**Remarks on " COLORING RANDOM TRIANGULATION"**

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**Abstract**

We transform the two-matrix model, studied by P.Di Francesco and al. in [1], into a normal one-matrix model by identifying a "formal" integral used by these authors as a proper integral. We show also, using their method, that the results obtained f or the resolvent and the density are not reliable.

# 1 THE MODEL

In a recent paper P. Di Francesco, B. Eynard and E. Guitter [1] discuss a model of two $n \times n$ hermitean matrices $M$ and $R$ with a partition function

$$Z(p,q,g;N) = \int dMdR \exp\{-NTr[p\log(1-M) + q\log(1-R) + gMR]\} \tag{1}$$

where $g$ is later replaced by

$$g = \frac{1}{t} \tag{2}$$

and the strong coupling limit is of primary interest: $t \to 0$. Applying the Itzykson-Zuber integral identity, the partition function is transformed into an integral over the

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eigenvalues $\{m_i\}_{i=1,..n}$ and $\{r_i\}_{i=1,..n}$ of $M$ and $R$ respectively.

$$Z(p,q,g;N) \sim \int \Delta(m)\Delta(r) \prod_{i=1}^{n} [\exp\{-N[p\log(1-m_i) \\ +q\log(1-r_i) + g m_i r_i]\} dr_i dm_i] \tag{3}$$

where $\Delta$ is the Vandermonde determinant.

However, an integral such as

$$\int_{\mathbb{R}^2} dxdy \exp\{-N[p\log(1-x) + q\log(1-y) + gxy]\} p_n(x)\tilde{p}_m(y) \tag{4}$$

where $p_n(\tilde{p}_m)$ are polynomials of degree $n(m)$ does not exist.

In later parts of the paper the authors want to ascribe a meaning to such integrals by the "formal integral" (see eq. (4.2))

$$\int_{\mathbb{R}^2} dxdy\ x^\alpha y^\beta \exp[-\frac{N}{t}xy] \;=\; \alpha!\ \delta_{\alpha\beta}\ (\frac{t}{N})^{\alpha+1} \qquad (\alpha,\beta \in \mathbb{Z}_+) \tag{5}$$

Special cases of such "formal integral" are already used earlier in the text. It is, however, easy to see that the "formal integral" is a proper integral

$$2\int_{\mathbb{R}^2} dxdy\ w^\alpha \bar{w}^\beta \exp[-\frac{N}{t}w\bar{w}] \;=\; \alpha!\ \delta_{\alpha\beta}\ (\frac{t}{N})^{\alpha+1} \qquad (w = x+iy \quad , \quad \bar{w} = x - iy) \tag{6}$$

It is also easy to go back and replace all "formal integrals" by proper integrals, then for (3) we obtain

$$Z(p,q,g;N) \sim \int \mid \Delta(w) \mid^2 \prod_{i-1}^{n} [\exp\{-N[p\log(1-w_i) \\ +q\log(1-\bar{w}_i) + g w_i \bar{w}_i]\} dx_i dy_i] \tag{7}$$

where we have to restrict the $w_i$ to, say

$$0 \leq\mid w_i \mid\leq 1 \tag{8}$$

and

$$Re(Np) < 1, Re(Nq) < 1 \tag{9}$$

if we want to give an analytic meaning to Z in the variable $t = g^{-1}$, but we can integrate over the whole complex plane if we are interested only in the formal power expansion in $t$ for $t \to 0$. Finally, without referring to the Itzykson-Zuber formula we can integrate over $U(n)$ to get

$$Z(p,q,g;N) = 2^n \int dMdM^\dagger \exp\{-NTr[p\log(1-M) + q\log(1-M^\dagger) + gMM^\dagger]\} \tag{10}$$

where $M$ is a normal matrix

$$[M, M^\dagger] = 0 \tag{11}$$

Thus the model actually evaluated is not a hermitean two-matrix model but a normal one-matrix model.

