Value Preserving Strategies and a General Framework for Local Approaches to Optimal Portfolios

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Abstract: We present some new general results on the existence and form of value preserving portfolio strategies in a general semimartingale setting. The concept of value preservation will be derived via a mean-variance argument. It will also be embedded into a framework for local approaches to the problem of portfolio optimisation.

1. Introduction

The main approach to portfolio optimisation is the expected utility approach. More precisely, in searching for optimal actions the investor has to decide how many shares of which security he has to hold at each time instant to maximise his expected utility of consumption and/or terminal wealth. In a continuous-time setting, one can roughly distinguish between two solution methods for the portfolio problem:

- the stochastic control approach of Merton (see Merton (1990) for a survey over Merton's work),
- the martingale approach of Cox and Huang (1989), Karatzas, Lehoczky and Shreve (1987) and Pliska (1986) (see Korn (1997a) for an introduction to the martingale approach).

In contrast to the expected utility approach the principle of value preservation underlies a non-utility based approach to economic decision making. It was introduced by Hellwig (1987) for valuing economic resources. Since then it has been applied to many economic areas including portfolio selection in discrete-time models (see e.g. Hellwig (1993), Wiesemann (1995)) and in continuous-time models (see Korn (1997b), (1998)).

Although value preservation stems from a totally different background, it has very close relations to the concepts of growth-optimum portfolios, numeraire portfolios and the minimal martingale measure. In a discrete-time setting one can consult Schäl (1995), Schäl (1998) and Korn and Schäl (1999) for surveys of these relations. Korn (1998) gives a complete description of the relation between the so-called value preserving martingale measure and the minimal martingale measure in a continuous-time market model with continuous asset prices.

In this paper, the value preserving approach is embedded into a general framework for local approaches to optimal portfolios. Further, we present some new general results on the existence and form of value preserving portfolio strategies in a general semimartingale setting.

In Section 2 we will give a brief review of the portfolio problem and introduce the market setting. Section 3 will contain the basic definitions of the concept of value preservation. Section 4 presents the main existence and uniqueness results for both the value preserving martingale measure and the value preserving portfolio strategy.

2. The market model and some basics of portfolio optimisation

We consider a securities market consisting of one bond with price $S_0(t) \equiv 1$ and d risky assets with prices $S(t) = (S_1(t), .., S_d(t))'$ traded continuously in time on a finite interval $[0,T]$. We assume the price processes of the risky assets to be special semimartingales with canonical decomposition

$$S_k(t) = S_k(0) + M_k(t) + A_k(t) \tag{1}$$

defined on a complete probability space (Ω, f, P), where $\{f_t\}_{t\in[0,T]}$ is a filtration satisfying the usual conditions, and where for simplicity we assume that f_0 contains only sets of P-measure zero or one. We further assume $f = f_T$.

Let

$$X(t) = \sum_{k=0}^{n} \varphi_k(t) S_k(t) \tag{2}$$

be the wealth process of an investor with trading strategy φ, and let U_1, U_2 be two utility functions. Then the usual procedure of portfolio optimisation consists of finding a pair (φ, c) of a trading strategy and a consumption process which maximises the expected utility of consumption over the trading interval $[0, T]$ and/or of terminal wealth at the time horizon T:

$$E\left(\int_0^T U_1(t, c(t))dt + U_2(X(T))\right). \tag{3}$$

The advantages of this approach are a well-developed theory for diffusion-type models (see e.g. Karatzas/Shreve (1998), Korn (1997a) or Merton (1990)) and its justification by decision theoretic arguments. The criticisms against this expected utility approach range from ethic arguments ("use of time-additive functionals prevents intergenerational justice" (see Hellwig (1993) for a discussion of this topic) over practical problems ("what utility function shall I choose ?") to theoretical shortcomings as there is not much work done in portfolio optimisation for the general semimartingale case. A very noteworthy exception to this last argument is work by Kramkov and Schachermayer (1997) which will also indirectly enter Section 4.

The main purpose of this paper is to present an alternative local approach to portfolio optimisation and in particular its special case, the approach of value preserving portfolio strategies. To do so, we first give a precise definition of trading strategies and consumption processes. Of course, we assume that our model is free of arbitrage and let Q be an equivalent martingale measure.

