General Continuous Multicriteria Location Problems

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Abstract

In this paper we deal with the determination of the whole set of Pareto-solutions of location problems with respect to Q general criteria. These criteria include as particular instances median, center or cent-dian objective functions. The paper characterizes the set of Pareto-solutions of all these multicriteria problems. An efficient algorithm for the planar case is developed and its complexity is established. Extensions to higher dimensions as well as to the non-convex case are also considered. The proposed approach is more general than the previously published approaches to multicriteria location problems and includes almost all of them as particular instances.

Keywords: Location Theory, Multicriteria Optimization, Algebraic Optimization, Geometrical Algorithms

1 Introduction

In the process of locating a new facility usually more than one decision maker is involved. This is due to the fact that typically the cost connected to this decision is relatively high. Of course, different persons may (or will) have different (conflicting) objectives. On other occasions, different scenarios must be compared in order to be implemented, or simply uncertainty in the parameters leads to consider different replications of the objective function. If only one objective has to be taken into account a broad range of models is available in the literature (see Chapter 11 in [Dre95]). In contrast to that only a few papers looked at (more realistic) models for facility location, where more than one objective is involved (see [FP95, HN96]). One of the main deficiencies of the existing approaches is that only a few number (in most papers 1) of different types of objectives can be considered and solution approaches depend very much on a specific chosen metric. Also a detailed complexity analysis is missing in most of the papers.

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On the other hand there is a clear need for flexible models where the complexity status is known, since these are prerequisites for a successful implementation of a decision support system for location planning which can really be used by decision-makers. In this paper we present a model for continuous multicriteria location problems which fulfills the requirement of flexibility with respect to the choice of objective functions. To this end we present a new type of objective function (called ordered Weber function), developed in [PF95, RCNPF96, JP96], which includes most of the classical location objective functions as special cases, like for e.g. the Weber objective, the center objective, the cent-dian objective and the Weber objective with positive and negative weights.

Additionally, we allow the use of polyhedral gauges as distance functions in each objective function. It should be mentioned that by the polyhedral gauge approach we are able to approximate every gauge (see [Val64, WW85]). The outline of the rest of the paper is as follows:

In Section 2 the problem is formally introduced and basic tools and definitions are presented. Section 3 is devoted to the bicriteria case in the plane, while Section 4 extends these results to the general planar Q-criteria case. In Section 5 generalizations looking at the non-convex and at the n-dimensional case are discussed. The paper ends with some conclusions and an outlook to future research. Throughout the paper we keep track of the complexity of the presented algorithms.

2 Basic tools and definitions

First we restate some definitions which are needed throughout the paper.

Let $B_i \subset \mathbb{R}^n$ be a compact, convex set containing the origin in its interior, for $i \in \mathcal{M} := \{1, 2, \ldots, M\}$. The gauge with respect to B_i is defined as

$$\gamma_i \; : \; \mathbb{R}^n \; \to \; \mathbb{R} \quad , \quad \gamma_i(x) := \inf\{\, r > 0 \; : \; x \in rB_i \,\} \tag{1}$$

the polar set B_i^o of B_i is given by

$$B_i^o := \{\, p \in \mathbb{R}^n \; : \; \langle p, x \rangle \leq 1 \quad \forall \; x \in B_i \,\} \tag{2}$$

the normal cone to B_i at x is given by

$$N(B_i, x) := \{ p \in \mathbb{R}^n \; : \; \langle p, y - x \rangle \leq 0 \quad \forall \, y \in B_i \} \tag{3}$$

and the boundary of B_i is denoted by $bd(B_i)$.

The case we will mainly consider in this paper is where each γ_i with $i \in \mathcal{M}$ is a polyhedral gauge, which means B_i is a convex polytope with extreme points $Ext(B_i) := \{e_1^i, \ldots, e_{G_i}^i\}$. Let $G_{\max} := \max\{G_i \; : \; i \in \mathcal{M}\}$. In this case we define fundamental directions $d_1^i, \ldots, d_{G_i}^i$ as the half-lines defined by 0 and $e_1^i, \ldots, e_{G_i}^i$. Further, we define Γ_g^i as the cone generated by d_g^i and d_{g+1}^i (fundamental directions of B_i) where $d_{G_i+1}^i := d_1^i$.

Let $\pi = (p_i)_{i \in \mathcal{M}}$ be a family of elements of $\mathbb{R}^n$ such that $p_i \in B_i^o$ for each $i \in \mathcal{M}$ and let $\mathrm{C}_\pi = \bigcap_{i \in \mathcal{M}} (a_i + N(B_i^o, p_i))$. A nonempty convex set C is called an elementary convex set if there exists a family π such that $\mathrm{C}_\pi = \mathrm{C}$.

It should be noted that if the unit balls are polytopes we can obtain the elementary convex sets as intersection of cones generated by fundamental directions of these balls pointed at each demand point. Therefore each elementary convex set is a polyhedron whose vertices are called intersection points (see Figure 1). Finally, in the case of $\mathbb{R}^2$ there exists an upper bound of the number of elementary convex sets which is $O((MG_{\max})^2)$. [DM85] proved that the Weber problem is linear in each elementary

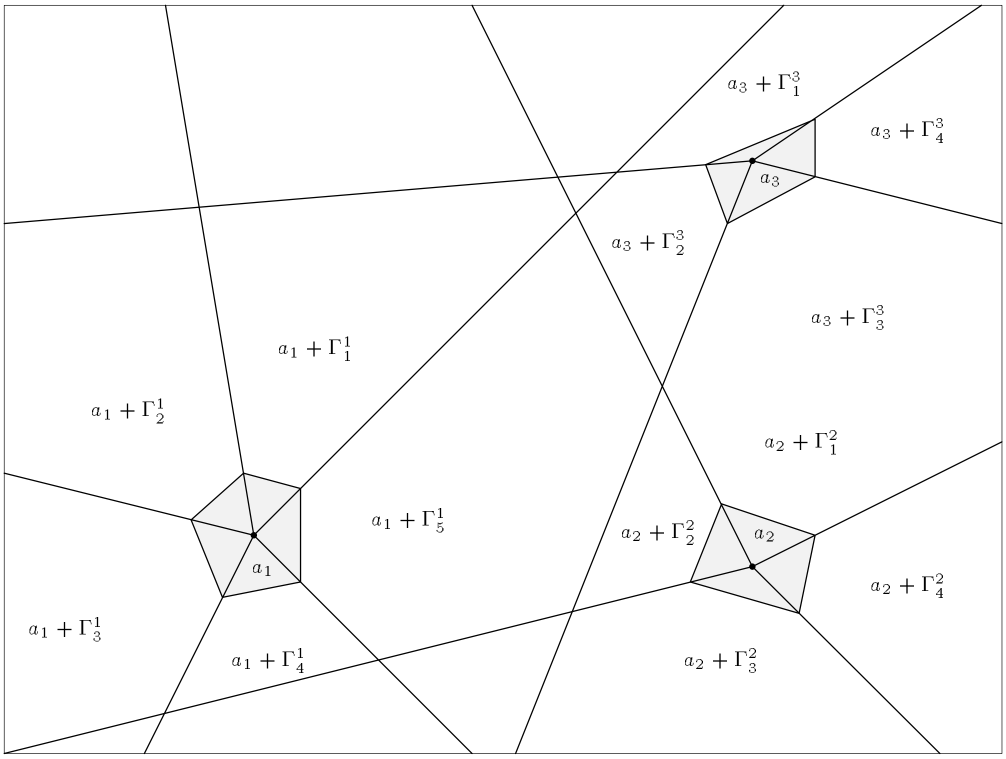

Figure 1: Existing facilities a_i, $i \in \mathcal{M}$, and their unit balls B_i

convex set. Therefore, if we consider polyhedral gauges, there always exists an optimal solution to the Weber problem in the set of intersection points.

2.1 A general approach : The ordered Weber problem

In this section we present a general location model, the ordered Weber Problem, introduced by [PF95] and later elaborated for the polyhedral case by [RCNPF96].

Let us consider a set of demand points $A := \{a_1, \ldots, a_M\}$ and corresponding gauges $\gamma_i(\cdot)$, $i \in \mathcal{M}$, and two sets of nonnegative scalars $\Omega := \{\omega_1, \ldots, \omega_M\}$ and $\Lambda := \{\lambda_1, \ldots \lambda_M\}$, where the element ω_i, $i \in \mathcal{M}$, is the weight of the importance given to the existing facility a_i, $i \in \mathcal{M}$, and the elements of Λ allow to choose among different kinds of objective functions. Given a permutation σ of $\mathcal{M}$ verifying

$$\omega_{\sigma(1)}\gamma_{\sigma(1)}(x - a_{\sigma(1)}) \leq \omega_{\sigma(2)}\gamma_{\sigma(2)}(x - a_{\sigma(2)}) \leq \ldots \leq \omega_{\sigma(M)}\gamma_{\sigma(M)}(x - a_{\sigma(M)})$$

we define $\gamma(x-A)_{(i)} := \omega_{\sigma(i)}\gamma_{\sigma(i)}(x-a_{\sigma(i)})$. The ordered Weber problem is given by the following formulation

$$\min_{x\in\mathbb{R}^n} F(x) = \sum_{i=1}^{M} \lambda_i \gamma(x-A)_{(i)} \tag{4}$$

The set of optimal solutions of this problem is called $\mathcal{X}^*(F)$ or simply by $\mathcal{X}^*$ if this is possible without causing confusion. This objective function looks very much like the objective function of the classical Weber problem, but in fact this function is pointwise defined and in general not convex as the following example shows.