## 2 THE SADDLE POINT EQUATION

The partition function is calculated in the limit $N \to \infty$ so that $n = zN$ and $z$ is kept fixed. In this limit the partition function is approximated by (see eq.(4.20) of [1])

$$Z = N^n \int_0^\infty d\alpha_i ... d\alpha_n \exp\{N^2 S(\alpha_1..\alpha_n, p, q, z)\} \tag{12}$$

with

$$\begin{aligned} S = \frac{1}{N}\sum_{i=1}^{n} & \Big[ \quad (\alpha_i + p - z)(\log(\alpha_i + p - z) - 1) \\ & + \quad (\alpha_i + q - z)(\log(\alpha_i + q - z) - 1) - 2\alpha_i(\log(\alpha_i) - 1) \\ & + \quad \frac{1}{N}\log(\gamma(\alpha_i N + 1, \frac{N}{t})) + \alpha_i \log(\frac{t}{N})\Big] \\ & + \quad \frac{1}{N^2}\sum_{i\neq j}\log(\mid \alpha_i - \alpha_j \mid) \\ & - \quad \int_0^z ds[(p-s)(\log(p-s) - 1) + (q-s)(\log(q-s) - 1) \\ & \quad\; + s(\log(s) - 1) + s\log(t)] \end{aligned} \tag{13}$$

where $\gamma(\alpha, \xi)$ is the incomplete $\gamma$-function [2]. In the limit $N \to \infty$ the term containing the incomplete $\gamma$-function is calculated as

$$\lim_{N\to\infty}\Big\{-\frac{t}{N}\log(\gamma(\xi\frac{N}{t} + 1, \frac{N}{t})) - \xi\log(\frac{t}{N})\Big\} = \Theta(\xi - 1) + \xi(1 - \log(\xi))\Theta(1 - \xi) \tag{14}$$

where $\xi = \alpha t$ and so that the derivative of the r.h.s is continuous at $\xi = 1$. The saddle point equation is then

$$\begin{aligned} 2P\int d\beta\frac{\rho(\beta)}{\beta - \alpha} &= \Theta(1 - \alpha t)\log\Big(t\frac{(\alpha + p - z)(\alpha + q - z)}{\alpha}\Big) \\ &+ \Theta(\alpha t - 1)\log\Big(\frac{(\alpha + p - z)(\alpha + q - z)}{\alpha^2}\Big) \end{aligned} \tag{15}$$

valid for $\alpha$ in the support of the density $\rho$ and with the normalisation

$$\int d\beta\rho(\beta) = z \tag{16}$$

We mention that the free energy also has an additional term compared with the expression obtained in the equation (4.29) of [1]

$$t\partial_t f = \int_0^\infty d\beta\beta\rho(\beta) - \frac{1}{2}z^2 + \frac{1}{t}\int_{t^{-1}}^\infty d\beta\rho(\beta) \tag{17}$$

We solve the saddle point equation by making the ansatz

$$\rho(\alpha) = \rho_1(\alpha) + \rho_2(\alpha) \tag{18}$$

$$\rho_i(\alpha) \geq 0 \quad , \quad \int d\beta \rho_i(\beta) = z_i \tag{19}$$

$$z_1 + z_2 = z \tag{20}$$

and

$$supp(\rho_1) = <\gamma_1, \gamma_2> \tag{21}$$

$$supp(\rho_2) = <\gamma_3, \gamma_4> \tag{22}$$

In order to solve (15) we have to make sure that

$$\gamma_2 \leq \frac{1}{t} \leq \gamma_3 \tag{23}$$

Next we introduce the resolvent

$$\omega_i(\alpha) = \int d\beta \frac{\rho_i(\beta)}{\alpha - \beta} \tag{24}$$

so that (15) induces

$$\omega_1(\alpha + i0) + \omega_1(\alpha - i0) = -\log\Big(t\frac{(\alpha + p - z)(\alpha + q - z)}{\alpha}\Big) \tag{25}$$

$$\omega_2(\alpha + i0) + \omega_2(\alpha - i0) = -\log\Big(\frac{(\alpha + p - z)(\alpha + q - z)}{\alpha^2}\Big) \tag{26}$$

and

$$\lim_{\alpha \to \infty} \alpha \omega_i(\alpha) = \int d\beta \rho_i(\beta) = z_i \tag{27}$$

We shall use the method of [1], and show that their result for $\omega_1, \rho_1$ is not reliable, in spite of the fact that the saddle point equation (25) is the same as the one obtained in [1]. For simplicity we set $z_1 = z, z_2 = 0$ from now on.

