H_0}$, $|\cdot|_H$, or $|\cdot|_{H_0'}$.

Proposition 5.1.1. *For $N \in \mathbb{N}$ we assume that $\mathcal{D}(\mathcal{E}^N)$ contains the coordinate projection $\pi_i(x) := x_i$, $x \in \mathbb{R}^{k_N}$, for each $i \in \{1,\dots,k_N\}$ and*

$$\sum_{i,j=1}^{k_N} x_i x_j \Gamma_{\mathcal{E}^N}(\pi_i,\pi_j) \le \gamma \sum_{i=1}^{k_N} x_i^2 \quad \text{for all } x \in \mathbb{R}^{k_N}$$

with a constant $\gamma \in (0,\infty)$ which is independent of N. If $(m_N \circ \Lambda_N^{-1})_{N\in\mathbb{N}}$ is a tight family of probability measures on H, then the family $(P_N \circ U_N^{-1})_{N\in\mathbb{N}}$ is tight on $C([0,\infty), H_0')$.

Proof. At first we fix $N \in \mathbb{N}$ and $\varphi \in H_0$. Let $(X_t)_{t\ge 0}$ denote the canonical process with $X_t(\omega) = \omega(t)$ for $\omega \in \Omega_N$. The time-reversal operator on Ω_N is defined by $r_T(\omega)(t) := \omega(T-t)$ for $0 \le t \le T$. This proof follows a well-known idea for the derivation of a tightness result in the context of symmetric Markov processes, which has been realized in [27, Theorem 6.1], [35, Theorem 5.1] or [34, Lemma 5.2] among others. There is a P_N-martingale $(M_t)_{t\ge0}$ such that

$$\langle\varphi, \Lambda_N X_t\rangle - \langle\varphi,\Lambda_N X_0\rangle = \frac{1}{2}M_t + \frac{1}{2}(M_{T-t}\circ r_T - M_T \circ r_T)\ P_N\text{-a.s.},\quad 0\le t\le T, \tag{5.1.7}$$

and whose quadratic variation is given by

$$<M>_t(\omega) = \int_0^t \Gamma_{\mathcal{E}^N}\big(\langle\varphi,\Lambda_N\,\cdot\rangle, \langle\varphi,\Lambda_N\,\cdot\rangle\big)\big(\omega(s)\big)\,\mathrm{d}s\ P_N\text{-a.s.}\quad \text{for } \omega\in\Omega,\ t\ge 0. \tag{5.1.8}$$

The formula of (5.1.7) is stated in [38, Thm. 5.7.1]. The identification of the quadratic variation in (5.1.8) follows from [38, Thm. 5.1.3. & 5.2.3]. From (5.1.8) and the assumption we obtain the bound

$$<M>_t - <M>_s \le N^{d-2}\gamma(t-s)\sum_{i=1}^{k_N}\langle\varphi,\xi_{N,i}\rangle^2\ P_N\text{-a.s.},\quad 0\le s\le t. \tag{5.1.9}$$

First applying (5.1.7) and the Minkowski inequality, then (5.1.8) the Burkholder-Davis-Gundy inequality, we estimate

$$\|\langle\varphi,(U_N X)_t - (U_N X)_s\rangle\|_{\mathcal{L}^4(P_N)}$$

$$\leq \frac{1}{2}\|M_{N^2t} - M_{N^2s}\|_{\mathcal{L}^4(P_N)} + \frac{1}{2}\|M_{T-N^2s} - M_{T-N^2t}\|_{\mathcal{L}^4(P_N)}$$
$$\leq \frac{C}{2}\Big(\int_{\Omega_N}\big(<M>_{N^2t} - <M>_{N^2s}\big)^2\,\mathrm{d}P_N\Big)^{\frac{1}{4}}$$
$$+\frac{C}{2}\Big(\int_{\Omega_N}\big(<M>_{T-N^2s} - <M>_{T-N^2t}\big)^2\,\mathrm{d}P_N\Big)^{\frac{1}{4}}, \quad 0 \leq s \leq t \leq T. \quad (5.1.10)$$

with a constant $C \in (0,\infty)$, independent of N. With (5.1.9), the Cauchy-Schwartz inequality and the estimate $\langle \xi_{N,i}, \xi_{N,i}\rangle \leq N^{-d}$, $i = 1,\dots,k_N$, we continue the estimate of (5.1.10) and get

$$\begin{aligned}
&\|\langle \varphi, (U_N X)_t - (U_N X)_s\rangle\|^2_{\mathcal{L}^4(P_N)} \\
&\leq C^2 N^d \gamma(t-s)\sum_{i=1}^{k_N}\langle \varphi, \xi_{N,i}\rangle^2 \leq C^2\gamma(t-s)\sum_{i=1}^{k_N}\langle 1_{\mathrm{supp}[\xi_{N,i}]}\varphi, \varphi\rangle \\
&\leq C^2\gamma(t-s)n^*\langle \varphi,\varphi\rangle. \qquad (5.1.11)
\end{aligned}$$

Now, we fix an orthonormal basis $\{\varphi_i|\ i \in \mathbb{N}\}$ of H_0. Using Minkowski inequality and then (5.1.11) we obtain

$$\begin{aligned}
&\big\||(U_N X)_t - (U_N X)_s|_{H_0'}\big\|^2_{\mathcal{L}^4(P_N)} \\
&= \Big\|\sum_{i\in\mathbb{N}}\langle \varphi_i, (U_N X)_t - (U_N X)_s\rangle^2\Big\|_{\mathcal{L}^2(P_N)} \\
&\leq \sum_{i\in\mathbb{N}}\big\|\langle \varphi_i, (U_N X)_t - (U_N X)_s\rangle\big\|^2_{\mathcal{L}^4(P_N)} \leq (t-s)\gamma n^* C^2\|J\|^2_{\mathrm{HS}}
\end{aligned}$$

or analogously $\big\||\omega(t)-\omega(s)|_{H_0'}\big\|_{\mathcal{L}^4(P_N\circ U_N^{-1})} \leq (t-s)^{\frac{1}{2}}(\gamma n^*)^{\frac{1}{2}}C\|J\|_{\mathrm{HS}}$. Hence, given an arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sup_{N\in\mathbb{N}} P_N \circ U_N^{-1}\Big(\Big\{\sup_{\substack{0\leq s,t<\infty \\ |s-t|\leq\delta}}|\omega(t)-\omega(s)|_{H_0'} \geq \varepsilon\Big\}\Big) \leq \varepsilon. \qquad (5.1.12)$$

Moreover, due to the stationarity of the one-dimensional distributions of P_N for each $N \in \mathbb{N}$, there exists a compact set $K_{\varepsilon,t}$ for each $t \geq 0$ such that

$$\inf_{N\in\mathbb{N}} P_N \circ U_N^{-1}(\{\omega(t) \in K_{\varepsilon,t}\}) \geq 1-\varepsilon. \qquad (5.1.13)$$

The tightness of $(P_N \circ U_N^{-1})_{N\in\mathbb{N}}$ on $C([0,\infty), H_0')$ now follows from a slightly modified version of [13, Theorem 7.2 of Chapter 3]. It can be proven analogous to the original version, only by changing the definition for a modulus of continuity, from the assignment of [13, Eq. (6.2) of Chapter 3], to

$$w(\omega,\delta,T) = \sup_{\substack{s,t\in[0,T] \\ |s-t|\leq\delta}}|\omega(t)-\omega(s)|_{H_0'}, \quad \omega \in C([0,\infty], H_0'),\ \delta > 0,\ T \in [0,\infty).$$

The analogue result of [13, Lemma 6.1 of Chapter 3], which is used in the proof of the original version, is provided in our case by the generalized Arzelà-Ascoli Theorem, stating that the set

$$A(K,\delta) := \Big\{\omega \in C([0,\infty), H_0') \,\Big|\, \text{there exist } 0 := t_0 < t_1 < \dots \\ \text{such that } t_i - t_{i-1} > \delta, \omega(t_i) \in K, \text{ and} \\ \omega\big|_{[t_{i-1},t_i]} \text{ can be extended to an affine linear function } \mathbb{R} \to H_0', \text{ for } i \in \mathbb{N}\Big\}$$

is relatively compact in $C([0,\infty], H_0')$ for any compact set $K \subset H_0'$ and $\delta > 0$. Analogous to the proof of [13, Theorem 7.2 of Chapter 3], we then obtain, using (5.1.12) and (5.1.13), that for given $\varepsilon_0 > 0$ there exists a compact set B in $C([0,\infty), H_0')$ such that

$$\inf_{N\in\mathbb{N}} P_N \circ U_N^{-1}\big(\{\omega \in C([0,\infty), H_0') \,|\, \mathrm{dist}(\omega, B) < \varepsilon_0\}\big) \geq 1 - \varepsilon_0.$$

Here, $\mathrm{dist}(\omega, B)$ denotes the distance of ω to the set B w.r.t. a suitable metric on $C([0,\infty), H_0')$ which induces the topology of local uniform convergence. In view of [13, Theorem 2.2 of Chapter 3] the proof is completed. □

The only assumption of Proposition 5.1.1 in which the shape of the height maps Λ_N play a role is the tightness of $(m_N \circ \Lambda_N^{-1})_{N\in\mathbb{N}}$. The question rises what would happen if we take a different admissible choice for an injective linear map from $\mathbb{R}^{k_N}$ into $\mathcal{B}_b([0,1]^d)$, say $\widetilde{\Lambda}_N$, instead of Λ_N. Can we specify some properties, i.e. further restrict the class of height maps we consider, such that the tightness of $(m_N \circ \Lambda_N^{-1})_{N\in\mathbb{N}}$ and the tightness of $(m_N \circ \widetilde{\Lambda}_N^{-1})_{N\in\mathbb{N}}$ are equivalent, given that Λ_N and $\widetilde{\Lambda}_N$ are in the specified class? In the literature concerning scaling limits of interface models, two different explicit choices for the height maps are very popular. In the first case, $\Lambda_N x$ is a piecewise linear, continuous function for each $x \in \mathbb{R}^{k_N}$. Such a framework is set up in [30]. In the second case, $\Lambda_N x$ is a piecewise constant step function for each $x \in \mathbb{R}^{k_N}$. [43, Chapter 4.2] considers such an approach. In both examples mentioned, $d = 1$ and accordingly

$$\Lambda_N x = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} x_i \xi_{N,i}, \quad x \in \mathbb{R}^N,$$
$$\text{with} \quad \xi_{N,i}(z) := \big((Nz - i + 1) \wedge (-Nz - i + 1)\big) \vee 0, \quad z \in [0,1],$$

in the first case and

$$\Lambda_N x = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} x_i \xi_{N,i}, \quad x \in \mathbb{R}^N, \quad \text{with} \quad \xi_{N,i}(z) := \mathbf{1}_{[i-1,i)}(Nz), \quad z \in [0,1],$$

in the second. Generally, it would be nice to give a class of height maps from $\mathbb{R}^{k_N}$ into $\mathcal{B}_b([0,1]^d)$ for which the tightness of $(m_N \circ \Lambda_N^{-1})_{N\in\mathbb{N}}$ is actually only an assumption on the family $(m_N)_N$ and not dependent on the particular choice of Λ_N. We now define that class of maps. For this purpose, we take a sequence of increasing cubes $Q_1 \subseteq Q_2 \subseteq \dots \subseteq [0,1]^d$, which exhaust $[0,1]^d$ up to a set of Lebesgue measure zero. We assume that Q_N is open w.r.t. the topology of $[0,1]^d$ and that

$$\big|Q_N \cap \frac{1}{N}\mathbb{Z}^d\big| = k_N.$$

In other words, there are decreasing sequences $(a_i^N)_{N\in\mathbb{N}}$ with $\inf_{N\in\mathbb{N}} a_i^N \leq 0$ for each $i = 1, \dots, d$, and increasing sequences $(b_i^N)_{N\in\mathbb{N}}$ with $\sup_{N\in\mathbb{N}} b_i^N \geq 1$ for each $i = 1, \dots, d$, such that

$$Q_N = (a_1^N, b_1^N) \times \dots \times (a_d^N, b_d^N) \cap [0,1]^d, \quad N \in \mathbb{N}.$$

So, we must have

$$\begin{aligned} k_N &= \left|\left\{\mathbf{k} \in \mathbb{Z}^d \cap [0,N]^d \,\middle|\, Na_i^N < \mathbf{k}_i < Nb_i^N, i = 1, \dots, d\right\}\right| \\ &= \prod_{i=1}^{d} \min\{N, \lceil Nb_i^N - 1\rceil\} - \max\{0, \lfloor Na_i^N \rfloor\}. \end{aligned}$$

We refer to $G_N := Q_N \cap \frac{1}{N}\mathbb{Z}^d$ as the set of grid points, or the set of sites. For each site $p \in G_N$ an elementary function $\xi_N^p(z) := \Xi(Nz - Np)$, $z \in [0,1]^d$, emerges as the shift and the re-scaling of one fixed function Ξ, which is of a certain class and is independent of N and p.