Example 2.1 *Consider two demand points $a_1 = (0,0)$ and $a_2 = (10,5)$, weights $\lambda_1 = 100$ and $\lambda_2 = 1$ with l_1-norm and $\omega_1 = \omega_2 = 1$. We obtain only two optimal solutions to Problem (4), lying in each demand point. Therefore the objective function is not convex since we have a non-convex optimal solution set.*

$$\begin{aligned} F(a_1) &= 100 \times 0 + 1 \times 15 = 15 \\ F(a_2) &= 100 \times 0 + 1 \times 15 = 15 \\ F(\frac{1}{2}(a_1+a_2)) &= 100 \times 7.5 + 1 \times 7.5 = 757.5 \end{aligned}$$

See Figure 2.

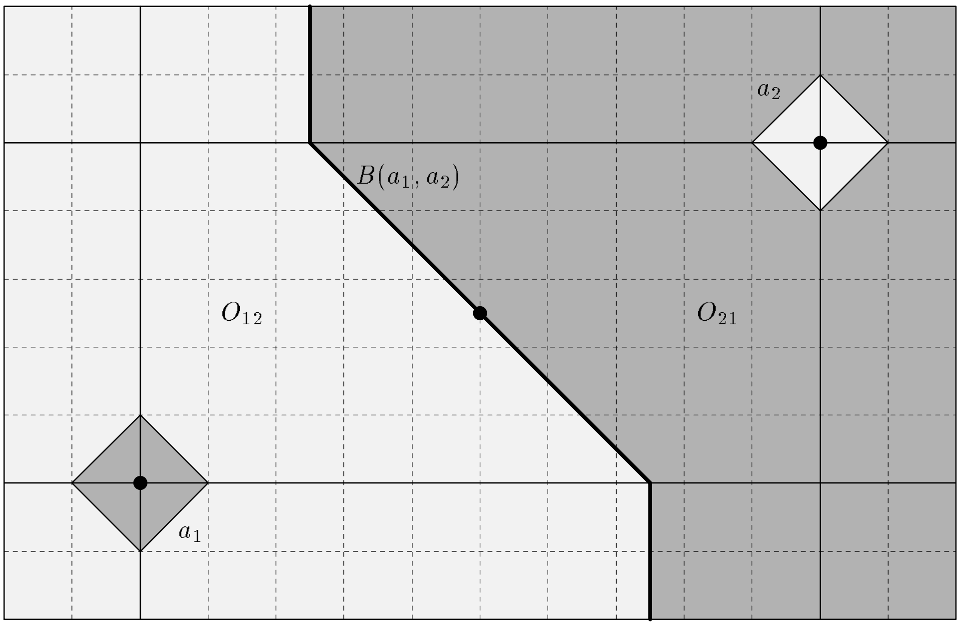

Figure 2: Illustration to Example 2.1

Nevertheless, if we assume that the weights are in increasing order we obtain that the objective function is convex (see [PF95] for more details).

In spite of its difficulty, the study of this model is very important because it provides a quite general framework to deal with continuous location problems, as the following theorem shows. To describe the different types of location problems we use a 5-Position classification scheme Pos1/Pos2/Pos3/Pos4/Pos5, which allows us to indicate the number of new facilities (Pos1), the type of the problem as planar, network-based, discrete etc. (Pos2), any assumptions and restriction such as $w_m = 1$ for all $m \in \mathcal{M}$, etc. (Pos3), the type of distance function such as l_p, general distance function d, etc. (Pos4), and the type of objective function (Pos5). For more details see [HNS96].

Theorem 2.1 *(see [NP96])*

1. *The classical Weber problem* $1/\mathbb{R}^n/\bullet/\gamma_i/\sum$ *with weights* ω_i, $i \in \mathcal{M}$, *is equivalent to the ordered Weber problem* $1/\mathbb{R}^n/\lambda_1 = \ldots = \lambda_M = 1/\omega_i\gamma_i/\sum_{ord}$.
2. *The center problem* $1/\mathbb{R}^n/\bullet/\gamma_i/\max$ *with weights* ω_i, $i \in \mathcal{M}$, *is equivalent to the ordered Weber problem* $1/\mathbb{R}^n/\lambda_1 = \ldots = \lambda_{M-1} = 0 \ \wedge\ \lambda_M = 1/\omega_i\gamma_i/\sum_{ord}$.
3. *The cent-dian problem* $1/\mathbb{R}^n/\bullet/\gamma_i/CD_\omega$ *with weights* ω_i, $i \in \mathcal{M}$, *is equivalent to the ordered Weber problem* $1/\mathbb{R}^n/\lambda_1 = \ldots = \lambda_{M-1} = \omega \ \wedge\ \lambda_M = 1/\omega_i\gamma_i/\sum_{ord}$.

It should be noted that the computation of $F(x)$ is not a trivial task. We do not have an explicit formula of F in $\mathbb{R}^n$, because we have different expressions for F depending on the order in the sequence of distances. Anyway, F behaves as the classical Weber function in a region where the order does not change. In this way, we need an extension of the concept of Voronoi diagrams to characterize these regions. To this end, we use the concept of ordered regions.

The set $B(a_i, a_j)$ for $i \neq j$ consisting of points

$$\{x \in \mathbb{R}^n \,:\, \omega_i\gamma_i(x - a_i) = \omega_j\gamma_j(x - a_j)\}$$

is called the bisector of a_i and a_j with respect to (ω_i, γ_i) and (ω_j, γ_j). Note that $B(a_i, a_j) = B(a_j, a_i)$. Given a permutation σ on the index set $\mathcal{M}$ the ordered region O_σ consists of the points (see Figure 3)

$$O_\sigma := \{x \in \mathbb{R}^n \,:\, \omega_{\sigma(1)}\gamma_{\sigma(1)}(x - a_{\sigma(1)}) \leq \ldots \leq \omega_{\sigma(M)}\gamma_{\sigma(M)}(x - a_{\sigma(M)})\} \qquad (5)$$

The set we obtain as intersection of an elementary convex set and an ordered region is called a generalized elementary convex set. The vertices of the generalized elementary convex set are called generalized intersection points and the set containing all the generalized intersection points is denoted by $\mathcal{GIP}$. The full dimensional generalized elementary convex sets are called cells. The set of all cells is denoted by $\mathcal{C}$.

[PF95] obtained that the objective function of the Ordered Weber problem is linear in each generalized elementary convex set. Therefore there exists an optimal solution of the Ordered Weber problem in $\mathcal{GIP}$. In the case of polyhedral gauges with at most $G_{\max}$ fundamental directions, [RCNPF96] obtained an upper bound of the number of generalized elementary convex sets which in $\mathbb{R}^2$ is $O(M^4 G_{max}^2)$.