Definition 1
a) A (d+1)-dimensional predictable process $\varphi(t) = (\varphi_0(t), .., \varphi_d(t))'$ which is integrable with respect to $S(t) = (S_0(t),.., S_d(t))'$ is called a trading strategy with wealth process $X(t)$ given by equation (2).
b) An RCLL-semimartingale $C(t)$ with $C(0) = 0$ is named a (cumulative) consumption process.
c) A pair (φ, C) consisting of a trading strategy and a consumption process is called a self-financing pair, if the corresponding wealth process satisfies

$$X(t) \;=\; x + \int_0^t \varphi(s-)\,dS(s) \;-\; C(t).\text{ [1]} \tag{4}$$

d) A self-financing pair (φ,C) will be called admissible if for every equivalent martingale measure Q for S, $X(t)+C(t)$ is a Q-local martingales and a Q^*-supermartingale for at least one equivalent martingale measure Q^*.

Note that the consumption process in the above sense can get negative. This feature will turn out to be quite natural in the setting of the next section. The possibility of a signed consumption process forces some comments on the two last parts of Definition 1. First, on can argue that relation (4) does not characterise a self-financing strategy as $C(t)$ is a signed process and thus also contains parts similar to an additional endowment stream. However, in contrast to the situation of the local risk minimisation approach in Föllmer and Schweizer (1991), we will not look at pairs (φ, C) which are mean-self-financing. This would correspond to a consumption process having zero expectation. In portfolio optimisation one looks for increasing consumption process (in a sense that will be made precise later). Therefore, mean-self-financing pairs are not of interest. We will therefore not introduce a new terminology but slightly misuse the term "self-financing pair". Another comment concerns part d) of the definition. As we do not require non-negativity of consumption and have not further specified integrability conditions for the trading strategies, the required local martingale and supermartingale properties put bounds on consumption and the wealth process. They are the replacement of the usual non-negativity requirements on both the wealth and the consumption process and the square integrability of the trading strategies. Indeed, it is easy to show that this requirement excludes arbitrage opportunities from our market setting. It should also be noted that without any change of the following results, we could also require a non-negative wealth process.

3. Portfolio value and value preserving strategies

The main idea of our local framework for portfolio optimisation lies in a separation between valuing future payments and immediate returns which are both consequences of portfolio and consumption decisions. While the value of the future returns will be mirrored in the so-called portfolio value, the portfolio return process is related to the immediate returns.

Definition 2 "Portfolio value and portfolio return"
a) Let Q be an equivalent martingale measure for $S(t)$. Then

$$V_t^Q \;:= V_t^Q(\varphi,C) := \; E^Q\left(X(T) + (C(T)-C(t)) \,|\, f_t\right) \tag{5}$$

is called the **portfolio value** of (φ,C) at time t (wrt. Q) .
b) Let V_t^Q and (φ,C) and be as in a). Then the stochastic process R_t^Q with

[1] In $\int\varphi dS$ we always identify φ with its last d components as we have $dS_0 = 0$. In using vector stochastic integrals we use the same notation as in Schweizer (1995).

$$dR_t^Q := \frac{d\left(C(t)+V_t^Q\right)}{V_{t-}^Q}\ , \quad R_0^Q = 0 \tag{6}$$

is called the **(rate of) portfolio return** (of the strategy (φ,C) with respect to Q).

Note that the portfolio value process can simply be interpreted as the price of the random future payments $X(T)$ and $C(T)-C(t)$. It is of course motivated by option pricing theory. As the future payments $X(T)$ and $C(T)-C(t)$ are by definition attainable via following the strategy (φ,C), the portfolio value process is actually independent of the choice of Q. However, in later sections a unique equivalent martingale measure Q will be determined by requirements on both the portfolio value process and the portfolio return. Further, the definition of the portfolio return captures the balancing problem between the two competing aims of having a lot of consumption now (i.e. a big value of $dC(t)$) and of saving money for higher future payouts (i.e. a positive value of dV_t^Q).

It can easily be shown that in our setting with a constant bond price and no additional endowment stream the portfolio value cannot exceed the initial capital x. For cases with portfolio values different from the initial capital and a non-constant bond price consider Korn (1997b). This source also contains the necessary modification of the definition of the portfolio value in the case of a non-constant bond price.

With the help of the above definition our local approach to portfolio optimisation can be summarised by the following "algorithm":

Local approach to portfolio optimisation:

Step 1: Look for good choices of the portfolio value process ("development of the future ability over time") and the portfolio return ("allowance for immediate consumption")!

Step 2: Realize the requirements of step 1 via the choice of (φ,C) (and Q) !

Of course, the vague formulation of these steps leaves a lot of room for various suggestions on what a "good choice" of portfolio value and portfolio return processes could be. A natural requirement would be that of an increasing portfolio return process. Another goal could be to look for an increasing portfolio value process while abandoning consumption.