We introduce two parameters $r, \delta$ and three hyperbolic angles (assumed to be all $\geq 0)\Phi_1, \Phi_2, \Phi_3$ by

$$\alpha = z - r - 2\delta\cosh(\Phi) \tag{28}$$

$$p = r + 2\delta\cosh(\Phi_1) \tag{29}$$

$$q = r + 2\delta\cosh(\Phi_2) \tag{30}$$

$$z = r + 2\delta\cosh(\Phi_3) \tag{31}$$

Then

$$\frac{t}{\alpha}(\alpha + p - z)(\alpha + q - z) = \delta t \frac{4(T_1^2 - T^2)(T_2^2 - T^2)(1 - T_3^2)}{(1 - T_1^2)(1 - T_2^2)(1 - T^2)(T_3^2 - T^2)} \tag{32}$$

where

$$T = \tanh(\frac{\Phi}{2}) \tag{33}$$

$$T_i = \tanh(\frac{\Phi_i}{2}) \geq 0 \tag{34}$$

with

$$\gamma_1 = z - r - 2\delta \ , \ \ \gamma_2 = z - r + 2\delta$$

Along the cut $< \gamma_1, \gamma_2 >$ the parameter $\Phi$ is assumed to vary as $+i\varphi$ $(0 \leq \varphi \leq \pi)$ for $Im(\alpha) \searrow 0$ and as $-i\varphi$ for $Im(\alpha) \nearrow 0$. On the real axis outside the cut, $\Phi$ is necessarily so that

$$T = \tanh(\frac{\Phi}{2}) \to -1 \quad \text{for} \quad \alpha \to \pm\infty \tag{35}$$

Taking into account (34) and (35) only one ansatz for $\omega_1(\alpha)$ is possible which has only the cut at $< \gamma_1, \gamma_2 >$, namely

$$\omega_1(\alpha) = -\log\Big(\frac{2(T_1 - T)(T_2 - T)(1 + T_3)}{(1 - T)(1 + T_1)(1 + T_2)(T_3 - T)}\Big) \tag{36}$$

provided we take

$$\begin{aligned} \delta t &= \frac{(1 - T_1)(1 - T_2)(1 + T_3)}{(1 + T_1)(1 + T_2)(1 - T_3)} \\ &= u_1 u_2 u_3^{-1} \end{aligned} \tag{37}$$

where

$$\begin{aligned} u_i &= \frac{1 - T_i}{1 + T_i} = e^{-\Phi_i} \\ u &= \frac{1 - T}{1 + T} = e^{-\Phi} \end{aligned}$$

Finally we obtain for the density

$$\rho_1(\alpha) = \frac{1}{2\pi}\Big(\varphi - 2(\Psi_1(\varphi) + \Psi_2(\varphi) - \Psi_3(\varphi))\Big) \tag{38}$$

where

$$\Psi(\alpha) = \arctan(\frac{\tan(\frac{\varphi}{2})}{T_i})$$

We can show that

$$\begin{aligned} z &= \int \rho_1(\beta) d\beta \\ &= \int_0^{\pi} d\varphi 2\delta \sin(\varphi) \rho_1(\beta(\varphi)) \\ &= \delta[u_3 - u_1 - u_2] \\ &= \lim_{\alpha \to +\infty} \alpha\omega_1(\alpha) \end{aligned} \tag{39}$$

as desired.

Positivity of the density function (38) is achieved if and only if

$$T_1 + T_2 - T_3 - 1 \geq 0 \tag{40}$$

This constraint follows from the observation that the neighborhood of $\pi$ in the variable $\varphi$ is critical, i.e. negative values of $\rho_1$ appear here first. Namely, if we write

$$\frac{\varphi}{2} = \frac{\pi}{2} - \chi \tag{41}$$

we have then

$$\Psi_i(\varphi) = \frac{\pi}{2} - T_i\chi + O(\chi^3) \tag{42}$$

and consequently

$$\rho_1(\alpha) = \frac{1}{\pi}(T_1 + T_2 - T_3 - 1)\chi + O(\chi^3) \tag{43}$$

# References


[1] P. Di Francesco, B. Eynard and E. Guitter, *Coloring Random Triangulation* , cond-mat/9711050

[2] M.Abramowitz and I.A.Stegun, *Handbook of Mathematical Functions*, Dover publications, INC., New York.