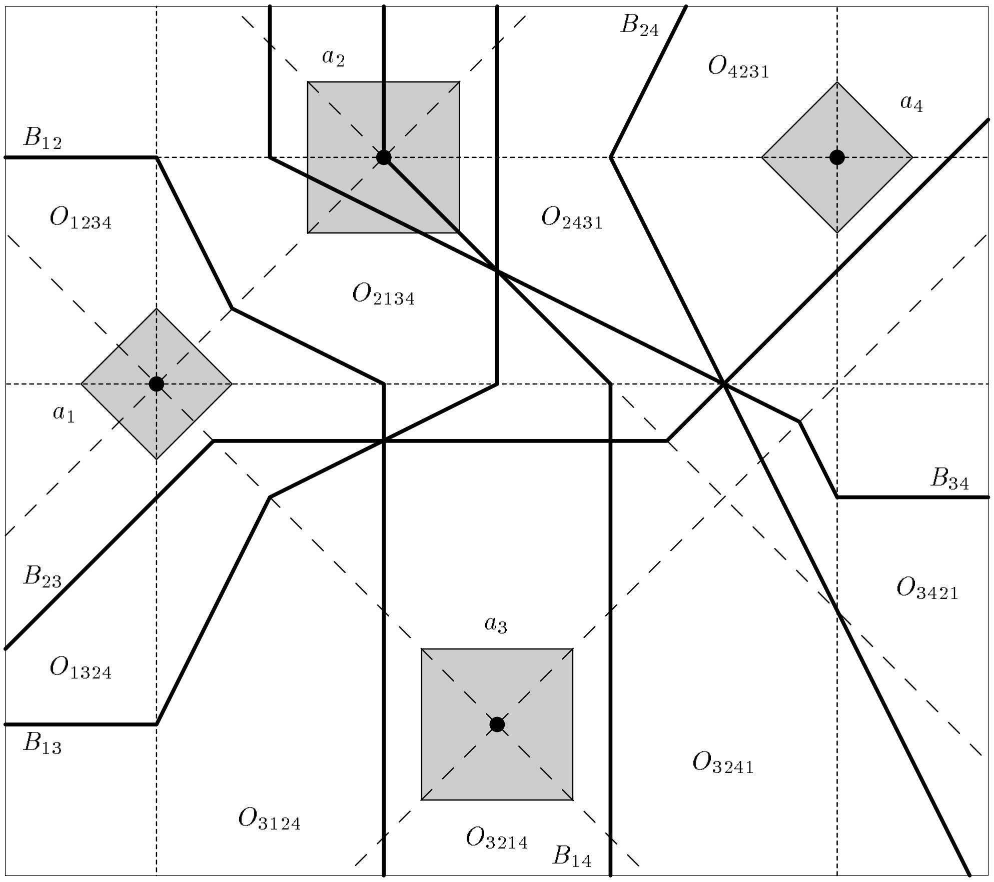

Figure 3: Bisector lines and ordered regions generated by 4 existing facilities $a_1, \ldots, a_4$ associated with the l_1-norm respectively the l_∞-norm for $\Omega := \{1, 1, 1, 1\}$

2.2 Multicriteria problems and level sets

Let $F^1, \ldots, F^Q$ be functions from $\mathbb{R}^n$ to $\mathbb{R}$. If we want to optimize simultaneously all these objective functions we get points in a Q-dimensional objective space and we do not have the canonical order of $\mathbb{R}$ anymore. In this section, we introduce two different orderings of $\mathbb{R}^Q$, the lexicographic and Pareto ordering, which can be given in the following way. Let $z, \bar{z} \in \mathbb{R}^Q$ and $\mathcal{Q} := \{1, \ldots, Q\}$, then

$$z \leq_{lex} \bar{z} \quad :\Leftrightarrow \quad z = \bar{z} \text{ or } z_q < \bar{z}_q \text{ for } q := \min\{i \in \mathcal{Q} \; : \; z_i \neq \bar{z}_i\}$$

$$z \preceq \bar{z} \quad :\Leftrightarrow \quad z_q \leq \bar{z}_q \quad \forall q \in \mathcal{Q} \text{ and } z_p < \bar{z}_p \text{ for some } p \in \mathcal{Q} \; .$$

A point $x \in \mathbb{R}^n$ is called a lexicographic location (or lex-optimal) if there exists a permutation π of the set $\mathcal{Q}$ such that

$$(F^{\pi(1)}(x), F^{\pi(2)}(x) \ldots, F^{\pi(Q)}(x)) \leq_{lex} (F^{\pi(1)}(y), F^{\pi(2)}(y) \ldots, F^{\pi(Q)}(y)) \quad \forall \, y \in \mathbb{R}^n \, .$$

We denote the set of lexicographic solutions by $\mathcal{X}^*_{lex}\left(F^1, \ldots, F^Q\right)$ or simply by $\mathcal{X}^*_{lex}$ if this is possible without causing confusion. The set of lexicographic solutions with respect to a fixed permutation π is denoted by $\mathcal{X}^*_{\pi(1),\ldots,\pi(Q)}$.

On the other hand a point $x \in \mathbb{R}^n$ is called a Pareto location (or Pareto-optimal) if there exists no $y \in \mathbb{R}^n$ such that

$$(F^1(y), F^2(y), \ldots, F^Q(y)) \preceq (F^1(x), F^2(x), \ldots, F^Q(x))$$

We denote the set of Pareto solutions by $\mathcal{X}^*_{par}\left(F^1, \ldots, F^Q\right)$ or simply by $\mathcal{X}^*_{par}$ if this is possible without causing confusion. Note that $\mathcal{X}^*_{lex} \subseteq \mathcal{X}^*_{par}$.

For technical reasons we will also use the concept of weak Pareto optimality, where a point $x \in \mathbb{R}^n$ is called a weak Pareto location (or weak Pareto-optimal) if there exists no $y \in \mathbb{R}^n$ such that

$$F^q(y) < F^q(x) \quad \forall\, q \in \mathcal{Q}\,.$$

We denote the set of weak Pareto solutions by $\mathcal{X}^*_{w-par}\left(F^1, \ldots, F^Q\right)$ or simply by $\mathcal{X}^*_{w-par}$ if this is possible without causing confusion. Note that $\mathcal{X}^*_{w-par} \subseteq \mathcal{X}^*_{par}$.

In order to obtain a geometrical characterization of a Pareto solution we use the concept of level sets.

For a function $F : \mathbb{R}^n \to \mathbb{R}$ the level set for a value $z \in \mathbb{R}$ is given by

$$L_{\leq}(F, z) := \{x \in \mathbb{R}^n \ : \ F(x) \leq z\}$$

and the level curve for a value $z \in \mathbb{R}^n$ is given by

$$L_{=}(F, z) := \{x \in \mathbb{R}^n \ : \ F(x) = z\}$$

Using the level sets and level curves [HN96] obtained that a point $x \in \mathbb{R}^n$ is a Pareto solution if and only if the following statement holds:

$$\bigcap_{q=1}^{Q} L_{\leq}(F^q, F^q(x)) = \bigcap_{q=1}^{Q} L_{=}(F^q, F^q(x))$$

Finally [War83] proved, that the set $\mathcal{X}^*_{par}$ is connected, provided that the objective functions are convex.

3 Bicriteria ordered Weber problems

3.1 Properties of $1/\mathbb{R}^2/\lambda_i \ :\nearrow /\gamma_i/2 - (\sum_{ord})_{par}$

In this and the following sections we restrict ourselves to the plane in order to use the geometric properties of the $\mathbb{R}^2$ to develop efficient algorithms. Moreover, we first consider the bicriteria case, since - as will be seen later - it is the basis for solving the Q-criteria case.

To this end, in this section we are looking for the Pareto solutions of the following vector optimization problem in $\mathbb{R}^2$

$$\min_{x \in \mathbb{R}^2} \begin{pmatrix} F^1(x) := \sum_{i=1}^{M} \lambda_i^1 \gamma^1 (x - A)_{(i)} \\ F^2(x) := \sum_{i=1}^{M} \lambda_i^2 \gamma^2 (x - A)_{(i)} \end{pmatrix}$$

where the weights λ_i^q are in increasing order with respect to the index i for each $q = 1, 2$, that is,

$$\lambda_1^q \leq \lambda_2^q \leq \ldots \leq \lambda_M^q\,, \quad q = 1, 2$$

and $\gamma^q(x - A)_{(i)}$ depends on the set Ω^q of importance given to the existing facilities by the q-th criteria, $q = 1, 2$. Therefore the previous vector optimization problem is convex, as was discussed in Section 2. In the classification scheme we use this problem is denoted by $1/\mathbb{R}^2/\lambda_i : \nearrow /\gamma_i/2 - (\sum_{ord})_{par}$.

Note that in a multicriteria setting each objective function F^q, $q \in \mathcal{Q}$, generates its own set of bisector lines. Therefore in the multicriteria case the generalized elementary convex sets are generated by all the fundamental directions d_g^i, $i = 1, \ldots, M$, $g = 1 \ldots G_i$, and the bisector lines $B^q(a_i, a_j)$, $q \in \mathcal{Q}$.

We are able to give a geometrical characterization of the set $\mathcal{X}^*_{par}$ by the following theorem.

Theorem 3.1 *$\mathcal{X}^*_{par}(F^1, F^2)$ is a connected chain from $\mathcal{X}^*(F^1)$ to $\mathcal{X}^*(F^2)$ consisting of facets or vertices of cells or complete cells.*

Proof:

First of all, we know that $\mathcal{X}^*_{par} \neq \emptyset$, so we can choose $x \in \mathcal{X}^*_{par}$. There exists at least one cell $\mathrm{C} \in \mathcal{C}$ with $x \in \mathrm{C}$. Hence three cases can occur:

1. $x \in int(\mathrm{C})$: Since $x \in \mathcal{X}^*_{par}$ we obtain

$$\bigcap_{q=1}^{Q} L_{\leq}(F^q, F^q(x)) = \bigcap_{q=1}^{Q} L_{=}(F^q, F^q(x))$$

and by linearity of the ordered Weber problem in each cell we have

$$\bigcap_{q=1}^{Q} L_{\leq}(F^q, F^q(y)) = \bigcap_{q=1}^{Q} L_{=}(F^q, F^q(y)) \quad \forall\, y \in \mathrm{C}$$

which means $y \in \mathcal{X}^*_{par}$ $\forall\, y \in \mathrm{C}$, hence $\mathrm{C} \subseteq \mathcal{X}^*_{par}$

2. $x \in \overline{ab} := conv\{a, b\} \subset bd(\mathrm{C})$ and $a, b \in Ext(\mathrm{C})$. We can choose $y \in int(\mathrm{C})$ and 2 cases can occur:

   (a) $y \in \mathcal{X}^*_{par}$. Hence we can continue as in Case 1.