Condition 5.1.2. (i) The archetype function Ξ is a non-negative, measurable function on $\mathbb{R}^d$ and characterized by the following properties:

$$\operatorname{supp}[\Xi] \subseteq [-1,1]^d, \qquad \int_{\mathbb{R}^d} \Xi(z)\,\mathrm{d}z = 1, \qquad \sum_{\mathbf{k}\in\mathbb{Z}^d} \Xi(z - \mathbf{k}) = 1, \quad z \in [0,1]^d.$$

(ii) Additionally, the strict inequality $\displaystyle\int_{\mathbb{R}^d} \Xi(z)^2\,\mathrm{d}z > \frac{1}{2}$ shall be satisfied.

(iii) The elementary functions $\xi_N^p(z) := \Xi(Nz - Np)$, $z \in [0,1]^d$, indexed by $p \in G_N$, correspond to the functions $\xi_{N,i} = \Lambda_N(N^{1-\frac{d}{2}}\mathbf{e}_i)$ from (5.1.3), with index $i = 1, \dots, k_N$. More precisely, we assume that there is a numbering $p_1, \dots, p_{k_N}$ of all sites such that $\xi_{N,i} = \xi_N^{p_i}$ for $i = 1, \dots, k_N$.

The condition specifies a class of height maps, of which kind we assume the map Λ_N to be. The parameters in that set-up, by which Λ_N is completely determined, are hence the archetype function Ξ, a k_N-sized set of grid points G_N, and a numbering of the grid points, i.e. a bijection $\{1, \dots, k_N\} \to G_N$. The advantage of restricting to that class of height maps is explained by Lemma 5.1.4. If m_N is a probability measure on $(\mathbb{R}^{k_N}, \mathcal{B}(\mathbb{R}^{k_N}))$ for $N \in \mathbb{N}$ and m is a probability measure on $(H, \mathcal{B}(H))$, then the validity of the statement $m_N \circ \Lambda_N^{-1} \Rightarrow m$ is independent from the particular choice of Ξ, as long as in accordance with Condition 5.1.2.

A short remark should be made about the second item of 5.1.2. It can be shown, actually, that (i) already implies

$$\frac{1}{2} \leq \int_{\mathbb{R}^d} \Xi(z)^2\,\mathrm{d}z \leq 1.$$

By demanding the strict inequality in (ii) of Condition 5.1.2, we ensure a property which is needed in the proof of 5.1.4.

Remark 5.1.3. Let $(\lambda_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$ such that $\lambda_{\mathbf{k}} = 0$ holds except for finitely many $\mathbf{k} \in \mathbb{Z}^d$. Further, let

$$A_{\mathbf{k},\mathbf{l}} := \int_{\mathbb{R}^d} \Xi(z - \mathbf{k})\Xi(z - \mathbf{l})\,\mathrm{d}z, \quad \mathbf{k}, \mathbf{l} \in \mathbb{Z}^d,$$

and $c := \frac{1}{2} - \int_{\mathbb{R}^d} \Xi(z)^2\,\mathrm{d}z$. It holds

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} \sum_{\mathbf{l}\in\mathbb{Z}^d} \lambda_{\mathbf{k}} A_{\mathbf{k},\mathbf{l}} \lambda_{\mathbf{l}} \geq 2c \sum_{\mathbf{k}\in\mathbb{Z}^d} \lambda_{\mathbf{k}}^2.$$

Proof. Using the equality

$$\frac{1}{2} = \sum_{\mathbf{l}\in\mathbb{Z}^d} A_{\mathbf{k},\mathbf{l}} - \frac{1}{2} = c + \sum_{\substack{\mathbf{l}\in\mathbb{Z}^d\\ \mathbf{l}\neq\mathbf{k}}} A_{\mathbf{k},\mathbf{l}}, \quad \mathbf{k}\in\mathbb{Z}^d,$$

to get to the third line, we have

$$\begin{aligned}
&\sum_{(\mathbf{k},\mathbf{l})\in\mathbb{Z}^d\times\mathbb{Z}^d} \lambda_{\mathbf{k}} A_{\mathbf{k},\mathbf{l}}\lambda_{\mathbf{l}}\\
&= \sum_{\mathbf{k}\in\mathbb{Z}^d} \frac{1}{2} A_{\mathbf{k},\mathbf{k}}\lambda_{\mathbf{k}}^2 + \sum_{\{\mathbf{k},\mathbf{l}\}\subset\mathbb{Z}^d} (A_{\mathbf{k},\mathbf{l}} + A_{\mathbf{k},\mathbf{l}})\lambda_{\mathbf{k}}\lambda_{\mathbf{l}} + \sum_{\mathbf{l}\in\mathbb{Z}^d} \frac{1}{2} A_{\mathbf{l},\mathbf{l}}\lambda_{\mathbf{l}}^2\\
&= \sum_{\mathbf{k}\in\mathbb{Z}^d} \frac{1}{4}\lambda_{\mathbf{k}}^2 + \sum_{\{\mathbf{k},\mathbf{l}\}\subset\mathbb{Z}^d} 2A_{\mathbf{k},\mathbf{l}}\lambda_{\mathbf{k}}\lambda_{\mathbf{l}} + \sum_{\mathbf{l}\in\mathbb{Z}^d} \frac{1}{4}\lambda_{\mathbf{l}}^2 + c\sum_{\mathbf{k}\in\mathbb{Z}^d}\lambda_{\mathbf{k}}^2\\
&= \sum_{\substack{(\mathbf{k},\mathbf{l})\in\mathbb{Z}^d\times\mathbb{Z}^d\\ \mathbf{l}\neq\mathbf{k}}} \frac{1}{2} A_{\mathbf{k},\mathbf{l}}\lambda_{\mathbf{k}}^2 + \sum_{\{\mathbf{k},\mathbf{l}\}\subset\mathbb{Z}^d} 2A_{\mathbf{k},\mathbf{l}}\lambda_{\mathbf{k}}\lambda_{\mathbf{l}} + \sum_{\substack{(\mathbf{k},\mathbf{l})\in\mathbb{Z}^d\times\mathbb{Z}^d\\ \mathbf{l}\neq\mathbf{k}}} \frac{1}{2} A_{\mathbf{k},\mathbf{l}}\lambda_{\mathbf{l}}^2 + 2c\sum_{\mathbf{k}\in\mathbb{Z}^d}\lambda_{\mathbf{k}}^2\\
&= \sum_{\{\mathbf{k},\mathbf{l}\}\subset\mathbb{Z}^d} A_{\mathbf{k},\mathbf{l}}(\lambda_{\mathbf{k}}+\lambda_{\mathbf{l}})^2 + 2c\sum_{\mathbf{k}\in\mathbb{Z}^d}\lambda_{\mathbf{k}}^2\\
&\geq 2c\sum_{\mathbf{k}\in\mathbb{Z}^d}\lambda_{\mathbf{k}}^2.
\end{aligned}$$

□

In Lemma 5.1.4 we consider another admissible choice $\widetilde{\Xi}$ for an archetype function in the sense of Condition 5.1.2. Then, we set $\widetilde{\xi}_{N,i} := \widetilde{\xi_N^{p_i}}(z) := \widetilde{\Xi}(Nz - Np_i)$, $z\in[0,1]^d$, $i = 1,\dots k_N$ and

$$\widetilde{\Lambda}_N x := N^{\frac{d}{2}-1}\sum_{i=1}^{k_N} x_i\widetilde{\xi}_{N,i}, \quad x\in\mathbb{R}^{k_N}.$$

Lemma 5.1.4.

Let m_N be a probability measure on $(\mathbb{R}^{k_N},\mathcal{B}(\mathbb{R}^{k_N}))$ for $N\in\mathbb{N}$ and m be a probability measure on $(H,\mathcal{B}(H))$. Referring to the weak convergence of measures on H it holds

$$m_N\circ\Lambda_N^{-1}\Rightarrow m \quad \text{implies} \quad m_N\circ\widetilde{\Lambda}_N^{-1}\Rightarrow m.$$

Proof. We start with two remarks. The first claims the inequalities

$$cN^{-d}\sum_{i=1}^{k_N} x_i^2 \leq \Big|\sum_{i=1}^{k_N} x_i\xi_{N,i}\Big|_H^2 \leq \Big(\frac{2}{N}\Big)^d\sum_{i=1}^{k_N} x_i^2 \quad \text{for } x\in\mathbb{R}^{k_N}, \tag{5.1.14}$$

where c is a positive constant, independent from N. A second remark concerns the asymptotic behaviour of the height map. More precisely, we have

$$\lim_{N\to\infty}\Big|\sum_{p\in G_N}\varphi(p)\xi_N^p - \varphi\Big|_H = 0 \quad \text{for} \quad \varphi\in C([0,1]^d). \tag{5.1.15}$$

The validity of (5.1.15) is easily check. Since $\operatorname{supp}[\Xi]\subseteq[-1,1]^d$, we have

$$\Big|\sum_{p\in(1/N)\mathbb{Z}^d}\varphi(p)\Xi(Nz-Np)-\varphi(z)\Big| = \Big|\sum_{p\in(1/N)\mathbb{Z}^d}(\varphi(p)-\varphi(z))\Xi(Nz-Np)\Big|$$
$$\leq \sup_{\substack{z'\in\mathbb{R}^d\\ |z'-z|_{\mathrm{euc}}\leq 2\sqrt{2}}}|\varphi(z')-\varphi(z)|\sum_{p\in(1/N)\mathbb{Z}^d}\Xi(Nz-Np) = \sup_{\substack{z'\in\mathbb{R}^d\\ |z'-z|_{\mathrm{euc}}\leq \frac{2\sqrt{2}}{N}}}|\varphi(z')-\varphi(z)|$$

for $\varphi\in C_b(\mathbb{R}^d)$ and $z\in[0,1]^d$. The right-hand side converges to zero for $N\to\infty$. Moreover, if $z\in(0,1)^d$ and $N_0\in\mathbb{N}$ is large enough such that $B^{(\mathrm{euc})}_{\frac{2\sqrt{2}}{N}}(z)\subset Q_{N_0}$, then it holds

$$\sum_{p\in G_N}\varphi(p)\xi_N^p(z) = \sum_{p\in(1/N)\mathbb{Z}^d}\tilde{\varphi}(p)\Xi(Nz-Np), \quad \varphi\in C_b([0,1]^d),\ N\geq N_0,$$

where $\tilde{\varphi}$ denotes a continuous, bounded function on $\mathbb{R}^d$, extending φ. Hence, (5.1.15) is fulfilled.

We move on to prove (5.1.14). We choose $\mathbf{k}^1,\dots,\mathbf{k}^{k_N}\in\mathbb{Z}^d$ such that $\{\frac{1}{N}\mathbf{k}^1,\dots,\frac{1}{N}\mathbf{k}^{k_N}\} = G_N$. Then, by virtue of 5.1.3, there is a positive constant c such that

$$\begin{aligned} c\sum_{\mathbf{k}\in NG_N}\lambda_{\mathbf{k}}^2 &\leq \int_{\mathbb{R}^d}\Big(\sum_{\mathbf{k}\in NG_N}\lambda_{\mathbf{k}}\Xi(z-\mathbf{k})\Big)^2\,\mathrm{d}z \\ &= N^d\int_{\mathbb{R}^d}\Big(\sum_{\mathbf{k}\in NG_N}\lambda_{\mathbf{k}}\Xi(Nz-\mathbf{k})\Big)^2\,\mathrm{d}z \\ &\leq N^d\Big|\Big(\sum_{\mathbf{k}\in NG_N}\lambda_{\mathbf{k}}\xi_N^{\frac{1}{N}\mathbf{k}}(z)\Big|_H^2 \quad \text{for } \lambda\in\mathbb{R}^{NG_N}. \end{aligned}$$

The left inequality of (5.1.14) now follows after choosing a suitable numbering $\{1,\dots,k_N\}\to G_N$, $i\mapsto p_i$, with $\xi_{N,i}=\xi_N^{p_i}$. The second inequality of (5.1.14) is immediate from the estimates

$$ab\langle\xi_{N,i},\xi_{N,j}\rangle \leq \frac{1}{2}(a^2+b^2)\big|\langle\xi_{N,i},\xi_{N,j}\rangle\big| \leq \frac{N^{-d}}{2}(a^2+b^2), \quad a,b\in\mathbb{R},\ i,j=1,\dots,k_N,$$

and

$$\big|\{j\,|\,j=1,\dots,k_N \text{ such that } \langle\xi_{N,i},\xi_{N,j}\rangle\neq 0\}\big| \leq 2^d, \quad i=1,\dots,k_N.$$