In this paper we will from now on focus on the case of value preserving strategies. Before we give a precise definition of a value preserving strategy, we try to explain why this is a desirable goal. In doing so, we take a different approach as in Korn (1997b, 1998). We start with:

Definition 3
An admissible pair (φ,C) is called **of constant value** (wrt. Q) if we have

$$V_t^Q(\varphi,C) = \text{const} \quad \forall\ t\in[0,T]. \tag{7}$$

Of course, pairs of constant value exist. A somewhat boring one is given by (φ,C) = $((x,0,...,0)', 0)$, but we will show that there are a lot more interesting ones. However, our first result will be a negative one. It is a natural aim to hope for a pair of constant value **and** an increasing consumption (or equivalently: an increasing portfolio return process), but there is only one such pair, the above boring one.

Proposition 4
Let $A(t)$ be an **increasing**, adapted RCLL process with $A(0)=0$. Let further (φ,C) be an admissible pair of constant value satisfying

$$R_t^Q = A(t), \qquad \forall\ t\in[0,T]. \tag{8}$$

Then we must have

$$V_t^Q(\varphi,C) = x,\ \ A(t) = 0 \quad \forall\ t\in[0,T] \tag{9}$$

and $(\varphi,C) = ((x,0,...,0)', 0)$.

Proof:
As (φ,C) is an admissible pair of constant value we have

$$X(T) = E^Q(X(T)\mid f_T) = V_T^Q(\varphi,C) = V_0^Q(\varphi,C) = E^Q(X(T) + C(T))$$

which implies $E^Q(C(T)) = 0$. As assumption (8) implies

$$dC(t) = V_0^Q(\varphi,C)\,dR_t^Q,$$

the vanishing expectation (with respect to Q) of the consumption yields

$$0 = R_t^Q = A(t) \ \text{ a.s. } \forall\ t\in[0,T].$$

Hence, we have

$$X(T) = V_T^Q(\varphi,C) = V_t^Q(\varphi,C) = E^Q(X(T)\mid f_t) = X(t) \quad \forall\ t\in[0,T],$$

which yields the remaining part of assertion (9) and the form of the trading strategy.

A pair of constant value with a strictly non-negative consumption process would have been an arbitrage opportunity. Hence, Proposition 4 is an expected result. As there is no non-trivial pair of constant value with an increasing portfolio return process, the next best thing would be to look for such a pair where the portfolio return process forms a submartingale (i.e. it is increasing in a mean value sense). Although that requirement seems to be fairly weak, it will add a lot of structure to the actual form of the portfolio return process.

Proposition 5
Let (φ,C) be a pair of constant value x with respect to Q such that

$$R_t^Q \ = M^Q(t) + A^Q(t) \tag{10}$$

is an RCLL **submartingale** (where (10) denotes the canonical Doob-Meyer decomposition of the portfolio return process with respect to P). Then:
a) R_t^Q is a Q-martingale.
b) In the (incomplete) "standard diffusion setting" let $Z(t)$ be the density process of Q with respect to P satisfying

$$dZ(t) = -Z(t)\,\theta^Q(t)\,'\,dW(t) \ \ , \ \ Z(0) = 1$$

for a suitable progressively measurable process $\theta^Q(t)$. Then there exists a unique progressively measurable (a.s. L^2) process $\xi(t)$ with

$$M^Q(t) + A^Q(t) = \int_0^t \xi(s)'dW(s) + \int_0^t \xi(s)'\theta^Q(s)ds \ . \tag{11}$$

Proof:
a) By assumption and in analogy to the proof of Proposition 4 we have $X(T) \ = x$. Together with assumption (10) this implies

$$\begin{aligned} x \ &= E^Q\,(X(T\,) + C(T) - C(t) \mid f_t) \\ &= E^Q\,(X(T)) + \ x(M^Q(T) + A^Q(T) - (M^Q(t) + A^Q(t)) \mid f_t) \\ &= x + x\,E^Q\,(M^Q(T) + A^Q(T) - (M^Q(t) + A^Q(t)) \mid f_t)\ . \end{aligned}$$

Hence, $M^Q(t) + A^Q(t)$ is a Q-martingale.
b) Note that under the above assumptions we have

$$dS_i(t) = S_i(t)\left(b_i(t)dt + \textstyle\sum_{j=1}^n \sigma_{ij}(t)dW_j(t)\right)$$

with an n-dimensional Brownian motion $W(t)$, $n \geq d$, and $\sigma(t)\sigma(t)'$ uniformly positive definite. Then due to part a) just proved and Itô's martingale representation theorem (with respect to Q !), there exists a unique progressively measurable (a.s. L^2) process $\xi(t)$ with

$$M^Q(t) + A^Q(t) = \int_0^t \xi(s)'dW^Q(s) \quad \forall\ t \in [0,T]$$

where now $W^Q(t)$ is a Q-Brownian motion given by

$$W^Q(t) = W(t) + \int_0^t \theta^Q(s)ds\ .$$

Here, $\theta^Q(t)$ is given by the density process $Z(t)$ of Q with respect to P satisfying

$$\begin{gathered} dZ(t) = -Z(t)\,\theta^Q(t)\,'\,dW(t) \ \ , \ \ Z(0) = 1 \\ \sigma(t)\ \theta^Q(t) \ = b(t) \quad \forall\ t \in [0,T]\ . \end{gathered}$$