   (b) $y \notin \mathcal{X}^*_{par}$. Therefore using the linearity we obtain first

$$\bigcap_{q=1}^{Q} L_{\leq}(F^q, F^q(z)) \neq \bigcap_{q=1}^{Q} L_{=}(F^q, F^q(z)) \quad \forall\, z \in int(\mathrm{C})$$

   and second, we have

$$\bigcap_{q=1}^{Q} L_{\leq}(F^q, F^q(z)) = \bigcap_{q=1}^{Q} L_{=}(F^q, F^q(z)) \quad \forall\, z \in \overline{ab}$$

   since $x \in \mathcal{X}^*_{par}$. Therefore we have that $\mathrm{C} \not\subseteq \mathcal{X}^*_{par}$ and $\overline{ab} \subseteq \mathcal{X}^*_{par}$.

3. $x \in Ext(\mathrm{C})$. We can choose $y \in int(\mathrm{C})$ and two cases can occur
   (a) If $y \in \mathcal{X}^*_{par}$, we can continue as in Case 1.
   (b) If $y \notin \mathcal{X}^*_{par}$, we choose $z_1, z_2 \in Ext(\mathrm{C})$ such that $\overline{xz_1}, \overline{xz_2}$ are faces of C,
      i. If z_1 or z_2 are in $\mathcal{X}^*_{par}$, we can continue as in Case 2.
      ii. If z_1 and z_2 are not in $\mathcal{X}^*_{par}$, then using the linearity in the same way as before we obtain that $(\mathrm{C} \setminus \{x\}) \cap \mathcal{X}^*_{par} = \emptyset$

Hence, we obtain that the set of Pareto solutions consists of complete cells, complete faces and vertices of these cells. Since we know that the set $\mathcal{X}^*_{par}$ is connected, the proof is completed. □

3.2 An algorithm for solving $1/\mathbb{R}^2/\lambda_i : \nearrow / \gamma_i / 2 - (\sum_{ord})_{par}$

In order to compute the chain of Pareto optimal solutions we use the concept of level sets and the linearity of the level sets in each cell. We present the following two results, whose proofs are quite lengthy and technical and can be found in [NW97] for the usual Weber problem.

Lemma 3.1 *Let* C *be a cell with extreme point x and let y and z be the adjacent extreme points to x such that we have in counterclockwise order $y\,x\,z$ (see Figure 4). If $x \in \mathcal{X}^*_{par}$, then 5 situations can occur:*

A : $\mathrm{C} \subseteq \mathcal{X}^*_{par}$ *, i.e.* C *is contained in the chain.*

B : $\overline{xy}$ *and* $\overline{xz}$ *are candidates for* $\mathcal{X}^*_{par}$.

C : $\overline{xy}$ *is candidate for* $\mathcal{X}^*_{par}$.

D : $\overline{xz}$ *is candidate for* $\mathcal{X}^*_{par}$.

E : *Neither* $\overline{xy}$ *nor* $\overline{xz}$ *are contained in* $\mathcal{X}^*_{par}$.

We denote by $\mathbf{sit}(\mathrm{C}, x)$ the situation appearing in cell C according to the extreme point x of C.

Lemma 3.2 *Let* $\mathrm{C}_1, \ldots, \mathrm{C}_{P_x}$ *be the cells containing the intersection point x, considered in counterclockwise order, and $y_1, \ldots, y_{P_x}$ the intersection points adjacent to x, considered in counterclockwise order, (see Figure 5). If $x \in \mathcal{X}^*_{par}$ and $i \in \{1, \ldots, P_x\}$, then the following holds:*

$$\overline{xy_{i+1}} \subseteq \mathcal{X}^*_{par} \quad \Longleftrightarrow \quad \left\{ \begin{array}{ll} & \mathbf{sit}(\mathrm{C}_i, x) = \mathbf{A} \\ or & \mathbf{sit}(\mathrm{C}_{i+1}, x) = \mathbf{A} \\ or & \left\{ \begin{array}{ll} & \mathbf{sit}(\mathrm{C}_i, x) \in \{\mathbf{B}, \mathbf{C}\} \\ and & \mathbf{sit}(\mathrm{C}_{i+1}, x) \in \{\mathbf{B}, \mathbf{D}\} \end{array} \right\} \end{array} \right\}$$

Applying these two results we describe the following algorithm.

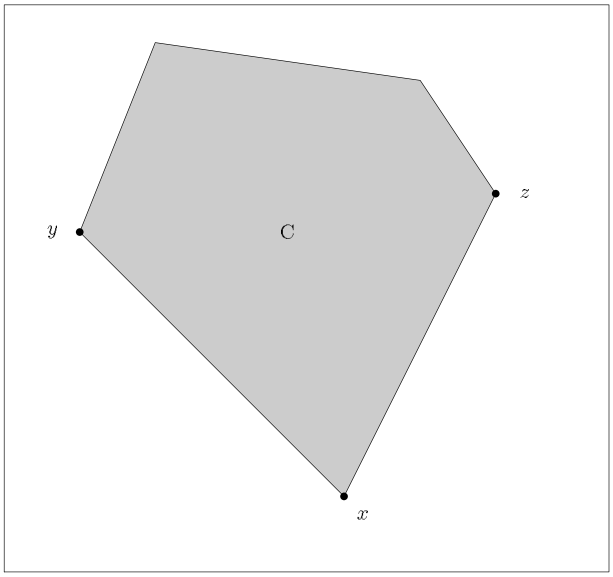

Figure 4: Illustration to Lemma 3.1: $y, x, z \in Ext(\mathrm{C})$ in counterclockwise order

ALGORITHM 3.1

(Solving $1/\mathbb{R}^2/\lambda_i : \nearrow /\gamma_i/2 - (\sum_{ord})_{par}$.)

Input:

1. *Demand points* $a_i \in \mathbb{R}^2$, $i \in \mathcal{M}$.
2. *Weights* λ_i^q , $i \in \mathcal{M}$, $q = 1, 2$ *satisfying* $0 \leq \lambda_1^q \leq \ldots \leq \lambda_M^q$ *for* $q = 1, 2$.
3. *Weights* ω_i^q , $i \in \mathcal{M}$, $q = 1, 2$ *satisfying* $\omega_i^q \geq 0$ *for* $i \in \mathcal{M}$, $q = 1, 2$.
4. *Polyhedral gauges* $\gamma_i : \mathbb{R}^2 \to \mathbb{R}$, $i \in \mathcal{M}$.

Output:

1. $\mathcal{X}^*_{par}(F^1, F^2)$.

Steps:

1. *Computation of the planar graph generated by the cells.*
2. *Compute the two sets of lexicographical solutions* $\mathcal{X}^*_{1,2}$, $\mathcal{X}^*_{2,1}$.
3. *IF* $\mathcal{X}^*_{1,2} \cap \mathcal{X}^*_{2,1} \neq \emptyset$
4. *THEN ($\star$ trivial case :* $\mathcal{X}^*(F^1) \cap \mathcal{X}^*(F^2) \neq \emptyset$ $\star$*)* $\mathcal{X}^*_{par} := co\{\mathcal{X}^*_{1,2}\}$
5. *ELSE ($\star$ non trivial case :* $\mathcal{X}^*(F^1) \cap \mathcal{X}^*(F^2) = \emptyset$ $\star$*)*
6. $\mathcal{X}^*_{par} := \mathcal{X}^*_{1,2} \cup \mathcal{X}^*_{2,1}$
7. *Choose* $x \in \mathcal{X}^*_{1,2} \cap \mathcal{GIP}$ *and* $i := 0$.