We close this first part of the proof with the remark that (5.1.14) and (5.1.15) remain valid, of course, if $\xi_{N,i}$ is replaced by $\widetilde{\xi}_{N,i}$ and ξ_N^p is replaced by $\widetilde{\xi}_N^p$, as they have the identical properties by assumption. Endowed with the necessary preliminaries now, we commit ourselves to the main part of the proof now. We assume $m_N\circ\Lambda_N^{-1}\Rightarrow m$ and show tightness of $(m_N\circ\widetilde{\Lambda}_N^{-1})_N$. The claim is that for each compact set $K_1\subset H$ there exists $N_0\in\mathbb{N}$ and a compact set $K_2\subset H$ such that

$$\bigcup_{N\geq N_0}\widetilde{\Lambda}_N\circ\Lambda_N^{-1}(K_1)\subseteq K_2. \tag{5.1.16}$$

To prove (5.1.16) we now show that the set on the left-hand-side is totally bounded. Let $\varepsilon>0$ and c be the constant from (5.1.14). Since K_1 is totally bounded and $C([0,1]^d)$ is dense in H we can, possibly after using the triangular inequality, find $m\in\mathbb{N}$ and $\varphi_1,\dots,\varphi_m\in C([0,1]^d)$ such that

$$K_1\subseteq\bigcup_{l=1}^m B_{(\varepsilon/3)\sqrt{(c/2^d)}}(\varphi_l).$$

Due to (5.1.15), we can choose $N_0 \in \mathbb{N}$ and $x_{N,l} \in \mathbb{R}^{k_N}$ for $l = 1, \dots, m$, $N \geq N_0$, such that

$$\widetilde{\Lambda}_N x_{N,l} \in B_{\varepsilon/3}(\varphi_l) \quad \text{and} \quad \Lambda_N x_{N,l} \in B_{(\varepsilon/3)\sqrt{(c/2^d)}}(\varphi_l), \quad l = 1, \dots, m,\ N \geq N_0.$$

From (5.1.14) we conclude

$$|\widetilde{\Lambda}_N \circ \Lambda_N^{-1} h|_H \leq \sqrt{\frac{2^d}{c}}\, |h|_H, \quad h \in \mathrm{Im}(\Lambda_N),\ N \in \mathbb{N}. \tag{5.1.17}$$

Now, if $y \in \Lambda_N^{-1}(B_{(\varepsilon/3)\sqrt{(c/2^d)}}(\varphi_l))$ for $N \geq N_0$ and some $l \in \{1, \dots, m\}$, then estimating with the triangular inequality and (5.1.17) yields

$$\begin{aligned} |\widetilde{\Lambda}_N y - \varphi_l|_H &= \big|\widetilde{\Lambda}_N \circ \Lambda_N^{-1}(\Lambda_N y - \Lambda_N x_{N,l}) + \widetilde{\Lambda}_N x_{N,l} - \varphi_l\big|_H \\ &\leq \sqrt{\frac{2^d}{c}}\, \big|\Lambda_N y - \varphi_l + \varphi_l - \Lambda_N x_{N,l}\big|_H + \big|\widetilde{\Lambda}_N x_{N,l} - \varphi_l\big|_H \\ &\leq \sqrt{\frac{2^d}{c}} \Big(\frac{\varepsilon}{3}\sqrt{\frac{c}{2^d}} + \frac{\varepsilon}{3}\sqrt{\frac{c}{2^d}}\Big) + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence,

$$\bigcup_{N \geq N_0} \widetilde{\Lambda}_N \circ \Lambda_N^{-1}(K_1) \subseteq \bigcup_{l=1}^{m} B_\varepsilon(\varphi_l).$$

and (5.1.16) is proven. W.l.o.g. we take $N_0 \geq 2$. The tightness of $(m_N \circ \widetilde{\Lambda}_N^{-1})_N$ now follows immediately using the tightness of $(m_N \circ \Lambda_N^{-1})_N$. Indeed, for $\delta > 0$ and a compact set $K_\delta \subset H$ such that

$$\inf_{N \in \mathbb{N}} m_N(\Lambda_N^{-1}(K_\delta)) \geq 1 - \delta,$$

we can choose a compact set $K'_\delta \subset H$ with $\widetilde{\Lambda}_N \circ \Lambda_N^{-1}(K_\delta) \subseteq K'_\delta$ for $N \geq N_0$, because of what has been shown. For each $N = 1, \dots, N_0 - 1$ we choose a compact set Q_δ^N, such that $m_N(\widetilde{\Lambda}_N^{-1}(Q_\delta^N)) \geq 1 - \delta$. Then, we have

$$\begin{aligned} &\inf_{N \in \mathbb{N}} m_N \circ \widetilde{\Lambda}_N^{-1}\big(Q_\delta^1 \cup \dots \cup Q_\delta^{N_0-1} \cup K'_\delta\big) \\ &\geq \min\Big\{m_1(\widetilde{\Lambda}_1^{-1}(Q_\delta^1)), \dots, m_{N_0-1}(\widetilde{\Lambda}_{N_0-1}^{-1}(Q_\delta^{N_0-1})), \inf_{N \geq N_0} m_N(\Lambda_N^{-1}(K_\delta))\Big\} \\ &\geq 1 - \delta \end{aligned}$$

verifying the tightness of $(m_N \circ \widetilde{\Lambda}_N^{-1})_N$. By virtue of [13, Theorem 4.5 of Chapter 3], the claim of this lemma is now shown with the statement

$$\lim_{N \to \infty} \int_{\mathbb{R}^{k_N}} F(\langle \varphi, \widetilde{\Lambda}_N x \rangle)\, \mathrm{d}m_N(x) = \int_H F(\langle \varphi, h \rangle)\, \mathrm{d}m(h),$$
$$\text{for} \quad F \in \mathrm{Lip}_b(\mathbb{R}),\ \varphi \in C([0,1]^d). \tag{5.1.18}$$

The last part of this proof is dedicated to prove 5.1.18. We have

$$\Big|\int_{\mathbb{R}^{k_N}} F(\langle \varphi, \widetilde{\Lambda}_N x \rangle)\, \mathrm{d}m_N(x) - \int_H F(\langle \varphi, h \rangle)\, \mathrm{d}m(h)\Big|$$

$$\leq \left| \int F(\langle \varphi, \widetilde{\Lambda}_N x\rangle - \langle \varphi, \Lambda_N x\rangle) \, \mathrm{d}m_N(x) \right| + \left| \int F(\langle \varphi, \Lambda_N x\rangle) \, \mathrm{d}m_N(x) - \int_H F(\langle \varphi, h\rangle) \, \mathrm{d}m(h) \right|.$$

Let $\varepsilon > 0$. A simple $\frac{\varepsilon}{3}$-argument will show the convergence of (5.1.18) by bounding the right-hand side of the inequality above. We can choose $N_0 \in \mathbb{N}$ large enough such that the second term takes a value smaller equal $\frac{\varepsilon}{3}$ for $N \geq N_0$, because of the assumption. Let L denote the Lipschitz constant of F and K be a compact set such that $\inf_{N\in\mathbb{N}} m_N \circ \widetilde{\Lambda}_N^{-1}(K) \geq 1 - \frac{\varepsilon}{3\|F\|_\infty}$. It follows

$$\left| \int F(\langle \varphi, \widetilde{\Lambda}_N x\rangle - \langle \varphi, \Lambda_N x\rangle) \, \mathrm{d}m_N(x) \right| \leq \int_{\widetilde{\Lambda}_N^{-1}(K)} L \big| \langle \varphi, \widetilde{\Lambda}_N x\rangle - \langle \varphi, \Lambda_N x\rangle \big| \, \mathrm{d}m_N(x) + \frac{\varepsilon}{3}.$$

We now apply the Cauchy-Schwartz inequality, the mean value theorem, and then (5.1.14) with $\xi_{N,i}$ replaced by $\widetilde{\xi}_{N,i}$, to obtain the estimate

$$\begin{aligned}
&L\big|\langle \varphi, \widetilde{\Lambda}_N x\rangle - \langle \varphi, \Lambda_N x\rangle\big| \\
&= L N^{\frac{d}{2}-1} \Big| \sum_{i=1}^{k_N} x_i \big(\langle \varphi, \widetilde{\xi}_{N,i}\rangle - \langle \varphi, \xi_{N,i}\rangle \big) \Big| \\
&\leq L N^{-\frac{d}{2}-1} \sum_{i=1}^{k_N} |x_i| \Big| N^d \int_{[0,1]^d} \varphi \widetilde{\xi}_{N,i} \, \mathrm{d}z - N^d \int_{[0,1]^d} \varphi \xi_{N,i} \, \mathrm{d}z \Big| \\
&\leq L N^{-\frac{d}{2}-1} \Big(\sum_{i=1}^{k_N} x_i^2 \Big)^{\frac{1}{2}} \sup_{\substack{z,z' \in [0,1]^d \\ |z-z'| < 2\sqrt{2}/N}} |\varphi(z) - \varphi(z')| \\
&\leq \frac{L \max_{h\in K} |h|_H}{\sqrt{c}} \sup_{\substack{z,z' \in [0,1]^d \\ |z-z'| < 2\sqrt{2}/N}} |\varphi(z) - \varphi(z')|, \quad x \in \widetilde{\Lambda}_N^{-1}(K).
\end{aligned}$$

After possibly choosing N_0 even larger, also this estimate is bounded by $\frac{\varepsilon}{3}$ if $N \geq N_0$. This concludes the proof. □

5.2 Convergence results

5.2.1 Perturbation with densities

All discussion in this section so far is based on underlying probability measures m_N on $(\mathbb{R}^{k_N}, \mathcal{B}(\mathbb{R}^{k_N}))$ for $N \in \mathbb{N}$. The relevant ones for the purpose of this section are of a certain type. They have a density w.r.t. a non-degenerate, centred Gaussian measure μ_N on $\mathbb{R}^{k_N}$. The relevant density function has the form $\exp\big(\sum_{i=1}^{k_N} f_N(x_i)\big)$, $x \in \mathbb{R}^{k_N}$, where $f_N : \mathbb{R} \to \mathbb{R}$ is a function of bounded variation. So, we are interested in the question whether the weak convergence of the image measures $\mu_N \circ \Lambda_N^{-1} \Rightarrow \mu$ on H, where μ is a non-degenerate, centred Gaussian measure on H, imply the weak convergence of the images of weighted measures $(\rho_N \mu_N) \circ \Lambda_N^{-1} \Rightarrow \rho\mu$ on H, with densities ρ_N of such a type. We want to scale in such a way, that a function $\exp\big(\int_{[0,1]^d} f \circ h \, \mathrm{d}z\big)$, $h \in H$, where $f : \mathbb{R} \to \mathbb{R}$ again is a function of bounded variation, appears as the limiting density. This is clearly a bounded function, which gives a well-defined value for each $h \in H$. It is, however, not continuous on H, as the easy example of choosing $f := \mathbf{1}_{[0,\infty)}$ can show.

With the decomposition $A_1 \cup A_2 = [0,1]^d$ for suitable $A_1, A_2 \in \mathcal{B}([0,1]^d)$, $A_1 \cup A_2 = \emptyset$, we can write

$$\exp\big(\int_{[0,1]^d} f \circ h \,\mathrm{d}z\big) = \exp\big(\int_{A_1} f \circ h \,\mathrm{d}z\big) \exp\big(\int_{A_2} f \circ h \,\mathrm{d}z\big), \quad h \in H.$$

The Jordan decomposition for a function f of bounded variation on a compact interval, as derived e.g. in [15, Chapter 5.2], yields the representation as the difference of two bounded, monotone increasing functions. This generalizes to a function $f : \mathbb{R} \to \mathbb{R}$, which is globally of bounded variation. Then the Jordan decomposition states the existence of $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ such that

$$f = f_1 - f_2, \quad \|f_1\|_\infty + \|f_2\|_\infty \le \|f\|_\infty + \|f\|_{\mathrm{TV}}, \quad f_1, f_2 \text{ monotone increasing.} \tag{5.2.1}$$

Here,

$$\|f\|_{\mathrm{TV}} := \sup_{\substack{\{x_1,\dots,x_m\}\subset\mathbb{R} \\ x_1 \le \dots \le x_m}} \sum_{i=1}^{m-1} |f(x_{i+1}) - f(x_i)|$$

denotes the total variation of f. We now define the notion of convergence which fits our purpose. We recall d^* from Section 3.2.1 and specify a notion for convergence $f_N \longrightarrow f$ through the following conditions.

Condition 5.2.1. Let f, f_N for $N \in \mathbb{N}$, be functions of bounded variation from $\mathbb{R}$ to $\mathbb{R}$. We assume that

$$f = f_1 - f_2, \quad f_N = f_1^{(N)} - f_2^{(N)},$$

is a decomposition in the sense of (5.2.1) for f , respectively for f_N for each $N \in \mathbb{N}$, and that the following conditions hold:

(i) $\sup_N \|f_N\|_\infty + \|f_N\|_{\mathrm{TV}} < \infty$.

(ii) $\lim_{N\to\infty} d^*(f_1^{(N)}, f_1) = \lim_{N\to\infty} d^*(f_2^{(N)}, f_2) = 0$.