The definition of $W^Q(t)$ and the above martingale representation lead to

$$M^Q(t) + A^Q(t) = \int_0^t \xi(s)'dW(s) + \int_0^t \xi(s)'\theta^Q(s)ds$$

which is the assertion to prove.

Note that part b) of the above proposition indicates a close connection between the aimed form of the portfolio return process and the equivalent martingale measure Q given by its density process $Z(t)$. Of course, for a given martingale measure Q, an arbitrary P-submartingale need not have the form (11). Also, a given P-submartingale need not have a Doob-Meyer decomposition of the form (11) for an arbitrary equivalent martingale measure Q.

As a consequence of part b) the diffusion coefficient $\theta^Q(t)$ enters the instantaneous mean (or return) of R_t^Q while it does not enter its instantaneous variance. As our attitude towards the future payments is already included in the requirement of the constant portfolio value process, we now have to specify the portfolio return process more closely to complete step 1 of the above "algorithm". For this, we take on a classical mean-variance view on the choice of the instantaneous mean rate $\xi(t)'\theta^Q(t)$ and the instantaneous variance $\xi(t)'\xi(t)$ of R_t^Q. More precisely, as for given Q we can only choose $\xi(t)$ (by requiring a special form of the portfolio return), we look for that parameter ξ which at time t solves the following mean-variance type problem:

$$\boxed{\begin{aligned} &\max_{\xi \in R^d} \xi' \theta^Q(t) \\ &\text{such that } \xi'\xi \le \theta^Q(t)'\theta^Q(t)\,. \end{aligned}} \tag{12}$$

The solution to this problem is easily seen to be

$$\xi = \theta^Q(t). \tag{13}$$

This corresponds to a myopic choice of the portfolio return. However, existence of this solution does not necessarily imply existence of an admissible pair (φ, C) with the desired return process.

Note that another choice for upper bound in problem (12) would simply lead to an optimal value for ξ which would be a positive multiple of $\theta^Q(t)$. Economically, this means that the relative weights **among** the risky assets remain unchanged, only the riskless bond gets more weight. Hence, in this sense the upper bound in (12) seems to be quite a general one. If we now indeed require a portfolio return process of the form (11) with an "optimal value" of ξ as in equation (13) then it satisfies the stochastic differential equation

$$dR_t^Q \;=\; Z(t)d\left(\frac{1}{Z(t)}\right) = \theta^Q(t)dW(t) + \left\|\theta^Q(t)\right\|^2 dt \tag{14}$$

where $\|.\|$ denotes the Euclidean norm.

We will use this equation as the basis for the requirement on the evolution of the portfolio return process in our definition of a value preserving strategy.

Definition 6
a) A pair (φ,C) of constant portfolio value with initial wealth x is called a **value preserving portfolio strategy** (wrt. Q) if it satisfies

$$V_t^Q(\varphi,C) = x \quad \forall\ t\in[0,T], \tag{15}$$

$$dR_t^Q = Z(t-)d\left(\frac{1}{Z(t)}\right) \tag{16}$$

where $Z(t)$ is the density process of Q with respect to P.
b) If there exists a value preserving portfolio strategy wrt. Q in the sense of a) then Q is called a **value preserving martingale measure**.

Up to now we hope to have indicated that a value preserving strategy is an attractive one. However, it is not at all clear that value preserving strategies really exist. Note in particular that unless P is already a martingale measure itself, the "boring strategy" of Proposition 4 is **not** a value preserving one. Thus, the question of existence of value preserving strategies has at least no obvious solution. We will devote the next section to the existence question and explicit calculation of value preserving strategies in the general semimartingale setting.

In Korn (1997b, 1998) requirement (16) was introduced without the mean-variance optimality considerations preceding Definition 6. There, it was mainly motivated by the fact that $Z(t)$ can be viewed as a "risk-adjusted discount factor" which is maximal in the sense of

$$E\big(Z(t)X(t)\big) \le x$$

for all non-negative wealth processes $X(t)$ corresponding to an admissible trading strategy φ and an initial wealth of x. In that sense it seems a desirable goal for the investor's wealth process to have the same evolution as $1/Z(t)$, the corresponding "risk-adjusted accumulation factor". As the rate of return of $1/Z(t)$ can then be looked at as a "random interest rate", a strategy with requirement (16) is called an "interest rate oriented" one in Korn (1997b, 1998).