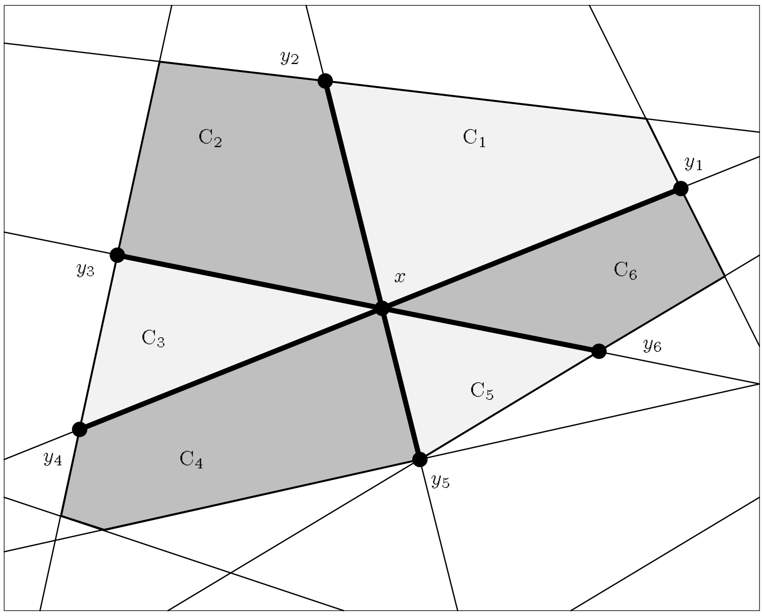

Figure 5: Illustration to Lemma 3.2 with $P_x = 6$

WHILE $x \notin \mathcal{X}^*_{2,1}$ *DO*

BEGIN

REPEAT

$i := i + 1$

IF $i > P_x$ *THEN* $i := i - P_x$

UNTIL $\mathbf{sit}(\mathrm{C}_i, x) := \mathbf{A}$ OR ($\mathbf{sit}(\mathrm{C}_i, x) \in \{\mathbf{B},\mathbf{C}\}$ AND $\mathbf{sit}(\mathrm{C}_{i+1}, x) \in \{\mathbf{B},\mathbf{D}\}$)

IF $\mathbf{sit}(\mathrm{C}_i, x) := \mathbf{A}$

THEN (⋆ *We have found a bounded cell.* ⋆) $\mathcal{X}^*_{par} := \mathcal{X}^*_{par} \cup \mathrm{C}_i$

ELSE (⋆ *We have found a bounded face.* ⋆) $\mathcal{X}^*_{par} := \mathcal{X}^*_{par} \cup \overline{xy_i}$

$temp := x$

$x := y_i$

$i := i_x(temp) - 1$ (⋆ Where $i_x(temp)$ is the index of $temp$ in the list
of adjacent generalized intersection points
to the generalized intersection point x. ⋆).

END □

[BO79] proved by using a line-sweep-principle that the computation of a planar graph induced by n line segments in the plane, can be computed in $O((n+s)log\, n)$ time, where s is the number of intersection points of the line segments.

In order to obtain the complexity of the scan-line-principle in our problem we need line segments instead of halflines. The following lemma shows that we only have to look for the solutions in a bounded region. Therefore all halflines defining the cells are transformed into segments.

Lemma 3.3

$$l_2(x) \le 2 \max_{i \in \mathcal{M}} \left\{ l_2(a_i) \frac{\max_{e \in Ext(B_i)}\{l_2(e)\}}{\min_{e \in Ext(B_i)}\{l_2(e)\}} \right\} := L \quad \forall\, x \in \mathcal{X}^*_{par}\left(F^1, \ldots, F^Q\right)$$

where $l_2(.)$ is the Euclidean norm. That means $\mathcal{X}^*_{par}\left(F^1, \ldots, F^Q\right)$ is included in the Euclidean ball with center 0 and radius L.

Proof:

By a simple generalization of the result obtained by [PF95] we know

$$\mathcal{X}^*_{par}\left(F^1, \ldots, F^Q\right) \subseteq \mathcal{X}^*_{w-par}\left(\gamma_1(. - a_1), \ldots, \gamma_M(. - a_M)\right) .$$

So, it is sufficient to prove the claim for

$$x \in \mathcal{X}^*_{w-par}\left(\gamma_1(. - a_1), \ldots, \gamma_M(. - a_M)\right) .$$

Since $x \in \mathcal{X}^*_{w-par}\left(\gamma_1(. - a_1), \ldots, \gamma_M(. - a_M)\right)$, there exists no point $y \in \mathbb{R}^n$ which is strictly dominating x. That means, especially for $y = 0$, there exists an index $i(x) \in \mathcal{M}$ with

$$\gamma_{i(x)}(x - a_{i(x)}) \le \gamma_{i(x)}(0 - a_{i(x)}) = \gamma_{i(x)}(-a_{i(x)})$$

Using the triangular inequality this means

$$\gamma_{i(x)}(x) \le \gamma_{i(x)}(a_{i(x)}) + \gamma_{i(x)}(-a_{i(x)})$$

Using the two inequalities

$$\begin{aligned} l_2(x) &\le \gamma_i(x) \max_{e \in Ext(B_i)}\{l_2(e)\} \quad \forall x \in \mathbb{R}^n \\ \gamma_i(x) &\le l_2(x) \frac{1}{\min_{e \in Ext(B_i)}\{l_2(e)\}} \quad \forall x \in \mathbb{R}^n \end{aligned}$$

which are valid for each polyhedral gauge γ_i with $i \in \mathcal{M}$, we obtain

$$\begin{aligned} l_2(x) &\le \gamma_{i(x)}(x) \max_{e \in Ext(B_{i(x)})}\{l_2(e)\} \\ &\le \left(\gamma_{i(x)}(a_{i(x)}) + \gamma_{i(x)}(-a_{i(x)})\right) \max_{e \in Ext(B_{i(x)})}\{l_2(e)\} \\ &\le 2\, l_2(a_{i(x)}) \frac{\max_{e \in Ext(B_{i(x)})}\{l_2(e)\}}{\min_{e \in Ext(B_{i(x)})}\{l_2(e)\}} \end{aligned}$$

The right side of the previous inequality depends on $i(x) \in \mathcal{M}$, to avoid this we have to consider the maximum over all indices which leads to the inequality :

$$l_2(x) \leq 2 \max_{i \in \mathcal{M}} \left\{ l_2(a_i) \frac{\max_{e \in Ext(B_i)}\{l_2(e)\}}{\min_{e \in Ext(B_i)}\{l_2(e)\}} \right\}$$

□

This lemma implies that in the case of the ordered Weber problem the computation of the planar graph generated by the fundamental directions and bisector lines can be done in $O((M^2 G_{\max} + M^4 G_{\max}^2) \log (M^2 G_{\max})) = O(M^4 G_{\max}^2 \log (M G_{\max}))$.

The evaluation of the ordered Weber function for one point needs $O(M \log(M G_{\max}))$, therefore we obtain $O(M^5 G_{\max}^2 \log (M G_{\max}))$ for the computation of lexicographic solutions. At the end, the complexity for computing the chain is $O(M^5 G_{\max}^2 \log (M G_{\max}))$, since we have to consider at most $O(M^4 G_{max}^2)$ cells and the determination of $\mathbf{sit}(\,.\,,\,.\,)$ can be done in $O(M \log(M G_{max}))$. The overall complexity is $O(M^5 G_{\max}^2 \log (M G_{\max}))$.

4 The Q-Criteria Case

In this section we turn to the Q-criteria case and we develop an efficient algorithm for computing $\mathcal{X}_{par}^* \left(F^1, \ldots, F^Q\right)$ using the results of the bicriteria case. First of all we notice that testing if a point is Pareto in the Q-criteria case, $Q > 3$, can be reduced to study the 3-criteria case by the following theorem.

Theorem 4.1 ([EN]) *For Q functions $F^1, \ldots, F^Q$ and a point $x \in \mathbb{R}^2$ it is sufficient, to consider at most 3 functions $F^{q_1}, F^{q_2}, F^{q_3}$ simultaneously to decide, whether*

$$x \in \mathcal{X}_{par}^* \left(F^1, \ldots, F^Q\right) \quad or \quad x \notin \mathcal{X}_{par}^* \left(F^1, \ldots, F^Q\right) .$$

In the following we will extend this theorem in the sense that we describe the whole set of Pareto locations by means of two and three criteria subproblems. To this for, we need some definitions. A cell C is called a collapsed cell with respect to $i, j, k \in \mathcal{Q}$, pairwise different , if $C \in \mathcal{X}_{par}^* (F^i, F^j)$ and $L_=(F^k, F^k(x)) \cap C$ is not collinear with $L_=(F^i, F^i(x)) \cap C = L_=(F^j, F^j(x)) \cap C$ for $x \in int(C)$.