Let the height map Λ_N, $N \in \mathbb{N}$, be of the type considered in Lemma 5.1.4.

Proposition 5.2.2. *Let μ_N be a probability measure on $\mathbb{R}^{k_N}$ and $f_N : \mathbb{R} \to \mathbb{R}$ be a function of bounded variation, $N \in \mathbb{N}$. Moreover, let μ be a non-degenerate, centered Gaussian measure on H, i.e. a Gaussian measure with mean zero and* $\operatorname{supp}[\mu] = H$, *and $f : \mathbb{R} \to \mathbb{R}$ again be a function of bounded variation. If $f_N \longrightarrow f$ in the sense of Condition 5.2.1 and $\mu_N \circ \Lambda_N^{-1} \Rightarrow \mu$ on H, then*

$$\Big[\exp\Big(N^{-d} \sum_{i=1}^{k_N} f_N(N^{d/2-1} x_i)\Big) \,\mathrm{d}\mu_N(x)\Big] \circ \Lambda_N^{-1}$$
$$\Rightarrow \quad \exp\Big(\int_{(0,1)^d} f \circ h(z)\,\mathrm{d}z\Big)\,\mathrm{d}\mu(h) \tag{5.2.2}$$

as $N \to \infty$, referring to the weak convergence of measures on H.

Proof. We start by giving a general statement concerning the level sets of a non-degenerate, centered Gaussian measure on H. The claim is that the level set $\{z \in (0,1)^d \mid h(z) = a\}$ has Lebesgue measure zero for μ-a.e. $h \in H$ and every $a \in \mathbb{R}$. More precisely, the $\mathrm{d}z$-class defined by the composition $\mathbf{1}_{\{a\}} \circ h$ vanishes in $\mathrm{d}z$-a.e. sense for each $h \in H \setminus \mathcal{N}_a$ and $\mathcal{N}_a$ is a set in $\mathcal{B}(H)$ with $\mu(\mathcal{N}_a) = 0$. This is shown as follows.

Let $\varphi \in H$ such that the image measure of μ under a shift $\tau_{s\varphi}h := h + s\varphi$, $h \in H$, $s \in \mathbb{R}$, is absolutely continuous w.r.t. μ itself, i.e. $\mu \circ \tau_{s\varphi}^{-1} \ll \mu$. In that case, we immediately have $\mu \ll \mu \circ \tau_{s\varphi}^{-1}$, hence the equivalence $\mu \sim \mu \circ \tau_{s\varphi}^{-1}$, for $s \in \mathbb{R}$. Moreover, defining a measure

$$\sigma_\varphi(A) := \int_{\mathbb{R}} \mu \circ \tau_{s\varphi}^{-1}(A)\, \mathrm{d}s, \quad A \in \mathcal{B}(H),$$

it holds $\sigma_\varphi \sim \mu$. Now, let $A \subset [0,1]^d$ be a Borel measurable set such that $\varphi \neq 0$ a.e. on A. Since,

$$\int_H u \,\mathrm{d}\mu = \int_H \int_{\mathbb{R}} u(h + s\varphi) \frac{\mathrm{d}\mu}{\mathrm{d}\sigma_\varphi}(h + s\varphi)\, \mathrm{d}s\, \mathrm{d}\mu(h) \quad \text{for all } u \in \mathcal{B}_+(H)$$

and each singleton is negligible w.r.t. the Lebesgue measure, Fubini's theorem yields

$$\begin{aligned}\int_H \int_A \mathbf{1}_a(h(z))\, \mathrm{d}z\, \mathrm{d}\mu(h) &= \int_H \int_{\mathbb{R}} \int_A \mathbf{1}_a\big(h(z) + s\varphi(z)\big)\, \mathrm{d}z \frac{\mathrm{d}\mu}{\mathrm{d}\sigma_\varphi}(h + s\varphi)\, \mathrm{d}s\, \mathrm{d}\mu(h) \\ &= \int_H \int_A \int_{\mathbb{R}} \mathbf{1}_{\frac{a-h(z)}{\varphi(z)}}(s) \frac{\mathrm{d}\mu}{\mathrm{d}\sigma_\varphi}(h + s\varphi)\, \mathrm{d}s\, \mathrm{d}z\, \mathrm{d}\mu(h) = 0. \quad (5.2.3)\end{aligned}$$

Due to the Cameron-Martin formula, the space $\{\varphi \in H \mid \mu \circ \tau_{s\varphi}^{-1} \ll \mu \text{ for } s \in \mathbb{R}\}$ is dense in H, as is shown in [19, Theorems 3.1.2 & 3.1.3 of Chapter II]. Hence, using (5.2.3) we can find an orthonormal basis $\varphi_1, \varphi_2, \ldots$ in H and Borel measurable subsets $A_1, A_2, \ldots$ of $[0,1]^d$, such that $A_i = \{z \in [0,1]^d \,|\, \tilde{\varphi}_i(z) \neq 0\}$ for some $\mathrm{d}z$-version $\tilde{\varphi}_i$ of φ_i and

$$\int_H \int_{A_i} \mathbf{1}_a(h(z))\, \mathrm{d}z\, \mathrm{d}\mu(h) = 0, \quad i \in \mathbb{N}.$$

Let $B := [0,1]^d \setminus \big(\bigcup_{i\in\mathbb{N}} A_i\big)$. Since $\langle \mathbf{1}_B, \varphi_i \rangle = 0$ for every $i \in \mathbb{N}$, the set B has Lebesgue measure zero. Therefore,

$$\begin{aligned}\int_H \int_{[0,1]^d} \mathbf{1}_a(h(z))\, \mathrm{d}z\, \mathrm{d}\mu(h) &= \int_H \int_{\bigcup_{i\in\mathbb{N}} A_i} \mathbf{1}_a(h(z))\, \mathrm{d}z\, \mathrm{d}\mu(h) \\ &\leq \sum_{i=1}^{\infty} \int_H \int_{A_i} \mathbf{1}_a(h(z))\, \mathrm{d}z\, \mathrm{d}\mu(h) = 0.\end{aligned}$$

In particular, there exists $\mathcal{N}_a \in \mathcal{B}(H)$ with $\mu(\mathcal{N}_a) = 0$, such that the $\mathrm{d}z$-class defined by the composition $\mathbf{1}_{\{a\}} \circ h$ vanishes in $\mathrm{d}z$-a.e. sense for each $h \in H \setminus \mathcal{N}_a$.

Now, let m_N, m be finite measures on H, $N \in \mathbb{N}$, such that

$$\int_{\left\{z\in(0,1)^d \,\middle|\, h(z)=a\right\}} \mathrm{d}z = 0 \ \mathrm{d}m(h)\text{-a.e.}, \quad a \in \mathbb{R}. \quad (5.2.4)$$

Moreover, let g_N, g be monotone increasing functions on $\mathbb{R}$, $\mathbb{N} \in \mathbb{N}$, such that $\lim_{N\to\infty} d^*(g_N, g) = 0$. By virtue of Lemma 4.2.2 it holds

$$m_N \Rightarrow m \quad \text{implies} \quad e^{\int_{(0,1)^d} g_N \circ h\, \mathrm{d}z}\, \mathrm{d}m_N(h) \quad \Rightarrow \quad e^{\int_{(0,1)^d} g \circ h\, \mathrm{d}z}\, \mathrm{d}m(h)$$

$$\text{and also } e^{-\int_{(0,1)^d} g_N \circ h \, \mathrm{d}z} \, \mathrm{d}m_N(h) \quad \Rightarrow \quad e^{-\int_{(0,1)^d} g \circ h \, \mathrm{d}z} \, \mathrm{d}m(h) \tag{5.2.5}$$

Until the end of this proof, $\widetilde{\Lambda}_N$, $N \in \mathbb{N}$, shall denote the height map which emerges from the choice $\widetilde{\Xi} = \mathbf{1}_{[0,1)^d}$ as the archetype function, instead of the function Ξ associated to Λ_N. In other words, if $p_1, \dots, p_{k_N}$ is a numbering such that $\xi_{N,i} = \Xi(N \, \cdot \, - Np_i)$ and $\Lambda_N x = N^{\frac{d}{2}-1} \sum_{i=1}^{k_N} x_i \xi_{N,i}$, $x \in \mathbb{R}^{k_N}$, then we define

$$\widetilde{\Lambda}_N x := N^{\frac{d}{2}-1} \sum_{i=1}^{k_N} x_i \mathbf{1}_{[0,1)^d}(N \, \cdot \, - Np_i) = N^{\frac{d}{2}-1} \sum_{i=1}^{k_N} x_i \mathbf{1}_{p_i + [0, \frac{1}{N})^d}, \quad x \in \mathbb{R}^{k_N}.$$

We now argue why (5.2.5) implies the claim of this proposition. Indeed, by using a Jordan decomposition from (5.2.1) for f_N, $N \in \mathbb{N}$, and f as in the assumptions of this proposition, (5.2.5) yields the statement

$$m_N \Rightarrow m \quad \text{implies} \quad e^{\int_{(0,1)^d} f_N \circ h(z) \, \mathrm{d}z} \, \mathrm{d}m_N(h) \Rightarrow e^{\int_{(0,1)^d} f \circ h(z) \, \mathrm{d}z} \, \mathrm{d}m(h), \tag{5.2.6}$$

given that m_N, m are finite measures on H with (5.2.4). Since

$$N^{-d} \sum_{i=1}^{k_N} f_N(N^{\frac{d}{2}-1} x_i) = \int_{(0,1)^d} f_N(\widetilde{\Lambda}_N x(z)) \, \mathrm{d}z, \quad x \in \mathbb{R}^{k_N}, \tag{5.2.7}$$

assuming $\mu_N \circ \widetilde{\Lambda}_N^{-1} \Rightarrow \mu$ and (5.2.6), one can conclude

$$\begin{aligned}
&\lim_N \int_{\mathbb{R}^{k_N}} F(\widetilde{\Lambda}_N x) \exp\Big(N^{-d} \sum_{i=1}^{k_N} f_N(N^{d/2-1} x_i)\Big) \, \mathrm{d}\mu_N(x) \\
&= \lim_N \int_H F(h) e^{\int_{(0,1)^d} f_N \circ h \, \mathrm{d}z} \, \mathrm{d}(\mu_N \circ \widetilde{\Lambda}_N^{-1})(h) \\
&= \int_H F(h) e^{\int_{(0,1)^d} f \circ h \, \mathrm{d}z} \, \mathrm{d}\mu(h) \quad \text{for all } F \in C_b(H).
\end{aligned} \tag{5.2.8}$$

We note that $\mu_N \circ \widetilde{\Lambda}_N^{-1} \Rightarrow \mu$ is equivalent to $\mu_N \circ \Lambda_N^{-1} \Rightarrow \mu$ because of 5.1.4. However, (5.2.8) is equivalent to (5.2.2) again by virtue of 5.1.4. So, the claim of the proposition follows from (5.2.5) indeed. □

5.2.2 Convergence of finite-dimensional distributions

We continue to work in the setting of Proposition 5.2.2. There, a statement about the weak convergence of measures has been derived. Now, we go a step further and show Mosco convergence of the corresponding gradient Dirichlet forms. We make the same assumptions, except that now also the approximating sequence of reference measures are required to be non-degenerate, centred Gaussian, just like their limit. So, we take f, f_N for $N \in \mathbb{N}$, functions of bounded variation, and define the densities

$$\rho_N(x) := \exp\Big(N^{-d} \sum_{i=1}^{k_N} f_N(N^{\frac{d}{2}-1} x_i)\Big), \quad x \in \mathbb{R}^{k_N},$$

and

$$\rho(h) := \exp\Big(\int_{(0,1)^d} f(h(z)) \, \mathrm{d}z\Big), \quad h \in H.$$

The reference measures are non-degenerate, centred Gaussian measures μ_N on $\mathbb{R}^{k_N}$ for $N \in \mathbb{N}$ and a non-degenerate, centred Gaussian measure μ on H. We assume

$$f_N \longrightarrow f \quad \text{as in Condition 5.2.1 and} \quad \mu_N \circ \Lambda_N^{-1} \Rightarrow \mu \quad \text{on } H, \tag{5.2.9}$$

where the height map Λ_N, $N \in \mathbb{N}$, is as in Lemma 5.1.4. For the purpose of this section we additionally assume $\rho\mu(H) = 1$ and $\rho_N \mu_N(\mathbb{R}^{k_N}) = 1$ for $N \in \mathbb{N}$. We remark that normalizing the densities defined above would still be incorporated in our scheme, as one could simply consider functions $f_N - \frac{N^d \ln Z_N}{k_N}$ replacing f_N and $f - \ln Z$ replacing f, where

$$Z_N := \int_{\mathbb{R}^{k_N}} \exp\Big(N^{-d} \sum_{i=1}^{k_N} f_N(N^{\frac{d}{2}-1} x_i)\Big) \, \mathrm{d}\mu_N(x)$$

and

$$Z := \int_H \exp\Big(\int_{(0,1)^d} f(h(z)) \, \mathrm{d}z\Big) \, \mathrm{d}\mu(h).$$

Indeed, we have $\frac{N^d}{k_N} \to 1$ and $Z_N \to Z$ because of Proposition 5.2.2. Therefore we still have $f_N - \frac{N^d \ln Z_N}{k_N} \longrightarrow f - \ln Z$ in the sense of Condition 5.2.1.