A first consequence of Definition 6 can be obtained immediately and is already given in Korn (1997b):

Corollary 7
a) If (φ,C) is value preserving (wrt. Q) then $C(t)$ is given by

$$dC(t) = x\,\frac{1}{1/Z(t-)}d\left(\frac{1}{Z(t)}\right). \tag{17}$$

b) If there exists a value preserving martingale measure then the corresponding density process $Z(t)$ satisfies

$$0 = E^{Q}\Big(\int_t^T Z(s-) d\left(\frac{1}{Z(s)}\right) \Big| f_t \Big) \quad \text{a.s..} \tag{18}$$

i.e. it is a Q-martingale.

Thus, a value preserving strategy always contains a consumption process with a non-vanishing local martingale part. Further, it is uniquely determined by the value preserving martingale measure Q (if it exists at all !).

4. Existence of value preserving portfolio strategies

In this section we turn to the question of existence of value preserving strategies and their explicit form. We will first quote the main result of Korn (1998) which gives a (nearly) complete answer to that question for the case of continuous semimartingale asset prices. For the definition and for properties of the minimal martingale measure occurring in the following theorem see, e.g., Schweizer (1995).

Theorem 8

Let $S(t)$ be a continuous semimartingale with a bounded mean-variance trade-off process

$$\mathrm{K(t)} := \left\langle \int \lambda \, dM \right\rangle(t) \tag{19}$$

satisfying the following structure condition:

i) $\qquad A_i(.) \ll \langle M_i \rangle$.

(SC) ii) $\exists$ predictable, R^d-valued process $\lambda(.) \in L^2_{loc}(M)$ with $\Sigma(t)\lambda(t) = \eta(t)$
where for a fixed increasing process $D(.)$ with $D(0)=0$, $\langle M_i \rangle \ll D(.)$ we have

$$A_i(t) = \int_0^t \eta_i(s) dD(s) \; , \qquad \left\langle M_i, M_j \right\rangle(t) = \int_0^t \Sigma_{ij}(s) dD(s) \, .$$

Then we have:

a) The unique value preserving martingale measure Q for $S(.)$ coincides with the minimal martingale measure Q^* for $S(.)$.

b) The unique value preserving strategy (φ, C) is given by

$$(\varphi_1(t), \ldots, \varphi_d(t))' = x \, \lambda(t) \, , \tag{20}$$

$$\varphi_0(t) = x \left(1 - \sum_{i=1}^d \lambda_i(t) S_i(t) \right) , \tag{21}$$

$$C(t) = x \int_0^t \lambda(s)' dS(s) \, . \tag{22}$$

Popular applications of Theorem 8 are incomplete Black-Scholes models or general incomplete diffusion type models as given in Korn (1997b) (see also Proposition 5 b) for such incomplete diffusion models). In these cases, the processes $\lambda(.)$, $D(.)$, $\Sigma(.)$, and $\eta(.)$ are explicitly given by

$$D(t) = t,\ \ \eta(t) = \mathrm{diag}(S(t))\, b(t),\ \ \Sigma(t) = \mathrm{diag}(S(t))\sigma(t)\sigma(t)'\mathrm{diag}(S(t)),$$
$$\lambda(t) = (\mathrm{diag}(S(t)))^{-1}(\sigma(t)\sigma(t)')^{-1}b(t)$$

and the requirement $\Sigma(t)\lambda(t) = \eta(t)$ boils down to the existence of the inverse of $\sigma(t)\sigma(t)'$, i.e. to the requirement of $\sigma(t)$ having full rank. Then the value preserving strategy is given by

$$(\varphi_1(t)S_1(t),...,\varphi_d(t)S_d(t))' = x\ (\sigma(t)\sigma(t)')^{-1}b(t).$$

Note that this trading strategy coincides with the optimal one for the case of the negative exponential utility function (with the choice of a risk-aversion parameter of one !). **Also**, the corresponding portfolio process coincides with the log-optimal one under the requirement of a non-negative consumption ! The difference between these two problems and the value preserving strategy, however, lies in the way how the gains from trading are used, i.e. in the choice of the consumption process.