Denote by $Seg(C)$ the set of all segments defining the boundary of C. The set $PR_{ijk}(C) := \{s \in Seg(C) \,:\, \exists\, x \in ri(s) \cap \mathcal{X}_{par}^* \left(F^i, F^j, F^k\right)\}$ is called the *Pareto remainder* of C with respect to $i, j, k \in \mathcal{Q}$, where $ri(s)$ denotes the relative interior of s. Denote $\mathcal{X}_{par}^*(2) := \bigcup_{\substack{i,j \in \mathcal{Q} \\ i \neq j}} \mathcal{X}_{par}^* \left(F^i, F^j\right)$, the union of all bicriteria chains, by $Col(\mathcal{X}_{par}^*(2))$ the set of collapsed cells in $\mathcal{X}_{par}^*(2)$ and by $PR(\mathcal{X}_{par}^*(2))$ the set of all Pareto remainder in $\mathcal{X}_{par}^*(2)$. Let $\mathcal{X}_i^*$ be the set of optimal solutions of the single criterion problem with objective function F^i. Given $i, j, k \in \mathcal{Q}$, denote by

$$\mathcal{X}_{par}^* \left(\mathcal{X}_i^*; F^i, F^j, F^k\right) := \mathcal{X}_{par}^* \left(F^i, F^j, F^k\right) \cap \mathcal{X}_i^*,$$

$$\mathcal{X}_{par}^* \left(F^i\right) := \mathcal{X}_{par}^* \left(F^1, \ldots, F^Q\right) \cap \mathcal{X}_i^*,$$

$$S_{ijk} := \bigcup_{l=i,j,k} \mathcal{X}^*_{par}\left(\mathcal{X}^*_l; F^i, F^j, F^k\right).$$

The next result reduces the determination of the Pareto optimal set for the 3-criteria case to the study of the bicriteria case. Let us denote by

$$g_\varphi(x) := \{y \in \mathbb{R}^2 \ : \ (y_1, y_2) = (x_1, x_2) + \omega(\cos\varphi, \sin\varphi), \ \omega \geq 0\} \text{ for } \varphi \in [0, 2\pi).$$

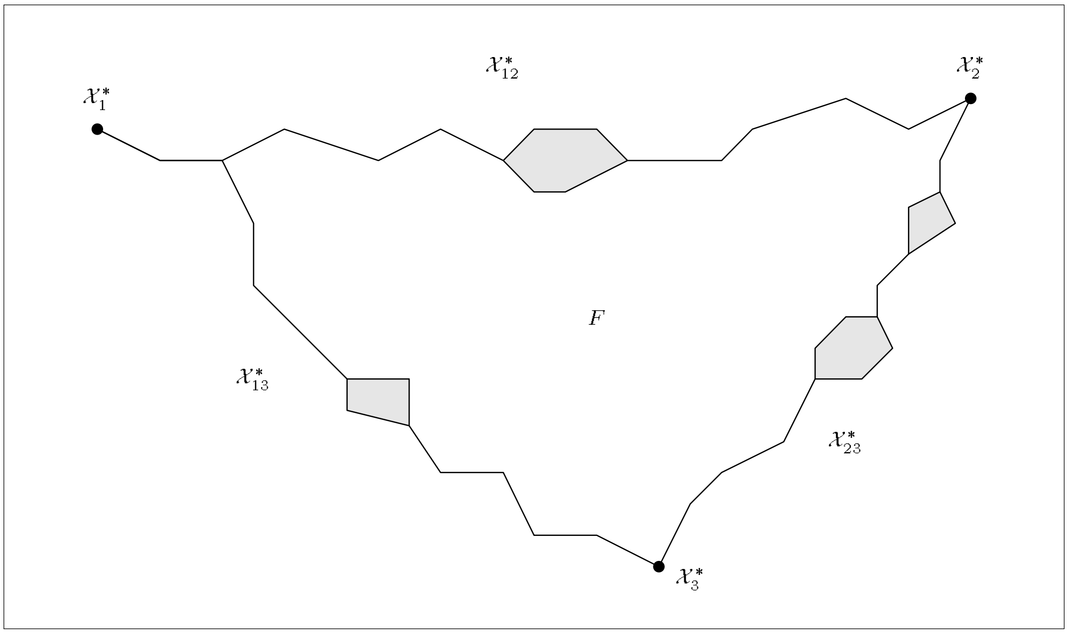

Figure 6: Illustration to Lemma 4.1

Lemma 4.1
Let

$$F' := (\mathcal{X}^*_{par}(2) \setminus Col(\mathcal{X}^*_{par}(2))) \cup PR(\mathcal{X}^*_{par}(2)) \cup S_{123}$$

and

$$F := \{x \in \mathbb{R}^2 \ : \ g_\varphi(x) \cap F' \neq \emptyset \quad \forall \varphi \in [0, 2\pi)\}.$$

Then

$$\mathcal{X}^*_{par}\left(F^1, F^2, F^3\right) = F' \cup F.$$

Proof:
We prove this lemma by double inclusion. First, we prove $F' \cup F \subset \mathcal{X}^*_{par}(F^1, F^2, F^3)$. Then, we shall prove $\mathcal{X}^*_{par}(F^1, F^2, F^3) \subset F' \cup F$.

Indeed, let $x \in F' \cup F$. If $x \in F'$ then x is an Pareto-optimal solution because by construction these points are Pareto-optimal solutions. Otherwise, $x \in F \setminus F'$. Then, there exists a continuous curve of Pareto-optimal points all around x. Since the set $\mathcal{X}^*_{par}(F^1, F^2, F^3)$ is connected [War83] x must belong to $\mathcal{X}^*_{par}(F^1, F^2, F^3)$.

Conversely, let $x \in \mathcal{X}^*_{par}(F^1, F^2, F^3)$. One of the following cases must occur:

1. The slopes of the three objectives functions at x are parallel. This case implies that x is alternate Pareto-optimal with respect to two pairs of objective functions. Therefore, x belongs to an Pareto-optimal cell with respect to two Pareto-optimal chains for two criteria, which is not collapsed. Consequently $x \in F'$.

2. There exists only two slopes of the objective functions at x being parallel. This case implies that x is alternate Pareto-optimal for two criteria but that considering the third one the corresponding cell collapses to a part of its boundary. Therefore, $x \in PR(\mathcal{X}^*_{par}(2)) \subset F'$.

3. The slopes of the objective functions at x are not pairwise parallel.

Let us denote $I_{ij}(x) := L_{\leq}(F^i, F^i(x)) \cap L_{\leq}(F^j, F^j(x))\,,\ i < j\ ,\ i,j = 1,2,3$. Since x is Pareto-optimal $\bigcap_{\substack{i,j \in Q \\ i \neq j}} I_{ij}(x) = \{x\}$. This implies that $(L_{\leq}(F^1, F^1(x)) \setminus (I_{12}(x) \cup I_{13}(x))$, $(L_{\leq}(F^2, F^2(x)) \setminus (I_{12}(x) \cup I_{23}(x))$ and $(L_{\leq}(F^3, F^3(x)) \setminus (I_{13}(x) \cup I_{23}(x))$ are pairwise disjoint in a neighborhood of x. As a straightforward consequence of the definitions of $I_{ij}(x)$ and $\mathcal{X}^*_{par}(\mathcal{X}^*_i; F^1, F^2, F^3)$ one can obtain the following statements:

(i) $\mathcal{X}^*_{par}(F^i, F^j) \cap I_{ij}(x) \neq \emptyset$.

(ii) $\mathcal{X}^*_{par}(F^i, F^j) \cap L_{=}(F^k, F^k(x))$ has to be a connected set with respect to the relative topology induced in $L_{=}(F^i, F^i(x))$ for $k = i, j$.

(iii) $\mathcal{X}^*_{par}(\mathcal{X}^*_i; F^1, F^2, F^3) \cap \mathcal{X}^*_{par}(F^i, F^j) \neq \emptyset$ for $j \neq i$, $j = 1,2,3$.

By (iii) $\mathcal{X}^*_{par}(2) \cup S_{123}$ defines a bounded region in $\mathbb{R}^2$. By (i) and (ii) $\mathcal{X}^*_{par}(F^i, F^j)$ cannot intersect $L_{=}(F^i, F^i(x))$ (or $L_{=}(F^j, F^j(x))$) only in the interior of $L_{\leq}(F^k, F^k(x))$ with $i \neq j \neq k$. Therefore, either $x \in \mathcal{X}^*_{par}(F^i, F^j)$ for some pair i, j or x belongs to some set $\mathcal{X}^*_{par}(\mathcal{X}^*_i, F^1, F^2, F^3)$ for some $i = 1,2,3$ or x is enclosed in the region defined by $\mathcal{X}^*_{par}(2) \cup S_{123}$. In the first and second cases $x \in PR(\mathcal{X}^*_{par}(2))$ and in the later case $x \in F'$. Hence, the result is proved. □

Denote by

$$\begin{aligned} \mathcal{X}^*_{par}(3) &:= \bigcup_{\substack{i,j,k \in Q \\ i \neq j \neq k}} \mathcal{X}^*_{par}\left(F^i, F^j, F^k\right) \\ encl(\mathcal{X}^*_{par}(3)) &:= \{x \in \mathbb{R}^2 \ :\ g_\varphi(x) \cap \mathcal{X}^*_{par}(3) \neq \emptyset \quad \forall \varphi \in [0, 2\pi)\}\ . \end{aligned}$$

Theorem 4.2

$$\mathcal{X}^*_{par}\left(F^1, F^2, \ldots, F^Q\right) := \mathcal{X}^*_{par}(3) \cup encl(\mathcal{X}^*_{par}(3))$$

Proof:

Let us denote $F := \mathcal{X}^*_{par}(3) \cup encl(\mathcal{X}^*_{par}(3))$. Assume that $x \in F$. If $x \in \mathcal{X}^*_{par}(3)$ then x also belongs to $\mathcal{X}^*_{par}\left(F^1, \ldots, F^Q\right)$ because $\mathcal{X}^*_{par}(3) \subset \mathcal{X}^*_{par}\left(F^1, \ldots, F^Q\right)$. If $x \in encl(\mathcal{X}^*_{par}(3))$ there exists a continuous curve of points of $\mathcal{X}^*_{par}(3)$ around x and by connectivity arguments $x \in \mathcal{X}^*_{par}\left(F^1, \ldots, F^Q\right)$.