We now take a closer look at the condition $\mu_N \circ \Lambda_N^{-1} \Rightarrow \mu$. Since μ is a non-degenerate, centred Gaussian measures on H, the covariance

$$\mathrm{Cov}_\mu(h_1, h_1) := \int_H \langle h_1, k\rangle \langle h_2, k\rangle \, \mathrm{d}\mu(k), \quad h_1, h_2 \in H,$$

defines an inner product on H which makes $(H, \mathrm{Cov}_\mu(\cdot,\cdot))$ a pre-Hilbert space. Taking the abstract completion of H w.r.t. the norm induced by Cov_μ we construct a new Hilbert space $(H_\mu, \langle\cdot,\cdot\rangle_\mu)$, in which $(H, \langle\,\cdot\,,\,\cdot\,\rangle)$ is densely and continuously included. The self-adjoint generator $(A, \mathcal{D}(A))$ of the symmetric closed form $\langle\cdot,\cdot\rangle$ with domain H on $(H_\mu, \langle\cdot,\cdot\rangle_\mu)$ is characterized by

$$H = \mathcal{D}(A^{\frac{1}{2}}), \quad \langle Au, v\rangle_\mu = \langle u, v\rangle \quad \text{for } u \in \mathcal{D}(A),\ v \in H.$$

The spectrum of A is a pure point spectrum which consists of real, positive eigenvalues $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_i \le \ldots$. In fact, the inverse A^{-1} is a trace class operator from H into H_μ.

Analogous structures exists w.r.t. μ_N for each $N \in \mathbb{N}$. We take into account the support of $\mu_N \circ \Lambda_N^{-1}$. Since Λ_N is an isomorphism from $\mathbb{R}^{k_N}$ into the Hilbert space $(\mathrm{Im}(\Lambda_N), \langle\cdot,\cdot\rangle)$, the image measure $\mu_N \circ \Lambda_N^{-1}$ is a non-degenerate, centered Gaussian measure on $\mathrm{Im}(\Lambda_N)$. However, since we are in the finite-dimensional case, there is no need to do a completion. We define an inner product

$$\langle h_1, h_2\rangle_{\mu_N \circ \Lambda_N^{-1}} := \int_{\mathbb{R}^{k_N}} \langle h_1, \Lambda_N x\rangle \langle h_2, \Lambda_N x\rangle \, \mathrm{d}\mu_N(x), \quad h_1, h_2 \in \mathrm{Im}(\Lambda_N).$$

There is a symmetric, positive operator A_N on $\mathrm{Im}(\Lambda_N)$ with

$$\langle A_N u, v\rangle_{\mu_N \circ \Lambda_N^{-1}} = \langle u, v\rangle \quad \text{for } u, v \in \mathrm{Im}(\Lambda_N).$$

The spectrum of A_N comprises real, positive eigenvalues $0 < \lambda_1^N \le \cdots \le \lambda_{k_N}^N < \infty$. We choose an orthonormal basis $\{\varphi_i^N \mid i = 1, \ldots, k_N\}$ of $(\mathrm{Im}(\Lambda_N), \langle\cdot,\cdot\rangle_{\mu_N \circ \Lambda_N^{-1}})$ such that $A_N \varphi_i^N = \lambda_i^N$ for $i = 1, \ldots, k_N$. We reformulate a well-known consequence of

$\mu_N \circ \Lambda_N^{-1} \Rightarrow \mu$, the convergence of the spectral structure of A_N towards that of A, in the next Lemma. The reader should be aware though, that the exact condition which characterizes the weak convergence of Gaussian measures in terms of their covariance operators in a frame as ours, uses a stronger notion for the convergence of operators, the one induced by the nuclear norm. For further reading on that topic, we refer to [10] and [14]. The statement of the Lemma is just enough for our purpose and turns out crucial in the proof of the subsequent Theorem. We set $\lambda_i^N := 0$ and $\varphi_i^N := 0$ for $i > k_N$.

Lemma 5.2.3. *(i)* $\lim_{N\to\infty} \lambda_i^N = \lambda_i$ *for all* $i \in \mathbb{N}$.

(ii) There is a subsequence $(\mu_{N_l})_l$ *of* $(\mu_N)_N$ *and an orthonormal basis* $\{\varphi_i \mid i \in \mathbb{N}\}$ *of* H_μ *such that*

$$\lim_{l\to\infty} \varphi_i^{N_l} = \varphi_i \quad \textit{and} \quad \varphi_i \in \mathcal{D}(A) \textit{ with } A\varphi_i = \lambda_i \quad \textit{for all} \quad i \in \mathbb{N}.$$

The convergence takes place strongly in H.

Proof. The statement is exactly the content of [28, Corollary 2.5]. So, we understand $(\mathrm{Im}(\Lambda_N), \langle\cdot,\cdot\rangle_{\mu_N\circ\Lambda_N^{-1}})$, $N \in \mathbb{N}$ as a sequence of converging Hilbert spaces with asymptotic space $(H_\mu, \langle\cdot,\cdot\rangle_\mu)$, denoting the disjoint union of these Hilbert spaces with $\mathcal{H}^{\mathrm{Cov}}$. There is a canonical way to do this, considering the convergence of the second moments

$$\lim_{N\to\infty} \int_{\mathbb{R}^{k_N}} \langle k, \Lambda_N x\rangle^2 \,\mathrm{d}\mu_N(x) = \int_H \langle k, h\rangle^2 \,\mathrm{d}\mu(h), \quad k \in H. \tag{5.2.10}$$

Indeed, if $\boldsymbol{pr}_N$ denote the orthogonal projection in $(H, \langle\cdot,\cdot\rangle)$ with image $\mathrm{Im}(\Lambda_N)$ for $N \in \mathbb{N}$, then (5.2.10) means that

$$\lim_{N\to\infty} \langle \boldsymbol{pr}_N k, \boldsymbol{pr}_N k\rangle_{\mu_N\circ\Lambda_N^{-1}} = \langle k, k\rangle_\mu, \quad k \in H.$$

Moreover, as a consequence of (5.1.15), we have

$$\begin{aligned}\lim_{N\to\infty} \langle A_N^{\frac12} \boldsymbol{pr}_N\varphi, A_N^{\frac12} \boldsymbol{pr}_N\varphi\rangle_{\mu_N\circ\Lambda_N^{-1}} &= \lim_{N\to\infty} \langle \boldsymbol{pr}_N\varphi, \boldsymbol{pr}_N\varphi\rangle \\ &= \langle\varphi,\varphi\rangle = \langle A^{\frac12}\varphi, A^{\frac12}\varphi\rangle_\mu, \quad \varphi \in C([0,1]^d).\end{aligned} \tag{5.2.11}$$

The assumptions of the cited corollary require the compact convergence of the spectral structure of A_N towards that of A for $N \to \infty$. In view of 5.2.11, it is enough to show the following. Given $u_N \in \mathrm{Im}(\Lambda_N)$ for $N \in \mathbb{N}$ and $u \in H$, then the weak convergence of $(u_N)_N$ in $(H, \langle\cdot,\cdot\rangle)$ towards u implies the strong convergence of $(u_N)_N$ towards u in the topology of $\mathcal{H}^{\mathrm{Cov}}$. In other words, if u_N, u are as declared, with $u_N \rightharpoonup u$ weakly in H, then we have to show

$$\lim_{N\to\infty} \int_H \langle u_N, h\rangle^2 \,\mathrm{d}(\mu_N \circ \Lambda_N^{-1})(h) = \int_H \langle u, h\rangle^2 \,\mathrm{d}\mu(h)$$

and

$$\lim_{N\to\infty} \int_H \langle u_N, h\rangle\langle k, h\rangle \,\mathrm{d}(\mu_N \circ \Lambda_N^{-1})(h) = \int_H \langle u, h\rangle\langle u, k\rangle \,\mathrm{d}\mu(h), \quad k \in H.$$

Both of these integral convergences, however, can be shown with a simple $\varepsilon/3$-argument, as made explicit in [40, Theorem 2.2]. To check the assumptions of the cited theorem, it is enough to state the facts that for every $\varepsilon > 0$ there exists $r \in (0, \infty)$ such that

$$\sup_{N\in\mathbb{N}} \int_{\{h\in H \,|\, |h|_H > r\}} |h|_H^2 \,\mathrm{d}(\mu_N \circ \Lambda_N^{-1}) < \varepsilon, \tag{5.2.12}$$

and that the convergence of the linear functional $\langle u_N, \cdot\rangle\big|_K \longrightarrow \langle u, \cdot\rangle\big|_K$ takes place uniformly on each compact set K contained in H. With the remark that (5.2.12) is an immediate consequence of [10, Theorem 1], which states that

$$\sup_{N\in\mathbb{N}} \int_H \exp(a|h|_H^2) \,\mathrm{d}(\mu_N \circ \Lambda_N^{-1}) < \infty$$

for some $a > 0$, we conclude this proof. □

The Dirichlet forms whose asymptotic behaviour we are going to analyse in the proof of Theorem 5.2.6 below are standard gradient forms on the Euclidean space with a weight function. Let $N \in \mathbb{N}$ be fixed. The weight equals the product of the function ρ_N and the Radon-Nikodym derivative $\frac{\mathrm{d}\mu_N}{\mathrm{d}x}$ of the Gaussian measure μ_N. We set

$$\mathcal{E}^N(u,v) := \sum_{i=1}^{k_N} \int_{\mathbb{R}^{k_N}} \partial_i u \,\partial_i v \rho_N \,\mathrm{d}\mu_N, \quad u, v \in C_b^1(\mathbb{R}^{k_N})^{\sim}.$$

Since for each compact set $K \subset L^2(\mathbb{R}^{k_N}, \mu_N)$, there is a constant c_K such that $\rho_N(x)\frac{\mathrm{d}\mu_N}{\mathrm{d}x}(x) \geq c_K$ $\mathrm{d}x$-a.e. on K, the form $\mathcal{E}^N$ with domain $C_b^1(\mathbb{R}^{k_N})^{\sim}$ is a classical example of a closable pre-Dirichlet form on $L^2(\mathbb{R}^{k_N}, \rho_N\mu_N)$, as treated in Section 3.1.2. We denote its closure by $(\mathcal{E}^N, \mathcal{D}(\mathcal{E}^N))$. Let $(p_t^N)_{t\geq 0}$ be the $\rho_N\mu_N$-symmetric Markov transition function on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ whose induced semigroup $(T_t^N)_{t\geq 0}$ of symmetric contraction operators on $L^2(\mathbb{R}^{N_k}, \rho_N\mu_N)$ is associated with $\mathcal{E}^N$, i.e. it holds

$$\mathcal{D}(\mathcal{E}^N) = \Big\{u \in L^2(\mathbb{R}^{k_N}, \rho_N\mu_N) \,\Big|\, \sup_{t>0} \frac{1}{t} \int_{\mathbb{R}^{k_N}} (u - T_t^N u) u \rho_N \,\mathrm{d}\mu_N < \infty\Big\}$$
$$\text{with} \quad \mathcal{E}^N(u,v) = \lim_{t\to 0} \frac{1}{t} \int_{\mathbb{R}^{k_N}} (u - T_t^N u) v \rho_N \,\mathrm{d}\mu_N \quad \text{for } u, v \in \mathcal{D}(\mathcal{E}^N).$$

Using the method of Mosco convergence we are able to identify the accumulation points of the laws $(P_N)_N$, where P_N for $N \in \mathbb{N}$ denotes the unique probability measure on $\Omega_N := C([0,\infty), \mathbb{R}^{k_N})$ whose finite-dimensional distributions can be expressed through $(p_t^N)_{t\geq 0}$ as in (5.1.1).