Theorem 8 states that in the case of continuous asset prices the value preserving and the minimal martingale measures are the same. However, this does **not** mean that the two concepts underlying the origins of these measures coincide. Both concepts are local concepts where the introduction of the minimal martingale measure was motivated by the approach of local risk minimisation for option hedging in incomplete markets (see Föllmer and Schweizer (1991)). The signed cost process in the local risk minimisation approach can be seen as an analogue to the signed consumption process in our local approach. This cost process takes into account the necessity of payment or extraction of additional amounts of money to deal with the non-hedgeable components of the contingent claim in the incomplete market setting. However, it must be pointed out that a non-zero cost process in the local risk-minimisation approach only occurs in incomplete markets while the signed consumption process in the value preservation approach is an intrinsic feature of this approach.

The structure condition (SC) looks very restrictive at first sight. However, as pointed out in Schweizer (1995) it is a mild form of a no arbitrage requirement and thus a very weak condition.

In the more general case of not necessarily continuous semimartingale prices we can make use of the relation between the numeraire portfolio and value preservation and can take advantage of some existence and uniqueness results for the numeraire portfolio. Most of these relations are already given in a discrete-time setting in Schäl (1995), Schäl (1998) and Korn and Schäl (1999). For this we first recall the notion of the numeraire portfolio.

Definition 9

A self-financing portfolio process $\pi(t)$ is called a numeraire portfolio if for the corresponding wealth process $X^{\pi}(t) > 0$ with representation

$$X^{\pi}(t) = 1 + \sum_{i=1}^{d}\int_0^t \theta_i(s-)dS_i(s) \tag{23}$$

the discounted asset price processes

$$\hat{S}_i(t) := \frac{S_i(t)}{X^{\pi}(t)}, \quad i = 0,..., d, \tag{24}$$

are martingales with respect to the original probability measure *P*.

The use of the wealth process of a numeraire portfolio as discount factor ("numeraire") circumvents the change of measure from *P* to an equivalent martingale measure *Q* for valuing contingent claims. With this discount factor, the discounted stock prices are already martingales with respect to the original measure *P*. However, it will turn out that a numeraire portfolio is always unique if it exists at all. Thus, the decision to use the wealth process of a numeraire portfolio as numeraire is already a decision about which equivalent martingale measure to choose in an incomplete market.

The notion of the numeraire portfolio was introduced in Long (1990). A very recent treatment of the numeraire portfolio in the general continuous-time semimartingale setting is given in Becherer (1999).

The following two theorems clarify the relation between the numeraire portfolio and the value preserving measure (and the value preserving strategy).

Theorem 10 "Numeraire portfolio $\Rightarrow$ value preserving strategy"
Let π(t) be a numeraire portfolio with wealth process given by relation (23). Then:

a)
$$\frac{dQ}{dP} = \frac{1}{X^{\pi}(T)} \tag{25}$$

defines an equivalent martingale measure for $S(.)$ on f_T.
b) Let *Q* be the equivalent martingale measure of a). If

$$Y(t) := \int_0^t \frac{1}{X^{\pi}(s-)} dX^{\pi}(s) \tag{26}$$

is a *Q*-martingale then a value preserving strategy (φ, C) is given by

$$\varphi_0(t) = x - \textstyle\sum_{i=1}^{d} \varphi_i(t) S_i(t) \,, \tag{27}$$

$$(\varphi_1(t),...,\varphi_d(t))' = \frac{x}{X^{\pi}(t)} \theta(t) \,, \tag{28}$$

$$C(t) = x\, Y(t) \,. \tag{29}$$

Proof:
a) The strict positivity of $X^{\pi}(t)$, the martingale property of $1/X^{\pi}(t)$ and in particular

$$E\left(1 \middle/ X^{\pi}(T) \right) = 1$$

imply that *Q* as defined in a) is a probability measure on f_T and equivalent to *P*. From this, the *Q*-martingale property of $\hat{S}(t)$ and the definition of *Q* we obtain

$$E^Q\big(S_i(t) \mid f_s \big) = E^Q\Big(\frac{S_i(t)}{X^{\pi}(T)} \mid f_s \Big) \Big/ E^Q\Big(\frac{1}{X^{\pi}(T)} \mid f_s \Big)$$

$$= E^{Q}\left(\hat{S}_i(t) \mid f_s\right) / E^{Q}\left(\frac{1}{X^{\pi}(T)} \mid f_s\right)$$

$$= \frac{\hat{S}_i(s)}{1/X^{\pi}(s)} = S_i(s) \qquad \forall\, 0 \le s \le t \le T\,.$$

b) Choose the pair (φ, C) according to (27)-(29). Then by construction the corresponding wealth process $X(t)$ satisfies:

$$X(t) = \sum_{i=0}^{d} \varphi_i(t) S_i(t) = x,$$

$$dX(t) = d(x) = 0 = x\left(\sum_{i=0}^{d} \frac{\theta_i(t-)}{X^{\pi}(t-)} dS_i(t) - \sum_{i=1}^{d} \frac{\theta_i(t-)}{X^{\pi}(t-)} dS_i(t)\right)$$

$$= \sum_{i=0}^{d} \varphi_i(t-) dS_i(t) - dC(t)\,.$$

Hence, (φ, C) is a self-financing trading strategy with a constant wealth process. Further, by construction and assumption (26) $X(t)$ and $C(t)$ are Q-martingales. Finally, we have

$$E^{Q}\left(X(T) + C(T) - C(t) \middle| f_t\right) = x \left(1 + E^{Q}\left(Y(T) - Y(t) \middle| f_t\right)\right) = x.$$

Therefore, (φ, C) is a value preserving portfolio strategy.

Thus, the question of existence of a value preserving strategy can be answered by the existence of a numeraire portfolio. One immediate corollary to Theorem 11 can therefore be drawn from Examples 1 and 2 of Becherer (1998) where the existence of a numeraire portfolio is proved in a general complete market setting and in the case where there are only finitely many values for the stock prices:

Corollary 11
Assume the notation of Theorem 10. Then:
a) In a complete market there exists a value preserving strategy given by (26)-(28).
b) Let $|\Omega| < \infty$ and assume that the market model is free of arbitrage. Then there exists a value preserving strategy given by (26)-(28).

In more general markets, Becherer (1998) contains existence results for a weaker notion of the numeraire portfolio. To use these results in particular cases one has to check if the "weak" numeraire portfolio is actually one in the sense of Definition 9. We can also show that both the value preserving strategy and the value preserving measure are unique:

Theorem 12 "Uniqueness"
a) A value preserving measure and the corresponding value preserving portfolio strategy are unique (if they exist at all !).

b) If a value preserving portfolio strategy exists then there exists a numeraire portfolio.

Proof:
Let us first prove part b) above. Therefore, b) let Q be the value preserving measure with density process $Z(t)$. Then $Z(t)$ is a strictly positive P-martingale with $Z(0) = 1$ such that $Z(t)S_i(t)$ are P-martingales for i=0,...,d. Define

$$X^{\pi}(t) := \frac{1}{Z(t)} .$$

Due to Corollary 7 we have

$$dC(t) = x\ Z(t-)d\left(\frac{1}{Z(t)}\right) ,$$

and as (φ,C) is a value preserving strategy the following equation holds:

$$dC(t) = \sum_{i=1}^{d} \varphi_i(t-)dS_i(t) .$$

By using these two identities and defining the process $\theta(t)$ via

$$\theta_0(t) = X^{\pi}(t) - \sum_{i=1}^{d} \theta_i(t)S_i(t)$$

$$(\theta_1(t),...,\theta_d(t))' = \frac{X^{\pi}(t)}{x}\left(\varphi_1(t),\ldots,\varphi_d(t)\right)$$

we obtain

$$\sum_{i=1}^{d} \theta_i(t-)dS_i(t) = \frac{X^{\pi}(t-)}{x}\sum_{i=1}^{d} \varphi_i(t-)dS_i(t)$$

$$= X^{\pi}(t-)\ Z(t-)d\left(\frac{1}{Z(t)}\right) = d\left(\frac{1}{Z(t)}\right) = d\,X^{\pi}(t) .$$

Hence, $X^{\pi}(t)$ is the wealth process of the self-financing trading strategy $\theta(t)$ or, equivalently, of the corresponding portfolio process. As further, $X^{\pi}(t)$ starts with initial value of 1, and as by construction of $X^{\pi}(t)$ the quotients

$$S_i(t) \big/ X^{\pi}(t) \qquad i=1,...,d$$

are P-martingales then $X^{\pi}(t)$ is the wealth process corresponding to a numeraire portfolio. Therefore, note also that $X^{\pi}(t)$ is a Q^*-local martingale for all equivalent martingale measures Q^* due to

$$dX^{\pi}(t) = dC(t) = X^{\pi}(t) \sum_{i=1}^{d} \varphi_i(t-)dS_i(t) .$$

As $X^{\pi}(t)$ is non-negative it is then also a Q^*-supermartingale. Hence´, part b) is proved. As for the proof of part a), note that it is well-known that a numeraire portfolio is unique (see e.g. Conze and Viswanathan (1991)). This can easily be seen in the following way. Let $\pi(t),\pi_1(t)$ be two numeraire portfolios with corresponding wealth processes $X(t)$, $X_1(t)$. Then by part a) of Theorem 10 both quotients $X(t)/X_1(t)$ and $X_1(t)/X(t)$ are non-negative local martingales, hence supermartingales. By the strict Jensen inequality this can only be the case if both quotients are almost surely equal to 1. Then, by the uniqueness of the numeraire portfolio and the way we have constructed it out of both the value preserving measure and value preserving portfolio strategy in the above proof of part b), the assertions of part a) follow.