Conversely, let $x \in \mathcal{X}^*_{par}\left(F^1, \ldots, F^Q\right)$ one and only one of the following cases can occur:

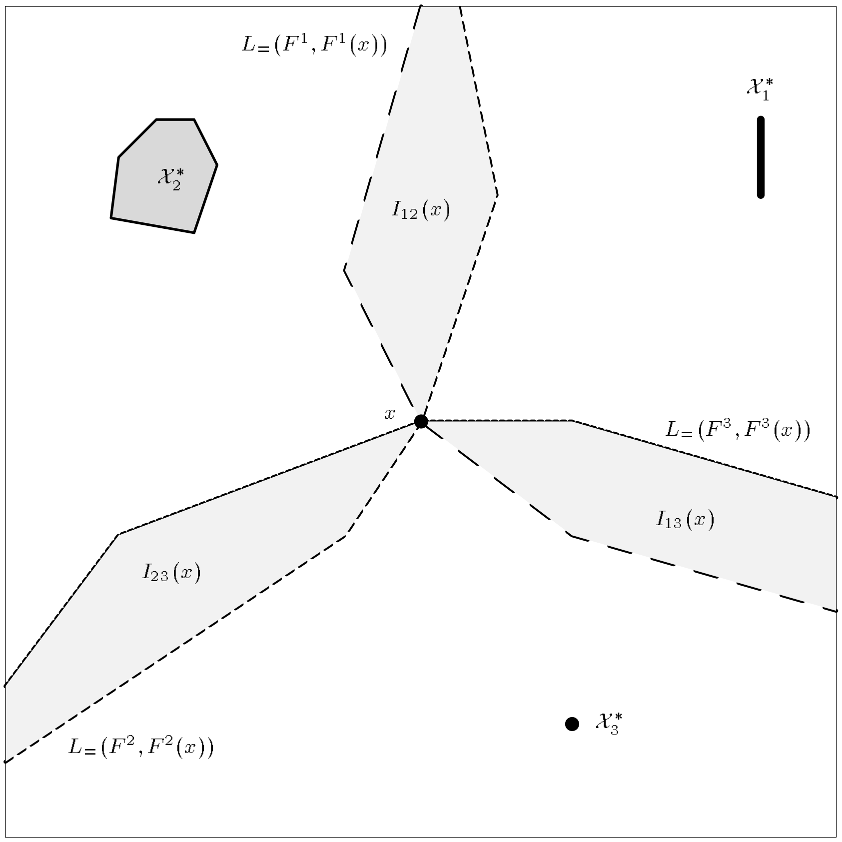

Figure 7: Illustration to the proof of Lemma 4.1

1. There are at least 3 pairwise non-parallel slopes corresponding with F^i, F^j, F^k at x. Hence $x \in \mathcal{X}^*_{par}\left(F^i, F^j, F^k\right) \subset \mathcal{X}^*_{par}(3)$.

2. There are at least two exhaustive subsets of parallel slopes (parallel to each other). In this case x belongs at least to two chains of bicriteria Pareto solutions. This implies that x also belongs at least to a set of Pareto solutions for three criteria.

3. There are two exhaustive subsets of parallel slopes which are non-parallel to each other. In this case, x belongs to a cell C (with respect to the cell structure generated by two objective functions F^i, F^j) which has been collapsed at least twice with respect to three criteria. Indeed, we can choose the objective functions F^i, F^j out of the first subset having opposite slopes in C. Then, we can also choose two more objective functions F^k, F^l out of the second subset, also with opposite slopes between them. The collapsing process of C with respect to i, j, k and i, j, l includes the whole boundary of C, i.e. $PR_{ijk}(C) \cup PR_{ijl}(C) = Seg(C)$. Therefore x belongs to $encl(\mathcal{X}^*_{par}(3))$.

Since in any case $x \in F$ the proof is complete. □

From the previous results we can derive an algorithm for solving the Q-criteria case. The idea of the procedure is to start with the bicriteria chains and merge them together using the results of this section to get the Q-criteria solutions.

ALGORITHM 4.1

(Solving $1/\mathbb{R}^2/\lambda_i : \nearrow /\gamma_i/Q - (\sum_{ord})_{par}$.)

<u>*Input:*</u>

1. *Demand points* $a_i \in \mathbb{R}^2$, $i \in \mathcal{M}$.
2. *Weights* λ_i^q , $i \in \mathcal{M}$, $q = 1, 2, \ldots, Q$ *satisfying* $0 \le \lambda_1^q \le \ldots \le \lambda_M^q$ *for* $q \in \mathcal{Q}$.
3. *Weights* ω_i^q , $i \in \mathcal{M}$, $q = 1, 2$ *satisfying* $\omega_i^q \ge 0$ *for* $i \in \mathcal{M}$, $q \in \mathcal{Q}$.
4. *Polyhedral gauges* $\gamma_i : \mathbb{R}^2 \to \mathbb{R}$, $i \in \mathcal{M}$.

<u>*Output:*</u>

1. $\mathcal{X}^*_{par}\left(F^1, \ldots, F^Q\right)$.

<u>*Steps:*</u>

1. *Compute the planar graph generated by the cells.*
2. *For all* $i, j \in \mathcal{Q} := \{1, 2, \ldots, Q\}$, $i < j$, *compute the bicriteria lexicographic solutions* $\mathcal{X}^*_{ij}$.
3. *Compute* $Ext(conv\{\bigcup_{i=1}^{Q} \mathcal{X}^*_{par}(F^{q_i})\}) = \{y_1, \ldots, y_L\}$.
4. *Order* $y_1, \ldots, y_L$ *counterclockwise.*
5. *For* $i = 1$ *to* L *compute* $\mathcal{X}^*_{par}(F^{q_i}, F^{q_{i+1}})$ *where* $q_{L+1} := q_1$ *using the bicriteria algorithm.*
6. *Denote the temporary set* $\mathcal{X}^*_{par}$ *with* $\mathcal{X}^*_{temp}$ *and its boundary by* Bd_{temp}.
7. $\mathcal{X}^*_{temp} := \bigcup_{i=1}^{Q} \mathcal{X}^*_{par}(F^{q_i}, F^{q_{i+1}})$
8. *For* $i, j \in \mathcal{Q}$, $i < j$ *compute the chain* $\mathcal{X}^*_{par}(F^i, F^j)$.
9. *Build* P_{ij}, *the region enclosed* $\mathcal{X}^*_{par}(F^i, F^j)$ *and* Bd_{part}, *where* Bd_{part} *is the part of* Bd_{temp} *from the first to the last intersection point of* $\mathcal{X}^*_{par}(F^i, F^j)$ *and* Bd_{temp}. *Note that* P_{ij} *is (usually) a non simple polygon.*
10. $\mathcal{X}^*_{temp} := \mathcal{X}^*_{temp} \cup P_{ij}$
11. *For all cells in* bd_{temp} *check if the whole cell is Pareto for all* Q *criteria.*
    (This can be done in linear time see [FGFZ92] *)*
12. $\mathcal{X}^*_{par}\left(F^1, \ldots, F^Q\right) := \mathcal{X}^*_{temp}$

□

The generation of the planar graph is $O(Q^2M^4G^2_{\max}\log(QMG_{\max}))$ and the computation of the bicriteria lexicographic solutions for all pairs is $O(M^5Q^4G^2_{\max}\log(MG_{\max}))$. Therefore the complexity of these two steps is $O(M^5Q^4G^2_{\max}\log(QMG_{\max}))$.

The extreme points of the convex hull of the whole bicriteria lexicographic solutions can be found in $O(Q^2\log Q)$. And the computation of the chain of bicriteria Pareto solutions between each pair has complexity $O(M^5Q^2G^2_{\max}\log(MG_{\max}))$. The union of P_{ij} and $\mathcal{X}^*_{temp}$ (Steps 8-9) can be done in a similar way as the intersection of two non-convex polygons. Therefore the complexity of this step is $O(Q^4M^8G^4_{max})$, since $\mathcal{X}^*_{temp}$ and P_{ij} have at most $O(Q^2M^4G^2_{max})$ vertices. Finally, the overall complexity is $O(Q^4M^8G^4_{max})$.