We now define the Dirichlet form which turns out to be the Mosco limit of $(\mathcal{E}^N)_N$, as the proof of Theorem 5.2.6 below shows. Let $(L, \mathcal{D}(L))$ denote the Ornstein-Uhlenbeck operator on $L^2(H, \mu)$, as defined in [19, Chapter II]. For every element u in the Sobolev space $W(H, \mu) := \mathcal{D}(L^{\frac{1}{2}})$ we denote its weak gradient by ∇u, which is an element in $L^2(H, H, \mu)$, and set

$$\|u\|_{W(H,\mu)} := \Big(\int_H |\nabla u|_H^2 \,\mathrm{d}\mu\Big)^{\frac{1}{2}} + \Big(\int_H u^2 \,\mathrm{d}\mu\Big)^{\frac{1}{2}}.$$

Then, the set $\mathcal{C}$ of μ-classes of functions from $\mathcal{F}C_b^{\infty}(H)$ defined as the space

$$\Big\{F(\langle\cdot, \varphi_1\rangle, \ldots, \langle\cdot, \varphi_m\rangle) \,\Big|\, F \in C_b^{\infty}(\mathbb{R}^m),\ \varphi_1, \ldots, \varphi_m \in H,\ m \in \mathbb{N}\Big\}$$

form a dense linear subspace of $W(H,\mu)$ w.r.t. $\|\cdot\|_{W(H,\mu)}$. If $u \in \mathcal{C}$, $h \in H$, then the quantity $\langle \nabla u, h\rangle$ coincides with the Gâteaux derivative of u in the direction h. Since $\exp(-\|f\|_\infty) \le \rho(h) \le \exp(\|f\|_\infty)$, $h \in H$, we can define a Dirichlet form on $L^2(H,\rho\mu)$ by

$$\mathcal{E}(u,v) := \int_H \langle \nabla u, \nabla v\rangle \rho \,\mathrm{d}\mu, \quad u,v \in \mathcal{D}(\mathcal{E}) = W(H,\mu).$$

$(\mathcal{E},\mathcal{D}(\mathcal{E}))$ corresponds to the minimal gradient form, which has been addressed in Section 4.1. We denote its associated strongly continuous contraction semigroup on $L^2(H,\rho\mu)$ by $(T_t)_{t\ge 0}$. The elements in $W(H,\mu)$ can be characterized by the existence of a sufficiently large collection of directional derivatives and a sumability condition. Recalling the definition of the one-component form from Section 4.1, we define $D_k := \mathcal{D}_{\max}(\mathcal{E}_k)$ for short, given a μ-admissible element $k \in H$. Of course, the perturbation ρ we consider doesn't change the notion of admissibility.

The definition of admissibility given in [19, Section 4.1 of Chapter II] differs slightly from the one we used in Section 4.1. In the cited textbook, an element $k \in H$ is called admissible if the image measure $\mu \circ \tau_{sk}^{-1}$ under a shift $\tau_{sk}h := h + sk$, $h \in H$, $s \in (0,\infty)$, is equivalent to the measure μ. We remark that in this case, also the measure

$$\sigma_k(A) := \int_{\mathbb{R}} \mu \circ \tau_{sk}^{-1}(A)\,\mathrm{d}s, \quad A \in \mathcal{B}(H),$$

is equivalent to μ. The Radon-Nikodym derivative $\frac{\mathrm{d}\mu}{\mathrm{d}\sigma_k}$ is characterized by the equation

$$\int_H u\,\mathrm{d}\mu = \int_H \int_{\mathbb{R}} u(h+sk)\frac{\mathrm{d}\mu}{\mathrm{d}\sigma_k}(h+sk)\,\mathrm{d}s\,\mathrm{d}\mu(h)$$

for all non-negative functions $u \in \mathcal{B}(H)$. Moreover, by [17, Proposition 4.2], it holds

$$\begin{aligned} \int_H u\,\mathrm{d}\mu &= \int_H \int_{\mathbb{R}} u(h+sk)\frac{\mathrm{d}\mu}{\mathrm{d}\sigma_k}(h+sk)\,\mathrm{d}s\,\mathrm{d}\nu_k(h) \\ \text{with} \quad & \nu_k := \mu \circ \pi_k^{-1}, \\ \text{and} \quad & \pi_k h := h - \frac{\langle h,k\rangle}{\langle k,k\rangle} h \end{aligned} \tag{5.2.13}$$

for all non-negative functions $u \in \mathcal{B}(H)$. There is a stronger notion of admissibility for a direction k, the strict admissibility, which states

$$\Big(\frac{\mathrm{d}\mu}{\mathrm{d}\sigma_k}(h + \cdot\, k)\Big)^{-1} \quad \text{is locally integrable on } \mathbb{R}, \ \mu\text{-a.e.} \quad h \in H.$$

It is shown in [17], however, that all $k \in H \setminus \{0\}$, which are strictly admissible, in fact are also μ-admissible in the sense we used in Section 4.1. Strict admissibility therefore ensures that the weak derivative ∂_k in the direction of k may be regarded as a closed linear operator on $L^2(H,\mu)$ with domain D_k. We sum up relevant properties for the proof below, taken from [19, Section 4 of Chapter II] and [17, Sections 4 & 5].

Remark 5.2.4. (i) Let $k \in H$ be μ-admissible and $u \in D_k$. A shift in the direction of k commutes with the operator ∂_k, i.e.

$$\nabla_k(u \circ \tau_{sk})(h) = \tilde{u}(h + \cdot\, k)'(s) = (\nabla_k u)(h + sk)\ (\mu \times \mathrm{d}s)\text{-a.e.} \quad h \in H.$$

(ii) Let φ_i be as in Lemma 5.2.3 for $i \in \mathbb{N}$. Normalizing in H we set $\hat{\varphi}_i := \frac{1}{\sqrt{\lambda_i}}\varphi_i$, $i \in \mathbb{N}$. It holds

$$\frac{\mathrm{d}\mu}{\mathrm{d}\sigma_{\hat{\varphi}_i}} = \sqrt{\frac{\lambda_i}{2\pi}} \exp\Big(-\frac{1}{2}\lambda_i \langle \,\cdot\,, \hat{\varphi}_i\rangle^2\Big). \tag{5.2.14}$$

In particular, $(\hat{\varphi}_i)_{i\in\mathbb{N}}$ is an orthonormal basis of admissible elements in H. We have

$$W(H,\mu) = \Big\{ u \in L^2(H,\mu) \cap \Big(\bigcap_{i\in\mathbb{N}} D_{\hat{\varphi}_i}\Big) \,\Big|\, \sum_{i=1}^{\infty} (\nabla_{\hat{\varphi}_i} u)^2 \in L^1(H,m)\Big\}$$

and moreover,

$$\nabla u = \sum_{i=1}^{\infty} (\nabla_{\hat{\varphi}_i} u)\hat{\varphi}_i \quad \text{in } L^2(H,\mu), \quad u \in W(H,\mu).$$

(iii) Let $u \in L^2(H,\mu)$. By (i) and (ii) and we have $u \in W(H,\mu)$ if and only if $u \in \bigcap_{i\in\mathbb{N}} D_{\hat{\varphi}_i}$ and

$$\sum_{i=1}^{\infty} \sqrt{\frac{\lambda_i}{2\pi}} \int_H \int_{\mathbb{R}} \big(u'(h + \cdot\,\hat{\varphi}_i)\big)^2(s) \exp\Big(-\frac{1}{2}\lambda_i\big(\langle h, \hat{\varphi}_i\rangle + s\big)^2\Big)\, \mathrm{d}s\, \mathrm{d}\mu(h) < \infty$$

In that case,

$$\begin{aligned}
&\sum_{i=1}^{\infty} \sqrt{\frac{\lambda_i}{2\pi}} \int_H \int_{\mathbb{R}} \big(u'(h + \cdot\,\hat{\varphi}_i)\big)^2(s)\rho(h + s\hat{\varphi}_i) \\
&\qquad\qquad \times \exp\Big(-\frac{1}{2}\lambda_i\big(\langle h, \hat{\varphi}_i\rangle + s\big)^2\Big)\, \mathrm{d}s\, \mathrm{d}\mu(h) \\
&= \sum_{i=1}^{\infty} \sqrt{\frac{\lambda_i}{2\pi}} \int_H \int_{\mathbb{R}} \nabla_{\hat{\varphi}_i} u(h + s\hat{\varphi}_i)^2 \rho(h + s\hat{\varphi}_i) \\
&\qquad\qquad \times \exp\Big(-\frac{1}{2}\lambda_i\big(\langle h, \hat{\varphi}_i\rangle + s\big)^2\Big)\, \mathrm{d}s\, \mathrm{d}\mu(h) \\
&= \int_H \sum_{i=1}^{\infty} (\nabla_{\hat{\varphi}_i} u)^2 \rho \,\mathrm{d}\mu = \mathcal{E}(u,u).
\end{aligned}$$

Remark 5.2.5. In the finite-dimensional case, every direction is strictly admissible. Hence, if $k \in \mathrm{Im}(\Lambda_N)$ and $u \in \mathcal{B}(H)$ is non-negative, then

$$\begin{aligned}
&\int_H u \,\mathrm{d}(\mu_N \circ \Lambda_N^{-1}) \\
&\qquad = \sqrt{\frac{\langle A_N k, k\rangle}{2\pi}} \int_H \int_{\mathbb{R}} u(h + sk) e^{-\frac{\langle h + sk, A_N k\rangle^2}{2\langle A_N k, k\rangle}} \,\mathrm{d}s\, \mathrm{d}(\mu_N \circ \Lambda_N^{-1})(h).
\end{aligned}$$

In the theorem below, (5.2.9) is assumed and $\Omega := C([0,\infty), H_0')$. Moreover, we recall U_N from (5.1.5). The probability P_N is defined as above, directly following the definition of the form $\mathcal{E}^N$, for each $N \in \mathbb{N}$. The tightness of $(P_N \circ U_N^{-1})_{N\in\mathbb{N}}$ w.r.t. the topology of weak convergence of measures on Ω follows from Proposition 5.1.1. We are now prepared to state the main convergence result of this chapter. For $t > 0$ and an element $v \in L^\infty(H,\mu)$ the symbol $T_t(\,\cdot\, v)$ denotes the assignment $u \mapsto T_t(uv)$, which maps $L^\infty(H,\mu)$ into itself. For bounded linear operators $S_1, \ldots, S_m$ on $L^\infty(H,\mu)$, $m \in \mathbb{N}$, we write $\prod_{i=1}^m S_i$ for the composition $S_1 \circ \cdots \circ S_m$.

Theorem 5.2.6. *Let P^* be an accumulation point of $(P_N \circ U_N^{-1})_{N\in\mathbb{N}}$ w.r.t. the topology of weak convergence of measures on Ω.*

The canonical process on Ω is a Markov process under P^* with

$$P^*(\{\omega(t) \in H\}) = 1, \quad t \geq 0.$$

Its transition function is a version of $(T_t)_{t>0}$, i.e.

$$\begin{aligned}\int_\Omega & F_1(\omega(t_1)) \cdot \dots \cdot F_m(\omega(t_m)) \, \mathrm{d}P^*(\omega) \\ &= \int_H \Big(\prod_{i=0}^{m-2} T_{t_{m-i}-t_{m-i-1}}(\ \cdot \ F_{m-i}) \Big) (T_{t_1} F_1) \rho \, \mathrm{d}\mu, \\ & F_1, \dots, F_m \in C_b(H), \quad 0 \leq t_1 \leq \dots \leq t_m < \infty, \quad m \in \mathbb{N}_{\geq 2}. \end{aligned} \tag{5.2.15}$$

In particular, $(P_N \circ U_N^{-1})_{N\in\mathbb{N}}$ is a sequence of weakly convergent measures on Ω.

Proof. Let $N \in \mathbb{N}$. Define $S_t^N u(\Lambda_N x) := T_{N^2t}^N(u \circ \Lambda_N)(x)$ for $u \in L^2(H, (\rho_N\mu_N) \circ \Lambda_N^{-1})$. Then $(S_t^N)_{t\geq 0}$ forms a strongly continuous contraction semigroup on $L^2(H, (\rho_N\mu_N) \circ \Lambda_N^{-1})$. Due to (5.1.1) we have

$$\begin{aligned}\int_\Omega & F_1(\omega(t_1)) \cdot \dots \cdot F_m(\omega(t_m)) \, \mathrm{d}(P_N \circ U_N^{-1})(\omega) \\ &= \int_{\Omega_N} F_1(\Lambda_N \omega(N^2 t_1)) \cdot \dots \cdot F_m(\Lambda_N \omega(N^2 t_m)) \, \mathrm{d}P_N(\omega) \\ &= \int_{\mathbb{R}^{k_N}} \prod_{i=0}^{m-2} T^N_{N^2 t_{m-i} - N^2 t_{m-i-1}} \big(\ \cdot \ (F_{m-i} \circ \Lambda_N)\big) \big(T^N_{N^2 t_1}(F_1 \circ \Lambda_N)\big) \rho_N \, \mathrm{d}\mu_N \\ &= \int_H \prod_{i=0}^{m-2} S^N_{t_{m-i}-t_{m-i-1}}(\ \cdot \ F_{m-i})(S^N_{t_1} F_1) \, \mathrm{d}(\rho_N \mu_N \circ \Lambda_N^{-1}), \\ & \quad F_1, \dots, F_m \in C_b(H), \quad 0 \leq t_1 \leq \dots \leq t_m < \infty, \quad m \in \mathbb{N}_{\geq 2}.\end{aligned}$$