Further worked out examples such as stochastic volatility models or hyperbolic models are left for a future paper.

5. Further Relations and Implications for Practical Use

The relations between the values preserving strategies, the minimal martingale measure and the numeraire portfolio are not the only relevant ones. In fact, there is also a close relation to the so-called growth optimal portfolio (i.e. the portfolio that maximizes the expected log-utility from terminal wealth simultaneously at all future time instants). This connection is examined in Korn and Schäl (1999) in a discrete-time framework.

Another aspect is to consider value preserving portfolios under constraints on both the portfolio and the wealth process. This topic is treated in Korn (1997b).

Other aspects for future research can be different requirements on the evolution of the portfolio value process than that of a constant one. There are a lot of possibilities that will fit into our local approach of Section 3. Some ideas are already given in Hellwig (1998). There, instead of value preserving strategies it is also looked at strategies with a sustaining value (i.e. strategies with a non-decreasing portfolio value process), a weakly sustaining value (i.e. strategies with a non-decreasing mean portfolio value process) and weakly value preserving strategies.

Finally, one should comment on the practical implications of value preserving strategies. A typical application could be the management of an investment fund. As the portfolio value process describes the evolution of the ability of the fund compared to the evolution of the market, the necessity for negative consumption has the following interpretation: If the fund performs better than the market then money can be taken out without losing the position of the fund in the market. If however the fund performs poorly, then additional money from the investors is needed to ensure the funds position compared to the performance of the market. Of course, the investors can refuse to invest additional money if they take into account a lower ranking of the fund compared to other market participants than the fund had before the necessity for negative consumption occurred.

References

Becherer, D. (1999) *The numeraire portfolio for unbounded semimartingales*, working paper, TU Berlin.

Conze A., Viswanathan (1991) *Probability measures and numeraires*, working paper.

Cox J., Huang C. (1989) *Optimum consumption and portfolio policies when asset prices follow a diffusion process*, J. Econ.Th. 49, 33-83.

Föllmer H., Schweizer M. (1991) *Hedging of contingent claims under incomplete information*, in: *Applied stochastic analysis* eds. M.H.A. Davis and R.J. Elliot, Gordon and Breach, London.

Hellwig K. (1987) *Bewertung von Ressourcen* , Physica Verlag, Heidelberg.

Hellwig K. (1993) *Intertemporal choice reexamined*, Discussion Papers in Economics, Univ. Ulm.

Hellwig K. (1998) *Creating value*, Discussion Papers in Economics, Univ. Ulm.

Karatzas I., Lehoczky J., Shreve S. (1987) *Optimal portfolio and consumption decisions for a small investor on a finite horizon*, SIAM J.Con.Opt. 25, 1557-1586.

Karatzas I., Shreve S. (1998) *Methods of Mathematical Finance*, Springer.

Korn R. (1997a) *Optimal Portfolios*, World Scientific, Singapore.

Korn R. (1997b) *Value preserving portfolio strategies in continuous-time models*, MMOR 45(1), 1-43.

Korn R. (1998) *Value preserving portfolio strategies and the minimal martingale measure*, MMOR 47(2), 169-179.

Korn R, Schäl M. (1999) *On value preserving and growth optimal portfolios*, to appear in MMOR 50(2).

Kramkov D., Schachermayer W. (1997) *The asymptotic elasticity of utility functions and optimal investment in incomplete markets*, working paper.

Long J. (1990) *The numeraire portfolio*, J. Finance 44, 205-209.

Merton R. (1990) *Continuous Time Finance*, Blackwell.

Pliska S. (1986) *A stochastic calculus model for continuous trading: optimal portfolios*, MOR 11, 371-382.

Schäl M. (1995) *A market numeraire for a jump process*, working paper, Univ. Bonn.

Schäl M. (1998) *On the numeraire portfolio and the interest-rate-oriented portfolio in discrete-time financial markets*, Proceedings of the GAMM-meeting 1998, Bremen.

Schweizer M. (1995) *On the minimal martingale measure and the Föllmer-Schweizer decomposition*, Stoch. Anal. Appl. 13, 573-599.

Wiesemann T. (1995) *Wertorientiertes Portfoliomanagement: Ein Modell zur intertemporalen Portfoliowerterhaltung* , Doctoral Dissertation, Univ. Ulm.