5 Extensions

5.1 Multicriteria Ordered Weber Problems with Attraction and Repulsion

If we allow the weights ω_i^q , $i\in\mathcal{M}$, $q\in\mathcal{Q}$, to be positive or negative, we cannot apply the procedures presented in the preceding sections. Especially, we do not have the following properties anymore:

- Convexity of the objective functions F^q , $q\in\mathcal{Q}$.
- Connectivity of the set of Pareto optimal points $\mathcal{X}^*_{par}(F^1,\dots,F^q)$.

As a consequence a solution algorithm for the multicriteria ordered Weber problem with attraction and repulsion, classified as $1/\mathrm{I\!R}^2/\omega_i^q \not\geq 0/\gamma_i/Q-(\sum_{ord})_{par}$, has to have a completely different structure than the algorithm for the convex case $1/\mathrm{I\!R}^2/\lambda_i\ :\nearrow/\gamma_i/Q-(\sum_{ord})_{par}$.

Note that for negative ω_i^q we cannot write $\omega_i^q\gamma_i(x)=\gamma_i(\omega_i^q x)$ anymore. Instead we have $\omega_i^q\gamma_i(x)=-\gamma_i(|\omega_i^q|x)$. Therefore, the increasing order of the weights $\lambda_1^q,\dots,\lambda_M^q$ cannot be maintained and we drop the assumption $0\le\lambda_1^q\le\dots\le\lambda_M^q$.

However the following properties are still fulfilled:

- The cell structure remains the same, since fundamental directions and bisector lines do not depend on λ_i^q and the sign of ω_i^q .
- Moreover we still have the linearity of the objective functions F^q inside each cell.

Consequently, we can compute the local Pareto solutions with respect to a single cell as described in the case of $1/\mathrm{I\!R}^2/\lambda_i\ :\nearrow/\gamma_i/Q-(\sum_{ord})_{par}$. Of course we cannot be sure that the local Pareto solutions remain globally Pareto. Therefore to obtain the set of global Pareto solutions all local Pareto solutions have to be compared.

A schematic approach for solving $1/\mathrm{I\!R}^2/\omega_i^q \not\geq 0/\gamma_i/Q-(\sum_{ord})_{par}$ would be:

1. Compute the local Pareto solutions for each cell $C\in\mathcal{C}$.
2. Compare all solutions of step 1 and get $\mathcal{X}^*_{par}(F^1,\dots,F^q)$.
3. Output: $\mathcal{X}^*_{par}(F^1,\dots,F^q)$.

In general Step 2 might become very time consuming because we have to compare in the Q-criteria case $O(Q^2 M^4 G_{\max}^2)$ cells. However for more special cases efficient algorithms can be developed. In the single criteria case solution algorithms can be found in [JP96, ND97]. If we restrict ourselves to the bicriteria case, we can do a procedure similar to the one used in [HNL96] for network locations:

After having finished step 1 we project all locally Pareto sets into the 2-dimensional objective space. We get a set of $L \in O(M^4 G_{\max}^2)$ line segments. We apply the algorithm of [Her89], with the modification described in [HNL96], to get the global pareto solution in the objective space in $O(L \log L)$ time. Afterwards we transform back the solutions to the decision space.

5.2 Higher Dimensions

The problem $1/\mathbb{R}^n/\lambda_i : \nearrow /\gamma_i/Q - (\sum_{ord})_{par}$ has essentially the same structure as the problem previously considered in $\mathbb{R}^2$. Therefore, similar approaches to the ones given in Section 4 could be applied. The unique difference is that in $\mathbb{R}^n$ we should check for $n+1$ criteria at each time (see [EN]), so that instead of considering bicriteria chains we must consider n-criteria chains in Step 2. Nevertheless, although theoretically possible this approach is computationally very time consuming, since many situtations may occur when one intersects n level curves or level sets.

Alternatively, we can use a different approach based on checking for local Pareto-optimality using convex analysis tools. This approach is just based on the null vector condition:

$$x^* \text{ is Pareto-solution} \quad \Longleftrightarrow \quad 0 \in ri\left(conv\left(\bigcup_{q \in \mathcal{Q}} \partial F^q(x^*)\right)\right)$$

where $\partial F^q(x^*)$ stands for the subdifferential set of F^q at x^* (see [FP95]).

In fact, it can easily be seen that the approach used for the planar case is nothing else than a geometrical interpretation of this null vector condition exploiting the additional properties of the plane. It is straightforward to see that in the same way as it was done for the planar case, there exists a subdivision $\mathcal{C}$ of $\mathbb{R}^n$, such that in each element $C \in \mathcal{C}$ the objective functions $F^1, \ldots, F^Q$ are linear. Therefore, in the interior of each cell $C \in \mathcal{C}$, $\partial F^q(x^*)$ is constant and equals to the vector defining the linear representation of F^q. This means that we can check the null vector condition in each cell in $O(Q \log Q)$ using [FGFZ92]. Then using the connectivity of the set $\mathcal{X}_{par}^*$ we can get this set by just following a backtrack search in the subdivision C.

In order to obtain the set of Pareto solutions in each cell C, a scheme testing generalized elementary convex sets following a non-increasing order on its dimension can be performed.

An schematic procedure for solving $1/\mathbb{R}^n/\lambda_i : \nearrow /\gamma_i/Q - (\sum_{ord})_{par}$ is given in what follows. The following notation is used:

1. $A(C)$ set of generalized elementary convex sets of any dimension adjacent to C.
2. $\mathcal{X}_{par}^*(C)$ set of Pareto-optimal solutions on C.
3. τ auxiliary set.

ALGORITHM 5.1
(Solving $1/\mathbb{R}^n/\lambda_i : \nearrow /\gamma_i/Q - (\sum_{ord})_{par}$.)

<u>*Input:*</u>

1. *Demand points* $a_i \in \mathbb{R}^n$, $i \in \mathcal{M}$.
2. *Weights* λ_i^q , $i \in \mathcal{M}$, $q = 1, 2, \ldots, Q$ *satisfying* $0 \leq \lambda_1^q \leq \ldots \leq \lambda_M^q$ *for* $q = 1, 2, \ldots, Q$.
3. *Weights* ω_i^q , $i \in \mathcal{M}$, $q = 1, 2$ *satisfying* $\omega_i^q \geq 0$ *for* $i \in \mathcal{M}$, $q = 1, 2, \ldots, Q$.
4. *Polyhedral gauges* $\gamma_i : \mathbb{R}^n \to \mathbb{R}$, $i \in \mathcal{M}$.

<u>*Output:*</u>

1. $\mathcal{X}^*_{par}\left(F^1, \ldots, F^Q\right)$.

<u>*Steps:*</u>

1. *Solve* $1/\mathbb{R}^n/\lambda_i^1, \omega_i^1/\gamma_i/\sum_{ord}$. *Let* D *be a generalized elemantary convex set containing optimal solutions of* $1/\mathbb{R}^n/\lambda_1/\gamma_i/\sum_{ord}$.
   Initialize $\mathcal{X}^*_{par} = \mathcal{X}^*_{par}(D)$, $\tau = \emptyset$.
2. *WHILE* C *exists such that* $C \cap (\mathcal{X}^*_{par} \setminus \tau) \neq \emptyset$ *DO*
3.   *BEGIN*
4.   *Compute* $A(C)$, *set* $C_0 = C$.
5.   *REPEAT*
6.   *Choose* $C_1 \in A(C_0)$.
7. $A(C_0) = A(C_0) \setminus C_1$
8.   *Compute* $\mathcal{X}^*_{par}(C_1)$.
9.   *IF* $\mathcal{X}^*_{par}(C_1) \nsubseteq \mathcal{X}^*_{par}$
10.   *THEN*
11.   *BEGIN*
12. $\mathcal{X}^*_{par} = \mathcal{X}^*_{par} \cup \mathcal{X}^*_{par}(C_1)$
13. $C = C_1$
14.   *EXITREPEAT*
15.   *END*
16.   *UNTIL* $A(C) = \emptyset$

17. *IF* $A(C_0) = \emptyset$ *THEN* $\tau = \tau \cup C_0$

18. *END*

19. *Output* $\mathcal{X}^*_{par}$

□

The reader can realize that the hardest part of this algorithmic scheme is the computation of the set $A(C)$ of adjacent generalized elementary convex sets to C. Once this step is efficiently done all the remaining steps can be performed with minor effort.

6 Conclusions

In this paper we showed the usefulness of ordered Weber problems for modelling multicriteria locational decision problems. We developed efficient algorithms and proved structural results. Also a detailed complexity analysis of these algorithms is provided. Extensions to the multifacility case as well as improvements for the complexity results for special cases are under research. Also a more detailed discussion of the problems mentioned in Section 5 is planned. Furthermore, we are working on an implementation of ordered Weber problems in **LOLA** (Library of Location Algorithms , [HKNS96]).

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