So, the finite-dimensional distributions of $P_N \circ U_N^{-1}$ are given through the semigroup $(S_t)_{t\geq 0}$. The statement of the theorem now follows from

$$\lim_{N\to\infty} \int_{\mathbb{R}^{k_N}} T^N_{N^2 t}(F \circ \Lambda_N)(x) G(\Lambda_N x) \rho_N(x) \, \mathrm{d}\mu_N(x) = \int_H T_t F(h) G(h) \rho(h) \, \mathrm{d}\mu(h)$$
$$\text{for all } t > 0 \text{ and } F, G \in C_b(H), \tag{5.2.16}$$

because this would imply the weak convergence of the finite-dimensional distributions of $P_N \circ U_N^{-1}$ towards the right-hand side of (5.2.15). The tightness of $(P_N \circ U_N^{-1})_N$ on Ω is proven in Proposition 5.1.1. The proof of (5.2.16) runs via Mosco convergence. At first, we identify the Dirichlet form on $L^2(H, (\rho_N\mu_N) \circ \Lambda_N^{-1})$ which is associated with $(S_t)_{t\geq 0}$. We have

$$\frac{1}{t} \int_H (u - S_t u) u \, \mathrm{d}(\rho_N \mu_N) \circ \Lambda_N^{-1} = \frac{1}{t} \int_H \big(u \circ \Lambda_N - T_{N^2 t}(u \circ \Lambda_N)\big)(u \circ \Lambda_N) \rho_N \, \mathrm{d}\mu_N,$$

for $t > 0$ and $u \in L^2(H, (\rho_N\mu_N) \circ \Lambda_N^{-1})$ and hence

$$\lim_{t\downarrow 0} \frac{1}{t} \int_H (u - S_t u) u \, \mathrm{d}(\rho_N \mu_N) \circ \Lambda_N^{-1}$$

$$= \lim_{t'\downarrow 0} \frac{N^2}{t'} \int_H \big(u\circ\Lambda_N - T_{t'}(u\circ\Lambda_N)\big)(u\circ\Lambda_N)\rho_N \,\mathrm{d}\mu_N$$
$$= N^2\mathcal{E}^N(u\circ\Lambda_N, u\circ\Lambda_N) \quad \text{if } u\circ\Lambda_N \in \mathcal{D}(\mathcal{E}^N). \tag{5.2.17}$$

The limit of the left-hand side exists if and only if $u\circ\Lambda_N \in \mathcal{D}(\mathcal{E}^N)$, see [38, Section 3 of Capter I]. The Dirichlet form associated with $(S_t)_{t\geq 0}$ is thus the image form $N^2{\Lambda_N}^*\mathcal{E}^N$ of $\mathcal{E}^N$ under $N^2\Lambda_N$. However, for technical reasons we would rather like to work with the form $N^2{\widetilde{\Lambda}_N}^*\mathcal{E}^N$, which is the image form of $\mathcal{E}^N$ under $N^2\widetilde{\Lambda}_N$, instead of $N^2{\Lambda_N}^*\mathcal{E}^N$. Here, as already has been a convention in the proof of Proposition 5.2.2, $\widetilde{\Lambda}_N$ denote the height map which emerges from the choice $\widetilde{\Xi} = \mathbf{1}_{[0,1)^d}$ as the archetype function, instead of the function Ξ associated to Λ_N. More precisely, if $p_1,\dots,p_{k_N}$ is a numbering such that $\xi_{N,i} = \Xi(N\,\cdot\,-Np_i)$ and $\Lambda_N x = N^{\frac{d}{2}-1}\sum_{i=1}^{k_N} x_i\xi_{N,i}$, $x\in\mathbb{R}^{k_N}$, then we define

$$\widetilde{\Lambda}_N x := N^{\frac{d}{2}-1}\sum_{i=1}^{k_N} x_i \mathbf{1}_{[0,1)^d}(N\,\cdot\,-Np_i) = N^{\frac{d}{2}-1}\sum_{i=1}^{k_N} x_i \mathbf{1}_{p_i+[0,\frac{1}{N})^d}, \quad x\in\mathbb{R}^{k_N}.$$

By the analogue of (5.2.17) with Λ_N replaced by $\widetilde{\Lambda}_N$, the semigroup $(\widetilde{S}_t)_{t\geq 0}$ on $L^2(H,(\rho_N\mu_N)\circ\widetilde{\Lambda}_N^{-1})$ which is associated with $N^2{\widetilde{\Lambda}_N}^*\mathcal{E}^N$ reads

$$\widetilde{S}_t^N u(\widetilde{\Lambda}_N x) := T^N_{N^2t}(u\circ\widetilde{\Lambda}_N)(x) \quad \text{for} \quad u\in L^2(H,(\rho_N\mu_N)\circ\widetilde{\Lambda}_N^{-1}),\ t>0.$$

We now argue, why we can work with $N^2{\widetilde{\Lambda}_N}^*\mathcal{E}^N$ as well. Applying 5.1.4 to the sequence $(T^N_{N^2t}(F\circ\Lambda_N)\rho\,\mathrm{d}\mu_N)_N$ for fixed $F\in C_b(H)$ we see that (5.2.16) is equivalent to

$$\lim_{N\to\infty}\int_{\mathbb{R}^{k_N}} T^N_{N^2t}(F\circ\Lambda_N)(x)G(\widetilde{\Lambda}_N x)\rho_N(x)\,\mathrm{d}\mu_N(x) = \int_H T_tF(h)G(h)\rho(h)\,\mathrm{d}\mu(h)$$
$$\text{for all } t>0 \text{ and } F,G\in C_b(H). \tag{5.2.18}$$

Next, we fix $G\in C_b(H)$, use the symmetry of the operators $T^N_{N^2t}$ and apply 5.1.4 to the sequence $(T^N_{N^2t}(G\circ\widetilde{\Lambda}_N)\rho\,\mathrm{d}\mu_N)_N$ to see that (5.2.18), in turn, is equivalent to

$$\lim_{N\to\infty}\int_{\mathbb{R}^{k_N}} T^N_{N^2t}(F\circ\widetilde{\Lambda}_N)(x)G(\widetilde{\Lambda}_N x)\rho_N(x)\,\mathrm{d}\mu_N(x) = \int_H T_tF(h)G(h)\rho(h)\,\mathrm{d}\mu(h)$$
$$\text{for all } t>0 \text{ and } F,G\in C_b(H). \tag{5.2.19}$$

It is thus equivalent, whether we consider the converging Hilbert spaces

$$L^2(H,(\rho_N\mu_N)\circ\Lambda_N^{-1}) \overset{N\to\infty}{\longrightarrow} L^2(H,\rho\mu),$$

denoting their disjoint union by $\mathcal{H}$, or we consider

$$L^2(H,(\rho_N\mu_N)\circ\widetilde{\Lambda}_N^{-1}) \overset{N\to\infty}{\longrightarrow} L^2(H,\rho\mu),$$

denoting their disjoint union by $\widetilde{\mathcal{H}}$, and show the Mosco convergence of $(N^2{\Lambda_N}^*\mathcal{E}^N)_N$ towards $\mathcal{E}$ in $\mathcal{H}$ in the former case, or show Mosco convergence of $(N^2{\widetilde{\Lambda}_N}^*\mathcal{E}^N)_N$ towards $\mathcal{E}$ in $\widetilde{\mathcal{H}}$ in the second case. As already mentioned, the latter is the way we go.

Let $N \in \mathbb{N}$ and $\boldsymbol{pr}_N$ denote the orthogonal projection of H onto $\operatorname{Im}(\widetilde{\Lambda}_N)$. We start with a remark which is due to the chain rule. For $u \in C_b^1(H)$ and $x \in \mathbb{R}^{k_N}$ it holds

$$N^2 \sum_{i=1}^{k_N} \left| \partial_i (u \circ \widetilde{\Lambda}_N)(x) \right|^2 = \sum_{i=1}^{k_N} \left| \langle \nabla u(\widetilde{\Lambda}_N x), N^{\frac{d}{2}} \widetilde{\xi}_{N,i} \rangle \right|^2 = \left| \boldsymbol{pr}_N \nabla u(\widetilde{\Lambda}_N x) \right|_H^2.$$

So,

$$N^2 \widetilde{\Lambda}_N{}^* \mathcal{E}^N(u,u) = \int_H \left| \boldsymbol{pr}_N \, \nabla u \right|_H^2 \mathrm{d}(\rho_N \mu_N) \circ \widetilde{\Lambda}_N^{-1}, \quad u \in C_b^1(H). \tag{5.2.20}$$

Before we go on, we quickly want to remark that the space

$$\begin{aligned} &\mathcal{F}C_b^\infty(\operatorname{Im}(\widetilde{\Lambda}_N)) \\ &\quad := \left\{ F(\langle \cdot \,, \varphi_1 \rangle, \dots, \langle \cdot \,, \varphi_m \rangle) \,\middle|\, F \in C_b^\infty(\mathbb{R}^m),\, \varphi_1, \dots, \varphi_m \in \operatorname{Im}(\widetilde{\Lambda}_N),\, m \in \mathbb{N} \right\} \end{aligned}$$

is densely included in the domain of $N^2 \widetilde{\Lambda}_N{}^* \mathcal{E}^N$. Indeed, denoting by $\mathbf{e}_1, \dots, \mathbf{e}_{k_N}$ the unit vectors of $\mathbb{R}^{k_N}$, the equality

$$\begin{aligned} &N^2 \widetilde{\Lambda}_N{}^* \mathcal{E}^N \big(u - F(\langle \cdot \,, N^2 \widetilde{\Lambda}_N \mathbf{e}_1 \rangle, \dots, \langle \cdot \,, N^2 \widetilde{\Lambda}_N \mathbf{e}_{k_N} \rangle) \big) \\ &\quad + \int_H \left| u - F(\langle \cdot \,, N^2 \widetilde{\Lambda}_N \mathbf{e}_1 \rangle, \dots, \langle \cdot \,, N^2 \widetilde{\Lambda}_N \mathbf{e}_{k_N} \rangle) \right|^2 \mathrm{d}(\rho_N \mu_N) \circ \widetilde{\Lambda}_N^{-1} \\ &= N^2 \mathcal{E}^N \big(u \circ \widetilde{\Lambda}_N - F, u \circ \widetilde{\Lambda}_N - F \big) \\ &\quad + \int_{\mathbb{R}^{k_N}} |u \circ \widetilde{\Lambda}_N - F|^2 \rho_N \, \mathrm{d}\mu_N, \quad u \in \mathcal{D}(N^2 \widetilde{\Lambda}_N{}^* \mathcal{E}^N),\, F \in C_b^\infty(\mathbb{R}^{k_N}), \end{aligned}$$

allows to define a suitable approximation for any element in the domain of $N^2 \widetilde{\Lambda}_N{}^* \mathcal{E}^N$, since $C_b^\infty(\mathbb{R}^{k_N})$ is a form core of $\mathcal{E}^N$ by definition. Moreover, as $\nabla u(h) \in \operatorname{Im}(\widetilde{\Lambda}_N)$ for $u \in \mathcal{F}C_b^\infty(\operatorname{Im}(\widetilde{\Lambda}_N))$, $h \in H$, (5.2.20) simplifies to

$$N^2 \widetilde{\Lambda}_N{}^* \mathcal{E}^N(u,u) = \int_H \left| \nabla u \right|_H^2 \mathrm{d}(\rho_N \mu_N) \circ \widetilde{\Lambda}_N^{-1}, \quad u \in \mathcal{F}C_b^\infty(\operatorname{Im}(\widetilde{\Lambda}_N)).$$

Until the end of this proof, let $u_N \in \mathcal{F}C_b^\infty(\operatorname{Im}(\widetilde{\Lambda}_N))$, $N \in \mathbb{N}$, and $u \in L^2(H, \rho\mu)$ such that

$$\begin{aligned} &\sup_{N \in \mathbb{N}} \int_H u_N^2 \, \mathrm{d}(\rho_N \mu_N) \circ \widetilde{\Lambda}_N^{-1} < \infty \\ \text{and} \quad &\lim_N \int_H u_N v \, \mathrm{d}(\rho_N \mu_N) \circ \widetilde{\Lambda}_N^{-1} = \int_H u v \rho \, \mathrm{d}\mu, \quad v \in C_b(H). \end{aligned}$$

We assume $\liminf_{N \to \infty} N^2 \widetilde{\Lambda}_N{}^* \mathcal{E}^N(u_N, u_N) < \infty$ and specify two claims:

(a) $u \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}(u,u) \le \liminf_{N \to \infty} N^2 \widetilde{\Lambda}_N{}^* \mathcal{E}^N(u_N, u_N)$.

(b) $\lim_{N \to \infty} N^2 \widetilde{\Lambda}_N{}^* \mathcal{E}^N(v,v) = \mathcal{E}(v,v), \quad v \in C_b^1(H)$.

The proof of Mosco convergence is completed through the verification of (a) and (b). We start with property (a). Let A_N be the symmetric positive operator on $\operatorname{Im}(\widetilde{\Lambda}_N)$ with

$$\langle A_N u, v \rangle_{\mu_N \circ \widetilde{\Lambda}_N^{-1}} = \langle u, v \rangle \quad \text{for } u, v \in \operatorname{Im}(\widetilde{\Lambda}_N).$$

The spectrum of A_N is given by real, positive numbers $0 < \lambda_1^N \leq \cdots \leq \lambda_{k_N}^N < \infty$. We choose an orthonormal basis $\{\varphi_i^N \mid i = 1, \ldots, k_N\}$ of $(\mathrm{Im}(\widetilde{\Lambda}_N), \langle \cdot, \cdot \rangle_{\mu_N \circ \widetilde{\Lambda}_N^{-1}})$ such that $A_N \varphi_i^N = \lambda_i^N$ for $i = 1, \ldots, k_N$. Normalizing the eigenvectors w.r.t. $|\cdot|_H$, we set $\hat{\varphi}_i^N := \frac{1}{\sqrt{\lambda_i^N}} \varphi_i^N$ for $N \in \mathbb{N}$. The key is to prove:

$$\text{For } i \in \mathbb{N}: \quad u \in D_{\hat{\varphi}_i}, \quad \int_H \int_{\mathbb{R}} \big(u(h + \cdot \hat{\varphi}_i)'\big)^2(s) \varrho_i(h,s)\, \mathrm{d}s\, \mathrm{d}\mu(h)$$
$$\leq \liminf_{N \to \infty} \int_H \int_{\mathbb{R}} \big(u_N(h + \cdot \hat{\varphi}_i^N)'\big)^2(s) \varrho_i^N(h,s)\, \mathrm{d}s\, \mathrm{d}(\mu_N \circ \widetilde{\Lambda}_N^{-1})(h). \quad (5.2.21)$$

with

$$\varrho_i(h,s) := \sqrt{\frac{\lambda_i}{2\pi}} \rho(h + s\hat{\varphi}_i) \exp\Big(-\frac{1}{2} \lambda_i \big(\langle h, \hat{\varphi}_i \rangle + s\big)^2 \Big), \quad h \in H,\ s \in \mathbb{R}$$

and

$$\varrho_i^N(h,s) := \sqrt{\frac{\lambda_i^N}{2\pi}} \exp\Big(\int_{(0,1)^d} f_N(h(z) + s\hat{\varphi}_i^N(z))\, \mathrm{d}z - \frac{1}{2} \lambda_i^N \big(\langle h, \hat{\varphi}_i^N \rangle + s\big)^2 \Big)$$

for $h \in H$ and $s \in \mathbb{R}$. For $i, N \in \mathbb{N}$ with $k_N \geq i$, we have

$$u_N(h + \cdot \hat{\varphi}_i^N)'(s) = \frac{u_N}{\partial \hat{\varphi}_i^N}(h + s\hat{\varphi}_i^N), \quad h \in H,\ s \in \mathbb{R}.$$

The reason why a verification of (5.2.21) is enough to conclude this proof, is as follows. Summing up over $i \in \mathbb{N}$, applying first 5.2.4 (iii) and then Fubini's Lemma together with 5.2.5, we obtain the estimate

$$\mathcal{E}(u,u) \leq \sum_{i \in \mathbb{N}} \liminf_{N \to \infty} \int_H \int_{\mathbb{R}} \big(\frac{\partial u_N}{\partial \hat{\varphi}_i^N}(h + s\hat{\varphi}_i^N)\big)^2 \varrho_i^N(h,s)\, \mathrm{d}s\, \mathrm{d}(\mu_N \circ \widetilde{\Lambda}_N^{-1})(h)$$
$$= \sum_{i \in \mathbb{N}} \liminf_{N \to \infty} \mathbf{1}_{\{1,\ldots,k_N\}}(i)$$
$$\int_H \int_{\mathbb{R}} \big(\frac{\partial u_N}{\partial \hat{\varphi}_i^N}(h + s\hat{\varphi}_i^N)\big)^2 e^{\int_{(0,1)^d} f_N(h(z) + s\hat{\varphi}_i^N(z))\, \mathrm{d}z}$$
$$\times e^{-\frac{1}{2}\lambda_i^N (\langle h, \hat{\varphi}_i^N \rangle + s)^2}\, \mathrm{d}s\, \mathrm{d}(\mu_N \circ \widetilde{\Lambda}_N^{-1})(h)$$
$$\leq \liminf_{N \to \infty} \sum_{i \in \mathbb{N}}^{k_N} \int_H \big(\frac{\partial u_N}{\partial \hat{\varphi}_i^N}(h)\big)^2 e^{\int_{(0,1)^d} f_N(h(z))\, \mathrm{d}z}\, \mathrm{d}(\mu_N \circ \widetilde{\Lambda}_N^{-1})(h).$$

We continue by using (5.2.7) and arrive at

$$\mathcal{E}(u,u) \leq \liminf_{N \to \infty} \sum_{i \in \mathbb{N}}^{k_N} \int_{\mathbb{R}^{k_N}} \big(\frac{\partial u_N}{\partial \hat{\varphi}_i^N}(\widetilde{\Lambda}_N x)\big)^2 \rho_N(x)\, \mathrm{d}\mu_N(x)$$
$$= \liminf_{N \to \infty} \sum_{i \in \mathbb{N}}^{k_N} \int_H \big(\frac{\partial u_N}{\partial \hat{\varphi}_i^N}\big)^2\, \mathrm{d}(\rho_N \mu_N) \circ \widetilde{\Lambda}_N^{-1}$$
$$= \liminf_{N \to \infty} \int_H |\nabla u_N|_H^2\, \mathrm{d}(\rho_N \mu_N) \circ \widetilde{\Lambda}_N^{-1} = \mathcal{E}^N(u_N, u_N),$$

which would verify property (a). The latter, together with

$$\begin{aligned}\mathcal{E}(v,v) &= \lim_{N\to\infty}\int_H |\nabla v|_H^2 \,\mathrm{d}(\rho_N\mu_N)\circ\widetilde{\Lambda}_N^{-1} \\ &\geq \limsup_{N\to\infty}\int_H |\boldsymbol{pr}_N \nabla v|_H^2 \,\mathrm{d}(\rho_N\mu_N)\circ\widetilde{\Lambda}_N^{-1} \\ &= \limsup_{N\to\infty} N^2\widetilde{\Lambda}_N{}^*\mathcal{E}^N(v,v), \quad v\in C_b^1(H),\end{aligned}$$

would imply the equation $\mathcal{E}(v,v) = \lim_{N\to\infty} N^2\widetilde{\Lambda}_N{}^*\mathcal{E}^N(v,v)$, $v\in C_b^1(H)$, and hence (b).

Only (5.2.21) is left to show. Let $i\in\mathbb{N}$. We use (5.2.13), which of course holds for any Gaussian measure and hence also for the approximants. Below $\nu_{\hat{\varphi}_i^N}$ denotes the image of $\mu_N\circ\widetilde{\Lambda}_N$ under $\pi_{\hat{\varphi}_i^N}$ and $\nu_{\hat{\varphi}_i}$ denotes the image of μ under $\pi_{\hat{\varphi}_i}$. By (5.2.13), it suffices to show the following.

$$\begin{aligned}\text{For } i\in\mathbb{N}: \quad u\in D_{\hat{\varphi}_i}, \quad &\int_H\int_{\mathbb{R}} \big(u(h+\cdot\hat{\varphi}_i)'\big)^2(s)\varrho_i(h,s)\,\mathrm{d}s\,\mathrm{d}\nu_{\hat{\varphi}_i}(h) \\ &\leq \liminf_{N\to\infty}\int_H\int_{\mathbb{R}} \big(u_N(h+\cdot\hat{\varphi}_i^N)'\big)^2(s)\varrho_i^N(h,s)\,\mathrm{d}s\,\mathrm{d}\nu_{\hat{\varphi}_i^N}(h)(h). \quad (5.2.22)\end{aligned}$$

with

$$\varrho_i(h,s) := \sqrt{\frac{\lambda_i}{2\pi}}\rho(h+s\hat{\varphi}_i)\exp\big(-\tfrac{1}{2}\lambda_i s^2\big), \quad h\in H,\ s\in\mathbb{R}$$

and

$$\varrho_i^N(h,s) := \sqrt{\frac{\lambda_i^N}{2\pi}}\exp\Big(\int_{(0,1)^d} f_N(h(z)+s\hat{\varphi}_i^N(z))\,\mathrm{d}z - \frac{1}{2}\lambda_i^N s^2\Big)$$

for $h\in H$ and $s\in\mathbb{R}$.

We want to apply the convergence result of Proposition 4.1.1 in combination with the perturbation result of Theorem 4.2.4. To this purpose, we choose a decomposition

$$f = f_1 - f_2, \quad f_N = f_1^{(N)} - f_2^{(N)},$$

according to Condition 5.2.1. The densities factorize into

$$\begin{aligned}\varrho_i(h,s) := \sqrt{\frac{\lambda_i}{2\pi}}&\exp\Big(\int_{(0,1)^d} f_1(h(z)+s\hat{\varphi}_i(z))\,\mathrm{d}z\Big) \\ &\times\exp\Big(\int_{(0,1)^d} f_2(h(z)+s\hat{\varphi}_i(z))\,\mathrm{d}z\Big)\exp\big(-\tfrac{1}{2}\lambda_i s^2\big)\end{aligned}$$

and

$$\begin{aligned}\varrho_i^N(h,s) := \sqrt{\frac{\lambda_i^N}{2\pi}}&\exp\Big(\int_{(0,1)^d} f_1^{(N)}(h(z)+s\hat{\varphi}_i^N(z))\,\mathrm{d}z\Big) \\ &\times\exp\Big(\int_{(0,1)^d} f_2^{(N)}(h(z)+s\hat{\varphi}_i^N(z))\,\mathrm{d}z\Big)\exp\big(-\tfrac{1}{2}\lambda_i^N s^2\big).\end{aligned}$$

Hence, after a double application of Proposition 4.1.1, we further simplified (5.2.21). It suffices to show

$$\text{For } i \in \mathbb{N}: \quad u \in D_{\hat{\varphi}_i}, \quad \int_H \int_{\mathbb{R}} \big(u(h + \cdot\,\hat{\varphi}_i)'\big)^2(s) \exp\left(-\tfrac{1}{2}\lambda_i s^2\right) \mathrm{d}s \, \mathrm{d}\nu_{\hat{\varphi}_i}(h)$$
$$\leq \liminf_{N\to\infty} \int_H \int_{\mathbb{R}} \big(u_N(h + \cdot\,\hat{\varphi}_i^N)'\big)^2(s) \exp\left(-\tfrac{1}{2}\lambda_i^N s^2\right) \mathrm{d}s \, \mathrm{d}\nu_{\hat{\varphi}_i^N}(h)(h).$$

The latter however, is immediately clear from Proposition 4.1.1. Indeed, applying Remark 4.1.2 in our case, it is ensured that the following three equations are enough to seal everything with Proposition 4.1.1. For one thing,

$$\lim_{N\to\infty} \int_H \Big| \int_{\mathbb{R}} g(s) \exp\left(-\tfrac{1}{2}\lambda_i^N s^2\right) \mathrm{d}s \Big|^2 \mathrm{d}\nu_{\hat{\varphi}_i^N}(h)$$
$$= \int_H \Big| \int_{\mathbb{R}} g(s) \exp\left(-\tfrac{1}{2}\lambda_i s^2\right) \mathrm{d}s \Big|^2 \mathrm{d}\nu_{\hat{\varphi}_i}(h), \quad g \in C_b(\mathbb{R}),$$

On top of that, for each $m \in \mathbb{N}$, we must have

$$\limsup_{k\to\infty} \sup_{N\in\mathbb{N}} \operatorname*{ess\,sup}_{s\in[-m,m]} \exp\left(\tfrac{1}{2}\lambda_i^N s^2\right) \cdot \big(\exp\left(-\tfrac{1}{2}\lambda_i^N (\,\cdot\,)^2\right)_{\mathrm{ave},\frac{1}{k}}(s)\big) < \infty$$

as well as

$$\lim_{k\to\infty} \limsup_{N\to\infty} \int_E \int_{[-m,m]} \big| \exp\left(\tfrac{1}{2}\lambda_i^N s^2\right) - \big(\exp\left(-\tfrac{1}{2}\lambda_i^N (\,\cdot\,)^2\right)_{\mathrm{ave},\frac{1}{k}}(s)\big) \big| \, \mathrm{d}s \nu_N(z)$$
$$\Big(= \lim_{k\to\infty} \limsup_{N\to\infty} \int_{[-m,m]} \big| \exp\left(\tfrac{1}{2}\lambda_i^N s^2\right) - \big(\exp\left(-\tfrac{1}{2}\lambda_i^N (\,\cdot\,)^2\right)_{\mathrm{ave},\frac{1}{k}}(s)\big) \big| \, \mathrm{d}s \Big) = 0.$$

All of these three equations hold, because $(\lambda_i^N)_{N\in\mathbb{N}}$ convergent sequence in $\mathbb{R}$ with strictly positive limit. This concludes the proof.

□

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